REPRESENTING FINITE CONVEX GEOMETRIES BY RELATIVELY CONVEX SETS

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Abstract. A closure system with the anti-exchange axiom is called a convex geometry. One geometry is called a sub-geometry of the other if its closed sets form a sublattice in the lattice of closed sets of the other. We prove that convex geometries of relatively convex sets in n-dimensional vector space and their finite sub-geometries satisfy the n-Carousel Rule, which is the strengthening of the n-Carathéodory property. We also find another property, that is similar to the simplex partition property and does not follow from 2-Carusel Rule, which holds in sub-geometries of 2-dimensional geometries of relatively convex sets.

1. Introduction

A closure system $\mathbf{A} = (A, -)$, i.e. a set A with a closure operator $- : 2^A \to 2^A$ defined on A, is called *a convex geometry* (see [3]), if it is a zero-closed space (i.e. $\overline{\emptyset} = \emptyset$) and it satisfies *the anti-exchange axiom*, i.e.

$$x \in \overline{X \cup \{y\}} \text{ and } x \notin X \text{ imply that } y \notin \overline{X \cup \{x\}}$$
$$\text{for all } x \neq y \text{ in } A \text{ and all closed } X \subseteq A.$$

A convex geometry $\mathbf{A} = (A, -)$ is called finite, if set A is finite.

Very often, a convex geometry can be represented by its collection of closed sets. There is a convenient description of those collections of subsets of a given finite set A, which are, in fact, the closed sets of a convex geometry on A: if $\mathcal{A} \subseteq 2^A$ satisfies
(1) $\emptyset \in \mathcal{A}$;
(2) $X \cap Y \in \mathcal{A}$, as soon as $X, Y \in \mathcal{A}$;
(3) $X \in \mathcal{A}$ and $X \neq A$ implies $X \cup \{a\} \in \mathcal{A}$, for some $a \in A \setminus X$,
then $\mathcal{A}$ represents the collection of closed sets of a convex geometry $\mathbf{A} = (A, \mathcal{A})$.

A reader can be referred to [8],[9] for the further details of combinatorial and lattice-theoretical aspects of finite convex geometries.

For convex geometries $\mathbf{A} = (A, -)$ and $\mathbf{B} = (B, \tau)$, one says that $\mathbf{A}$ is a sub-geometry of $\mathbf{B}$, if there is a one-to-one map ϕ of closed sets of $\mathbf{A}$ to closed sets of $\mathbf{B}$ such that $\phi(X \cap Y) = \phi(X) \cap \phi(Y)$, and $\phi(\overline{X \cup Y}) = \tau(\phi(X) \cup \phi(Y))$, where $X, Y \subseteq A$, $\overline{X} = X$, $\overline{Y} = Y$. In other words, the lattice of closed subsets of $\mathbf{A}$ is a sublattice of the lattice of closed sets of $\mathbf{B}$. When geometries $\mathbf{A}$ and $\mathbf{B}$ are defined on the same set $X = A = B$, we also call $\mathbf{B}$ *a strong extension* of $\mathbf{A}$. Extensions of finite convex geometries were considered in [4] and [3], the more systematic treatment of extensions of finite lattices was given in [15].

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Given any class $\mathcal{L}$ of convex geometries, we will call it *universal*, if an arbitrary finite convex geometry is a sub-geometry of some geometry in $\mathcal{L}$.

One of main results in [3] proves that a specially designed class of convex geometries $\mathcal{AL}$ is universal. Namely, $\mathcal{AL}$ consists of convex geometries of the form $Sp(A)$, each of which is built on a carrier set of an algebraic and dually algebraic lattice A and whose closed sets are all complete lower subsemilattices of A closed with respect to taking joins of non-empty chains. At the same time, a subclass of all *finite* geometries from class $\mathcal{AL}$ cease to be universal, see [2] and [3].

In this paper, we want to consider another conveniently designed class of convex geometries, in fact, even an infinite hierarchy of classes.

Given a set of points A in Euclidean n-dimensional space $\mathbb{R}^n$, one defines a closure operator $- : 2^A \to 2^A$ on A as follows: for any $Y \subseteq A$, $\overline{Y} = ch(Y) \cap A$, where ch stands for *the convex hull*. One easily verifies that such an operator satisfies the anti-exchange axiom. Thus, $(A, -)$ is a convex geometry, which also will be denoted as $\mathbf{C}o(\mathbb{R}^n, A)$. We will call such convex geometry *a geometry of relatively convex sets* (assuming that these are convex sets "relative" to A). The convex geometries of relatively convex sets were studied in [13],[7] and [1].

For any geometry $C = \mathbf{C}o(\mathbb{R}^m, A)$, we will call $n \in \mathbb{N}$ *a dimension* of C, if n is the smallest number such that C could be represented as $\mathbf{C}o(\mathbb{R}^n, A)$, for appropriate $A \subseteq R^n$. In particular, $n \leq m$, and $n \leq p - 1$, if A is a finite non-empty set of cardinality $p > 1$.

Let $\mathcal{C}_n$ be the class of convex geometries of relatively convex sets of dimension $\leq n$, and let $\mathcal{C}$ be the the class of of all convex geometries of relatively convex sets of finite dimension (thus, including $\mathcal{C}_n$, $n \in \mathbb{N}$, as subclasses). By $\mathcal{C}_B$ we denote a subclass of $\mathcal{C}$ that consists of geometries of convex sets relative to bounded sets, i.e. $\mathbf{C}o(\mathbb{R}^n, A)$, for some n and $A \subseteq B$, where B is a ball in R^n. By $\mathcal{C}_f$ we denote a subclass if *finite* convex geometries in $\mathcal{C}$.

It is known that none of $\mathcal{C}_n$ is universal, due to the n-Carathéodory property that holds on any sub-geometry of geometry from $\mathcal{C}_n$ (see, for example, [7]), but fails on any geometry of dimension $n + 1$. We introduce a stronger property called the n-Carousel Rule and show that it holds on sub-geometries of $\mathcal{C}_n$. It allows to build a series of finite convex geometries C_n such that C_n satisfies the n-Carathéodory property, but cannot be a sub-geometry of any geometry in $\mathcal{C}_n$. On the other hand, C_n is a sub-geometry of some geometry in $\mathcal{C}_{n+1}$. We also prove that the so-called Sharp Carousel Rule holds in all sub-geometries in $\mathcal{C}_2$, a slight modification of *the simplex partition property* from [14].

It was shown in [7] that *every finite* closure system can be embedded into some geometry in the class $\mathcal{C}$, in particular, this class is universal for all finite convex geometries. This observation is a direct consequence of deep and complex result proved in [16] that every finite lattice is a sublattice of a finite partition lattice. Thus, class $\mathcal{C}$ can not be considered as specific to finite convex geometries. It is worth noting that the construction in [7] uses convex sets relative to A which is the collection of lines, in particular, A is always an unbounded set.

This leaves the following open questions:

Problem 1.1. *Is class $\mathcal{C}_B$ of geometries of convex sets relative to bounded sets universal? Is the class $\mathcal{C}_f$ of finite geometries of relatively convex sets universal?*

Note that the second question of two is a modification of Problem 3 from [3].

2. Carathéodory Property and Carousel Rule

We recall that a convex geometry $(A, -)$ satisfies the *n-Carathéodory property*, if $x \in \overline{S}$, $S \subseteq A$, implies $x \in \overline{\{a_0, \dots, a_n\}}$ for some $a_0, \dots, a_n \in S$. Besides, a_0 can be taken to be any pre-specified element of S.

Proposition 2.1. ([13, Lemma 3.2],[7, Proposition 25]) *For any $n \in \mathbb{N}$ and $A \subseteq \mathbb{R}^n$, convex geometry $\mathbf{C}o(\mathbb{R}^n, A)$ satisfies the n-Carathéodory property.*

Our aim is to formulate a stronger property, which we call the *n-Carousel Rule*, extending to arbitrary finite dimensions the 2-Carousel Rule introduced in [5].

Definition 2.2. A convex geometry $(A, -)$ satisfies the n-Carousel Rule, if $x, y \in \overline{S}$, $S \subseteq A$, implies $x \in \overline{\{y, a_1, \dots, a_n\}}$ for some $a_1, \dots, a_n \in S$.

Note that the n-Carathéodory property follows from the n-Carousel Rule. Indeed, if y is chosen among elements of S, and $x \in \overline{S}$, then, according to the n-Carousel Rule, $x \in \overline{\{y, a_1, \dots, a_n\}}$ for some $a_1, \dots, a_n \in S$, which is also a desired conclusion for the n-Carathéodory property.

Lemma 2.3. *For any $n \in \mathbb{N}$ and $A \subseteq R^n$, convex geometry $\mathbf{C}o(\mathbb{R}^n, A)$ satisfies the n-Carousel Rule.*

Proof. Consider $\mathbf{G} = \mathbf{C}o(\mathbb{R}^n, A)$, and let $x, y \in \overline{S}$, for some $S \subseteq A$.

Due to the n-Carathéodory property, $x \in \overline{\{c_0, c_1, \dots, c_n\}}$ and $y \in \overline{\{b_0, b_1, \dots, b_n\}}$ for some $c_0, b_0, \dots, c_n, b_n \in S$. In other words, points x, y belong to a convex polytope P in R^n with the vertices among $c_0, b_0, \dots, c_n, b_n$. Suppose $F_1, \dots, F_k$ are the faces of this polytope, i.e. they are at most $(n-1)$-dimensional convex polytopes. For arbitrary $y \in P$, we have $P \subseteq \bigcup_{i \leq k} P_i$, where $P_i = ch(y \cup F_i)$, $i = 1, \dots, k$. Hence, $x \in \overline{y \cup F_i}$ for some $i \leq k$. Now, due to the n-Carathéodory property, $x \in \overline{\{y, f_1, \dots, f_n\}}$ for some vertices $f_1, \dots, f_n$ of F_i, which are also elements of S. Thus, the conclusion of the n-Carousel Rule holds. □

Our next goal is to show that the n-Carousel Rule is preserved on finite subgeometries.

Lemma 2.4. *If geometry $\mathbf{H}$ satisfies the n-Carousel Rule, and $\mathbf{G}$ is a finite subgeometry of $\mathbf{H}$, then $\mathbf{G}$ satisfies the n-Carousel Rule.*

Proof. Suppose $\mathbf{H} = (H, -)$, $\mathbf{G} = (G, \tau)$ and ϕ is a one-to-one mapping from closed sets of $\mathbf{G}$ to closed sets of $\mathbf{H}$ that preserves the intersection and the closure of finite unions of sets.

Let assume that $\mathbf{G}$ does not satisfy the n-Carousel Rule. It means that, for some $x, y \in G$ and $S \subseteq G$, we have $x, y \in \overline{S}$, but $x \notin \overline{\{y, s_1, \dots, s_n\}}$, for any $s_1, \dots, s_n \in S$. In any finite convex geometry $(A, -)$, for any $a \in A$, the subset $\overline{a} \setminus a$ is closed. Hence, set $X = \overline{x} \setminus x$ is closed in $\mathbf{G}$. According to our assumption, $\overline{x} \cap \overline{\overline{y} \cup \overline{s_1} \cup \dots \cup \overline{s_n}} \subseteq X$, for any $s_1, \dots, s_n \in S$.

Take $x' \in \phi(\overline{x}) \setminus \phi(X)$ and $y' \in \phi(\overline{y})$. Note that $\overline{S} = \bigcup\{\overline{s} : s \in S\}$, hence $S' = \phi(\overline{S}) = \tau(\bigcup(\phi(\overline{s}) : s \in S))$. Since $x', y' \in S' = \tau(\bigcup(\phi(\overline{s}) : s \in S))$ and $\mathbf{H}$ satisfies the n-Carousel Rule, we have $x' \in \tau(\{y', s'_1, \dots, s'_n\})$, for some $s'_i \in \phi(\overline{s_i})$, $s_i \in S$. It follows $x' \in \phi(\overline{x}) \cap \tau(\phi(\overline{y}) \cup \phi(\overline{s_1}) \cup \dots \cup \phi(\overline{s_n})) = \phi(\overline{x}) \cap \phi(\overline{\overline{y} \cup \overline{s_1} \cup \dots \cup \overline{s_n}})$, which means $\phi(\overline{x} \cap \overline{\overline{y} \cup \overline{s_1} \cup \dots \cup \overline{s_n}}) \not\subseteq \phi(X)$, a contradiction. □

3. Convex geometries C_n

Using the n-Carousel Rule, it will not be difficult to built an example of a finite convex geometry that cannot be a sub-geometry of relatively convex sets of dimension $\leq n$.

Consider a point configuration in $\mathbb{R}^n$ that consists of extreme points $a_0, \dots, a_n$, equivalently, the vertices of a n-dimensional polytope P, and inner points x, y of P. Besides, choose x, y so that x belongs to only one of polytopes $P_i = ch(\{y\} \cup D \setminus a_i)$ and y belongs to only one of polytopes $Q_j = ch(\{x\} \cup D \setminus a_j)$, where $D = \{a_0, \dots, a_n\}$, $i, j \leq n$.

Let $\mathbf{D}_n = \mathbf{C}o(\mathbb{R}^n, D \cup \{x, y\})$.

According to our assumption, $\{y\} \cup D \setminus a_i$ and $\{x\} \cup D \setminus a_j$ are not closed sets in convex geometry D_n, for some unique $i, j \leq n, i \neq j$.

Consider closure space $\mathbf{C}_n = (D \cup \{x, y\}, \mathcal{D})$, where a family of closed sets $\mathcal{D}$ is defined as a collection of all closed sets of convex geometry $\mathbf{D}_n$, plus sets $\{y\} \cup D \setminus a_i$ and $\{x\} \cup D \setminus a_j$. These are, indeed, the closed sets of a closure operator, since the intersection of any members of $\mathcal{D}$ is again in $\mathcal{D}$. For this, it is enough to note that any subset of $\{y\} \cup D \setminus a_i$ and $\{x\} \cup D \setminus a_j$ is a closed set of convex geometry $\mathbf{D}_n$. We can claim more, namely:

Lemma 3.1. *$\mathbf{C}_n$ is a (finite) convex geometry that satisfies the n-Carathéodory property.*

Proof. To show that $\mathbf{C}_n$ is a convex geometry, one needs to demonstrate that every closed set can be extended by one point to obtain another closed set. This is true for any closed set of $\mathbf{D}_n$, since it is a convex geometry itself. This is also true for additional sets $\{y\} \cup D \setminus a_i$ and $\{x\} \cup D \setminus a_j$: the first can be extended by x to obtain $\{x, y\} \cup D \setminus a_i$, a closed set of $\mathbf{D}_n$, the second can be extended by y to obtain $\{x, y\} \cup D \setminus a_j$, another closed set of $\mathbf{D}_n$. $\square$

Lemma 3.2. *$\mathbf{C}_n$ cannot be a sub-geometry of any geometry of relatively convex sets of dimension $\leq n$.*

Proof. Indeed, $\mathbf{C}_n$ does not satisfy the n-Carousel Rule, since x is not in a closure of y with any n points from D (similarly, y is not in a closure of x with any n points from D). Hence, the claim of this lemma follows from 2.3 and 2.4. $\square$

On the other hand, we can show that $\mathbf{C}_n$ is a sub-geometry of some $(n+1)$-dimensional geometry of relatively convex sets. Indeed, consider $\mathbb{R}^{n+1}$, and subspace $S_0 \subseteq \mathbb{R}^{n+1}$ of all points whose last projection is 0; correspondingly, let $S_1 \subseteq \mathbb{R}^{n+1}$ be a subspace of all points whose last projection is 1. Consider points $c_0, c_1, \dots, c_n \in S_0$ whose convex hull is n-dimensional polytope C, and take an inner point u of C. Let $b_0, b_1, \dots, b_n, v \in S_1$ be obtained from $c_0, c_1, \dots, c_n, u$, correspondingly, by replacing the last projection by 1. Let $K = \{c_0, b_0, c_1, b_1, \dots, c_n, b_n, u, v\}$ and $\mathbf{G}_{n+1} = \mathbf{C}o(\mathbb{R}^{n+1}, K)$.

Define a mapping ϕ from closed sets of $\mathbf{C}_n$ to closed sets of $\mathbf{G}_{n+1}$: $\phi(\{a_i\}) = \{c_i, b_i\}$, $i = 0, \dots, n$, $\phi(\{x\}) = \{u\}$, $\phi(\{y\}) = \{v\}$. For any closed set $S = \{s_1, \dots, s_k\}$, $k > 1$, of $\mathbf{C}_n$, it is straightforward to check that $\phi(s_1) \cup \dots \cup \phi(s_k)$ is closed in $\mathbf{G}_{n+1}$, thus, we may define $\phi(S) = \phi(s_1) \cup \dots \cup \phi(s_k)$, for any closed S in $\mathbf{C}_n$. Evidently, this mapping preserves intersections. As for the closure of a union of closed sets X, Y in $\mathbf{C}_n$, we observe that

$$\overline{X \cup Y} = \begin{cases} X \cup Y \cup \{x, y\}, & a_0, \dots, a_n \in X \cup Y; \\ \\ X \cup Y, & \text{otherwise.} \end{cases}$$

Similarly, in $\mathbf{G}_{n+1}$, u (symmetrically, v) is not in a closure of v (u) with any n sets $\{c_i, b_i\}$, $i = 0, \dots, n$, since u (v) is an inner point of n-dimensional polytope with vertices $c_0, \dots, c_n$ ($b_0, \dots, b_n$). Hence, we have in $\mathbf{G}_{n+1}$

$$\overline{\phi(X) \cup \phi(Y)} = \begin{cases} \phi(X) \cup \phi(Y) \cup \{u, v\}, & a_0, \dots, a_n \in X \cup Y; \\ \\ \phi(X) \cup \phi(Y), & \text{otherwise.} \end{cases}$$

Therefore, ϕ preserves the closure of the union of closed sets, too.

4. Sharpening Carusel Rule

It turns our that one can slightly strengthen the n-Carusel Rule, and we are going to illustrate it in case of 2-Carusel Rule.

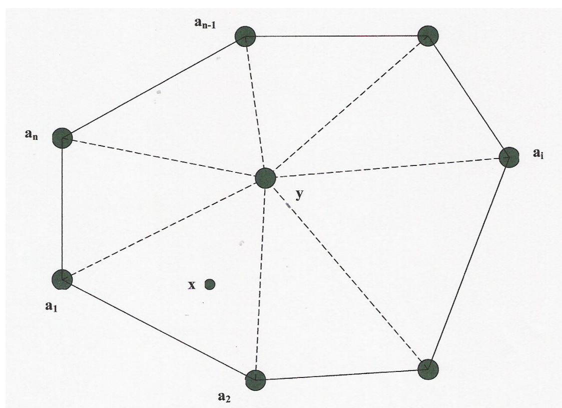

Figure 1.

First of all, let see the visual image of 2-Carusel Rule on Figure 1: if x, y are in the convex polygone generated by $a_1, \dots, a_n$, then x should be at least in one triangle generated by y and two points from $a_1, \dots, a_n$. In general, there might be multiple triangles of that sort containing x. On the other hand, if $n = 3$, i.e. x, y are inside the triangle defined by a_1, a_2, a_3, x can belong to maximum two triangles. In this case, x will be also on the segment containing y and one of points a_1, a_2, a_3. Indeed, if, say, $x \in \overline{\{y, a_1, a_2\}}$ and $x \in \overline{\{y, a_1, a_3\}}$, then $x \in \overline{\{y, a_1\}}$. Note that the property will hold even if y belongs to the boundary of triangle a_1, a_2, a_3.

The version of this property under additional assumption that the points on the plane are in the *general position*, i.e. no three of them are on the same line, is called *the simplex partition property* in [14]. In this case, one would say that x can be in exactly one of triangles $\overline{\{y, a_i, a_j\}}$, $i, j \in \{1, 2, 3\}$.

It turns out we can make the similar statement in any sub-geometry of 2-dimensional geometry, as long as we assume that y is not on the boundary of a_1, a_2, a_3.

Theorem 4.1. *Let $\mathbf{G} = (G, -)$ be any sub-geometry of 2-dimensional finite geometry $\mathbf{G}_0 = \mathbf{C}o(\mathbb{R}^2, G_0)$. Then the following implication holds for all x, y, a, b, c in G: if $y \in \overline{\{a, b, c\}}$, $\overline{y} \cap \overline{\{a, b\}} = \overline{y} \cap \overline{\{b, c\}} = \overline{y} \cap \overline{\{a, c\}} = \overline{y} \cap \overline{x} = \emptyset$, $x \in \overline{\{y, a, b\}}$ and $x \in \overline{\{y, a, c\}}$, then $\overline{x} \cap \overline{\{y, a\}} > \emptyset$.*

To prove Theorem we will need a few auxiliary statements.

Lemma 4.2. *Let $a_1, \dots, a_i, \dots, a_j, \dots, a_k, \dots, a_s, \dots, a_n$ be a circular order of vertices of some convex polygon on the plane. If s is a point of intersection of segments $[a_1, a_j]$ and $[a_i, a_k]$, then s is in triangle $\overline{\{a_s, a_i, a_j\}}$.*

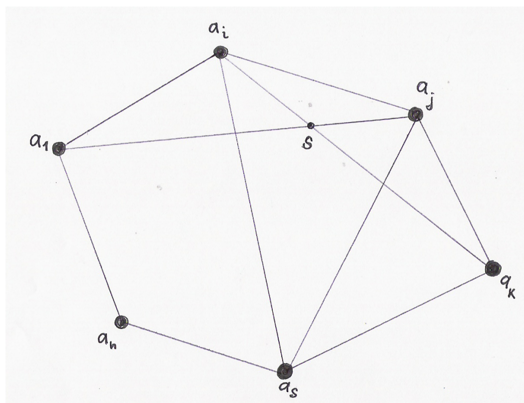

FIGURE 2.

Proof. It is true for any "diagonal" of a convex polygon $[a_1, a_j]$ that all the vertices between a_j and a_1 in their circular order belong to the same semi-plane generated by the line (a_1, a_j). In particular, $[a_1, a_j]$ and $[a_i, a_k]$, indeed, intersect at some point s, since the points a_i and a_k are separated by line (a_1, a_j).

In order to show that s is inside triangle $\overline{\{a_s, a_i, a_j\}}$, one needs to show that, for each side of a triangle, the third vertex and point s belong to the same semiplane generated by the line extending this side.

Take side $[a_i, a_j]$, then vertices a_k, a_s, a_1 are in the same semiplane generated by line (a_i, a_j), hence, both segments $[a_1, a_k]$ and $[a_i, a_j]$ are in that semiplane, implying that their intersection point s belongs there as well.

Take another side of triangle $[a_i, a_s]$. Then a_j, a_k are in the same semiplane generated by line (a_i, a_j). Since s is on segment $[a_i, a_k]$, it belongs to the same semiplane. Thus, s and a_j belong to the same semiplane generated by (a_i, a_s), which is needed. Similar is true for the side $[a_j, a_s]$ and points a_i and s. $\square$

Lemma 4.3. *Suppose the vertices of a convex polygon M with at least 4 vertices are split into three subsets A, B, C. If the vertices of one of these subsets are separated by the vertices of the others in the circular order, then every point of convex polygon M belongs to $\overline{A \cup B} \cup \overline{A \cup C} \cup \overline{B \cup C}$.*

Proof. Assume without loss of generality that vertices $a_1, a_2 \in A$ are separated by points either from B or C in the circular order of vertices of polygon M. If points

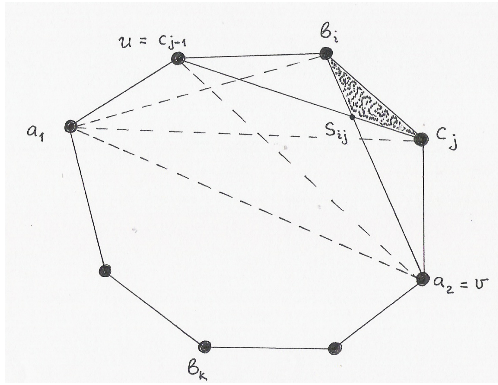

FIGURE 3.

from B all belong to one semi-plane generated by line (a_1, a_2), and all points from C are in the other semi-plane, then every point from M is in $\overline{A \cup B} \cup \overline{A \cup C}$. Thus, assume that there are points from both B and C in one of semi-planes, and points from, say, B are located in both semi-planes. Then the only points of M that do not belong to $\overline{A \cup B} \cup \overline{A \cup C}$ are the points of triangles of the form $\overline{\{b_i, c_j, s_{ij}\}}$, where $b_i \in B$, $c_j \in C$, c_j immediately follows b_i in the circular order of vertices of M, and s_{ij} is the point of intersection of lines (u, c_j), and (b_i, v), where u is closest from $A \cup C$ preceding point to b_i in the circular order, and v is the closest in circular order point from $A \cup B$ following c_j.

According to the assumption, there is vertex $b_k \in B$ that belongs to the other semi-plane generated by (a_1, a_2). Due to Lemma 4.2, when a_i is replaced by b_i, a_1 by u, a_j by c_j, a_k by v, s by s_{ij} and a_s by b_k, it follows that s_{ij} belongs to triangle $\overline{\{b_k, b_i, c_j\}}$. In particular, $\overline{\{b_i, c_j, s_{ij}\}} \subseteq \overline{\{b_k, b_i, c_j\}} \subseteq B \cup C$. □

Proof of Theorem 4.1. Due to Lemma 2.4, $\mathbf{G}$ satisfies 2-Carousel Rule, therefore, $y \in \overline{\{x, b, c\}}$. According to assumption that $\mathbf{G}$ is a subgeometry of $\mathbf{G}_0$, one can find an embedding ϕ of lattice of closed sets of $\mathbf{G}$ into lattice of closed sets of $\mathbf{G}_0$. Denote $U = \phi(\overline{u})$, for any $u \in \{\overline{a}, \overline{b}, \overline{c}, \overline{x}, \overline{y}\}$, and let $P = \phi(\emptyset)$. Then, according to conditions of theorem, $X, Y \subseteq \overline{A \cup B \cup C}$, $X \subseteq \overline{A \cup B \cup Y}$, $X \subseteq \overline{A \cup C \cup Y}$, $Y \subseteq \overline{B \cup C \cup X}$. Besides, $P = Y \cap \overline{A \cup B} = Y \cap \overline{A \cup C} = Y \cap \overline{B \cup C} = Y \cap X$.

Since points of $Y \setminus P$ are inside of convex polygon $\overline{A \cup B \cup C}$, but not in any $\overline{A \cup B}, \overline{A \cup C}, \overline{B \cup C}$, the vertices of $\overline{A \cup B \cup C}$ should appear in clusters, due to Lemma 4.3: elements from A should follow elements from C, which should follow elements of B, in their circular order. Figure 4 makes a sketch of arrangement, where a_1 and a_2 are end points of A-cluster, similarily, b_1, b_2 and c_1, c_2 are end-points of clusters B and C, correspondingly. Elements of $Y \setminus P$ are located inside triangle formed by points of intersection of lines (a_1, b_2), (b_1, c_2) and (c_1, a_2). We need to show that some point $x \in X \setminus P$ is in $\overline{Y \cup A}$.

Claim 1. *Let points of $\overline{A \cup C \cup Y}$ follow each other in the circular order $c_1, \dots, c_i, \dots, c_2, a_1, \dots, a_i, \dots, a_2, \dots, u, \dots, v \dots, c_1$. Then $v \in \overline{u \cup B \cup C}$.*

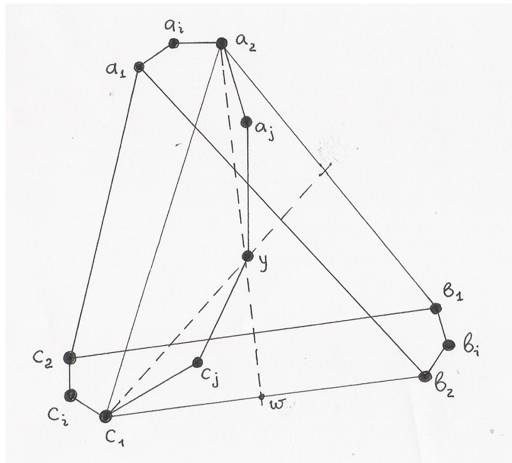

FIGURE 4.

One can use Figure 4 with possible identification of u as y, and v as c_j. We assume that $u \notin W = \overline{\{b_1, \ldots, b_i, \ldots, b_2, c_1, \ldots, c_i, \ldots, c_2\}}$, since otherwise the claim is obvious.

Draw the line (u, a_2), then v should be in the same semi-plane as c_1. Draw the line (c_1, u), then v should be in the semi-plane opposite to a_2. Since u is an inner point of triangle $\overline{\{a_2, b_1, c_2\}}$, line (a_2, u) crosses segment $[c_2, b_1]$ at some inner point. Hence, this line crosses another segment of convex polytope W, say, at point w (this point does not necessarily belong to configuration that generates G_0). Thus, v belongs to the convex polytope formed by u, c_1, w and all the vertices of the border of W between c_1 and w. In particular, $v \in \overline{u \cup B \cup C}$, as desired. End of proof of Claim 1.

Since $\overline{A \cup C} \subset \overline{Y \cup A \cup C}$, some vertices of $\overline{Y \cup A \cup C}$ should be from $Y \setminus \overline{A \cup C} = Y \setminus P = y \setminus \overline{B \cup C}$. Let y_1 be the first element from $Y \setminus P$ that appears after a_2 in the circular order of vertices of $\overline{Y \cup A \cup C}$ given in Claim 1. According to Claim 1, no point from C can appear between a_2 and y_1, since, otherwise, y_1 will be in $\overline{B \cup C}$. Thus, we have in the sequence from a_2 to y_1 only elements from A.

Similarly, let y_2 be the first element from $Y \setminus P = Y \setminus \overline{A \cup B}$ that appears in the circular order of vertices $b_2, \ldots, b_1, a_2, \ldots, a_1, \ldots, b_2$ of $\overline{Y \cup A \cup B}$ between a_1 and b_2. Then there is only elements from A in this sequence between a_1 and y_2.

Claim 2. *The sequences of segements forming the border of $\overline{Y \cup A \cup C}$ from c_1 to a_2 containing point y_1, and the border of $\overline{Y \cup A \cup B}$ from b_2 to a_1 containing point from y_2 intersect at some point (not necessarily the point of configuration forming G_0).*

If only one vertex in $\overline{A \cup B \cup C}$ is from A, i.e. $a_1 = a_2$, then a_1 might be the only point of intersection of $\overline{Y \cup A \cup B}$ and $\overline{Y \cup A \cup C}$. In this case we assume that this is point of intersection of borders of these two convex polygons stated in the claim.

Otherwise, points c_1 and a_2 are separated by line (a_1, b_2). If $c_1, t, \dots, s, w, u, v, \dots, a_2$ are the vertices of $\overline{Y \cup A \cup C}$, on the path from c_1 to a_2 that has point y_1, then one of the segments of this border, say, $[u, v]$, will cross $[a_1, b_2]$.

Now a_1 and b_2 are separated by line (u, v), hence, by all the lines (w, u),$(s, w) \dots$, (c_1, t). Therefore, the vertices of $\overline{Y \cup A \cup B}$ on the path from b_2 to a_1 and containing y_2, will cross each of those lines. It should cross line (c_1, t) on the "right" ray, i.e. on the ray with endpoint c_1 that contains t. On the other hand, it can cross line (u, v) only on the "left" ray, i.e. on the ray with the endpoint v that contains u. There should be a sequence of vertices in $\overline{Y \cup A \cup C}$, say, s, w, u, v, where the sequence of segements of $\overline{Y \cup A \cup B}$ will cross (s, w) on the "right" ray, while it will cross (u, v) on the "left" ray. This implies it will cross one of segments $[s, w], [w, u], [u, v]$. End of proof of claim.

Let us call a point of intersection from Claim 2 by O. Note again that, unlike points from A, B, C or Y, point O is just a geometrical location of intersection of some segments formed by points from $A \cup Y$. There are three possibilities for positioning of points y_1, y_2 and O (see Figure 5). In all three cases, $V = \overline{Y \cup A \cup B} \cap \overline{Y \cup A \cup C}$ is a convex polytope formed by points from A, point O and all the points prior to O on the path from a_1 to b_2 and on the path from a_2 to c_1. According to the assumption, $X \subseteq V$ and $Y \subseteq \overline{X \cup B \cup C}$. We need to show that some point from $X \setminus P$ belongs to $\overline{Y \cup A}$.

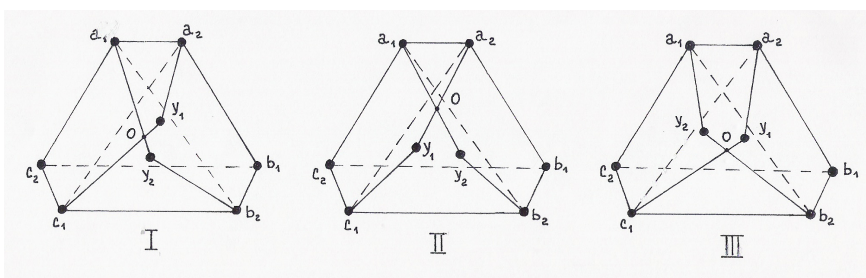

FIGURE 5.

(I) On the path from a_1 to b_2, point O occurs prior to y_2, but on the path from a_2 to c_1 point O occurs after y_1. Evidently, O belongs to $\overline{Y \cup A}$, since O is on a segment connecting two points from $A \cup y_2$. We want to show that any vertex of V between O and y_1 (which is also a vertex of $\overline{Y \cup A \cup C}$) cannot be from C. Indeed, if one vertex would be $c \in C$, then we can apply Claim 1 to vertex c in place of u, and any vertex of $\overline{Y \cup A \cup C}$ on the path from c to c_1 in place of v. Then $v \in \overline{B \cup C}$. In particular, O is in $\overline{B \cup C}$. We can apply now a symmetric statement of Claim 1 to the points on the border of $\overline{Y \cup A \cup B}$, identifying u with O and v with y_2. Then $y_2 \in \overline{B \cup C}$, a contradiction.

Thus, all the vertices of V must be in $\overline{Y \cup A}$, which proves $X \subseteq \overline{Y \cup A}$.

(II) Both y_1, y_2 occur after O on the corresponding paths. Then all the vertices of V are in $\overline{Y \cup A}$, which is needed.

(III) Both y_1, y_2 occur prior to O on the corresponding pathes. According to Claim 1, points of the path from a_2 to c_1 that appear after y_1 belong to $\overline{y_1 \cup B \cup C}$, in particular, $O \in \overline{y_1 \cup y_2 \cup B \cup C}$, thus, the part of polytope V formed by O, y_1, y_2

and all the vertices of both paths between y_1 and O, and y_2 and O, correspondingly, belong to $\overline{y_1 \cup y_2 \cup B \cup C}$. If all the points from $X \setminus P$ would be in that part of V, we would have $X \subseteq \overline{y_1 \cup y_2 \cup B \cup C}$. At least one of y_1, y_2 should be a vertex of $\overline{y_1 \cup y_2 \cup B \cup C}$. Then it can be in $\overline{X \cup B \cup C}$ only when it belongs to $X \cup B \cup C$. But then this points would be in P due to $Y \cap X = P = Y \cap \overline{B \cup C}$, a contradiction. It follows that at least one point from $X \setminus P$ should be in the part of V formed by points from $A \cup y_1 \cup y_2$. Thus, $X \cap \overline{Y \cup A} > P$. *End of proof of Theorem 4.1*

It follows from the proof of the theorem that the following property always holds in any geometry $G_0 = \mathbf{C}o(\mathbb{R}^2, A)$, hence, in any of its subgeometry:

$$\begin{gathered}\text{For all closed sets } X, Y, A, B, C,\\ \text{if } Y \subseteq \overline{A \cup B \cup C}\\ Y \cap \overline{A \cup B} = Y \cap \overline{B \cup C} = Y \cap \overline{A \cup C} = Y \cap X = P < Y, X\\ X \subseteq \overline{Y \cup A \cup B}, X \subseteq \overline{Y \cup A \cup C} \text{ and } Y \subseteq \overline{X \cup B \cup C}\\ \text{then } X \cap \overline{\{A \cup Y\}} > P.\end{gathered}$$

We will refer to this property as *the Sharp 2-Carousel Rule*.

In conclusion of this section, we give an example of the convex geometry that satisfies 2-Carousel Rule, but does not satisfy the Sharp 2-Carousel Rule.

Let $A = \{a, b, c, x, y\}$ and the collection of closed sets of $(A, -)$ include all one-element and two-element subsets; besides, three-element subsets are $\{x, a, b\}$, $\{x, a, c\}$, $\{y, b, c\}$, $\{x, y, w\}$, for $w \in \{a, b, c\}$, and four-element are $\{a, b, x, y\}$, $\{b, c, x, y\}$, $\{a, c, x, y\}$. This implies $x, y \in \overline{\{a, b, c\}}$, $x \in \overline{\{y, a, b\}}, \overline{\{y, a, c\}}$, and $y \in \overline{\{x, b, c\}}$. The Sharp 2-Carousel Rule fails since $x \notin \overline{\{y, a\}}$. Hence, $(A, -)$ is not a sub-geometry of any geometry of relatively convex sets.

5. Concluding remarks

Problem 1.1 asks whether any of classes $\mathcal{C}_B$, $\mathcal{C}_f$ is universal for *all* finite convex geometries. In fact, it is enough to check whether every finite *atomistic* convex geometry is a sub-geometry in one of those classes. Recall that a closure system $\mathbf{A} = (A, -)$ is called *atomistic*, if all one-element subsets of A are closed. This follows from the result proved in [3] (a different proof was given in [4]):

Proposition 5.1. *Every finite convex geometry has a strong atomistic extension. In particular, every finite convex geometry is a sub-geometry of some atomistic convex geometry.*

On the other hand, for the description of sub-geometries of class $\mathcal{C}_n$, the proposition above is not of great help, due to the fact the strong atomistic extension might not preserve the n-Carousel Rule.

Indeed, it is enough to give an example of an atomistic extension that does not preserve the n-Carathéodory Property.

Consider finite geometry $\mathbf{G} = (\{a, b, c, d, x\}, -)$ given by its collection of closed sets $\mathcal{G} = \{\emptyset, a, b, d, ab, ad, bd, cd, abd, acd, abx, adx, bcd, bcdx, acdx, abdx, abcdx\}$. In this convex geometry, for any closed sets $\overline{U} = U, \overline{V} = V$,

$$\overline{U \cup V} = \begin{cases} U \cup V \cup \{x\}, & a, b, c \in U \cup V; \\ \\ U \cup V, & \text{otherwise.} \end{cases}$$

In particular, this convex geometry satisfies the 3-Carathéodory Property.

Let $\mathbf{H} = (\{a, b, c, d, x\}, \tau)$ be another convex geometry on $\{a, b, c, d, x\}$, whose closed sets are all subsets of $\{a, b, c, d, x\}$, except $abcd$. One easily verifies that $\mathbf{G}$ is a sub-geometry of $\mathbf{H}$, therefore, $\mathbf{H}$ is an atomistic extension of $\mathbf{G}$. On the other hand, 3-Carathéodory property fails in $\mathbf{H}$, since $x \in \tau(abcd)$, but $x \notin \tau(abc) \cup \tau(abd) \cup \tau(acd) \cup \tau(bcd)$.

It would be interesting to describe necessary and sufficient properties of finite geometries which are sub-geometries of n-dimensional geometries of relatively convex sets. One of main results in [6] states that if a finite atomistic convex geometry with k extreme points $a_1, \dots, a_k$ and points x, y in the closure of $a_1, \dots, a_k$, satisfies the so-called Carousel Rule and Splitting Rule then it can be represented as $\mathbf{C}o(\mathbb{R}^2, A)$, with $A = \{a_1, \dots, a_k, x, y\}$ being some set of points on a plane. In this result the Carousel Rule is slightly more elaborate property than the 2-Carousel Rule (a version of the Carousel Rule was also formulated in [10], where the case of one point x in the closure of $a_1, \dots, a_k$ was investigated).

At the moment we are not aware of any example of a finite convex geometry satisfying the 2-Carousel Rule and the Sharp 2-Carousel Rule but not representable by relatively convex sets on the plane. Thus, we would like to ask:

Problem 5.2. *Is every finite convex geometry that satisfies* 2*-Carousel Rule and the Sharp* 2*-Carousel Rule a sub-geometry of some (finite) geometry* $\mathbf{C}o(\mathbb{R}^2, A)$*?*

In [5], the 2-Carousel Rule was essential in establishing the correspondence between two problems: the representation of an atomistic convex geometry as $\mathbf{C}o(\mathbb{R}^2, A)$ and the realization of an order type by point configuration on the plane. See [11] for the definition of an order type and [12] for the recent overview and references on the topic.

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