Boundary Element Method for Stokes Flow in Incompressible Newtonian Fluids

by

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Abstract

In this thesis, I present different discretization techniques for boundary integral method for Stokes flow in case of an incompressible Newtonian fluid. Boundary integral method (BIM) is one of many techniques that are used to solve Partial Differencial Equations (PDE) numerically. However, the basic advantage of the BIM is that it reduces the problem from n-dimensional domain to $n-1$; for example, the two-dimensional square-box that contains viscous liquid can be solved by using the values of an unkown function at the boundary of square. Nevertheless, the BIM exhibits some challenges in finding the Green's function for a particular domain or differential operator, solving the integral equations and, especially, in computing the values of a complex domain. The latter one is quite difficult because the flow diverges at corners (exhibits singularity).

The goal of this work is to derive general analytical solution for Stokes equation (in integral equations form) and to compute the discretized integral equations using different quadrature rules for cavity problem.

Thesis Supervisor: Yogi A. Erlangga
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Acknowledgments

First, I would like to express my deepest gratitude to my advisor, Professor Yogi Erlangga, whose knowledge and passion of applied mathematics inspired me to learn and practice mathematics more than I have ever before; my many years of training could not have been complete without his mastery, wisdom and mentorship. I would also like to mention my appreciation for teaching me a course MATH 351 Introduction to Numerical Methods with Applications, where I have been introduced with basic techniques in numerical analysis, which helped me to discover my passion to applied mathematics.

Next, I would like to express sincerest thanks to Eldar Akhmetgaliyev, PhD, who introduced the Boundary Integral Methods, which is contained in these pages and basis for the entire work I have done here. Moreover, I am thankful for his support and encouragement to apply for graduate school in the United States, in order to continue my education further.

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Contents

Chapter 1

Introduction

Stokes flow (also known as **creeping flow**) occurs in slowly moving and highly viscous fluids, which can be characterized by the conservation of momentum and mass [2]. The governing equations for a steady state can be expressed in the following form

$$\begin{aligned} -\mu\nabla^2\mathbf{u} + \nabla P - \rho\mathbf{F} = 0 \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \end{aligned} \tag{1.1}$$

where u is the velocity vector, ρ the fluid density, μ viscosity, $\mathbf{F}$ is a body force acting per unit mass, P is the pressure and Ω is a domain.

The divergence, $\nabla \cdot \mathbf{u}$ can be written in index notation as $\frac{\partial u_i}{\partial x_i}, i = 1, 2, 3$, then for our 2D case, the second part of (1.1) will be written as

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

The system (1.1) is called the **Stokes equations** for an incompressible Newtonian fluid. Its analytic solution may not be easy to obtain because of a conservation of mass.

1.1 Motivations for solving Stokes flow equations using boundary integral method

The Stokes equations can be derived from the general **Navier-Stokes Equations** [6]:

- Three momentum equations:

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla p = \mathbf{P} + \rho \mathbf{F}, \tag{1.2}$$

  where term $\mathbf{P}$ is defined as

$$\mathbf{P} = (\mu + \mu')\mathrm{grad}\,(\mathrm{div}\,\mathbf{u}) + \mu \nabla \mathbf{u},$$

  here μ and μ' are material constants.

- Continuity equation:

$$\rho_t + \mathbf{div}\,(\rho \mathbf{u}) = 0 \tag{1.3}$$

- Equation of state:

$$p = r(\rho) \tag{1.4}$$

If the flow is assumed to be incompressible, i.e. ρ is constant, (1.2) and (1.3) can be simplified to

$$\rho \frac{D\mathbf{u}}{Dt} + \mathbf{grad}\,p = \rho \mathbf{F}, \quad \mathbf{div}\,\mathbf{u} = 0 \tag{1.5}$$

and (1.4) will be dropped as it is no longer coupled with (1.2). Using the definition of total derivative, (1.5) can be written as [6][7]

$$\begin{aligned} \rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \Delta \mathbf{u} + \rho \mathbf{F}, \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \end{aligned} \tag{1.6}$$

but for creeping flow, $\frac{Du}{Dt} \approx 0$, so we get (1.1). Under incompressibility, the viscosity can be assumed to be constant.

The equations (1.2), (1.3) and (1.4) can be made non-dimensionalized (scaled transformations), by introducing the following transformations:

$$\mathbf{u}^* = \frac{\mathbf{u}}{U}, \quad \mathbf{x}^* = \frac{\mathbf{x}}{L}, \quad \mathbf{t}^* = t\frac{\mathbf{U}}{L}, \quad \mathbf{p}^* = \frac{\mathbf{pL}}{\mu U}.$$

where U is a reference velocity, L is a length scale and $\cdot^*$ denotes dimensionless quantities. Next, multiply the equation (1.6) by $L^2/(U\mu)$

$$\begin{aligned} \frac{\rho U L^2}{\mu L}\frac{\partial \mathbf{u}^*}{\partial t^*} + \frac{\rho U L}{\mu}\mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\nabla p^* + \Delta \mathbf{u}^* + \frac{\rho L^2}{U\mu}\mathbf{F}, \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u}^* = 0 \end{aligned} \tag{1.7}$$

Let $T = L/U$ be the time scale of flow and $\nu = \mu/\rho$. Define

$$\beta = \frac{L^2}{\nu T}, \quad Re = \frac{UL}{\nu}, \quad Fr = \frac{U}{\sqrt{gL}}.$$

Then we have,

$$\begin{aligned} \beta\frac{\partial \mathbf{u}^*}{\partial t^*} + Re\mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\nabla p^* + \Delta \mathbf{u}^* + \frac{Re\mathbf{F}}{Fr^2|\mathbf{F}|}, \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u}^* = 0 \end{aligned} \tag{1.8}$$

where Fr, Re and β are the Froude, Reynolds and Stokes numbers, respectively. The Stokes equations can be obtained by assuming $Re \ll 1$ and $\beta \ll 1$.

The countless experiments have confirmed this similarity law, thus, supports and justifies the basic assumptions for the Navier-Stokes Equations. PDEs in form as in (1.1) are easier to solve than Navier-Stokes Equations. The usage of Stokes flow in engineering problems is huge: from filtering (virus removal, air conditioning) to micro-organism propulsion [5], from blood flow [9] to ink-jet printing.

Since Stokes equations describe the motion of a viscous flows, they are studied by a subject called *Fluid Dynamics* and the numerical solution of the Stokes equations can be found in many textbooks of *Computational Fluid Dynamics*. Currently, there are three popular methods available for numerical solution such as (1) *Finite Element*

Methods, (2) *Finite Difference Methods* and (3) *Finite Volume Methods*. Recently, numerical methods other than those presented above have been also used, and one of them is *Boundary Element Methods* that exploit boundary integral equations in which only the boundaries of the domain are utilized to obtain approximate solutions [2].

For *Finite Element Methods*, first, a two-dimensional domain is discretized by simpler small domains such as triangular or quadrilateral elements as in Figure 1-1 and 1-2. The computational domains in both figures contain many subdomains or *finite elements*, within which the PDEs are approximated.

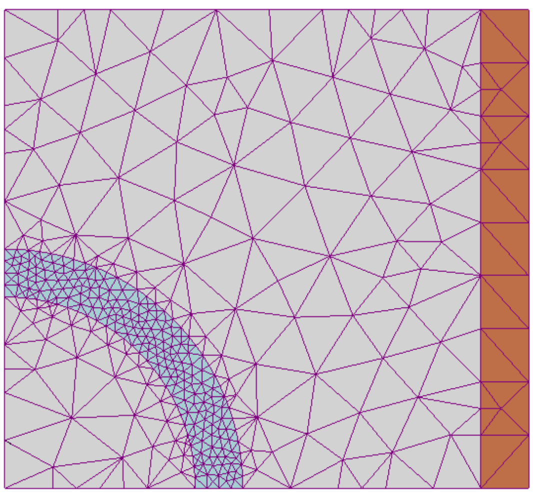

Figure 1-1: Triangular elements (from Wikipedia)

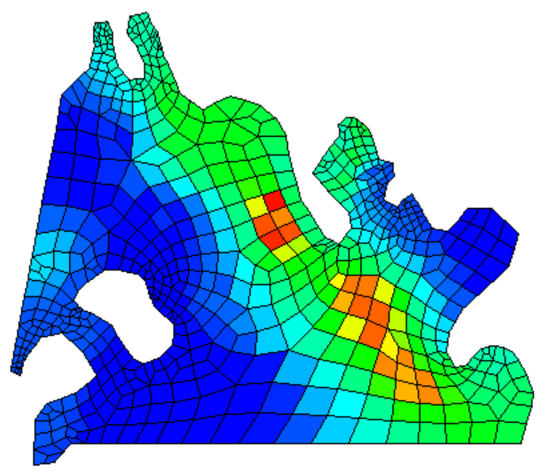

Figure 1-2: Quadrilateral elements

Boundary elements, however, is based on the finding solutions along the boundary of the domain. If we look at Figure 1-3, for two-dimensional domain only along the boundary curve the discretization is exploited, thus reducing the problem into one-dimensional. Since the solution we are seeking on the boundaries are the Green's function, that satisfies PDEs and boundary conditions, the solutions in the interior are calculated for this boundary data [2].

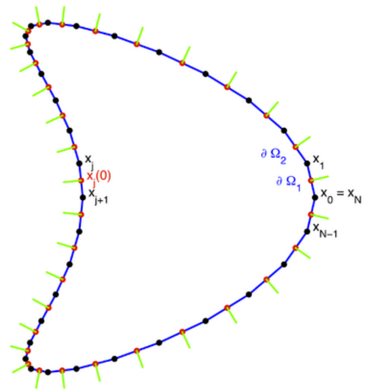

Figure 1-3: Boundary discretization of a kite (from Simon Fraser University)

1.2 Constructing Green's function for Stokes flow

As previously mentioned, the fundamental solution for Stokes can be expressed in a variety of ways depending on the dimension of the domain of Ω. The three-dimensional fundamental solution for Stokes flow is a type of $1/r$, while two-dimensional one is a logarithmic type [7]. Since the work is done on two-dimensional space we restrict the calculations only for this dimension. Also, the fundamental solution for Stokes flow differ according to whether the problem requires the free-space solution involves bounded domain.

Consider a point force acting in the direction $\mathbf{F}$ at the point $\mathbf{x}_0$ within highly viscous flow, here $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$. Including the forcing singular term in the Stokes equations (1.1), we have

$$-\nabla P + \mathbf{F}\delta(\mathbf{x} - \mathbf{x}_0) + \mu\nabla^2\mathbf{u} = 0, \tag{1.9}$$

where δ is the Dirac delta function. The general solution for equation (1.9) is assumed to take the form: for velocity,

$$u_i(\mathbf{x}) = \frac{1}{4\pi\mu}G_{ij}(\mathbf{x}, \mathbf{x}_0)F_j, \tag{1.10}$$

where the G_{ij} is the Green's function yet to be determined.[1], and for the pressure

$$P = \frac{1}{4\pi} p_j F_j. \tag{1.11}$$

Since the above solution u and P are rather arbitary, P_i and G_{ij} are linked via (1.9). In index notation, the momentum equation (1.1) becomes

$$0 = -\frac{\partial P}{\partial x_i} + F_i \delta(x_i - x_{0i}) + \mu \sum_{k=1}^{2} \frac{\partial^2 u_i}{\partial x_k^2} \quad i = 1, 2.$$

Substituting the equation (1.10) and equation of pressure P into (1.1), we obtain

$$-4\pi\delta(x_i - x_{0i}) = -\frac{\partial p_j}{\partial x_i} + \frac{\partial^2 G_{ij}}{\partial x_k^2}. \tag{1.12}$$

Thus, any possible choice of P and G_{ij} are considered to be a solution, if they satisfy (1.12). In the next subsection, we present the type of Green's function with corresponding p that meet the condition (1.12).

1.2.1 The stokeslet in 2D flow

The Green's function for Stokes flow in two-dimensional space is a solution of the singularity forced equation

$$-\nabla P + \mathbf{F}\delta(\mathbf{x} - \mathbf{x}_0) + \mu\nabla^2\mathbf{u} = \mathbf{0}. \tag{1.13}$$

Definition. The *stokeslet*, or a free-space Green's function for Stokes flow (sometimes called the *Oseen-Burgers tensor*), is a solution of (1.13) when there are no boundaries in the flow, that is when the point force at $\mathbf{x}_0$ acts in an unbounded fluid [8].

In the two-dimensional space, the stokeslet takes the form (for a derivation, see the Appendix A)

$$G_{ij} = \delta_{ij} \log r + \frac{\hat{x}_i \hat{x}_j}{r^2} \tag{1.14}$$

[1]The factor of $1/4\pi\mu$ is included purely for convenience [8]

where δ_{ij} is the *Kronecker delta*, $\hat{\mathbf{x}}_i = \mathbf{x} - \mathbf{x}_0$. This Green's function induced the Stokeslet pressure

$$p_j = 2\frac{\hat{x_j}}{r^2},$$

needed to compute in (1.11).

1.2.2 Layer Potentials

Let h be a continuous function on $\partial\Omega$ and $\Phi(x)$ is a fundamental solution for a given equation. The **the single layer potential with moment** h is defined as [4]:

$$\bar{u}(x) = -\int_{\partial\Omega} h(y)\Phi(x-y)\,dS(y) \tag{1.15}$$

The **double layer potential with moment** h is defined as

$$\bar{\bar{u}}(x) = -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial\nu_y}(x-y)\,dS(y) \tag{1.16}$$

For a moment, consider that for h a continuous function on $\partial\Omega$, $\bar{\bar{u}}$ defined as in (1.16). Now, by choosing h appropriately, such that for all $x_0 \in \partial\Omega$,

$$\lim_{x\in\Omega\to x_0} \bar{\bar{u}}(x) = g(x_0)$$

then solution for a given harmonic PDE can be found. However, lets first introduce the **Gauss' Lemma**.

Lemma 1. (Gauss' Lemma) Consider the double layer potential,

$$\bar{\bar{u}}(x) = -\int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu_y}(x-y)\,dS(y).$$

Then,

$$\bar{\bar{u}}(x) = \begin{cases} 0 & x \in \Omega^c \\ 1 & x \in \Omega \\ 1/2 & x \in \partial\Omega \end{cases} \tag{1.17}$$

where the Ω^c is a compliment of Ω.

Now, the single and double layer potentials for two dimensional Stokes flow can be introduced:

The single layer potential

$$I_j^{SL}(\mathbf{x}_0) = \int_C f_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0)\, dl(\mathbf{x}) \tag{1.18}$$

and **the double layer potential**

$$I_j^{DL}(\mathbf{x}_0) = \int_C u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k\, dl(\mathbf{x}) \tag{1.19}$$

where, G_{ij} is a Green's function for Stokes flow and $T_{ijk}(\mathbf{x}, \mathbf{x}_0)$ is a stress tensor corresponding to the two-dimensional stokeslet

$$T_{ijk}(\mathbf{x}, \mathbf{x}_0) = -4 \frac{\hat{x}_i \hat{x}_j \hat{x}_k}{r^4} \tag{1.20}$$

with notation $\mathbf{x} - \mathbf{x}_0 = \hat{\mathbf{x}}$ and $n_k = \dfrac{x_k - x_{k0}}{|x_k - x_{k0}|}$.

1.2.3 Stress tensor identity in two dimensions

For a closed contour C, and the unit normal $\mathbf{n}$ points out of C, the two-dimensional stress tensor identity is

$$\int_C T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_i(\mathbf{x})\, dl(\mathbf{x}) = -\begin{bmatrix} 4\pi \\ 2\pi \\ 0 \end{bmatrix} \delta_{jk}. \tag{1.21}$$

Also note that,

- 0, if $\mathbf{x}_0 \in \Omega^c$.
- 2π, if $\mathbf{x}_0 \in \partial\Omega$ or C.
- 4π, if $\mathbf{x}_0 \in \Omega$.

Finally, the stress tensor corresponding to the two-dimensional stokeslet can be described as following

$$T_{ijk}(\mathbf{x}, \mathbf{x}_0) = 4\frac{\hat{x}_i \hat{x}_j \hat{x}_k}{r^4}. \tag{1.22}$$

Chapter 2

The boundary integral equation for Stokes flow in 2D

2.1 The Driven Cavity Problem

As an example consider the driven cavity problem in fluid machenics. Take a two dimensional square box as in figure 2-1, which is filled with the viscous liquid with a given density ρ. Consider that the top of the box moves at a speed U along the positive x direction.

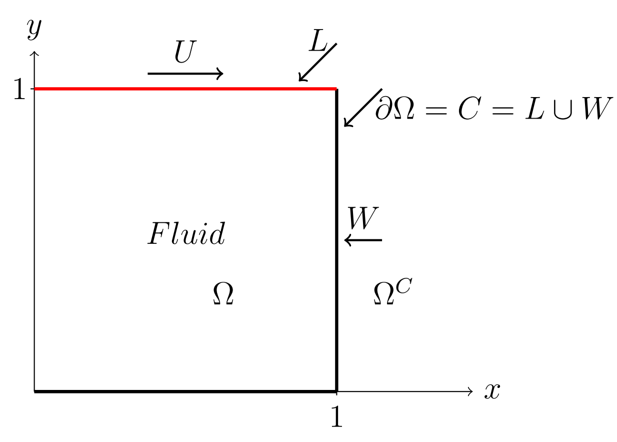

Figure 2-1: Cavity problem

Using the Gauss Lemma (see the Lemma 1 on section 1.2.2) for potentials, the boundary integral equations for two-dimensional flow bounded by a contour C are as follows:

- If $\mathbf{x}_0 \in \Omega$ (main domain), the boundary integral equation is

$$u_j(\mathbf{x}_0) = -\frac{1}{4\pi\mu}\int_C G_{ij}(\mathbf{x},\mathbf{x}_0) f_i(\mathbf{x})\, dl(\mathbf{x}) + \frac{1}{4\pi}\int_C u_i(\mathbf{x}) T_{ijk}(\mathbf{x},\mathbf{x}_0) n_k\, dl(\mathbf{x}) \quad (2.1)$$

- If $\mathbf{x}_0 \in \Omega^c$ (complement of the main domain), the boundary integral equation takes form as

$$-\frac{1}{4\pi\mu}\int_C G_{ij}(\mathbf{x},\mathbf{x}_0) f_i(\mathbf{x})\, dl(\mathbf{x}) + \frac{1}{4\pi}\int_C u_i(\mathbf{x}) T_{ijk}(\mathbf{x},\mathbf{x}_0) n_k\, dl(\mathbf{x}) = 0 \quad (2.2)$$

- Finally, if $\mathbf{x}_0 \in \partial\Omega$ or $\mathbf{x}_0 \in C$ (along the contour/boundary), the boundary integral equation becomes

$$\frac{1}{2}u_j(\mathbf{x}_0) = -\frac{1}{4\pi\mu}\int_C G_{ij}(\mathbf{x},\mathbf{x}_0) f_i(\mathbf{x})\, dl(\mathbf{x}) + \frac{1}{4\pi}\int_C u_i(\mathbf{x}) T_{ijk}(\mathbf{x},\mathbf{x}_0) n_k\, dl(\mathbf{x}) \quad (2.3)$$

Using (2.3) and taking the point $\mathbf{x}_0$ to lie on the walls of the box, we obtain

$$-\frac{1}{4\pi\nu}\int_C f_i(\mathbf{x}) G_{ij}(\mathbf{x},\mathbf{x}_0)\, dl(\mathbf{x}) + \frac{1}{4\pi}\int_C u_i(\mathbf{x}) T_{ijk}(\mathbf{x},\mathbf{x}_0) n_k\, dl(\mathbf{x}) = \begin{cases} \frac{1}{2}U\delta_{1j} & \text{if } \mathbf{x}_0 \text{ is on } L \\ 0 & \text{if } \mathbf{x}_0 \text{ is on } W. \end{cases} \quad (2.4)$$

where L is the lid and W the other three walls of the box, and $C = L \cup W$. The first case in (2.4) corresponds to moving lid L, and the second case corresponds to no-slip condition on the wall W.

Modifying the double layer potential integral,

$$\frac{1}{4\pi}\int_C u_i(\mathbf{x}) T_{ijk}(\mathbf{x},\mathbf{x}_0) n_k\, dl(\mathbf{x}) = \frac{1}{4\pi}U\delta_{1i}\int_L T_{ijk}(\mathbf{x},\mathbf{x}_0) n_k\, dl(\mathbf{x}).$$

Denoting

$$\mathscr{D}_j(\mathbf{x}_0) = \frac{1}{4\pi} U \int_L T_{1jk}(\mathbf{x}, \mathbf{x}_0) n_k \, dl(\mathbf{x}),$$

then

$$\frac{1}{4\pi\nu} \int_C f_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) \, dl(\mathbf{x}) = \begin{cases} \mathscr{D}_j(\mathbf{x}_0) - \frac{1}{2} U \delta_{1j} & \text{if } \mathbf{x}_0 \in L \\ \mathscr{D}_j(\mathbf{x}_0) & \text{if } \mathbf{x}_0 \in W. \end{cases} \tag{2.5}$$

in the above equations, we set (see previous chapter)

$$G_{ij} = \delta_{ij} \log r + \frac{\hat{x}_i \hat{x}_j}{r^2},$$

and the stress tensor is

$$T_{ijk}(\mathbf{x}, \mathbf{x}_0) = -4 \frac{\hat{x}_i \hat{x}_j \hat{x}_k}{r^4},$$

where, $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$ and $r = |\mathbf{x} - \mathbf{x}_0|$.

2.2 On Numerical Solution in $\mathbb{R}^2$

Now, by discretizing each four sides of the contour C using elements of equal length h such that the contour C is covered by N elements, then

$$h = \frac{N}{4}$$

where N (a multiple of 4) is the total number of elements. The starting point is the left of the top of the box, and label the elements clockwise.

So, (2.5) can be rewritten as

$$\frac{1}{4\pi\nu} \int_C f_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) \, dl(\mathbf{x}) = \begin{cases} \mathscr{D}_j(\mathbf{x}_0) - \frac{1}{2} U \delta_{1j} & \text{if } \mathbf{x}_0 \in E^k \text{ for } 1 \leq k \leq \frac{N}{4} \\ \mathscr{D}_j(\mathbf{x}_0) & \text{if } \mathbf{x}_0 \in E^k \text{ for } \frac{N}{4} + 1 \leq k \leq N \end{cases} \tag{2.6}$$

where E^k denotes the k^{th} element.

At this point, the integral on the left cannot be evaluated as $f_i(\mathbf{x})$ in general is not known. Approximation to this integral can be, however, be constructed by assuming

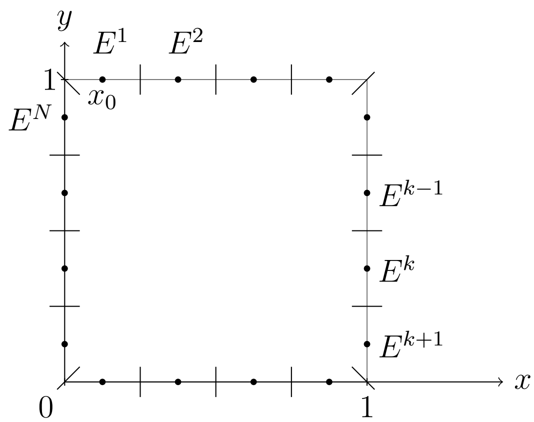

Figure 2-2: Boundary discretization

a function typically polynomial, that approximates f.

Now, denote

$$I_{ij}^{k}(\mathbf{x}_0) = \int_{E^k} G_{ij}(\mathbf{x}, \mathbf{x}_0)\, dl(\mathbf{x}) = \int_{E^k} \left[-\delta_{ij} \log r + \frac{\hat{x}_i \hat{x}_j}{r^2} \right] dl(\mathbf{x}). \tag{2.7}$$

This will help us when, we discretize equation (2.6). For each part of the boundary, the explicit form of $I_{ij}^{k}(\mathbf{x})$ can be computed. In this work, we consider f_i as a constant over E_i.

$$\begin{aligned} \int_C f_i(x) G_{ij}(\mathbf{x}, \mathbf{x}_0)\, d(\mathbf{x}) &= \\ \sum_{k=1}^{N} \int_{E}^{k} f_i^k(x) G_{ij}(\mathbf{x}, \mathbf{x}_0)\, dl(\mathbf{x}) &\approx \\ \approx \sum_{k=1}^{\infty} f_i^{(k)} \int_{E^k} G_{ij}(\mathbf{x}, \mathbf{x}_0)\, dl(\mathbf{x}) \end{aligned}$$

where $\int_{E^k} G_{ij}(\mathbf{x}, \mathbf{x}_0)\, dl(\mathbf{x}) = I_{ij}^{k}$.

- *Left vertical elements*

  For the left vertical side of the box (where $x = 0$), we obtain

$$I_{ij}^{k}(\mathbf{x}_0) = \int_{y_k}^{y_{k+1}} \left[-\delta_{ij} \log r + \frac{\hat{x}_i \hat{x}_j}{r^2} \right]_{x=0} dy,$$

  where

$$r^2 = (x - x_0)^2 + (y - y_0)^2 = x_0^2 + (y - y_0)^2$$

For instance, since $\delta_{ij} = 1$, when $i = j$ and $\delta_{ij} = 0$, when $i \neq j$, then

$$I_{11}^k(\mathbf{x}_0) = \int_{y_k}^{y_{k+1}} \left[-\frac{1}{2} \log[x_0^2 + (y - y_0)^2] + \frac{x_0^2}{x_0^2 + (y - y_0)^2} \right] dy$$

and

$$I_{12}^k(\mathbf{x}_0) = \int_{y_k}^{y_{k+1}} \frac{-x_0(y - y_0)}{x_0^2 + (y - y_0)^2} \, dy.$$

- *Right vertical elements*

  For the right vertical element parallel to the y axis at $x = 1$, we have

$$r^2 = (x - x_0)^2 + (y - y_0)^2 = (1 - x_0)^2 + (y - y_0)^2.$$

  As an example,

$$I_{11}^k(\mathbf{x}_0) = \int_{y_k}^{y_{k+1}} \left[-\frac{1}{2} \log[(1 - x_0)^2 + (y - y_0)^2] + \frac{(1 - x_0)^2}{(1 - x_0)^2 + (y - y_0)^2} \right] dy$$

  and

$$I_{12}^k(\mathbf{x}_0) = \int_{y_k}^{y_{k+1}} \frac{(1 - x_0)(y - y_0)}{(1 - x_0)^2 + (y - y_0)^2} \, dy.$$

- *Bottom horizontal elements*

  For the bottom horizontal side of the cavity (where $y = 0$), we obtain

$$r^2 = (x - x_0)^2 + (y - y_0)^2 = (x - x_0)^2 + y_0^2.$$

  For example,

$$I_{11}^k(\mathbf{x}_0) = \int_{x_k}^{x_{k+1}} \left[-\frac{1}{2} \log[(x - x_0)^2 + y_0^2] + \frac{(x - x_0)^2}{(x - x_0)^2 + y_0^2} \right] dx$$

  and

$$I_{12}^k(\mathbf{x}_0) = \int_{x_k}^{x_{k+1}} \frac{-(x - x_0)y_0}{(x - x_0)^2 + y_0^2} \, dx.$$

Chapter 3

The boundary element method for Stokes flow

3.1 Discretization methods

For general geometry of the boundary, the integral equation (2.7) cannot be evaluated analytically. An approximation to the intergal can be done via numerical intergration. As this approach requires a finite number of points on each element E^k, approximation leads to a discrete equation, which forms a system of linear equations that can be solved with a Gauss elimination procedure. Now, when the kernels have been derived for computing the numerical solution of Stokes flow, it is required to show the appropriated convertation methods from continuous integral equations into discrete forms. There are several methods that can be employed for this problem: (1) the *Nystrom method*, (2) the *collocation method* and (3) the *Galerkin method*, the Nystrom method is more practical for one-dimensional integral equations and requires less computation[3]. If Nystrom method requires the computation of only the kernel function, the Galerkin method demands numerical quadratures. Moreover, the Nystrom method is more stable, because it preserves the condition of the integral equations, while the collocation and the Galerkin methods depend on only basis and if the latter is chosen poorly, then the condition can be disturbed.

In this work, we shall focus only on the *quadrature type* methods. The quadrature method seeks for the solution of the integral by equation replacing the the integral with a weighted sum of integrals, evaluated at a finite number of points. In particular, we consider three quadratures: (1) the *Gauss-Legendre Quadrature*, (2) the *Chebyshev-Gauss Quadrature of the first kind* and (3) *Chebyshev-Gauss Quadrature of the second kind.*

3.1.1 Gaussian Quadrature

The general *Gaussian quadrature* for a piece-wise continuous function f is given as

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i) \tag{3.1}$$

with w_i be the weights of a function value at x_i. This equation approximately fail, if the integrating function $f(x)$ has a singularity in the integrating domain (like in our case). Nevertheless, if we are able to write integrating function as $f(x) = \omega(x)g(x)$, where $g(x)$ is a polynomial, then by introducing alternative weight function w_i' and points x_i' that depend on the $\omega(x)$, the error at the singularity can be controlled.

The **Gauss-Legendre** quadrature formula is constructed by approximating f with the *Legendre polynomials* of degree n.

$$\int_{-1}^{1} f(x)\,dx \approx \int_{-1}^{1} P_n(x)\,dx = \sum_{i=1}^{n} \frac{2}{(1-x_i^2)[P_n'(x_i)]^2} P_n(x_i), \tag{3.2}$$

In this case, the weights depend on the derivative of P_n at x_i. Next, the **Chebyshev-Gauss of the first kind** quadrature rule is

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \frac{\tilde{f(x)}}{\sqrt{1-x^2}}\,dx \approx \sum_{i=1}^{n} w_i g(x_i). \tag{3.3}$$

with $w_i = \frac{1}{\sqrt{1-x_i^2}}$ and $g(x_i) = \tilde{f}(x_i) = \sqrt{1-x_i^2} f(x_i)$, evaluated at x_i, the roots of *Chebyshev polynomials of the first kind.* The **Chebyshev-Gauss of the second**

Quadrature name	Interval	$\omega(x)$	Weights, w_i	Points, x_i
Legendre	[-1, 1]	1	$w_i = \frac{2}{(1-x_i^2)[P_n'(x_i)]^2}$	x_i is the $i-$th root of P_n
Chebyshev (first kind)	(-1, 1)	$\frac{1}{\sqrt{1-x^2}}$	$w_i = \frac{\pi}{n}$	$x_i = \cos\left(\frac{2i-1}{2n}\pi\right)$
Chebyshev (second kind)	[-1, 1]	$\sqrt{1-x^2}$	$w_i = \frac{\pi}{n+1}\sin^2\left(\frac{i}{n+1}\pi\right)$	$x_i = \cos\left(\frac{i}{n+1}\pi\right)$

Table 3.1: Quadrature types

kind quadrature is given by following

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \sqrt{1-x^2}u(x)\,dx \approx \sum_{i=1}^{n} w_i g(x_i), \tag{3.4}$$

where $u(x) = \frac{1}{\sqrt{1-x_i^2}} f(x)$ and x_i roots of *Chebyshev polynomials of the second kind.*

Table 3.1 illustrates the weight functions w_i and the correspoding points x_i, where the function g is defined.

As (3.2), (3.3) and (3.4) show, the intergating domain is between -1 and 1. Thus, we are required to introduce the change of interval formula

$$\int_{a}^{b} f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\,dx, \tag{3.5}$$

and when applied to Gaussian quadrature rule

$$\int_{a}^{b} f(x)\,dx \approx \frac{b-a}{2}\sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right). \tag{3.6}$$

3.1.2 System of Linear equations

Knowing (2.7) and using the quadrature methods, the left hand side of (2.6) becomes

$$\frac{1}{4\pi\nu}\sum_{k=1}^{N} f_i^k(\mathbf{x}) I_{ij}^k(\mathbf{x}_0).$$

Now, we can apply the discretized boundary integral elements at the midpoints of the elements, which means that $\mathbf{x}_0$ lies between k and $k+1$ elements, here $k = 1, ..., N$. Thus, we will obtain $2N$ linear algebraic equations with $2N$ unknowns with

$$\mathbf{D}_m = \begin{pmatrix} \mathscr{D}_1(\mathbf{x}_0) \\ \mathscr{D}_2(\mathbf{x}_0) \end{pmatrix},$$

$$\mathbf{I}_m^k(\mathbf{x}_0) = \begin{bmatrix} I_{11}^k(\mathbf{x}_0) & I_{12}^k(\mathbf{x}_0) \\ I_{21}^k(\mathbf{x}_0) & I_{22}^k(\mathbf{x}_0) \end{bmatrix},$$

which is a local matrix, and

$$\mathbf{f}^k = \begin{pmatrix} f_1^k \\ f_2^k \end{pmatrix},$$

and finally,

$$\frac{1}{4\pi\mu} \begin{bmatrix} \mathbf{I}_1^1 & \mathbf{I}_1^2 & \cdots & \mathbf{I}_1^N \\ \mathbf{I}_2^1 & \mathbf{I}_2^2 & \cdots & \mathbf{I}_2^N \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{I}_N^1 & \mathbf{I}_N^2 & \cdots & \mathbf{I}_N^N \end{bmatrix} \begin{pmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \vdots \\ \mathbf{f}^N \end{pmatrix} = \begin{pmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \\ \vdots \\ \mathbf{D}_N \end{pmatrix} - \frac{1}{2}U \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 1 \\ 0 \\ -- \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.7}$$

3.2 Computational results

After computing the wall tractions, the fluid velocity can be be determined at any point in the flow using (2.1) and (1.21).

$$u_j(\mathbf{x}_0) = -\frac{1}{4\pi\mu}\int_C G_{ij}(\mathbf{x}, \mathbf{x}_0) f_i(\mathbf{x})\, dl(\mathbf{x}) - \frac{U}{4\pi}\int_L T_{1j2}(\mathbf{x}, \mathbf{x}_0)\, dl(\mathbf{x}).$$

Figure 3-1 shows the discretization of the boundary of the cavity, while figure 3-2 illustrates the computed tractions using Gauss-Legendre quadrature, where $U = 2.0$

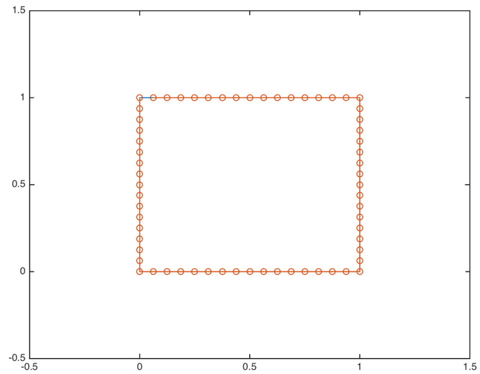

Figure 3-1: Boundary elements for cavity with $N = 16$

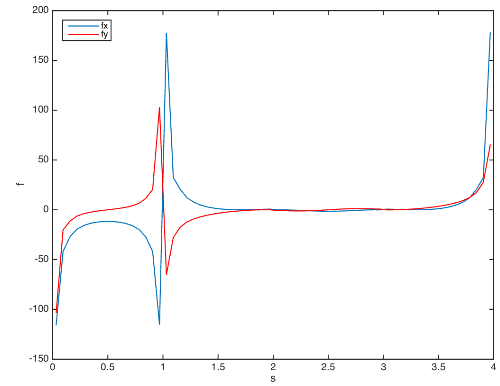

Figure 3-2: The computed tractions with $N = 64$

is the lid velocity and $\mu = 1.0$ is the fluid viscosity.

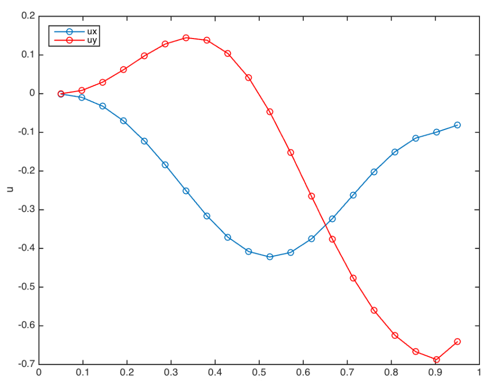

Figure 3-3: Velocity profiles along $y = x$

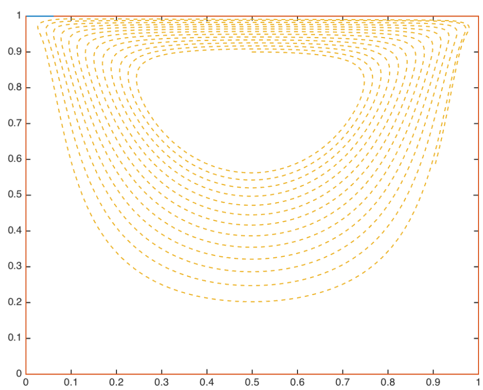

Figure 3-4: Streamlines using $N = 64$

Next, in figure 3-3 velocity profiles u_x and u_y, which stand for vertical and horizontal components, repectively, are plotted along the line $x = y$, while in figure 3-4 typical streamlines are shown (also using Gauss-Legendre quadrature).

Similar results can be obtained using Chebyshev-Gauss quadrature of first and second kinds' methods. Figures 3-5 and 3-6 illustrate the same problem with using Chebyshev-Gauss quadrature of the first kind.

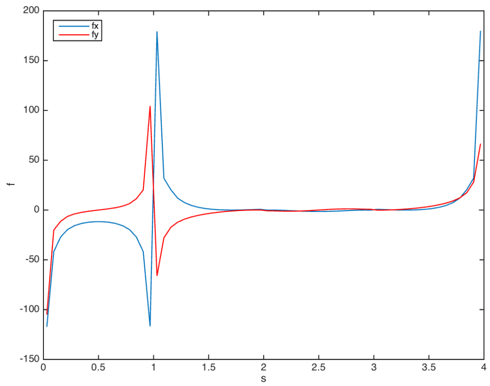

Figure 3-5: The computed tractions with $N = 64$ using Chebyshev-Gauss quad.

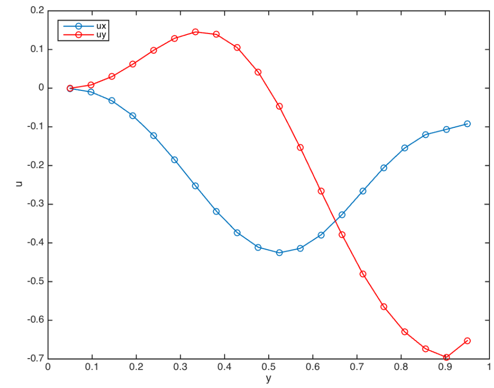

Figure 3-6: Velocity profiles along $y = x$ using Chebyshev-Gauss quad.

3.2.1 Convergence analysis

Now, denote u_{Ni} as a speed of a motion, where Ni is a fourth of a total number of elements (see subsection 2.2, i.e. $Ni = N/4$) and

$$u_{Ni}(\mathbf{x}_0) = \sqrt{u_x^2 + u_y^2},$$

i.e, write the relation between speed and velocity, and let

$$\epsilon_{Ni} = |u_{Ni} - u_{\frac{Ni}{2}}|$$

be error at point $\mathbf{x}_0$ with different values of Ni.

Figure 3-7 illustrates the error for the speed at two different points: (1) when $\mathbf{x}_0$ is far from the sides of the box, (2) when $\mathbf{x}_0$ is located closer to the corner. As we can see, for the first case the error gets small faster than for the second case, thus, making the points closer to the wall more dependent on the values at the boundary.

3.3 Conclusion

As we have observed, the Stokes flow equation is not explicitly time dependent, and it forms linear partial differential equations. For small *Reynolds number*, the Navier-

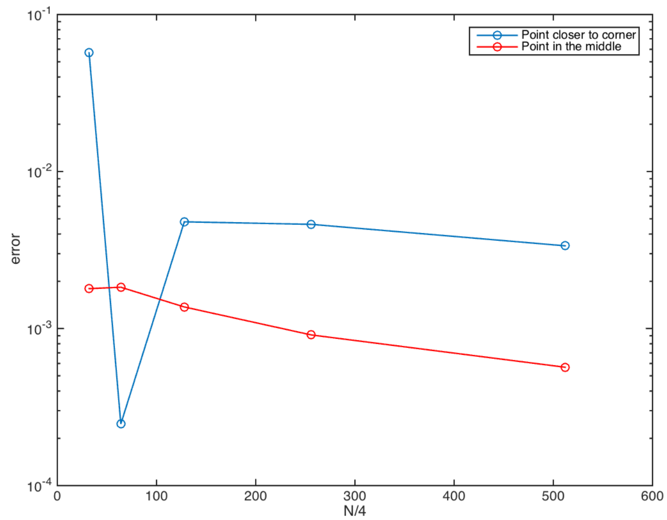

Figure 3-7: ϵ_{Ni} vs. $Ni = N/4$ using logarithmic scale

Stokes equation can be reduced to Stokes equations. Next, by using boundary element method, we can come up with intergral equations that can be solved only on boundary of the given domain. Considering well-known, numerical discretization methods such as Gaussian quadrature, continuous intergal equations can be transformed into the discrete values for a particular N.

As mentioned above that the boundary element method is quite difficult, because it is not possible to find a fundamental solution for a particular equation (in our case, Stokes equation) and for a particular domain (stokeslet) all the time. However, since the Green's function for Stokes equation has been derived before, we can easily implement numerical solution using boundary element method.

Coming to applications, the Stokes equation and its solution using BIM is used in small regions or small particles, such as blood flow, hemodialysis, virus removal because of it is easiness to implement for complicated geometries.

Appendix A

Vector calculus derivation of the Stokeslet

There are several derivations of the Stokeslet [7] such as the derivation using Fourier Transformation in [8] and the derivation based on vector calculus approach. In this appendix, we will use the latter one in order to obtain fundamental solution for Stokes flow in 2D [1].

First, rewrite the (1.13) in the following form

$$\nabla P - \mu \Delta \mathbf{u} = \mathbf{F} \delta(\mathbf{x} - \mathbf{x}_0). \tag{A.1}$$

Now denote $\Delta \mathbf{G} = \delta(\mathbf{x} - \mathbf{x}_0)$ and set $\Delta \mathbf{H} = \mathbf{G}$, then we obtain the biharmonic equation $\Delta\Delta\mathbf{H} = \delta(\mathbf{x} - \mathbf{x}_0)$. To proceed further, we should also know the fact that the divergence operator (∇) and the Laplacian commute, i.e. $\nabla(\Delta \mathbf{u}) = \Delta(\nabla \mathbf{u})$. If we apply the identity $\nabla \cdot (\mathbf{a}b) = b\nabla \cdot \mathbf{a} + a \cdot \nabla b$ to $\nabla \mathbf{F}\delta$, we obtain

$$\nabla(\nabla P) - \mu \nabla(\Delta \mathbf{u}) = \nabla(\mathbf{F}\delta).$$

Since $\nabla \cdot \mathbf{u} = 0$, we have

$$\Delta P = \mathbf{F} \nabla \delta,$$

which is equivalent to writing as

$$\Delta P = \mathbf{F}\nabla(\Delta \mathbf{G}),$$

and again using the commutative relation, we obtain

$$P = \mathbf{F} \cdot \nabla \mathbf{G}. \tag{A.2}$$

By substituting (A.2) into (A.1), we have

$$\mathbf{u} = \frac{1}{\mu}\mathbf{F}(\nabla\nabla - \mathbf{I}\Delta)\mathbf{H}. \tag{A.3}$$

Now it remains to solve the biharmonic equation for a point force at $\mathbf{x}_0$

$$\Delta\Delta\mathbf{H} = \delta.$$

Since, we have already known the fundamental solution for the Laplce equation, i.e. $\Phi(r) = -\ln(r)/2\pi$, where $r = |\mathbf{x} - \mathbf{x}_0|$. Thus, we are required to solve $\Delta\mathbf{H} = -\ln(r)/2\pi$. Let's express the Laplacian in polar coordinates

$$\Delta H = \mathbf{H}_{rr} + \frac{\mathbf{H}_r}{r} + \frac{1}{r^2}\mathbf{H}_{\theta\theta}. \tag{A.4}$$

Since, the angle dependency may be dropped, we are left with an ordinary differential equation

$$H''(r) + \frac{1}{r}H'(r) = -\frac{1}{2\pi}\ln r,$$

can be written as

$$rH'' + H' = -\frac{r}{2\pi}\ln r.$$

Then,

$$(rH')' = -\frac{r}{2\pi}\ln r$$

and after taking the integrals from both sides

$$rH' = -\frac{1}{4\pi}r^2 \ln r + \frac{1}{8\pi}r^2 + C_1.$$

Divide the last equation by r, and we obtain

$$H' = -\frac{1}{4\pi}r \ln r + \frac{1}{8\pi}r + \frac{C_1}{r}.$$

This is equivalent to writing as

$$\int dH = \int \left(-\frac{1}{4\pi}r \ln r + \frac{1}{8\pi}r + \frac{C_1}{r}\right) dr.$$

Evaluation of integrals gives us

$$H(r) = \frac{r^2}{8\pi}(1 - \ln r) + C_1 \ln r + C_2.$$

If we integrate the singularity forced equation over a circle with a radius of a centered at the singularity, we get

$$\int_S \nabla^4 H \, dS = \int_S \delta(\hat{\mathbf{x}}) = 1.$$

Now,

$$\nabla^4 = \nabla \cdot (\nabla \nabla^2 H),$$

so

$$\int_S \nabla \cdot (\nabla \nabla^2 H) \, dS = 1$$

and applying the divergence theorem

$$\int_l (\nabla \nabla^2 H) \cdot \mathbf{n} \, dl = 1,$$

but for circle $\mathbf{n} = \hat{\mathbf{r}}$, and so

$$\int_l \frac{\partial}{\partial r} \nabla^2 H \, dl = -1.$$

The last equation can be written as

$$\int_l \frac{\partial}{\partial r}\left(\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r}\right) dl = \int_l = \left(\frac{\partial^3 H}{\partial r^3} + \frac{1}{r}\frac{\partial^2 H}{\partial r^2} - \frac{1}{r^2}\frac{\partial H}{\partial r}\right) dl = -1$$

Substituting, the solution into the last equation

$$\int_l \left(\frac{2C_1}{r^3} - \frac{1}{4\pi r} - \frac{C_1}{r^3} - \frac{1}{8\pi r} - \frac{1}{4\pi r}\ln r - \frac{C_1}{r^3} - \frac{1}{8\pi r} + \frac{1}{4\pi r}\ln r\right) dl = -\frac{1}{2\pi a}2\pi a = -1.$$

As we can see, the values of C_1 and C_2 are immaterial, we will leave them $C_1 = 0$ and $C_2 = 0$, and obtain $H(r) = \frac{r^2}{8\pi}(1 - \ln r)$. Returning to (A.3), we have

$$\begin{aligned} \mathbf{u} &= \frac{1}{\mu}\mathbf{F}\cdot(\nabla\nabla - \mathbf{I}\Delta)\left(\frac{r^2}{8\pi}(1 - \ln r)\right) \\ &= \frac{1}{4\pi\mu}\mathbf{F}\cdot\left[\nabla\nabla\left(\frac{r^2}{2}(\ln r - 1)\right) + \frac{\delta_{ij}}{2} + \delta_{ij}\ln r\right] \\ &= \frac{1}{4\pi\mu}\mathbf{F}\cdot\left[-\frac{\delta_{ij}}{2} + \frac{\hat{x}_i\hat{x}_j}{r^2} + \frac{\delta_{ij}}{2} + \delta_{ij}\ln r\right]. \end{aligned}$$

Thus,

$$u_i = \frac{1}{4\pi\mu}G_{ij}(\mathbf{x}, \mathbf{x}_0)F_j, \qquad (A.5)$$

and the Stokeslet is

$$G_{ij} = \delta_{ij}\log r + \frac{\hat{x}_i\hat{x}_j}{r^2}. \qquad (A.6)$$

We can also write the pressure associated with the Stokeslet as

$$P = \frac{1}{4\pi}p_j F_j, \qquad (A.7)$$

where

$$p_j = 2\frac{\hat{x}_j}{r^2}.$$

Bibliography

[1] M. Blyth, *Boundary element and finite element methods.*

[2] T. J. Chung, *Computational Fluid Dynamics*, Cambridge University Press, Cambridge, UK, 2 ed., 2010.

[3] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, Berlin, Germany, 1992.

[4] G. B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, NJ, 2 ed., 1995.

[5] J. J. L. Higdon, A hydrodynamic analysis of flagellar propulsion, *Journal of Fluid Mechanics*, 90 (1979), pp. 685–711.

[6] H.-O. Kreiss and J. Lorenz, *Initial-Boundary Value Problems and the Navier-Stokes Equations*, Academic Press, Boston, MA, 1989.

[7] O. Marin, *Boundary integral methods for Stokes flow: Quadrature techniques and fast Eqald methods*, PhD thesis, Royal Institute of Technology, Stockholm, Sweden, 2012.

[8] C. Pozrikidis, *Boundary Integral and Singularity Methods for Linearised Vicsous Flow*, Cambridge University Press, Cambridge, UK, 1992.

[9] H. Zhao, A. H. G. Isfahani, L. N. Olson, and J. B. Freund, A spectral boundary integral method for flowing blood cells, *Journal of Computational Physics*, 229 (2010), pp. 3726–3744.