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L^P-CAFFARELLI–KOHN–NIRENBERG TYPE INEQUALITIES ON HOMOGENEOUS GROUPS

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Abstract

We prove L^p-Caffarelli–Kohn–Nirenberg type inequalities on homogeneous groups, which is one of most general subclasses of nilpotent Lie groups, all with sharp constants. We also discuss some of their consequences. Already in the Abelian cases of isotropic or anisotropic $\mathbb{R}^n$, our results provide new conclusions in view of the arbitrariness of the choice of the not necessarily Euclidean quasi-norm.

1. Introduction

Consider the following weighted Hardy–Sobolev type inequalities due to Caffarelli–Kohn–Nirenberg [5]: for all $f \in C_0^\infty(\mathbb{R}^n)$, it holds

$$\left(\int_{\mathbb{R}^n} \|x\|^{-p\beta} |f|^p \, \mathrm{d}x\right)^{\frac{2}{p}} \leq C_{\alpha,\beta} \int_{\mathbb{R}^n} \|x\|^{-2\alpha} |\nabla f|^2 \, \mathrm{d}x, \tag{1.1}$$

where, for $n \geq 3$

$$-\infty < \alpha < \frac{n-2}{2}, \quad \alpha \leq \beta \leq \alpha + 1, \quad \text{and} \quad p = \frac{2n}{n-2+2(\beta-\alpha)},$$

and, for $n = 2$

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$$-\infty < \alpha < 0, \quad \alpha < \beta \leq \alpha + 1, \text{ and } p = \frac{2}{\beta - \alpha},$$

and where $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the Euclidean norm. Nowadays, there is a lot of literature on Caffarelli–Kohn–Nirenberg type inequalities and their applications. In the case $p = 2$ (see for example [**25**]), for all $f \in C_0^\infty(\mathbb{R}^n)$, one has

$$\int_{\mathbb{R}^n} \|x\|^{-2(\alpha+1)} |f|^2 \, \mathrm{d}x \leq \widetilde{C}_\alpha \int_{\mathbb{R}^n} \|x\|^{-2\alpha} |\nabla f|^2 \, \mathrm{d}x, \tag{1.2}$$

with any $n \geq 2$ and $-\infty < \alpha < 0$, which in turn can be presented for any $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ as

$$\left\| \frac{1}{\|x\|^{\alpha+1}} |f| \right\|_{L^2(\mathbb{R}^n)} \leq C_\alpha \left\| \frac{1}{\|x\|^{\alpha}} |\nabla f| \right\|_{L^2(\mathbb{R}^n)}, \tag{1.3}$$

all $\alpha \in \mathbb{R}$.

Motivating the development of the analysis associated to homogeneous groups in [**12**], Folland and Stein raised an important question of determining which elements of the classical harmonic analysis do depend only on the group and the dilation structures. The natural setting for this kind of problem is that of homogeneous groups, in particular, including the cases of anisotropic structures on $\mathbb{R}^n$. In this paper, we show that the Caffarelli–Kohn–Nirenberg inequality continues to hold in the setting of homogeneous groups. In particular, it has to also hold on anisotropic $\mathbb{R}^n$, with a number of different consequences. Moreover, the quasi-norm $|\cdot|$ does not need to be Euclidean, but can be an arbitrary homogeneous quasi-norm on $\mathbb{R}^n$.

Recently, a homogeneous group version of the inequality (1.3) was obtained in the work [**21**], that is, it was proved that if $\mathbb{G}$ is a homogeneous group of homogeneous dimension Q, then, for all $f \in C_0^\infty(\mathbb{G} \backslash \{0\})$ and, for every homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$, we have

$$\frac{|Q - 2 - 2\alpha|}{2} \left\| \frac{f}{|x|^{\alpha+1}} \right\|_{L^2(\mathbb{G})} \leq \left\| \frac{1}{|x|^{\alpha}} \mathcal{R} f \right\|_{L^2(\mathbb{G})}, \quad \forall \alpha \in \mathbb{R}, \tag{1.4}$$

where $\mathcal{R}$ is defined by (2.4). Note that if $\alpha \neq \frac{Q-2}{2}$, then the constant in (1.4) is sharp for any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$. In the abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $Q = n$, $e(x) = x = (x_1, \ldots, x_n)$, so for each $\alpha \in \mathbb{R}$ with $\alpha \neq \frac{n-2}{2}$ and for any homogeneous quasi-norm $|\cdot|$ on $\mathbb{R}^n$, the inequality (1.4) implies a new inequality with the optimal constant:

$$\frac{|n - 2 - 2\alpha|}{2} \left\| \frac{f}{|x|^{\alpha+1}} \right\|_{L^2(\mathbb{R}^n)} \leq \left\| \frac{1}{|x|^{\alpha}} \frac{x}{|x|} \cdot \nabla f \right\|_{L^2(\mathbb{R}^n)}. \tag{1.5}$$

In turn, by using Schwarz's inequality with the standard Euclidean distance $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$, this implies the L^2 Caffarelli–Kohn–Nirenberg inequality [**5**] for $\mathbb{G} \equiv \mathbb{R}^n$ with the optimal constant:

$$\frac{|n-2-2\alpha|}{2}\left\|\frac{f}{\|x\|^{\alpha+1}}\right\|_{L^2(\mathbb{R}^n)} \leq \left\|\frac{1}{\|x\|^{\alpha}}\nabla f\right\|_{L^2(\mathbb{R}^n)}, \quad \forall\, \alpha \in \mathbb{R}, \tag{1.6}$$

for all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$. Here optimality of the constant $\frac{|n-2-2\alpha|}{2}$ was proved in [6, Theorem 1.1. (ii)]. In addition, we can also note that the analysis in these type inequalities and improvements of their remainder terms has a long history, initiated by Brezis and Nirenberg in [3] and then in [1] for Sobolev inequalities, and in [4] for Hardy inequalities, see also [2], with many subsequent works in this subject.

The L^2-inequality (1.3) in the anisotropic setting as well as in the more abstract setting of homogeneous groups was analyzed in [23] by using an explicit formula for the remainder that is available in the case of L^2-spaces. Such a remainder formula fails in the scale of L^p-spaces for $p \neq 2$ and, therefore, in this paper, we approach these inequalities by a different method.

Thus, the main aim of this paper is to extend the above inequality (1.4) to the general L^p case for all $1 < p < \infty$ by a different approach. Here, the first result of this paper is: for each $f \in C_0^\infty(\mathbb{G} \backslash \{0\})$, $1 < p < \infty$, and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$, we have

$$\frac{|Q-\gamma|}{p}\left\|\frac{f}{|x|^{\frac{\gamma}{p}}}\right\|^p_{L^p(\mathbb{G})} \leq \left\|\frac{1}{|x|^{\alpha}}\mathcal{R}f\right\|_{L^p(\mathbb{G})}\left\|\frac{f}{|x|^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{G})}, \quad \forall\, \alpha,\ \beta \in \mathbb{R}, \tag{1.7}$$

where $\gamma = \alpha + \beta + 1$, and the constant $\dfrac{|Q-\gamma|}{p}$ is sharp if $\gamma \neq Q$. Here $\mathcal{R}$ is the radial derivative operator on $\mathbb{G}$ defined by (2.4).

All above inequalities are generalizations of the classical Hardy inequality, which takes the form

$$\left\|\frac{f(x)}{\|x\|}\right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p}\|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad n \geq 2, \quad 1 \leq p < n, \tag{1.8}$$

where ∇ is the standard gradient in $\mathbb{R}^n$, $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and the constant $\frac{p}{n-p}$ is known to be sharp.

We refer to a recent interesting paper of Hoffmann-Ostenhof and Laptev [14] on this subject for inequalities with weights, to [13] for many-particle versions, to Ekholm–Kovařík–Laptev [10] for p-Laplacian interpretations, and to many further references therein and otherwise. We also refer to more recent preprints [18–23] and references therein for the story behind Hardy-type inequalities on nilpotent Lie groups. Moreover, we also note that we do not work with the 'horizontal' gradients since there may not be any for general homogeneous groups. If the group is stratified, also the horizontal versions are possible, and we refer to [24] for this subject.

Before giving preliminaries for stating our results, let us mention another observation that the Hardy inequality (1.8) can be sharpened to the inequality

$$\left\|\frac{f(x)}{\|x\|}\right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p}\|\mathcal{R}f\|_{L^p(\mathbb{R}^n)}, \quad n \geq 2, \quad 1 \leq p < n. \tag{1.9}$$

Here $\mathcal{R}$ is the radial derivative operator on $\mathbb{G}$ defined by (2.4). It is clear that (1.9) implies (1.8) since the function $\frac{x}{\|x\|}$ is bounded. The remainder terms for (1.9) have been analyzed by Ioku–Ishiwata–Ozawa [15], see also Machihara–Ozawa–Wadade [17].

One of the results in [21] was that if $\mathbb{G}$ is any homogeneous group and $|\cdot|$ is a homogeneous quasi-norm on $\mathbb{G}$, as an analogue of (1.9), we obtain the following generalized L^p-Hardy inequality:

$$\left\|\frac{f}{|x|}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p}\|\mathcal{R}f\|_{L^p(\mathbb{G})}, \quad 1<p<Q, \tag{1.10}$$

for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$. Here $\mathcal{R}$ is the radial derivative operator on $\mathbb{G}$ defined by (2.4). Let A be the n-diagonal matrix

$$A = \operatorname{diag}(\nu_1, \ldots, \nu_n), \tag{1.11}$$

where ν_k is the homogeneous degree of X_k, and

$$Q = \operatorname{Tr} A = \nu_1 + \cdots + \nu_n$$

is the homogeneous dimension of $\mathbb{G}$. We note that the exponential mapping $\exp_{\mathbb{G}}: \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism and the vector $e(x) = (e_1(x), \ldots, e_n(x))$ is the decomposition of its inverse $\exp_{\mathbb{G}}^{-1}$ with respect to the basis $\{X_1, \ldots, X_n\}$, namely $e(x)$ is determined by

$$\exp_{\mathbb{G}}^{-1}(x) = e(x)\cdot\nabla \equiv \sum_{j=1}^{n} e_j(x) X_j.$$

For $p=n$ or $p=Q$, the inequalities (1.8) and (1.10) fail for any constant. The critical versions of (1.9) with $p=n$ were investigated by Ioku–Ishiwata–Ozawa [16]. Their generalizations as well as a number of other critical (logarithmic) Hardy inequalities on homogeneous groups were obtained in recent works [18, 21, 22]. Here, we only mention a related family of logarithmic Hardy inequalities

$$\sup_{R>0}\left\|\frac{f-f_R}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{p-1}\left\|\frac{1}{|x|^{\frac{Q}{p}-1}}\mathcal{R}f\right\|_{L^p(\mathbb{G})}, \tag{1.12}$$

for all $1<p<\infty$, where $f_R = f(R\frac{x}{|x|})$. We refer to [18] for further explanations and extensions but only mention here that for $p=Q$, the inequality (1.12) gives a critical case of Hardy's inequalities (1.10).

In Section 2, we give some necessary tools on homogeneous groups and fix the notation. In Section 3, we present L^p-Caffarelli–Kohn–Nirenberg type inequalities on the homogeneous group $\mathbb{G}$ and then discuss their consequences and proofs. In Section 4, we discuss higher order cases.

2. Preliminaries

In this standard preliminary section, we very shortly recall some basics details of homogeneous groups. The general analysis on homogeneous groups was developed by Folland and Stein in their book [12], and we also refer for more recent developments to the monograph [11] by Véronique Fischer and the second named author.

It is known that a dilation family of a Lie algebra $\mathfrak{g}$ has the following representation:

$$D_\lambda = \operatorname{Exp}(A \ln \lambda) = \sum_{k=0}^{\infty} \frac{1}{k!}(\ln(\lambda)A)^k,$$

where A is a diagonalizable positive linear operator on $\mathfrak{g}$, and each family of linear mappings D_λ is a morphism of $\mathfrak{g}$, that is a linear mapping from $\mathfrak{g}$ to itself with the property

$$\forall X, Y \in \mathfrak{g}, \quad \lambda > 0, \quad [D_\lambda X, D_\lambda Y] = D_\lambda [X, Y],$$

where $[X, Y] := XY - YX$ is the Lie bracket. Shortly, a *homogeneous group* is a connected and simply connected Lie group whose Lie algebra is equipped with dilations. It induces the dilation structure on $\mathbb{G}$ which we continue to denote by $D_\lambda(x)$ or simply by λx. We denote by

$$Q := \operatorname{Tr} A$$

the homogeneous dimension of $\mathbb{G}$. We also recall that the standard Lebesque measure on $\mathbb{R}^N$ is the Haar measure for $\mathbb{G}$ (see, for example [11, Proposition 1.6.6]).

Let us fix a basis $\{X_1, \ldots, X_n\}$ of the Lie algebra $\mathfrak{g}$ of the homogeneous group $\mathbb{G}$ such that

$$AX_k = \nu_k X_k$$

for each $1 \leq k \leq n$, so that A can be taken to be

$$A = \operatorname{diag}(\nu_1, \ldots, \nu_n). \tag{2.1}$$

Then each X_k is homogeneous of degree ν_k and also

$$Q = \nu_1 + \cdots + \nu_n, \tag{2.2}$$

which is called a homogeneous dimension of $\mathbb{G}$. Since homogeneous groups are nilpotent, the exponential mapping $\exp_{\mathbb{G}}: \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism. In addition, the decomposition of $\exp_{\mathbb{G}}^{-1}(x)$ in the Lie algebra $\mathfrak{g}$ defines the vector

$$e(x) = (e_1(x), \ldots, e_n(x))$$

by the formula

$$\exp_{\mathbb{G}}^{-1}(x) = e(x) \cdot \nabla \equiv \sum_{j=1}^{n} e_j(x) X_j,$$

where $\nabla = (X_1, \ldots, X_n)$. Alternatively, this means the equality

$$x = \exp_{\mathbb{G}}(e_1(x)X_1 + \cdots + e_n(x)X_n).$$

Using homogeneity, we have

$$rx := D_r(x) = \exp_{\mathbb{G}}(r^{\nu_1}e_1(x)X_1 + \cdots + r^{\nu_n}e_n(x)X_n),$$

that is,

$$e(rx) = (r^{\nu_1}e_1(x), \ldots, r^{\nu_n}e_n(x)).$$

Thus, since $r > 0$ is arbitrary, without loss of generality taking $|x| = 1$, we obtain

$$\frac{d}{\mathrm{d}|rx|} f(rx) = \frac{d}{\mathrm{d}r} f(\exp_{\mathbb{G}}(r^{\nu_1}e_1(x)X_1 + \cdots + r^{\nu_n}e_n(x)X_n)). \tag{2.3}$$

Denoting by

$$\mathcal{R} := \frac{d}{\mathrm{d}r}, \tag{2.4}$$

for all $x \in \mathbb{G}$, this gives the equality

$$\frac{d}{\mathrm{d}|x|} f(x) = \mathcal{R}f(x), \tag{2.5}$$

for each homogeneous quasi-norm $|x|$ on a homogeneous group $\mathbb{G}$. That is, the operator $\mathcal{R}$ plays the role of the radial derivative on $\mathbb{G}$. It follows from (2.4) that $\mathcal{R}$ is homogeneous of order -1. It is known that every homogeneous group $\mathbb{G}$ admits a homogeneous quasi-norm $|\cdot|$. The $|\cdot|$-ball centred at $x \in \mathbb{G}$ with radius $R > 0$ can be defined by

$$B(x, R) := \{y \in \mathbb{G}\colon |x^{-1}y| < R\}.$$

The following polar decomposition was established in [**12**] (see also [**11**, Section 3.1.7]).

PROPOSITION 2.1 *Let $\mathbb{G}$ be a homogeneous group equipped with a homogeneous quasi-norm $|\cdot|$. Then there is a (unique) positive Borel measure σ on the unit quasi sphere*

$$\mathfrak{S} := \{x \in \mathbb{G}\colon\ |x| = 1\}, \tag{2.6}$$

such that for all $f \in L^1(\mathbb{G})$, we have

$$\int_{\mathbb{G}} f(x)\,\mathrm{d}x = \int_0^\infty \int_{\mathfrak{S}} f(ry)\,r^{Q-1}\,\mathrm{d}\sigma(y)\,\mathrm{d}r. \tag{2.7}$$

3. L^p-Caffarelli–Kohn-Nirenberg type inequalities and consequences

In this section and in the sequel, we adopt all the notation introduced in Section 2 concerning homogeneous groups and the operator $\mathcal{R}$. We formulate the following L^p-Caffarelli–Kohn–Nirenberg type inequalities on the homogeneous group $\mathbb{G}$ and then discuss their consequences and proofs.

Theorem 3.1 *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $\alpha, \ \beta \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, $1 < p < \infty$ and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$, we have*

$$\frac{|Q-\gamma|}{p}\left\|\frac{f}{|x|^{\frac{\gamma}{p}}}\right\|^p_{L^p(\mathbb{G})} \le \left\|\frac{1}{|x|^\alpha}\mathcal{R}f\right\|_{L^p(\mathbb{G})}\left\|\frac{f}{|x|^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{G})}, \tag{3.1}$$

where $\gamma = \alpha + \beta + 1$. If $\gamma \neq Q$, then the constant $\dfrac{|Q-\gamma|}{p}$ is sharp.

In the abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $Q = n$, $e(x) = x = (x_1, \ldots, x_n)$, so for any homogeneous quasi-norm $|\cdot|$ on $\mathbb{R}^n$ (3.1) implies a new inequality with the optimal constant

$$\frac{|n-\gamma|}{p}\left\|\frac{f}{|x|^{\frac{\gamma}{p}}}\right\|^p_{L^p(\mathbb{R}^n)} \le \left\|\frac{1}{|x|^\alpha}\frac{df}{d|x|}\right\|_{L^p(\mathbb{R}^n)}\left\|\frac{f}{|x|^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{R}^n)}, \tag{3.2}$$

which in turn, by using Schwarz's inequality with the standard Euclidean distance $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$, implies the L^p-Caffarelli–Kohn–Nirenberg type inequality (see [**8**, **9**]) for $\mathbb{G} \equiv \mathbb{R}^n$ with the sharp constant

$$\frac{|n-\gamma|}{p}\left\|\frac{f}{\|x\|^{\frac{\gamma}{p}}}\right\|^p_{L^p(\mathbb{R}^n)} \le \left\|\frac{1}{\|x\|^\alpha}\nabla f\right\|_{L^p(\mathbb{R}^n)}\left\|\frac{f}{\|x\|^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{R}^n)}, \tag{3.3}$$

for all $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$.

When $\alpha = 0$, $\beta = p-1$ and $1 < p < Q$, the inequality (3.1) gives the homogeneous group version of L^p-Hardy inequality

$$\left\|\frac{1}{|x|}f\right\|_{L^p(\mathbb{G})} \le \frac{p}{Q-p}\,\|\mathcal{R}f\|_{L^p(\mathbb{G})}, \tag{3.4}$$

again with $\frac{p}{Q-p}$ being the best constant (see [**18**–**24**] for weighted, critical, higher order cases, horizontal cases and their applications in different settings). Note that in comparison with stratified (Carnot) group versions, here the constant is best for any homogeneous quasi-norm $|\cdot|$.

In the abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, $n \geq 3$, we have $Q = n$, $e(x) = x = (x_1, \ldots, x_n)$, so for any quasi-norm $|\cdot|$ on $\mathbb{R}^n$, (3.4) implies the new inequality

$$\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \left\| \frac{df}{d|x|} \right\|_{L^p(\mathbb{R}^n)}. \tag{3.5}$$

In turn, by using Schwarz's inequality with the standard Euclidean distance $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$, it implies the classical Hardy inequality for $\mathbb{G} \equiv \mathbb{R}^n$

$$\left\| \frac{f}{\|x\|} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \|\nabla f\|_{L^p(\mathbb{R}^n)},$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. When $|x| \equiv \|x\|$ is the Euclidean distance the remainder terms for (3.5), that is, the exact formulae of the difference between the right-hand side and the left-hand side of the inequality have been analyzed by Ioku–Ishiwata–Ozawa [**15**], see also Machihara–Ozawa–Wadade [**17**] as well as [**16**].

The inequality (3.4) also implies the following Heisenberg–Pauli–Weyl type uncertainly principle on homogeneous groups (see for example [**7**, **19**, **20**] for versions of abelian and stratified groups): for each $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$, using Hölder's inequality and (3.4), we have

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{G})} &\leq \left\| \frac{1}{|x|} f \right\|_{L^p(\mathbb{G})} \||x|f\|_{L^{\frac{p}{p-1}}(\mathbb{G})} \\ &\leq \frac{p}{Q-p} \|\mathcal{R}f\|_{L^p(\mathbb{G})} \||x|f\|_{L^{\frac{p}{p-1}}(\mathbb{G})}, \quad 1 < p < Q, \end{aligned} \tag{3.6}$$

that is,

$$\|f\|^2_{L^2(\mathbb{G})} \leq \frac{p}{Q-p} \|\mathcal{R}f\|_{L^p(\mathbb{G})} \||x|f\|_{L^{\frac{p}{p-1}}(\mathbb{G})}, \quad 1 < p < Q. \tag{3.7}$$

In the abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, taking $Q = n$ and $e(x) = x$, we obtain that (3.7) with $p = 2$ implies the uncertainly principle with any quasi-norm $|x|$

$$\left(\int_{\mathbb{R}^n} |u(x)|^2 \, \mathrm{d}x \right)^2 \leq \left(\frac{2}{n-2} \right)^2 \int_{\mathbb{R}^n} \left| \frac{\mathrm{d}u(x)}{\mathrm{d}|x|} \right|^2 \mathrm{d}x \int_{\mathbb{R}^n} |x|^2 |u(x)|^2 \, \mathrm{d}x. \tag{3.8}$$

In turn, it implies the classical uncertainty principle for $\mathbb{G} \equiv \mathbb{R}^n$ with the standard Euclidean distance $\|x\|$

$$\left(\int_{\mathbb{R}^n} |u(x)|^2 \, \mathrm{d}x \right)^2 \leq \left(\frac{2}{n-2} \right)^2 \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}^n} \|x\|^2 |u(x)|^2 \, \mathrm{d}x,$$

which is the Heisenberg–Pauli–Weyl uncertainly principle on $\mathbb{R}^n$.

On the other hand, directly from the inequality (3.1), we can obtain a number of Heisenberg–Pauli–Weyl type uncertainly inequities which have various consequences and applications. For example, when $\alpha p = \alpha + \beta + 1$, we have

$$\frac{|Q - \alpha p|}{p} \left\| \frac{f}{|x|^{\alpha}} \right\|^{p}_{L^p(\mathbb{G})} \le \left\| \frac{\mathcal{R}f}{|x|^{\alpha}} \right\|_{L^p(\mathbb{G})} \left\| |x|^{\frac{1}{p-1}} \frac{f}{|x|^{\alpha}} \right\|^{p-1}_{L^p(\mathbb{G})}, \tag{3.9}$$

and, on the other hand, if $0 = \alpha + \beta + 1$ and $\alpha = -p$, then

$$\frac{Q}{p} \|f\|^p_{L^p(\mathbb{G})} \le \| |x|^p \mathcal{R} f \|_{L^p(\mathbb{G})} \left\| \frac{f}{|x|} \right\|^{p-1}_{L^p(\mathbb{G})}, \tag{3.10}$$

all with sharp constants.

Proof of Theorem 3.1. We may assume that $\gamma \neq Q$ since for $\gamma = Q$, there is nothing to prove. Introducing polar coordinates $(r, y) = \left(|x|, \frac{x}{|x|}\right) \in (0, \infty) \times \mathfrak{S}$ on $\mathbb{G}$, where $\mathfrak{S}$ is the pseudo-sphere in (2.6), and using Proposition 2.1, one calculates

$$\begin{aligned}
\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\gamma}}\, dx &= \int_0^{\infty} \int_{\mathfrak{S}} \frac{|f(ry)|^p}{r^{\gamma}} r^{Q-1}\, d\sigma(y)\, dr \\
&= \frac{1}{Q-\gamma} \int_0^{\infty} \int_{\mathfrak{S}} |f(ry)|^p \frac{dr^{Q-\gamma}}{dr}\, d\sigma(y)\, dr \\
&= -\frac{1}{Q-\gamma} \operatorname{Re} \int_0^{\infty} \int_{\mathfrak{S}} p f(ry) |f(ry)|^{p-2} \left(\overline{\frac{df(ry)}{dr}} \right) \frac{1}{r^{\gamma-1}} r^{Q-1}\, d\sigma(y)\, dr \\
&= -\frac{p}{Q-\gamma} \operatorname{Re} \int_{\mathbb{G}} f(x) \frac{|f(x)|^{p-2}}{|x|^{\gamma-1}} \left(\overline{\frac{d}{d|x|} f(x)} \right) dx \\
&\le \left| \frac{p}{Q-\gamma} \right| \int_{\mathbb{G}} \frac{|f(x)|^{p-1}}{|x|^{\gamma-1}} \left| \frac{d}{d|x|} f(x) \right| dx \\
&= \left| \frac{p}{Q-\gamma} \right| \int_{\mathbb{G}} \frac{|f(x)|^{p-1}}{|x|^{\alpha+\beta}} |\mathcal{R} f(x)|\, dx \\
&\le \left| \frac{p}{Q-\gamma} \right| \left(\int_{\mathbb{G}} \frac{|\mathcal{R} f(x)|^p}{|x|^{\alpha p}}\, dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\frac{\beta p}{p-1}}}\, dx \right)^{\frac{p-1}{p}},
\end{aligned}$$

where we have used Hölder's inequality. Thus, we arrive at

$$\left| \frac{Q-\gamma}{p} \right| \int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\gamma}}\, dx \le \left(\int_{\mathbb{G}} \frac{|\mathcal{R} f(x)|^p}{|x|^{\alpha p}}\, dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\frac{\beta p}{p-1}}}\, dx \right)^{\frac{p-1}{p}}. \tag{3.11}$$

Now let us show the sharpness of the constant. We need to examine the equality condition in above Hölder's inequality as in the Euclidean case (see [9]). Consider the function

$$g(x)=\begin{cases} e^{-\frac{C}{\lambda}|x|^{\lambda}}, & \lambda := \alpha-\dfrac{\beta}{p-1}+1\neq 0,\\ \dfrac{1}{|x|^{C}}, & \alpha-\dfrac{\beta}{p-1}+1=0, \end{cases} \tag{3.12}$$

where $C=\left|\frac{Q-\gamma}{p}\right|$ and $\gamma\neq Q$. Then, it can be checked that

$$\left|\frac{p}{Q-\gamma}\right|^{p}\frac{|\mathcal{R}g(x)|^{p}}{|x|^{\alpha p}}=\frac{|g(x)|^{p}}{|x|^{\frac{\beta p}{p-1}}}, \tag{3.13}$$

which satisfies the equality condition in Hölder's inequality. This shows that the constant $C=\left|\frac{Q-\gamma}{p}\right|$ is sharp. □

4. Higher order cases

In this section, we shortly discuss that by iterating the established L^{p}-Caffarelli–Kohn–Nirenberg type inequalities, one can get inequalities of higher order. To start let us consider in (3.1), the case

$$\beta=\gamma\left(1-\frac{1}{p}\right),$$

that is, taking $\beta=(\alpha+1)(p-1)$, the inequality (3.1) implies that $\gamma=p(\alpha+1)$ and

$$\left\|\frac{f}{|x|^{\alpha+1}}\right\|_{L^{p}(\mathbb{G})}\leq\frac{p}{|Q-p(\alpha+1)|}\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}f\right\|_{L^{p}(\mathbb{G})},\quad 1<p<\infty, \tag{4.1}$$

for any $f\in C_{0}^{\infty}(\mathbb{G}\backslash\{0\})$ and all $\alpha\in\mathbb{R}$ with $\alpha\neq\frac{Q}{p}-1$.

Now putting $\mathcal{R}f$ instead of f and $\alpha-1$ instead of α in (4.1), we consequently have

$$\left\|\frac{\mathcal{R}f}{|x|^{\alpha}}\right\|_{L^{p}(\mathbb{G})}\leq\frac{p}{|Q-p\alpha|}\left\|\frac{1}{|x|^{\alpha-1}}\mathcal{R}^{2}f\right\|_{L^{p}(\mathbb{G})},$$

for $\alpha\neq\frac{Q}{p}$. Combining it with (4.1), we get

$$\left\|\frac{f}{|x|^{\alpha+1}}\right\|_{L^{p}(\mathbb{G})}\leq\frac{p}{|Q-p(\alpha+1)|}\frac{p}{|Q-p\alpha|}\left\|\frac{1}{|x|^{\alpha-1}}\mathcal{R}^{2}f\right\|_{L^{p}(\mathbb{G})}, \tag{4.2}$$

for each $\alpha\in\mathbb{R}$ such that $\alpha\neq\frac{Q}{p}-1$ and $\alpha\neq\frac{Q}{p}$. This iteration process gives

$$\left\| \frac{f}{|x|^{\theta+1}} \right\|_{L^p(\mathbb{G})} \le A_{\theta,k} \left\| \frac{1}{|x|^{\theta+1-k}} \mathcal{R}^k f \right\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \tag{4.3}$$

for any $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ and all $\theta \in \mathbb{R}$ such that $\prod_{j=0}^{k-1} |Q - p(\theta + 1 - j)| \neq 0$, and

$$A_{\theta,k} := p^k \left[\prod_{j=0}^{k-1} |Q - p(\theta + 1 - j)| \right]^{-1}.$$

Similarly, we have

$$\left\| \frac{\mathcal{R} f}{|x|^{\vartheta+1}} \right\|_{L^p(\mathbb{G})} \le A_{\vartheta,m} \left\| \frac{1}{|x|^{\vartheta+1-m}} \mathcal{R}^{m+1} f \right\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \tag{4.4}$$

for any $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ and all $\vartheta \in \mathbb{R}$ such that $\prod_{j=0}^{m-1} |Q - p(\vartheta + 1 - j)| \neq 0$, and

$$A_{\vartheta,m} := p^m \left[\prod_{j=0}^{m-1} |Q - p(\vartheta + 1 - j)| \right]^{-1}.$$

Now putting $\vartheta + 1 = \alpha$ and $\theta + 1 = \frac{\beta}{p-1}$ into (4.4) and (4.3), respectively, from (3.1), we obtain

PROPOSITION 4.1 *Let* $1 < p < \infty$. *For any* $k, m \in \mathbb{N}$, *we have*

$$\frac{|Q-\gamma|}{p} \left\| \frac{f}{|x|^{\frac{\gamma}{p}}} \right\|^p_{L^p(\mathbb{G})} \le \widetilde{A}_{\alpha,m} \widetilde{A}_{\beta,k} \left\| \frac{1}{|x|^{\alpha-m}} \mathcal{R}^{m+1} f \right\|_{L^p(\mathbb{G})} \left\| \frac{1}{|x|^{\frac{\beta}{p-1}-k}} \mathcal{R}^k f \right\|^{p-1}_{L^p(\mathbb{G})}, \tag{4.5}$$

for any complex-valued function $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, $\gamma = \alpha + \beta + 1$, *and* $\alpha \in \mathbb{R}$ *such that* $\prod_{j=0}^{m-1} |Q - p(\alpha - j)| \neq 0$, *and*

$$\widetilde{A}_{\alpha,m} := p^m \left[\prod_{j=0}^{m-1} |Q - p(\alpha - j)| \right]^{-1},$$

as well as $\beta \in \mathbb{R}$ *such that* $\prod_{j=0}^{k-1} \left| Q - p\left(\frac{\beta}{p-1} - j \right) \right| \neq 0$, *and*

$$\widetilde{A}_{\beta,k} := p^{k(p-1)} \left[\prod_{j=0}^{k-1} \left| Q - p\left(\frac{\beta}{p-1} - j \right| \right) \right]^{-(p-1)}.$$

We also highlight the case $p = 2$. In this case, an interesting feature is that when we have the exact formula for the remainder which yields the sharpness of the constants as well. We first recall the following estimate and formula.

THEOREM 4.2 ([21]). *Let* $Q \geq 3$, $\alpha \in \mathbb{R}$ *and* $k \in \mathbb{N}$ *be such that we have* $\prod_{j=0}^{k-1} \left|\frac{Q-2}{2} - (\alpha + j)\right| \neq 0$. *Then for all complex-valued functions* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, *we have*

$$\left\|\frac{f}{|x|^{k+\alpha}}\right\|_{L^2(\mathbb{G})} \leq \left[\prod_{j=0}^{k-1} \left|\frac{Q-2}{2} - (\alpha + j)\right|\right]^{-1} \left\|\frac{1}{|x|^\alpha}\mathcal{R}^k f\right\|_{L^2(\mathbb{G})}, \tag{4.6}$$

where the constant above is sharp, and is attained if and only if $f = 0$.

Moreover, for all $k \in \mathbb{N}$ *and* $\alpha \in \mathbb{R}$, *the following equality holds*:

$$\begin{aligned} \left\|\frac{1}{|x|^\alpha}\mathcal{R}^k f\right\|^2_{L^2(\mathbb{G})} &= \left[\prod_{j=0}^{k-1} \left(\frac{Q-2}{2} - (\alpha + j)\right)^2\right] \left\|\frac{f}{|x|^{k+\alpha}}\right\|^2_{L^2(\mathbb{G})} \\ &+ \sum_{l=1}^{k-1}\left[\prod_{j=0}^{l-1} \left(\frac{Q-2}{2} - (\alpha + j)\right)^2\right] \left\|\frac{1}{|x|^{l+\alpha}}\mathcal{R}^{k-l} f \right. \\ &\left. + \frac{Q - 2(l+1+\alpha)}{2|x|^{l+1+\alpha}}\mathcal{R}^{k-l-1} f\right\|^2_{L^2(\mathbb{G})} \\ &+ \left\|\frac{1}{|x|^\alpha}\mathcal{R}^k f + \frac{Q-2-2\alpha}{2|x|^{1+\alpha}}\mathcal{R}^{k-1} f\right\|^2_{L^2(\mathbb{G})}. \end{aligned} \tag{4.7}$$

When $p = 2$, Theorem 3.1 can be restated that for each $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$, we have

$$\frac{|Q-\gamma|}{2} \left\|\frac{f}{|x|^{\frac{\gamma}{2}}}\right\|^2_{L^2(\mathbb{G})} \leq \left\|\frac{1}{|x|^\alpha}\mathcal{R} f\right\|_{L^2(\mathbb{G})} \left\|\frac{f}{|x|^\beta}\right\|_{L^2(\mathbb{G})}, \quad \forall \alpha, \ \beta \in \mathbb{R}, \tag{4.8}$$

where $\gamma = \alpha + \beta + 1$. Combining (4.8) with (4.6) (or (4.7)), one can obtain a number of inequalities with sharp constants, for example

$$\frac{|Q-\gamma|}{2} \left\|\frac{f}{|x|^{\frac{\gamma}{2}}}\right\|^2_{L^2(\mathbb{G})} \leq C_j(\beta, k) \left\|\frac{1}{|x|^\alpha}\mathcal{R} f\right\|_{L^2(\mathbb{G})} \left\|\frac{1}{|x|^{\beta-k}}\mathcal{R}^k f\right\|_{L^2(\mathbb{G})}, \tag{4.9}$$

for $\gamma = \alpha + \beta + 1$ and all $\alpha, \ \beta \in \mathbb{R}$ and $k \in \mathbb{N}$, such that,

$$C_j(\beta, k) := \left[\prod_{j=0}^{k-1} \left|\frac{Q-2}{2} - (\beta - k + j)\right|\right]^{-1} \neq 0,$$

as well as

$$\frac{|Q-\gamma|}{2}\left\|\frac{f}{|x|^{\frac{\gamma}{2}}}\right\|^2_{L^2(\mathbb{G})} \leq C_j(\alpha,k)\left\|\frac{1}{|x|^{\alpha-k}}\mathcal{R}^{k+1}f\right\|_{L^2(\mathbb{G})}\left\|\frac{f}{|x|^{\beta}}\right\|_{L^2(\mathbb{G})}, \tag{4.10}$$

for $\gamma=\alpha+\beta+1$ and all $\alpha,\beta\in\mathbb{R}$ and $k\in\mathbb{N}$, such that,

$$C_j(\alpha,k) := \left[\prod_{j=0}^{k-1}\left|\frac{Q-2}{2}-(\alpha+k+j)\right|\right]^{-1} \neq 0.$$

It follows from (4.7) that these constants $C_j(\beta,k)$ and $C_j(\alpha,k)$ in (4.9) and (4.10) are sharp.

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