

The lumped model parameters approach for static and dynamic power-law beam problems

by

Madi Begzhigitov

Submitted to the Department of Mathematics
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Author
Department of Mathematics
Apr 29, 2019

Certified by
Piotr Skrzypacz
Assistant Professor
Thesis Supervisor

Accepted by
Vassilios D. Tourassis
Dean, School of Science and Technology

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Abstract

It is important to estimate the natural frequencies of the structural elements in the design of mechanical or electromechanical structures. There is a wide use of single lumped-parameter spring-mass models in the industry for materials. Their behaviour is linear by Hooke's law within the geometric and loading conditions. In this work, the lumped-parameter theory is generalized for Hollomon's power-law materials and the lumped-parameters for the corresponding nonlinear restoring force in the spring-like model for the standard geometric and loading conditions of the power-law Euler-beams are provided. For each case in the given lumped-parameter model the corresponding effective mass is also calculated. Then, the resulting spring-mass system is solved to validate the solutions as approximations to the corresponding beam system. Numerical validations of the proposed lumped models for the cantilever beam with circular and rectangular cross-sections are presented.

Thesis Supervisor: Piotr Skrzypacz

Title: Assistant Professor

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I am also grateful for the support of Professor Dongming Wei who has shared the knowledge from the main references [1], [8], [10] and [11] authored by his NU ORAU grant research team members. Another great person to thank is Professor Daulet Nurakhmetov for his supportive help. The results of this thesis is based on the the previous work of the team.

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Chapter 1

Introduction

The power-law defined as

$$\sigma = K |\varepsilon|^{n-1} \varepsilon$$

is a well-known constitutive equation which is commonly used in the modeling of the axial stress-strain relations for annealed metals, where stress is denoted by σ , the strain by ε , and K and n ($0 < n < 1$) are material constants. The above power-law equation is used to model the work-hardening or strain-hardening materials which is sometimes referred to as Ludwick or Hollomon materials. For some frequent annealed metals the values of K and n can be taken from the engineering literature or undergraduate textbooks, e.g., see [2] - [4].

These materials and the power-law equation are broadly presented to the sophomore level undergraduate engineering students. There are more complex beam equations that model the power-law materials but their accessibility is limited to the students, see, e.g., [5]- [7]. Nevertheless, in the literature there is a shortage of mathematical models providing simple bench mark analytic or numerical solutions for modeling mechanical structures made of these materials.

In the literature about power-law Euler-Bernoulli beams, only the cantilever and micro-bridge beam designs have been obtained. The following work can be useful for design and construction of MEMS. In [18] authors provide the pull-in voltage analysis of electrostatically actuated structures of beams with clamped-clamped and

clamped-free end conditions. In the manuscript by Skrzypacz, Nurakhmetov and Wei [8] lumped model parameters have been obtained for several power-law beams. Wei and Liu [1] also derived some analytic and finite element solutions for the power-law Euler-Bernoulli beams. In addition, many attempts and assumptions are being made to make the design of beams as simple as possible.

In this work, the nonlinear beam equation for the power-law materials is considered and the lumped-parameter modeling elements useful for MEMS are studied.

The deflection of Euler-Bernoulli beams made of power-law materials can be described by the following nonlinear differential equation:

$$\frac{d^2}{dx^2} \left(K I_n \left| \frac{d^2 v}{dx^2} \right|^{n-1} \frac{d^2 v}{dx^2} \right) - f(x) = 0, \quad 0 < x < l, \quad (1.1)$$

where x is the axial location, $v(x)$ is the transversal deflection of the beam, $f(x)$ is the external force per unit length which acts on the beam, and

$$I_n = \int_A |y|^{n+1} dy dx \quad (1.2)$$

is the generalized second moment of inertia of the beam with cross section A , see [10]. Furthermore, it is assumed that I_n is independent of the axial location x .

The study of the lumped-parameter modeling elements includes stiffness and effective mass coefficients for various types of Euler-Bernoulli beams. The stiffness and effective mass parameters are calculated using static solutions to power-law Euler-Bernoulli beams. These valuable parameters will be used to study dynamic behaviour of micro power-law Euler-Bernoulli beams. Hence, the objective of this work is to obtain the stiffness and effective mass coefficients for different types of beams in order to study their dynamic behaviour. The intensive symbolic calculations will be supported by the computer algebra system Maple. The outcomes of this work can be used by researchers to predict more precisely the sensitivity and frequencies of MEMS.

Chapter 2

The model equation for power-law Euler-Bernoulli beams

In the following, the model equation for the power-law Euler-Bernoulli beams will be derived based on the results from [10]. Let

$$\begin{aligned}\epsilon_x &= \frac{\partial u}{\partial x}, \\ \epsilon_y &= \frac{\partial v}{\partial y}, \\ \epsilon_z &= \frac{\partial w}{\partial z}, \\ \gamma_{xy} &= \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \\ \gamma_{yz} &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), \\ \gamma_{zx} &= \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)\end{aligned}$$

be the strain components,

$$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$$

indicate the relative components of stress and let $\mathbf{u} = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$ be the displacement vector. Let

$$D(\mathbf{u}) = \begin{pmatrix} \epsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_z \end{pmatrix}$$

and

$$|D(\mathbf{u})| = \sqrt{\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 + 2\gamma_{xy}^2 + 2\gamma_{yz}^2 + 2\gamma_{zx}^2}$$

A general form of the Hollomon equation can be written in the form of [?]:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = C \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \quad (2.1)$$

where

$$C = \frac{K |D(\mathbf{u})|^{n-1}}{(1+\nu)(1-2\nu)}$$

with ν and K denoting material constants. In the case of $n = 1$, K equals the Young's modulus of linear elasticity and ν the appropriate Poisson's ratio. The relation (2.1), with the assumption that $\sigma_z = \sigma_{yz} = \sigma_{zx} = 0$ for plane stress problems can be simplified to

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{K}{1-\nu^2} |D(\mathbf{u})|^{n-1} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}. \quad (2.2)$$

If $\epsilon_z = \epsilon_{yz} = \epsilon_{zx} = 0$, then identical version can be also obtained for a plane strain. Now, let us introduce $I(\mathbf{u})$ which refers to the Lagrangian energy functional for a power-law elastoplastic material occupying a three dimension body V . This can be

defined by the difference of kinetic energy and the elastoplastic potential energy added to the work done by external forces. It can be written in the following form:

$$I(\mathbf{u}) = \frac{1}{2} \int_V \rho \dot{\mathbf{u}} \dot{\mathbf{u}}^\tau dV - \frac{1}{n+1} \int_V \sigma \epsilon^\tau dV + \int_V \mathbf{f} \mathbf{u}^\tau dV + \int_{\partial V} \mathbf{t} \mathbf{u}^\tau dS \quad (2.3)$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$, and $\boldsymbol{\epsilon} = (\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})$, $\mathbf{f} = (f_x, f_y, f_z)$ is the body force, $\dot{\mathbf{u}} = (\dot{u}, \dot{v}, \dot{w})$ the velocity, ρ the density, $\mathbf{t} = (t_x, t_y, t_z)$ the surface force. It is assumed that for the power-law Euler-Bernoulli beam the components of the displacement satisfy

$$\begin{aligned} u &= -y \frac{\partial v}{\partial x}, \\ v &= v(x, t), \\ w &= 0, \\ \mathbf{f} &= (0, r(x, t), 0), \\ \mathbf{t} &= (0, 0, 0). \end{aligned}$$

Consequently,

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = -y \frac{\partial^2 v}{\partial x^2}, \\ \epsilon_{xy} &= \frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) = 0, \\ \epsilon_y &= \epsilon_{xz} = \epsilon_{yz} = \epsilon_z = 0. \end{aligned}$$

Then, the Lagrangian for the power-law beam is defined as follows

$$I(v) = \frac{1}{2} \int_0^l \rho \dot{v}^2 + \frac{1}{n+1} \int_0^l K I_n \left| \frac{\partial^2 v}{\partial x^2} \right|^{n+1} dx - \int_0^l f(x) v(x) dx, \quad (2.4)$$

where $I_n = \int_A |y|^{n+1} dy dz$ denotes the beam's generalized moment of inertia. Here, A stands for the cross sectional area of the beam. By the use of principle of virtual work, the following fourth-order PDE describing the vertical beam displacement $v(t, x)$ at

the time t and at the axial location x can be derived

$$\rho \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2}{\partial x^2} \left(K I_n \left| \frac{\partial^2 v}{\partial x^2} \right|^{n-1} \frac{\partial^2 v}{\partial x^2} \right) - f(x) = 0, \quad 0 < x < l, \quad (2.5)$$

where $f(x)$ is the vertical load.

Chapter 3

Methods for calculating the lumped parameters

3.1 Stiffness coefficient

In order to model the behaviour and analyze the motion of micro-structures it is important to define their stiffness properties. In the case of Hook's materials ($n=1$), this means defining the coefficient

$$k = \frac{F}{|v_{max}|}$$

where F is load and v_{max} is the maximum deflection. This indicates the equivalent stiffness coefficient of the micro-structure and it can be used as a spring constant in the model. There are several ways of obtaining the stiffness coefficient: experimental, analytical and computational methods, see [13]. If possible, all the available and possible techniques should be used and then compared among them to come-up with a precise estimation. The stiffness coefficient depends on moment or the kind of a force affecting the micro-structure. For instance, k for a cantilever beam with a point load at the tip is different from that with distributed load. Thus, when obtaining k using experimental, analytical or computational technique it is essential to account to this fact in comparison with what the device experiences under the actual operating

conditions. But, anyway, due to different loads the extracted coefficients can be related and by knowing the other one can infer one. For example, k of a cantilever beam with distributed load (pressure) can be defined given k of the beam with a point load at the tip. Another fact to note is that the influence of residual stresses can cardinally affect the stiffness of micro-structures of immovable edges, such as clamped-clamped beams, and consequently need to be calculated precisely in the estimation of stiffness.

In the case of beams made of power-law materials, the following stiffness coefficients is introduced

$$k_n = \frac{F}{|v_{max}|^{n-1} v_{max}}. \quad (3.1)$$

3.2 Effective mass

Another essential coefficient for analyzing the motion of micro-structures is the effective mass coefficient which implies the fact that not all of the portions of a flexible micro-structure of distributed mass may co-act in a particular mode of motion nor do they essentially co-act in the same ratio. For instance, the mass near the support of the cantilever beam mostly does not contribute to the motion. The pre-knowledge of the stiffness coefficient of a micro-structure and the system's natural frequency near the mode of interest to identify an effective mass for the system is one of the methods to construct a spring-mass model. The natural frequency of the micro-structure can be identified experimentally, by Rayleigh's quotient methods and by finite-element methods, see [17]. Knowing the motion of each point in the system allows the use of the energy method for distributed mass or multimass systems. In systems where masses are linked by levers, gears, or rigid links, the motion of different masses can be shown via the trajectory $X(t)$ of some certain point and the system is simply one of a single degree-of-freedom (DOF), because there is only one necessary coordinate. The kinetic energy at the time t can be written in the following form:

$$E_{kin} = \frac{1}{2} m_{eff} \dot{X}^2, \quad (3.2)$$

where m_{eff} denotes an equivalent lumped mass or effective mass at the specified point. Also, knowing the stiffness coefficient at that point, the natural frequency can be found from the simple equation

$$\omega_n = \sqrt{\frac{k}{m_{eff}}}. \quad (3.3)$$

In the systems of distributed mass such as beams and springs, a pre-knowledge of the vibration amplitude's distribution becomes essential before the calculation of kinetic energy. Rayleigh showed that it is possible with a justified assumption from the shape of the vibration amplitude, the masses that were previously ignored could be taken into account and for the fundamental frequency arrive at the better estimate, see [19].

3.3 Computing the effective mass

Let us consider the cantilever beam having mass m and let $X(t)$ denote its deflection at the free end. The goal in this section is to demonstrate how to calculate the effective mass for the mass lumped model of the power-law Euler-Bernoulli cantilever beams made of the well-known power-law materials which are indicated in Table 6.2, [3]. First the following initial value problem for the deflection of the cantilever beam at its tip is considered

$$m_{eff}(n)\ddot{X} + k_n |X|^{n-1} X = 0 \quad (3.4)$$

subject to the initial conditions

$$\begin{aligned} X(0) &= \left(\frac{P}{2KI_n} \right)^{\frac{1}{n}} \frac{nl_n^{\frac{2}{n}+2}}{2(n+1)}, \\ \dot{X}(0) &= 0. \end{aligned} \quad (3.5)$$

The external force F at the tip acts only at $t = 0$ and is directly released. Here, $m_{eff}(n)$ denotes the effective mass, and k_n is the stiffness coefficient. To identify the equivalent mass $m_{eff} = m_{eff}(n)$ for the mass lumped model of the cantilever beam the Rayleigh method will be used. To this end, let the kinetic energy of the cantilever

beam have a form of

$$E_{kin} = \frac{1}{2} m_{eff} v_t^2(t, l), \quad (3.6)$$

where $v_t(t, l)$ stands for the velocity of the cantilever tip which has a point load at time t .

In section 4.1 it will be shown that, the deflection of the beam $v(x)$ can be written as

$$v(x) = v(l) \frac{2(n+1)}{nl^{\frac{2}{n}+2}} \left\{ \frac{(l-x)^{\frac{2}{n}+2}}{\left(\frac{2}{n}+1\right)\left(\frac{2}{n}+2\right)} + \frac{l^{\frac{2}{n}+1}x}{\frac{2}{n}+1} - \frac{l^{\frac{2}{n}+2}}{\left(\frac{2}{n}+1\right)\left(\frac{2}{n}+2\right)} \right\}.$$

In the method of Rayleigh [13], it is assumed that

$$v(t, x) = v(t, l) \frac{2(n+1)}{nl^{\frac{2}{n}+2}} \left\{ \frac{(l-x)^{\frac{2}{n}+2}}{\left(\frac{2}{n}+1\right)\left(\frac{2}{n}+2\right)} + \frac{l^{\frac{2}{n}+1}x}{\frac{2}{n}+1} - \frac{l^{\frac{2}{n}+2}}{\left(\frac{2}{n}+1\right)\left(\frac{2}{n}+2\right)} \right\}. \quad (3.7)$$

Then, (3.7) is differentiated with respect to time while keeping x fixed:

$$v_t(t, x) = v_t(t, l) \frac{2(n+1)}{nl^{\frac{2}{n}+2}} \left\{ \frac{(l-x)^{\frac{2}{n}+2}}{\left(\frac{2}{n}+1\right)\left(\frac{2}{n}+2\right)} + \frac{l^{\frac{2}{n}+1}x}{\frac{2}{n}+1} - \frac{l^{\frac{2}{n}+2}}{\left(\frac{2}{n}+1\right)\left(\frac{2}{n}+2\right)} \right\}. \quad (3.8)$$

Next, the the kinetic energy of the whole beam of length l can be determined as

$$E_{kin} = \int_0^l \frac{1}{2} v_t^2(t, x) \hat{m}(x) dx, \quad (3.9)$$

where $\hat{m}(x) = \hat{m} = \frac{m}{l}$ stands for the constant mass per unit length. The numeric and symbolic computing environment Maple is used to find the above integral. This results in the following form of the kinetic energy

$$E_{kin} = \frac{9n^3 + 22n^2 + 17n + 4}{90n^3 + 177n^2 + 114n + 24} m v_t^2(t, l) \quad (3.10)$$

Then, the outcomes of (3.6) and (3.10) are equated which leads to

Corollary 1 *The effective mass for the cantilever power-law beam of mass m and with a proof mass at the tip is given by*

$$m_{eff}(n) = \mu_{eff}(n) m ,$$

where

$$\mu_{eff}(n) = \frac{18n^3 + 44n^2 + 34n + 8}{90n^3 + 177n^2 + 114n + 24} . \quad (3.11)$$

is the effective mass coefficient. □

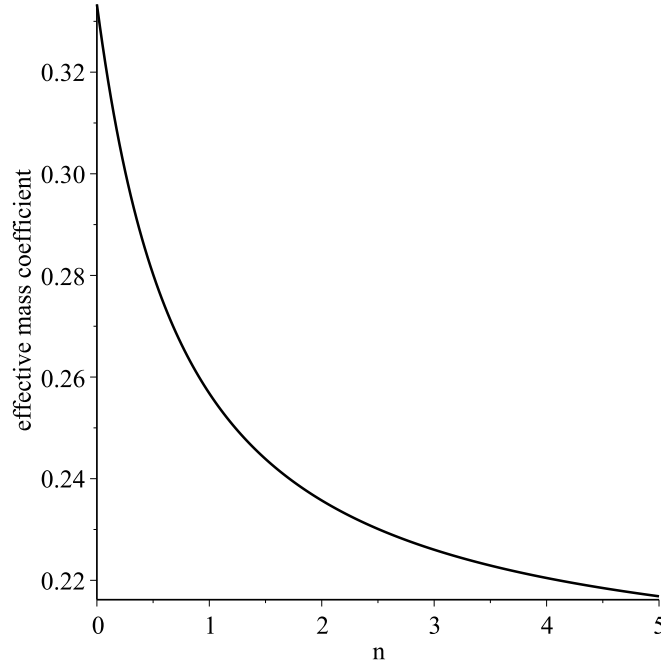


Figure 3-1: The effective mass coefficient μ_{eff} as a function of power-law index.

Notice that the effective mass m_{eff} for the case of $n = 1$ becomes

$$m_{eff}(1) = \frac{104}{405} m ,$$

see [13].

Figure 3-1 shows the monotonicity of the effective mass coefficient $\mu_{eff}(n)$ defined in (3.11).

Notice that

$$\frac{1}{5}m < m_{eff} < \frac{1}{3}m$$

where

$$\lim_{n \rightarrow \infty} m_{eff}(n) = \frac{1}{5}m \quad \text{and} \quad \lim_{n \rightarrow 0^+} m_{eff}(n) = \frac{1}{3}m.$$

Chapter 4

Selected power-law beams

4.1 Cantilever beam under uniformly distributed pressure

Let us view the cantilever beam under uniformly distributed pressure. In Figure 1 $f(x) = P = F/l$ is constant with respect to x . Consequently, (1.1) has the following identical form

$$\frac{d^2}{dx^2} \left(\left| \frac{d^2 v}{dx^2} \right|^{n-1} \frac{d^2 v}{dx^2} \right) = \frac{P}{K I_n}, \quad 0 < x < l \quad (4.1)$$

with boundary conditions:

$$v(0) = 0, \quad v'(0) = 0 \quad (4.2)$$

and

$$v''(l) = 0, \quad v'''(l) = 0. \quad (4.3)$$

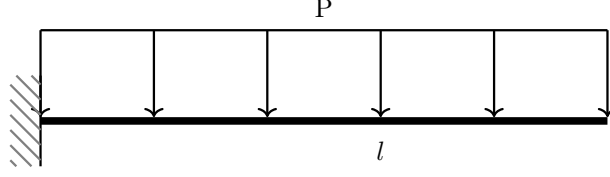


Figure 1: Cantilever beam under uniformly distributed pressure.

Next, (4.1) have been integrated twice to get

$$\left| \frac{d^2 v}{dx^2} \right|^{n-1} \frac{d^2 v}{dx^2} = \frac{P}{2KI_n} x^2 + C_1 x + C_2 \quad (4.4)$$

Then the derivative of (4.4) and the boundary conditions at $x = l$ are used to determine the integration constant C_1

$$\frac{d}{dx} \left(\left| \frac{d^2 v}{dx^2} \right|^{n-1} \frac{d^2 v}{dx^2} \right) \Big|_{x=l} = \left(n \left| \frac{d^2 v}{dx^2} \right|^{n-1} \frac{d^3 v}{dx^3} \right) \Big|_{x=l} = 0 = \frac{Pl}{KI_n} + C_1$$

Due to (4.4) and (4.3), $C_1 = -\frac{Pl}{KI_n}$ and $C_2 = \frac{Pl^2}{2KI_n}$. Then, the following equation shows (4.4) in the form of complete squares

$$\left| \frac{d^2 v}{dx^2} \right|^{n-1} \frac{d^2 v}{dx^2} = \frac{P}{2KI_n} (l - x)^2. \quad (4.5)$$

Two cases $\frac{d^2 v}{dx^2} \geq 0$ if $P > 0$ and $\frac{d^2 v}{dx^2} \leq 0$ if $P < 0$ arise from (4.5) Consequently, it implies that their solutions are built in a similar way. For the analytic solution $v(x)$ it follows that

$$\begin{aligned} \frac{d^2 v}{dx^2} &= \left(\frac{P}{2KI_n} \right)^{\frac{1}{n}} (l - x)^{\frac{2}{n}}, \\ &\implies \\ v(x) &= \left(\frac{P}{2KI_n} \right)^{\frac{1}{n}} \frac{(l - x)^{\frac{2}{n}+2}}{\left(\frac{2}{n} + 1 \right) \left(\frac{2}{n} + 2 \right)} + C_3 x + C_4 \end{aligned}$$

Integration constants C_3 and C_4 are found by boundary conditions (4.2):

$$C_3 = \left(\frac{P}{2KI_n} \right)^{\frac{1}{n}} \frac{l_n^{\frac{2}{n}+1}}{\frac{2}{n} + 1},$$

and

$$C_4 = - \left(\frac{P}{2KI_n} \right)^{\frac{1}{n}} \frac{l_n^{\frac{2}{n}+2}}{\left(\frac{2}{n} + 1 \right) \left(\frac{2}{n} + 2 \right)}.$$

Eventually, the following formula gives the exact solution of the boundary value problem (4.1)-(4.3), which is given by

$$v(x) = \left(\frac{P}{2KI_n} \right)^{\frac{1}{n}} \left\{ \frac{(l-x)^{\frac{2}{n}+2}}{\left(\frac{2}{n} + 1 \right) \left(\frac{2}{n} + 2 \right)} + \frac{l_n^{\frac{2}{n}+1}x}{\frac{2}{n} + 1} - \frac{l_n^{\frac{2}{n}+2}}{\left(\frac{2}{n} + 1 \right) \left(\frac{2}{n} + 2 \right)} \right\} \quad (4.6)$$

where $x \in (0, 1)$.

The solution can also have a form of a function of non-dimensional spatial variable $\hat{x} = x/l \in (0, 1)$:

$$v(\hat{x}) = \alpha l^{-\frac{1}{n}} \left\{ \frac{1}{\left(\frac{2}{n} + 1 \right) \left(\frac{2}{n} + 2 \right)} (1 - \hat{x})^{\frac{2}{n}+2} + \frac{1}{\frac{2}{n} + 1} \hat{x} - \frac{1}{\left(\frac{2}{n} + 1 \right) \left(\frac{2}{n} + 2 \right)} \right\} \quad (4.7)$$

where α is the predefined constant in the following form:

$$\alpha = l_n^{\frac{2}{n}+2} \left(\frac{F}{2KI_n} \right)^{\frac{1}{n}}$$

The stiffness coefficient

$$k_n = \frac{F}{|v_{max}|^{n-1} v_{max}} \quad (4.8)$$

is set by analogy to the linear case

$$k = \frac{F}{v_{max}}.$$

Here, v_{max} of the cantilever beam is

$$v_{max} = v(l) = \left(\frac{P}{2KI_n} \right)^{\frac{1}{n}} \frac{nl_n^{\frac{2}{n}+2}}{2(n+1)}.$$

Consequently, the stiffness coefficient in (4.8) is given by

$$k_n = \frac{2KI_n}{l^{1+2n}} \left(\frac{2(n+1)}{n} \right)^n. \quad (4.9)$$

If $n = 1$, the linear case is recovered

$$k_1 = \frac{8KI_1}{l^3}, \quad K = E, \quad v_{max} = v(l) = \frac{Pl^4}{8EI_1},$$

see [13, Chapter 4.1.3].

4.2 Hinged-hinged beam under uniformly distributed pressure load

Let us observe hinged-hinged beam under uniformly distributed pressure load which is illustrated in Figure 2. The governing equation for this beam has the form of

$$\frac{d^2}{dx^2} \left(\left| \frac{d^2v}{dx^2} \right|^{n-1} \frac{d^2v}{dx^2} \right) = \frac{P}{KI_n}, \quad 0 < x < l, \quad (4.10)$$

with boundary conditions

$$v(0) = 0, \quad v''(0) = 0 \quad (4.11)$$

$$v(l) = 0, \quad v''(l) = 0 \quad (4.12)$$

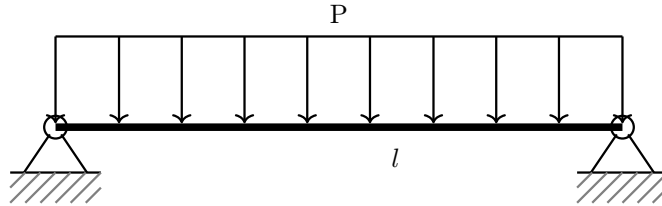


Figure 2: Hinged-hinged beam under uniformly distributed pressure load.

Integration of (4.10) yields

$$\left| \frac{d^2 v}{dx^2} \right|^{n-1} \frac{d^2 v}{dx^2} = \frac{P}{KI_n} \frac{x^2}{2} + C_1 x + C_2. \quad (4.13)$$

$v''(0) = v''(l) = 0$ from (4.11)-(4.12) gives $C_1 = -\frac{Pl}{2KI_n}$ and $C_2 = 0$.

Consequently, (4.13) can be written as

$$\left| \frac{d^2 v}{dx^2} \right|^{n-1} \frac{d^2 v}{dx^2} = \frac{P}{2KI_n} x(x-l), \quad 0 < x < l. \quad (4.14)$$

From (4.14) it can be seen that $\frac{d^2 v}{dx^2} < 0$ if $P > 0$, and $\frac{d^2 v}{dx^2} > 0$ if $P < 0$. Therefore, solutions can be constructed in a similar way. It brings to the following form

$$\frac{d^2 v}{dx^2} = - \left(\frac{P}{2KI_n} \right)^{\frac{1}{n}} (x(l-x))^{\frac{1}{n}}.$$

Next, after the following substitution $\hat{x} = \frac{x}{l}$, the above differential equation becomes

$$\frac{d^2 v}{d\hat{x}^2} = -\alpha \left(\frac{1}{l} \right)^{\frac{1}{n}} (\hat{x}(1-\hat{x}))^{\frac{1}{n}}. \quad (4.15)$$

Twice integration of (4.15) results in

$$v(\hat{x}) = -\alpha \left(\frac{1}{l} \right)^{\frac{1}{n}} \int_{s=0}^{\hat{x}} \int_{t=0}^s (t(1-t))^{\frac{1}{n}} dt ds + C_3 \hat{x} + C_4. \quad (4.16)$$

From $v(0) = v(l) = 0$ it follows that

$$C_3 = \alpha \left(\frac{1}{l} \right)^{\frac{1}{n}} \int_{s=0}^1 \int_{t=0}^s (t(1-t))^{\frac{1}{n}} dt ds$$

and $C_4 = 0$. The integral in C_3 can be written in terms of Euler-Beta integral

$$B(p, q) = \int_0^1 (1-t)^{p-1} t^{q-1} dt, \quad p, q > 0, \quad (4.17)$$

see [15]. Next, the order of integration is changed according to Fubini's theorem.

$$\int_{s=0}^1 \int_{t=0}^s (t(1-t))^{\frac{1}{n}} dt ds = \int_0^1 (1-t)^{\frac{1}{n}+1} t^{\frac{1}{n}} dt = B\left(\frac{1}{n}+2, \frac{1}{n}+1\right). \quad (4.18)$$

see [8]. It results in

$$v(\hat{x}) = -\alpha \left(\frac{1}{l}\right)^{\frac{1}{n}} \int_{s=0}^{\hat{x}} \int_{t=0}^s (t(1-t))^{\frac{1}{n}} dt ds + \alpha \left(\frac{1}{l}\right)^{\frac{1}{n}} B\left(\frac{1}{n}+2, \frac{1}{n}+1\right) \hat{x} \quad (4.19)$$

and

$$v(x) = l^{\frac{2}{n}+2} \left(\frac{P}{2KI_n}\right)^{\frac{1}{n}} \left\{ - \int_{s=0}^{\frac{x}{l}} \int_{t=0}^s (t(1-t))^{\frac{1}{n}} dt ds + \frac{1}{l} B\left(\frac{1}{n}+2, \frac{1}{n}+1\right) x \right\}. \quad (4.20)$$

Remark 2 Applying symbolic integration to (4.15) in Maple results in

$$v'(\hat{x}) = {}_{-2}F_1\left(1 + \frac{1}{n}, -\frac{1}{n}; 2 + \frac{1}{n}; \hat{x}\right) \alpha \left(\frac{1}{l}\right)^{\frac{1}{n}} + C_3 \quad (4.21)$$

and

$$v'(\hat{x}) = -\alpha \left(\frac{1}{l}\right)^{\frac{1}{n}} \hat{x}^{\frac{1}{n}+2} \frac{{}_2F_1\left(1 + \frac{1}{n}, -\frac{1}{n}; 3 + \frac{1}{n}; \hat{x}\right)}{\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)} + C_3 \hat{x} + C_4, \quad (4.22)$$

where ${}_2F_1\left(1 + \frac{1}{n}, -\frac{1}{n}; 2 + \frac{1}{n}; \hat{x}\right)$ is a Gauss hypergeometric function, see [22]. From $v(0) = 0$ and $v(1) = 0$ it follows that $C_3 = \alpha \left(\frac{1}{l}\right)^{\frac{1}{n}} \frac{{}_2F_1\left(1 + \frac{1}{n}, -\frac{1}{n}; 3 + \frac{1}{n}; 1\right)}{\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)}$ and $C_4 = 0$. Consequently, the exact solution can be also expressed as

$$v(\hat{x}) = -\alpha \left(\frac{1}{l}\right)^{\frac{1}{n}} \hat{x}^{\frac{1}{n}+2} \frac{{}_2F_1\left(1 + \frac{1}{n}, -\frac{1}{n}; 3 + \frac{1}{n}; \hat{x}\right)}{\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)} + \alpha \left(\frac{1}{l}\right)^{\frac{1}{n}} \frac{{}_2F_1\left(1 + \frac{1}{n}, -\frac{1}{n}; 3 + \frac{1}{n}; 1\right)}{\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)} \hat{x}.$$

□

Next, we refer to the following results from [8]

Lemma 3 *Let $n > 0$. Then, it holds true*

$$B\left(\frac{1}{n} + 2, \frac{1}{n} + 1\right) = \int_0^{1/2} t^{1/n} (1-t)^{1/n} dt.$$

□

Corollary 4 *Let $P > 0$. The maximum value of $v(x)$ occurs at $x = l/2$ and it is given by*

$$\begin{aligned} v_{max} &= v(l/2) \\ &= l^{\frac{2}{n}+1} \left(\frac{P}{2KI_n} \right)^{\frac{1}{n}} \left\{ - \int_{s=0}^{1/2} \int_{t=0}^s (t(1-t))^{\frac{1}{n}} dt ds + \frac{1}{2} B\left(\frac{1}{n} + 2, \frac{1}{n} + 1\right) \right\}. \end{aligned} \quad (4.23)$$

Alternatively, the maximum of $v(x)$ can be written in the following form

$$\begin{aligned} v_{max} &= l^{\frac{2}{n}+1} \left(\frac{F}{2KI_n} \right)^{\frac{1}{n}} \left\{ - \frac{{}_2F_1\left(1 + \frac{1}{n}, -\frac{1}{n}; 3 + \frac{1}{n}; \frac{1}{2}\right)}{\left(\frac{1}{n} + 1\right) \left(\frac{1}{n} + 2\right)} \left(\frac{1}{2}\right)^{\frac{1}{n}+2} \right. \\ &\quad \left. + \frac{1}{2} \frac{{}_2F_1\left(1 + \frac{1}{n}, -\frac{1}{n}; 3 + \frac{1}{n}; 1\right)}{\left(\frac{1}{n} + 1\right) \left(\frac{1}{n} + 2\right)} \right\}. \end{aligned}$$

□

It can be concluded from Lemma 4.2 that the corresponding stiffness coefficient is

$$\begin{aligned} k_n &= \frac{F}{|v_{max}|^{n-1} v_{max}} = \frac{2KI_n}{l^{2n+1}} \left(- \int_{s=0}^{1/2} \int_{t=0}^s (t(1-t))^{\frac{1}{n}} dt ds + \frac{1}{2} B\left(\frac{1}{n} + 2, \frac{1}{n} + 1\right) \right)^{-n} \\ &= \frac{2KI_n}{l^{2n+1}} \left(- \frac{{}_2F_1\left(1 + \frac{1}{n}, -\frac{1}{n}; 3 + \frac{1}{n}; \frac{1}{2}\right)}{\left(\frac{1}{n} + 1\right) \left(\frac{1}{n} + 2\right)} \left(\frac{1}{2}\right)^{\frac{1}{n}+2} + \frac{1}{2} \frac{{}_2F_1\left(1 + \frac{1}{n}, -\frac{1}{n}; 3 + \frac{1}{n}; 1\right)}{\left(\frac{1}{n} + 1\right) \left(\frac{1}{n} + 2\right)} \right)^{-n}. \end{aligned} \quad (4.24)$$

For the case $n = 1$ it follows that

$$v(x) = \frac{P}{24EI_1} x (x^3 - 2lx^2 + l^3),$$

and

$$v_{max} = v\left(\frac{l}{2}\right) = \frac{5Pl^4}{384EI_1},$$

and the corresponding stiffness coefficient will be

$$k_1 = \frac{Pl}{v_{max}} = \frac{384EI_1}{5l^3}, \quad (4.25)$$

see [13, Chapter 4.1.3].

4.3 Hinged-hinged beam under point load in the middle

Let us consider hinged-hinged beam under point load in the middle, as presented in Figure 3. The governing equation for $v(x)$ has the following form

$$\frac{d^2}{dx^2} \left(\left| \frac{d^2v}{dx^2} \right|^{n-1} \frac{d^2v}{dx^2} \right) = \frac{F}{KI_n} \delta\left(x - \frac{l}{2}\right), \quad 0 < x < l, \quad (4.26)$$

with boundary conditions

$$v(0) = 0, \quad v''(0) = 0, \quad (4.27)$$

$$v(l) = 0, \quad v''(l) = 0 \quad (4.28)$$

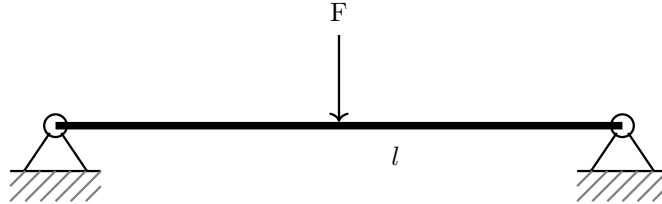


Figure 3: Hinged-hinged beam under point load in the middle.

The goal is to find $v(x)$ which is continuously differentiable. Integrating twice (4.26)

yields

$$-KI_n \left| \frac{d^2v}{dx^2} \right|^{n-1} \frac{d^2v}{dx^2} = \begin{cases} \frac{Fx}{2}, & 0 \leq x \leq \frac{l}{2}, \\ \frac{F(l-x)}{2}, & \frac{l}{2} \leq x \leq l \end{cases}$$

If $F > 0$, then $\frac{d^2v}{dx^2} < 0$. Then, it follows that

$$(-1)^n \left(\frac{d^2v}{dx^2} \right)^n = \begin{cases} \frac{Fx}{2KI_n}, & 0 \leq x \leq \frac{l}{2}, \\ \frac{F(l-x)}{2KI_n}, & \frac{l}{2} \leq x \leq l, \end{cases}$$

and from the above it can be inferred that

$$\frac{d^2v}{dx^2} = \begin{cases} -\left(\frac{F}{2KI_n}\right)^{\frac{1}{n}} x^{\frac{1}{n}}, & 0 \leq x \leq \frac{l}{2}, \\ -\left(\frac{F}{2KI_n}\right)^{\frac{1}{n}} (l-x)^{\frac{1}{n}}, & \frac{l}{2} \leq x \leq l. \end{cases}$$

Twice integration gives

$$v(x) = \begin{cases} -\left(\frac{F}{2KI_n}\right)^{\frac{1}{n}} \frac{n^2 x^{\frac{1}{n}+2}}{(n+1)(n+2)} + C_1 x + C_2, & 0 \leq x \leq \frac{l}{2}, \\ -\left(\frac{F}{2KI_n}\right)^{\frac{1}{n}} \frac{n^2 (l-x)^{\frac{1}{n}+2}}{(n+1)(n+2)} + C_3 x + C_4, & \frac{l}{2} \leq x \leq l. \end{cases} \quad (4.29)$$

The integration constants

$$C_2 = 0, \quad C_4 = -C_3 l, \quad C_1 + C_3 = 0$$

and

$$C_1 - C_3 = \left(\frac{F}{2}\right)^{\frac{1}{n}} \frac{2n \left(\frac{l}{2}\right)^{\frac{1}{n}+1}}{n+1}$$

are found from the boundary conditions $v(0) = 0$, $v(l) = 0$ and two continuity

conditions on $v(x)$ and $v'(x)$ at $x = l/2$. Thus,

$$C_1 = \left(\frac{F}{2KI_n} \right)^{\frac{1}{n}} \frac{n \left(\frac{l}{2} \right)^{\frac{1}{n}+1}}{n+1}, \quad C_2 = 0,$$

and

$$C_3 = - \left(\frac{F}{2KI_n} \right)^{\frac{1}{n}} \frac{n \left(\frac{l}{2} \right)^{\frac{1}{n}+1}}{n+1}, \quad C_4 = \left(\frac{F}{2KI_n} \right)^{\frac{1}{n}} \frac{n \left(\frac{l}{2} \right)^{\frac{1}{n}+1} l}{n+1}.$$

Consequently, the deflection of the hinged-hinged beam under the point load in the middle has the form of

$$v(x) = \begin{cases} \left(\frac{F}{2KI_n} \right)^{\frac{1}{n}} \left(-\frac{x^{\frac{1}{n}+2}}{(1+\frac{1}{n})(1+\frac{2}{n})} + \frac{(\frac{l}{2})^{\frac{1}{n}+1}}{1+\frac{1}{n}} x \right), & 0 \leq x \leq \frac{l}{2}, \\ \left(\frac{F}{2KI_n} \right)^{\frac{1}{n}} \left(-\frac{(l-x)^{\frac{1}{n}+2}}{(1+\frac{1}{n})(1+\frac{2}{n})} - \frac{(\frac{l}{2})^{\frac{1}{n}+1}}{1+\frac{1}{n}} x + \frac{(\frac{l}{2})^{\frac{1}{n}+1} l}{1+\frac{1}{n}} \right), & \frac{l}{2} \leq x \leq l \end{cases} \quad (4.30)$$

The following equation expresses the solution in the form of the function of non-dimensional variable $\hat{x} = x/l \in (0, 1)$:

$$v(\hat{x}) = \begin{cases} \alpha \left(-\frac{\hat{x}^{\frac{1}{n}+2}}{(1+\frac{1}{n})(1+\frac{2}{n})} + \frac{(\frac{1}{2})^{\frac{1}{n}+1}}{1+\frac{1}{n}} \hat{x} \right), & 0 \leq \hat{x} \leq \frac{1}{2}, \\ \alpha \left(-\frac{(1-\hat{x})^{\frac{1}{n}+2}}{(1+\frac{1}{n})(1+\frac{2}{n})} - \frac{(\frac{1}{2})^{\frac{1}{n}+1}}{1+\frac{1}{n}} \hat{x} + \frac{(\frac{1}{2})^{\frac{1}{n}+1}}{1+\frac{1}{n}} \right), & \frac{1}{2} \leq \hat{x} \leq 1. \end{cases} \quad (4.31)$$

The stiffness coefficient is given by

$$k_n = \frac{F}{|v_{max}|^{n-1} v_{max}} = 2KI_n \left(\frac{2}{l} \right)^{1+2n} \left(\frac{(n+1)(n+2)}{2n} \right)^n, \quad (4.32)$$

where

$$v_{max} = v \left(\frac{l}{2} \right) = \left(\frac{F}{2KI_n} \right)^{\frac{1}{n}} \left(\frac{l}{2} \right)^{\frac{1}{n}+2} \frac{2n}{(n+1)(n+2)}.$$

If $n = 1$, the linear case is recovered

$$k_1 = \frac{F}{v_{max}} = \frac{48KI_1}{l^3}, \quad K = E,$$

see [13, Chapter 4.1.3].

The deflection of the beam $v(x)$ by (4.30) can be expressed as

$$v(t, x) = \begin{cases} v(t, \frac{l}{2})E_n(x), & 0 \leq x \leq \frac{l}{2}, \\ v(t, \frac{l}{2})F_n(x), & \frac{l}{2} \leq x \leq l, \end{cases} \quad (4.33)$$

where

$$E_n(x) = \left(\frac{F}{2KI_n} \right)^{\frac{1}{n}} \frac{(n+1)(n+2)}{\left(\frac{l}{2}\right)^{\frac{1}{n}+2} 2n} \left(-\frac{x^{\frac{1}{n}+2}}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)} + \frac{\left(\frac{l}{2}\right)^{\frac{1}{n}+1}}{1 + \frac{1}{n}}x \right)$$

and

$$F_n(x) = \left(\frac{F}{2KI_n} \right)^{\frac{1}{n}} \frac{(n+1)(n+2)}{\left(\frac{l}{2}\right)^{\frac{1}{n}+2} 2n} \left(-\frac{(l-x)^{\frac{1}{n}+2}}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)} - \frac{\left(\frac{l}{2}\right)^{\frac{1}{n}+1}}{1 + \frac{1}{n}}x + \frac{\left(\frac{l}{2}\right)^{\frac{1}{n}+1}l}{1 + \frac{1}{n}} \right).$$

Next, differentiation of the above equation with respect to time while keeping x fixed yields

$$v_t(t, x) = \begin{cases} v(t, \frac{l}{2})E_n(x), & 0 \leq x \leq \frac{l}{2}, \\ v(t, \frac{l}{2})F_n(x), & \frac{l}{2} \leq x \leq l. \end{cases} \quad (4.34)$$

Figure 4-1 shows the graph of the effective mass coefficient $\mu_{eff}(n)$ defined in (4.36)

Thus, the kinetic energy can be written as

$$E_{kin} = \frac{n^4 + 12n^3 + 55n^2 + 30n + 4}{240n^2 + 156n + 24} m v_t^2 \left(t, \frac{l}{2} \right). \quad (4.35)$$

Next, equating the outcomes of (3.6) and (4.35) results in the effective mass coefficient

$$\mu_{eff} = \frac{n^4 + 12n^3 + 55n^2 + 30n + 4}{120n^2 + 78n + 12}. \quad (4.36)$$

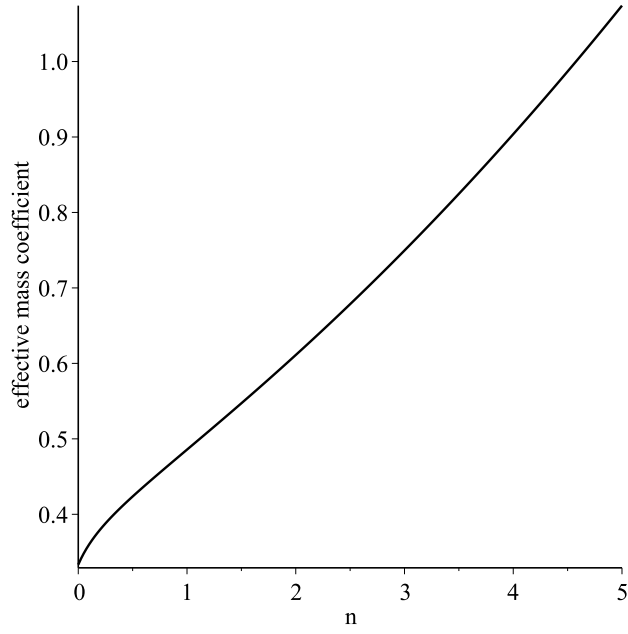


Figure 4-1: The effective mass coefficient μ_{eff} as a function of power-law index.

In the case $n = 1$, the effective mass is given by

$$m_{eff}(1) = \frac{17}{35}m.$$

Chapter 5

Galerkin approach for calculating the lumped parameters

In this chapter we want to present the alternative approach for calculating the lumped parameters.

5.1 Hinged-hinged beam under uniformly distributed pressure load

Let the Galerkin ansatz for the deflection of the beam to be written in the following form

$$v_a(x) = v_1 \varphi(x), \quad x \in (0, l),$$

where $v_1 > 0$. Multiplying (4.10) by $\varphi(x)$ and integrating twice, results in the Galerkin equation

$$\int_0^l v_1^n \left| \frac{d^2 \varphi(x)}{dx^2} \right|^{n-1} \frac{d^2 \varphi(x)}{dx^2} \frac{d^2 \varphi(x)}{dx^2} dx = \int_0^l \frac{P}{K I_n} \varphi(x) dx,$$

where the basic function $\varphi(x)$ is chosen as

$$\varphi(x) = \frac{16}{5l^4} x(l-x)(-x^2 + lx + l^2), \quad x \in (0, l).$$

Consequently,

$$v_{max} = v_a \left(\frac{l}{2} \right) = v_1 = \left(\frac{\frac{P}{KI_n} \int_0^l \varphi(x) dx}{\int_0^l \left| \frac{d^2 \varphi(x)}{dx^2} \right|^{n+1} dx} \right)^{\frac{1}{n}}.$$

By (4.24) it follows that the stiffness coefficient can be introduced as

$$\tilde{k}_n = \frac{Pl}{|v_{max}|^{n-1} v_{max}} = \frac{Pl}{v_{max}^n} = KI_n l \frac{\int_0^l \left| \frac{d^2 \varphi(x)}{dx^2} \right|^{n+1} dx}{\int_0^l \varphi(x) dx}, \quad v_{max} > 0.$$

Let us introduce the following transformations

$$\hat{x} = \frac{x}{l},$$

and define

$$\hat{\varphi}(\hat{x}) = \varphi(\hat{x}l).$$

Then,

$$\varphi(x) = \varphi(\hat{x}l) = \hat{\varphi}(\hat{x}) = \frac{16}{5} \hat{x}(1 - \hat{x})(-\hat{x}^2 + \hat{x} + 1), \quad (5.1)$$

and

$$\begin{aligned} \frac{d\varphi}{dx} &= \frac{d\hat{\varphi}}{d\hat{x}} \frac{d\hat{x}}{dx} = \frac{d\hat{\varphi}}{d\hat{x}} \frac{1}{l}, \\ \frac{d^2 \varphi}{dx^2} &= \frac{d^2 \hat{\varphi}}{d\hat{x}^2} \frac{1}{l^2}. \end{aligned} \quad (5.2)$$

Then, the stiffness coefficient reads as follows

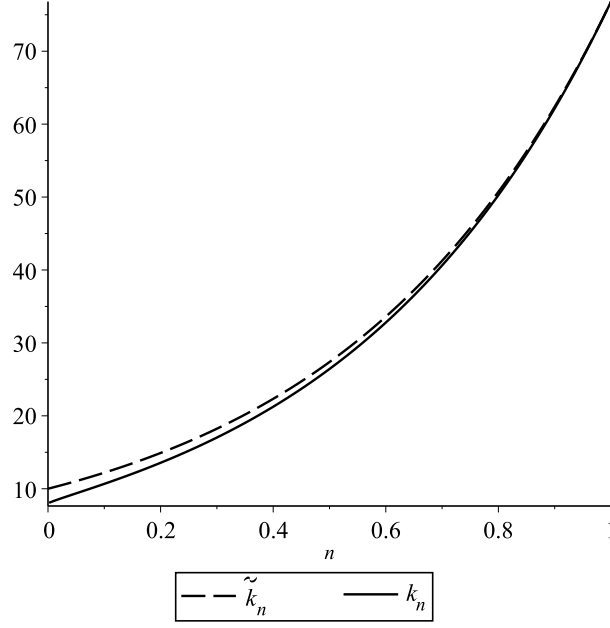


Figure 5-1: Exact and approximated stiffness coefficients for the hinged-hinged beam under uniformly distributed pressure load

$$\tilde{k}_n = K I_n l \frac{\int_0^1 \left| \frac{d^2 \hat{\varphi}(\hat{x})}{d\hat{x}^2} \right|^{n+1} \left| \frac{1}{l^2} \right|^{n+1} l d\hat{x}}{\int_0^1 \hat{\varphi}(\hat{x}) l d\hat{x}} = \frac{K I_n}{l^{2n+1}} \frac{\int_0^1 \left| \frac{d^2 \hat{\varphi}(\hat{x})}{d\hat{x}^2} \right|^{n+1} d\hat{x}}{\int_0^1 \hat{\varphi}(\hat{x}) d\hat{x}},$$

where

$$\hat{\varphi}(\hat{x}) = \frac{16}{5} \hat{x}(1 - \hat{x})(-\hat{x}^2 + \hat{x} + 1) > 0 \quad \text{for } \hat{x} \in (0, 1), \quad (5.3)$$

and

$$\frac{d^2 \hat{\varphi}(\hat{x})}{d\hat{x}^2} = -\frac{192}{5} \hat{x}(1 - \hat{x}) < 0 \quad \text{for } \hat{x} \in (0, 1).$$

Employing the symbolic integration in Maple, yields the approximate stiffness coefficient

$$\tilde{k}_n = \frac{25 K I_n}{16 l^{2n+1}} \frac{2^{4n+3} 3^{n+1} 5^{-n-1} \sqrt{\pi} \Gamma(n+2)}{\Gamma(\frac{5}{2} + n)}.$$

If $n = 1$, then

$$\tilde{k}_1 = \frac{384 E I_1}{5 l^3},$$

which coincides with the exact result from (4.25). Figure 5-1 shows the exact and approximated stiffness coefficient for the hinged-hinged beam under uniformly distributed pressure load.

Now,

$$v_1 = v_{max} = v_a \left(\frac{l}{2} \right) = \left(\frac{\frac{P}{KI_n} \int_0^1 \widehat{\varphi}(\widehat{x}) d\widehat{x}}{\frac{1}{l^{2n+2}} \int_0^1 \left| \frac{d^2 \widehat{\varphi}(\widehat{x})}{d\widehat{x}^2} \right|^{n+1} d\widehat{x}} \right)^{\frac{1}{n}} = l^{2+\frac{2}{n}} \left(\frac{P}{KI_n} \right)^{\frac{1}{n}} G^{\frac{1}{n}},$$

where

$$G = \frac{\int_0^1 \widehat{\varphi}(\widehat{x}) d\widehat{x}}{\int_0^1 \left| \frac{\partial^2 \widehat{\varphi}(\widehat{x})}{\partial \widehat{x}^2} \right|^{n+1} d\widehat{x}}.$$

Then, $v_a(x)$ can be rewritten as

$$v_a(x) = v_1 \widehat{\varphi}(\widehat{x}) = l^{2+\frac{2}{n}} \left(\frac{P}{KI_n} \right)^{\frac{1}{n}} G^{\frac{1}{n}} \widehat{\varphi}(\widehat{x}).$$

Next, the following approximation

$$v_a(t, x) = v_a(t, \frac{l}{2}) \widehat{\varphi}(\widehat{x}) = v_1(t) \widehat{\varphi}(\widehat{x}), \quad (5.4)$$

is considered, and the velocity of the beam at the axial position x is calculated by differentiating (5.4) with respect to time while keeping $\widehat{x}$ fixed

$$\partial_t v_a(t, x) = \partial_t v_a(t, \frac{l}{2}) \varphi(x) = v_1'(t) \varphi(x) = v_1'(t) \widehat{\varphi}(\widehat{x}),$$

Then, from (3.6) and (3.9) it follows that

$$\int_0^l \frac{1}{2} (v_1'(t))^2 \varphi^2(x) \frac{m}{l} dx = \frac{1}{2} (v_1'(t))^2 m_{eff}$$

from which the effective mass can be found as

$$m_{eff} = m \int_0^1 \widehat{\varphi}^2(\widehat{x}) d\widehat{x}.$$

In the case of $n = 1$ the calculated effective mass coefficient is

$$\mu_{eff}(1) = \frac{3968}{7875},$$

and the effective mass is equal to

$$m_{eff}(1) = \frac{3968}{7875} m.$$

5.2 Clamped-clamped power-law beam under uniformly distributed pressure

The governing equations for the clamped-clamped power-law beam under uniformly distributed pressure is:

$$\frac{d^2}{dx^2} \left(\left| \frac{d^2 v}{dx^2} \right|^{n-1} \frac{d^2 v}{dx^2} \right) = \frac{P}{K I_n}, \quad 0 < x < l \quad (5.5)$$

with boundary conditions

$$v(0) = 0, \quad v'(0) = 0, \quad (5.6)$$

$$v(l) = 0, \quad v'(l) = 0. \quad (5.7)$$

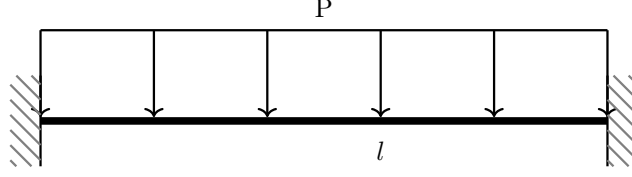


Figure 4: Clamped-clamped beam under uniformly distributed pressure.

Using the transformation (5.1), the calculations can be restricted to the beam of length $l = 1$.

Let the Galerkin ansatz for the deflection of the beam to be written in the following form

$$v_a(x) = v_1 \hat{\varphi}(\hat{x}), \quad \hat{x} \in (0, 1),$$

where $v_1 > 0$. Then, the Galerkin equation reads as follows

$$\int_0^1 v_1^n \left| \frac{d^2 \hat{\varphi}(\hat{x})}{d\hat{x}^2} \right|^{n-1} \frac{d^2 \hat{\varphi}(\hat{x})}{d\hat{x}^2} \frac{d^2 \hat{\varphi}(\hat{x})}{d\hat{x}^2} d\hat{x} = \int_0^1 \frac{P}{K I_n} \hat{\varphi}(\hat{x}) d\hat{x},$$

where

$$\hat{\varphi}(\hat{x}) = 16\hat{x}^2(1 - \hat{x})^2, \quad \hat{x} \in (0, 1).$$

Consequently,

$$v_1 = v_{max} = \left(\frac{\frac{P}{K I_n} \int_0^1 \hat{\varphi}(\hat{x}) d\hat{x}}{\int_0^1 \left| \frac{d^2 \hat{\varphi}(\hat{x})}{d\hat{x}^2} \right|^{n+1} d\hat{x}} \right)^{\frac{1}{n}},$$

and the stiffness coefficient is given by

$$\tilde{k}_n = \frac{Pl}{|v_{max}|^{n-1} v_{max}} = \frac{Pl}{v_{max}^n} = K I_n \frac{\int_0^1 \left| \frac{d^2 \hat{\varphi}(\hat{x})}{d\hat{x}^2} \right|^{n+1} d\hat{x}}{\int_0^1 \hat{\varphi}(\hat{x}) d\hat{x}}.$$

Finally, the stiffness coefficient for the beam of length l reads as

$$\tilde{k}_n = \frac{KI_n}{l^{2n+1}} \frac{\int_0^1 \left| \frac{d^2 \hat{\varphi}(\hat{x})}{d\hat{x}^2} \right|^{n+1} d\hat{x}}{\int_0^1 \hat{\varphi}(\hat{x}) d\hat{x}},$$

where

$$\hat{\varphi}(\hat{x}) = 16\hat{x}^2(1 - \hat{x})^2 > 0, \quad (5.8)$$

and

$$\frac{d^2 \hat{\varphi}(\hat{x})}{d\hat{x}^2} = 192\hat{x}^2 - 192\hat{x} + 32 < 0$$

for $\hat{x} \in (0, 1)$. The stiffness coefficient for the case $n = 1$ according to calculations in Maple is

$$\tilde{k}_1 = 384 \frac{KI_1}{l^3}.$$

By analogy to the previous case, the effective mass can be found as

$$m_{eff} = m \int_0^1 \hat{\varphi}^2(\hat{x}) d\hat{x}.$$

Then, the calculated effective mass coefficient is

$$\mu_{eff}(1) = \frac{128}{315},$$

and the effective mass is equal to

$$m_{eff}(1) = \frac{128}{315} m.$$

5.3 Clamped-clamped power-law beam with a point load in the middle

The governing equations for the clamped-clamped power-law beam with a point load in the middle is:

$$\frac{d^2}{dx^2} \left(\left| \frac{d^2 v}{dx^2} \right|^{n-1} \frac{d^2 v}{dx^2} \right) = \frac{F}{K I_n} \delta \left(x - \frac{l}{2} \right), \quad 0 < x < l \quad (5.9)$$

with boundary conditions

$$v(0) = 0, \quad v'(0) = 0, \quad (5.10)$$

$$v(l) = 0, \quad v'(l) = 0. \quad (5.11)$$

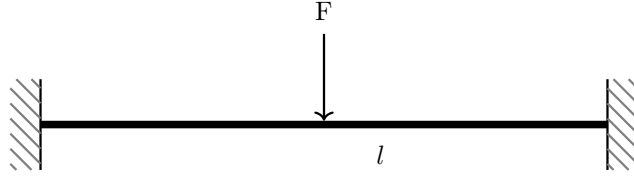


Figure 5: Clamped-clamped beam with a point load in the middle.

Let us write (5.9) in the following form

$$v^{IV}(x) = 0, \quad 0 < x \leq \frac{l}{2} - 0, \quad \frac{l}{2} + 0 \leq x < l, \quad (5.12)$$

with boundary conditions

$$\begin{aligned} v(0) &= 0, & v'(0) &= 0, \\ v(l) &= 0, & v'(l) &= 0, \end{aligned} \quad (5.13)$$

and three continuity conditions at $x = l/2$

$$\begin{aligned}\lim_{x \rightarrow l/2-0} v(x) &= \lim_{x \rightarrow l/2+0} v(x), \\ \lim_{x \rightarrow l/2-0} v'(x) &= \lim_{x \rightarrow l/2+0} v'(x), \\ \lim_{x \rightarrow l/2-0} v''(x) &= \lim_{x \rightarrow l/2+0} v''(x),\end{aligned}\tag{5.14}$$

and one jump condition at $x = l/2$

$$-EIv''' \left(\frac{l}{2} \right) = \begin{cases} -\frac{P}{2} & 0 < x \leq \frac{l}{2} - 0, \\ \frac{P}{2} & \frac{l}{2} + 0 \leq x < l. \end{cases}\tag{5.15}$$

Integrating equation (5.12), yields

$$\begin{aligned}v(x) &= C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4, \quad 0 < x < \frac{l}{2} - 0, \\ v(x) &= D_1 \frac{(x-l)^3}{6} + D_2 \frac{(x-l)^2}{2} + D_3(x-l) + D_4, \quad \frac{l}{2} + 0 < x < l.\end{aligned}$$

From (5.13) it follows that $C_3 = C_4 = D_3 = D_4 = 0$, and from (5.14) and (5.15) it can be concluded that

$$\begin{cases} C_1 - D_1 = -\frac{P}{EI}, \\ \left(D_1 - \frac{P}{EI}\right) \frac{l}{6} + C_2 + D_1 \frac{l}{6} - D_2 = 0, \\ C_2 + D_2 - \frac{Pl}{4EI} = 0, \\ \left(D_1 - \frac{P}{EI}\right) \frac{l}{2} + C_2 + D_1 \frac{l}{2} - D_2 = 0 \end{cases}$$

which implies $C_1 = -\frac{P}{2EI}$, $C_2 = \frac{Pl}{8EI}$, $D_1 = \frac{P}{2EI}$, $D_2 = \frac{Pl}{8EI}$. Consequently, the solution of the problem (5.12)-(5.15) is

$$v(x) = \begin{cases} \frac{Px^2(3l-4x)}{48EI}, & 0 < x \leq \frac{l}{2} - 0, \\ \frac{P(x-l)^2(4x-l)}{48EI}, & \frac{l}{2} + 0 \leq x < l. \end{cases} \quad (5.16)$$

The corresponding stiffness coefficient for the clamped-clamped beam with a point load in the middle is given by

$$k = \frac{P}{v_{max}(x)} = \frac{192EI}{l^3},$$

where

$$v_{max}(x) = v\left(\frac{l}{2}\right) = \frac{Pl^3}{192EI}.$$

The deflection of the beam by (5.16) can be approximated as

$$v(t, x) = \begin{cases} \frac{v\left(t, \frac{l}{2}\right) 192x^2(3l-4x)}{48l^3}, & 0 < x \leq \frac{l}{2}, \\ \frac{v\left(t, \frac{l}{2}\right) 192(x-l)^2(4x-l)}{48l^3}, & \frac{l}{2} \leq x < l. \end{cases} \quad (5.17)$$

Differentiation of the above equation with respect to time while keeping x fixed leads to the kinetic energy that can be written as

$$E_{kin} = \frac{13}{70}mv_t\left(t, \frac{l}{2}\right)^2. \quad (5.18)$$

Next, equating the outcomes of (5.18) and (3.6) brings us to the effective mass coefficient of the following form

$$\mu_{eff}(1) = \frac{13}{70},$$

and the effective mass

$$m_{eff}(1) = \frac{13}{70}m.$$

Chapter 6

The vibrating cantilever beam

Multiplication of the ODE (3.4) by $\dot{X}(t)$ followed by twice integration brings the implicit formula for the solution $X(t)$ as, see [11] for details

$$\int_0^{X(t)/A} \frac{ds}{\sqrt{1-s^{n+1}}} = \pm\omega_n + C, \quad (6.1)$$

where

$$\begin{aligned} A &= |X(0)|, \\ \omega_n &= \sqrt{\frac{2k_n A^{n-1}}{(n+1)m_{eff}(n)}}, \\ C &= \int_0^1 \frac{ds}{\sqrt{1-s^{n+1}}}. \end{aligned} \quad (6.2)$$

There is also explicit expression of the periodic solution $X(t)$ in terms of the generalized trigonometric functions

$$X(t) = A \sin_{2,n+1}(w_n t + C). \quad (6.3)$$

Here, $\sin_{2,n+1}(t)$ is the generalized sine function which is defined as a component of the solution $x(t)$ of the following initial value problem for the system of first order

ODEs

$$\begin{aligned}x' &= \phi(y), & x(0) &= 0, \\y' &= -\psi(x), & y(0) &= 1,\end{aligned}$$

where $\phi(y) = y$ and $\psi(y) = |x|^{n-1}$, see [11]. It should be noticed that unless the beam obeys Hook's law ($n = 1$) the angular frequency w_n depends on the initial deflection $X(0)$. The formula for the frequency of the oscillations is

$$f_n = \frac{w_n}{2\pi_{2,n+1}}, \quad (6.4)$$

where

$$\begin{aligned}\pi_{2,n+1} &= 2 \int_0^1 \frac{ds}{\sqrt{1-s^{n+1}}} \\&= \frac{2}{n+1} B\left(\frac{1}{n+1}, \frac{1}{2}\right).\end{aligned}$$

is the generalized Euclidean number π . The period T_{per} is found by the following formula

$$T_{per} = \frac{1}{f_n}.$$

Table 6.1 shows the values of f_n and T_{per} for different values of n . The formula

$$f_1 = \sqrt{\frac{k_1}{m_{eff}(1)}}$$

can be used in the case of Hook's materials to calculate the natural frequency. Let us

Table 6.1: The frequency of oscillations and period.

n	w_n	f_n	T_{per}
0.45	1079.89858	154.712678	0.0064635944
0.54	569.329722	83.2679994	0.0120094155
1	121.182980	19.2868703	0.0518487440

denote $v(t, x)$ as the deflection of the vibrating power-law beam at the axial position x and at the time $t > 0$ and consider the following problem

$$\left\{ \begin{array}{l} \varrho A v_{tt} + (K I_n |v_{xx}|^{n-1} v_{xx})_{xx} = 0, \\ v(t, 0) = 0, \quad v_x(t, 0) = 0, \quad v_{xx}(t, l) = 0, \quad v_{xxx}(t, l) = 0, \\ v(0, x) = \left(\frac{P}{2K I_n} \right)^{\frac{1}{n}} \left\{ \frac{(l-x)^{\frac{2}{n}+2}}{\left(\frac{2}{n}+1\right)\left(\frac{2}{n}+2\right)} + \frac{l^{\frac{2}{n}+1} x}{\frac{2}{n}+1} - \frac{l^{\frac{2}{n}+2}}{\left(\frac{2}{n}+1\right)\left(\frac{2}{n}+2\right)} \right\}, \\ v_t(0, x) = 0 \end{array} \right. \quad (6.5)$$

for $x \in (0, l)$, $t \in (0, T)$. In the case of the beams consisting of linear materials

Table 6.2: The parameters for the well-known power-law materials, see [2], [3].

Material	n	K(MPA)	ϱ (g/cm ³)
304 Stainless Steel, annealed	0.45	1275	8.00
Copper, annealed	0.54	315	8.91
Steel	1.00	$2 \cdot 10^5$	7.80

($n = 1$), the separation method can be used to obtain the solution in the form of Fourier series, see [20], [21],

$$v(t, \hat{x}) = \sum_{k=1}^{\infty} A_k Y_k(\hat{x}) \cos(w_k t), \quad \hat{x} = \frac{x}{l},$$

where

$$Y_k(\hat{x}) = \cosh(\lambda_k \hat{x}) - \cos(\lambda_k \hat{x}) - \frac{\cosh(\lambda_k) - \cos(\lambda_k)}{\sinh(\lambda_k) + \sin(\lambda_k)} (\sinh(\lambda_k \hat{x}) - \sin(\lambda_k \hat{x})),$$

and

$$w_k = \frac{\lambda_k^2}{l^2} \sqrt{\frac{E I_1}{A \varrho}}.$$

Table 6.3: The calculated lumped mass model parameters for the power-law beams of rectangular cross sections.

n	$k_n^\square$	$I_n^\square$
0.45	0.0000288251402	$8.40118254 \cdot 10^{-19}$
0.54	0.000026631019	$2.70134522 \cdot 10^{-19}$
1.00	0.0000168	$8.33333333 \cdot 10^{-22}$

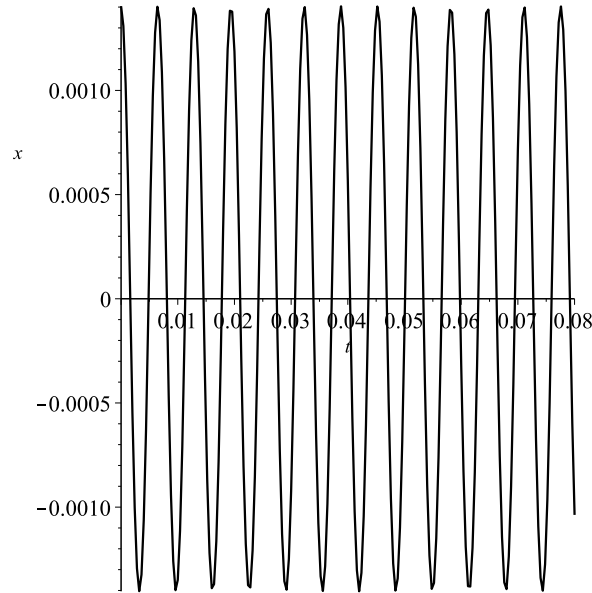


Figure 6-1: The solution of (3.4) for the case $n = 0.45$.

Here, A_k denotes the corresponding Fourier coefficients, and λ_k are positive roots of the equation

$$\cos(\lambda_k) \cosh(\lambda_k) = -1.$$

If $n \neq 1$, only the numerical approximations to (6.5) can be found, and the analytic solution to it remains an open problem.

In the following, power-law Euler-Bernoulli cantilever micro-beams of length l with rectangular cross-section A of height h and width b are considered. The initial load F is applied distributively to the cantilever beam and immediately released. The

following formula

$$I_n^\square = \frac{2b}{n+2} \left(\frac{h}{2} \right)^{n+2}$$

is applied to compute the constant moments of inertia defined in (1.2) for beams with rectangular cross-section A , see [12].

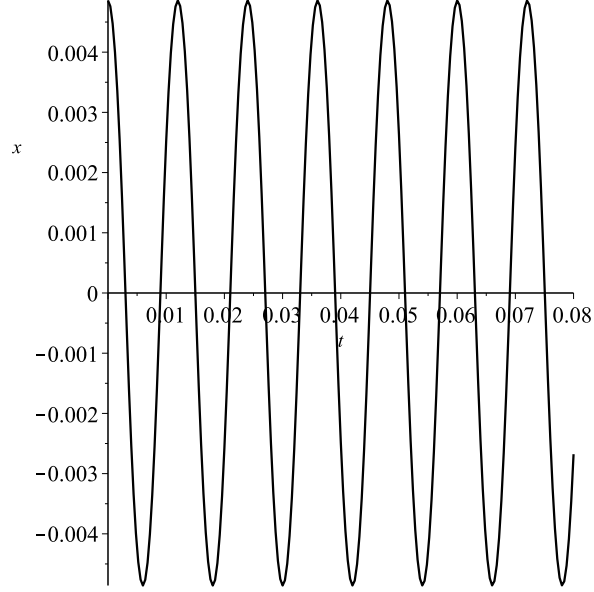


Figure 6-2: The solution of (3.4) for the case $n = 0.54$.

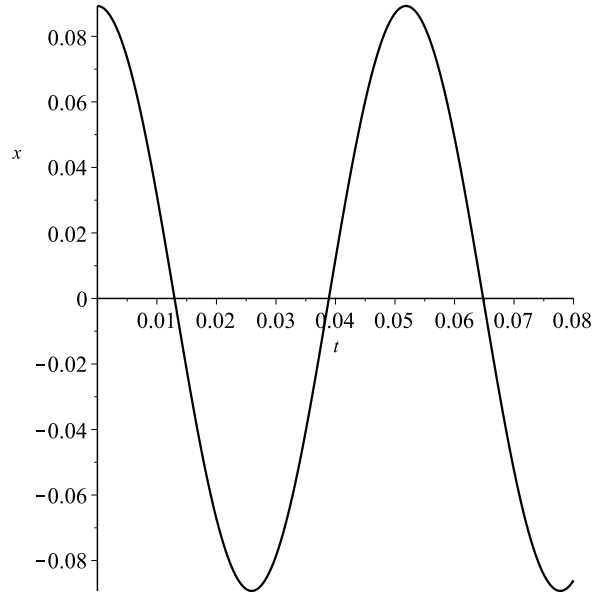


Figure 6-3: The solution of (3.4) for the case $n = 1$.

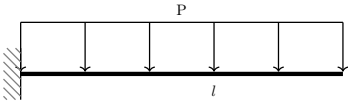
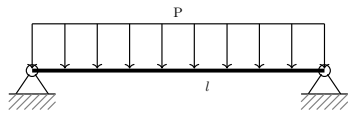
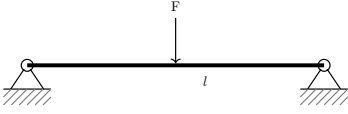
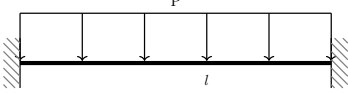
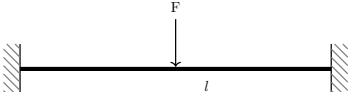
The results of simulations are presented for beams with length $l = 900 \mu m$ and rectangular cross-section, and made of different types of power-law materials whose parameters are showed in Tables 6.2-6.3. The height and width of the rectangular cross-section are $h = 10 \mu m$ and $b = 10 \mu m$, respectively. The load $F = 1.5 \cdot 10^{-6} N$ is applied uniformly to the beam at $t = 0$. The oscillations of the tip for the case of the cantilever beam under uniformly distributed pressure load are presented in Figures 6-2–6-3 for different values of n . The profiles for $X(t)$ represent the approximations for $v(t, l)$ from (6.5). Clearly, the frequency of oscillations depends on the power-law index n , as predicted by (6.2) and (6.4). It can be seen that the period increases with the increasing power-law index n .

Chapter 7

Conclusions and outlooks

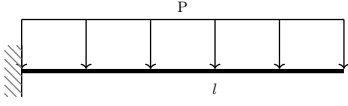
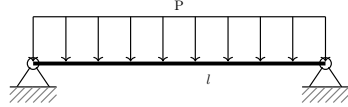
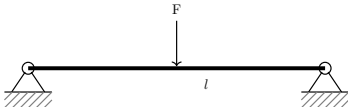
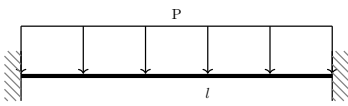
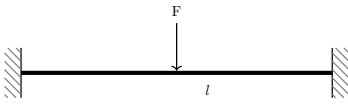
In this work, the nonlinear beam equation for the power-law materials is considered and lumped-parameter modeling elements useful for MEMS are studied.

Table 7.1: The mass lumped stiffness coefficients k_n for different types of beams of length l .

type of beam	stiffness coefficient k_n
	$\frac{2KI_n}{l^{1+2n}} \left(\frac{2(n+1)}{n} \right)^n$
	$\frac{2KI_n}{l^{2n+1}} \left(- \int_{s=0}^{1/2} \int_{t=0}^s (t(1-t))^{\frac{1}{n}} dt ds + \frac{1}{2} B \left(\frac{1}{n} + 2, \frac{1}{n} + 1 \right) \right)^{-n}$
	$2KI_n \left(\frac{2}{l} \right)^{1+2n} \left(\frac{(n+1)(n+2)}{2n} \right)^n$
	$384 \frac{KI_1}{l^3} \quad \text{for } n = 1$
	$\frac{192KI_1}{l^3} \quad \text{for } n = 1$

The stiffness and effective mass parameters for power-law materials have been defined and obtained using static solutions and Galerkin method for the power-law

Table 7.2: The effective mass $m_{eff}(n)$ for different types of beams of mass m .

type of beam	effective mass $m_{eff}(n)$
	$\frac{18n^3 + 44n^2 + 34n + 8}{90n^3 + 177n^2 + 114n + 24}m$
	$\frac{3968}{7875}m$ for $n = 1$
	$\frac{n^4 + 12n^3 + 55n^2 + 30n + 4}{120n^2 + 78n + 12}m$
	$\frac{128}{315}m$ for $n = 1$
	$\frac{13}{70}m$ for $n = 1$

Euler-Bernoulli beams. Computer algebra system Maple has been used for the intensive symbolic calculations. The solution to problem (6.5) have been discussed for the case of cantilever beam under uniformly distributed pressure load using the computed stiffness coefficient and effective mass values. The main results of this work are summarized in Tables 7.1 and 7.2.

The obtained lumped model parameters can be used for design and construction of mechanical structures, especially MEMS made of power-law materials.

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