Nazarbayev University
School of Sciences and Humanities

Capstone project

Finite Element Method for Retaining Walls Subjected to Swelling Pressure

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Abstract

Finite Element Method (FEM) is a widely used method of solving initial boundary-value problems from mechanical engineering. It allows addressing irregular domains and force terms, while enabling careful analysis of the approximated solutions. In this capstone project, a standard derivation of FEM from the mechanical engineering standpoint is presented, then all the necessary mathematical machinery is introduced to facilitate the discussion. Once the proper introduction to FEM is given, the paper dives into the two main subjects of the matter. First, the derivation of the Euler-Bernoulli beam equation is presented. It is a standard model for the analysis of retaining walls. Then a brief derivation of the swelling force, which models the effect of soil swelling onto the retaining walls, is given. Second, the finite element solution to the problem is derived, and the results are presented.

Keywords: finite element method, Euler-Bernoulli beam, swelling force, Hermite cubic elements, Newark method.

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Contents

1 Introduction to Finite Element method in 1d

Our ever-changing world is governed by differential equations (DEs). From the motion of stars to stock prices, from the population dynamics to the deformation of springs, numerous phenomena can be modeled using differential equations.

The analytic study of DEs is a very challenging topic, wherein we only have results about the simplest kinds of problems. But when it comes to Partial Differential Equations (PDEs) with irregular geometries and potential non-linearities, we simply don't have the means to approach them analytically in general (yet).

For that reason, we appeal to numerical schemes, which allow us to obtain approximations of the exact solutions using iterative techniques. There are many such schemes, like Finite Difference Method (FDM), Finite Volume Method (FVM), Boundary Element Method (BEM). In this project though, we are mostly concerned with the Finite Element Method (FEM). The reason why it is so widely used in engineering and mathematical modeling is because of its versatility. FEM can address

- Irregular domains of the problem,
- Irregular right-hand-side functions, like Dirac-Delta distribution,
- problems with analysis, like existence, uniqueness, and stability.

The study of this modern "subdomain" of Mathematics is by no means easy. One of its main challenges comes from the fact that it is a multi-disciplinary study. It takes expertise in different areas of mathematics, physics, and engineering from one to fully appreciate it. But in the end, it is only as challenging as it is rewarding.

1.1 A toy problem

In order to even start talking about FEM, we need first to introduce the notion of the *Weak Formulations* and *Weak solutions* to the boundary value problems. There are different ways to introduce these two concepts, we shall use a common one.

In this section, we will take a look at the physical (or engineering) derivation of the weak formulation and the weak solution for one particular problem from engineering. Even though it lacks generality, this small introduction gives a good physical justification to the procedures that we are going to employ in the later sections.

1.1.1 A problem from Mechanical Engineering

Consider an elastic bar in 1d, as shown below, Figure (1).
As you can see, both ends of the bar are fixed at $x = 0$ and $x = L$. Moreover, there is a force acting parallel to the axis of the bar. Our goal is to calculate the *pointwise dispacement* $u(x)$ (in the direction of x) of the elastic bar as a result of $f(x)$. Figure (2) illustrates what we mean by $u(x)$.

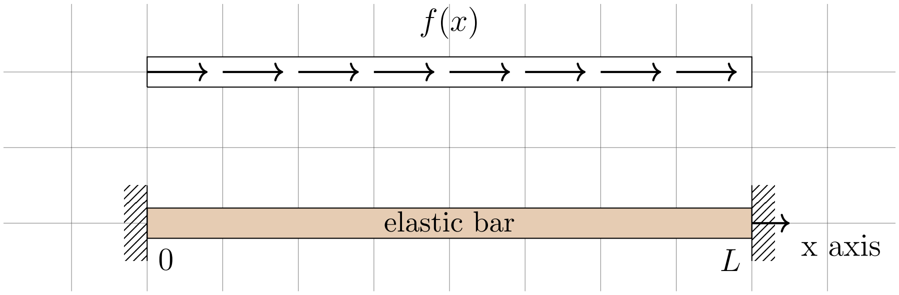

Figure 1: An elastic bar fixed at both ends. The force is acting along the direction of the bar.

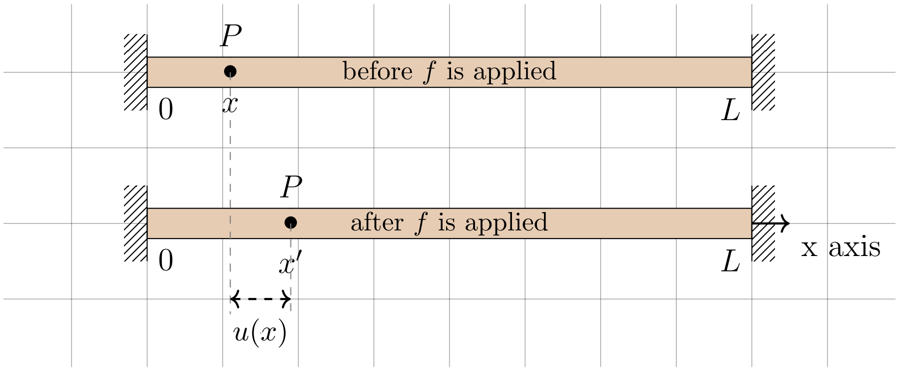

Figure 2: Displacement u of the point P (at x) is given by $u(x) = x' - x$.

Imagine holding a *weak* spring in your hands vertically. You will notice that gravity pulls each component of the spring downwards. Not only that, this deformation is not uniform, it is larger at the center than closer to the endpoints. If you did the same in zero-G, there will be no deformation. If we fix a point $x_0 \in (0, L)$ on the bar at zero-G by marking it, then check its position when it is acted on by gravity, this difference will be $u(x_0)$.

Our goal is, given $u(0), u(L)$ and $f(x)$, to find $u(x)$ on $(0, L)$.

1.1.2 Hooke's Law

Recall Hooke's Law. In simple terms, it states that the force (F) needed to extend or compress an elastic body by some (small) Δx is proptional to Δx, in particular,

$$F = k\Delta x, \tag{1}$$

where k is a spring constant. The same law can be rewritten in terms of stress (σ) and strain (ε), by dividing both sides of Eq. (1) by the cross-section area of the elastic body (A):

$$\sigma = E\varepsilon, \text{ where} \tag{2}$$

- σ is defined as F/A,

- E is Young's modulus, or modulus of elasticity, which depends on the material the body is made of, and
- ε is strain, a relative displacement of the body (ex. $(x' - x)/x$), where x is the length of the body before the deformation, and x' is the length of the body after the force is applied.

1.1.3 Constitutive relation $\sigma = E\frac{du}{dx}$

In fact, there are many ways to define strain. The definition in the previous section ($\varepsilon = (x' - x)/x$) is called *engineering strain*. We would want to apply Hooke's Law in the stress-strain from Eq. (2) to our model problem.

To do that, we first need to fix a point on the rod P located at $x_0 \in (0, L)$. Consider a neighbor of P (call it P_1) at the position $x_0 + \Delta x$. Let's say that P shifts from x_0 to x_0'. At the same time, the small Δx stretches (or compresses) to $\Delta x'$, see Figure (3).

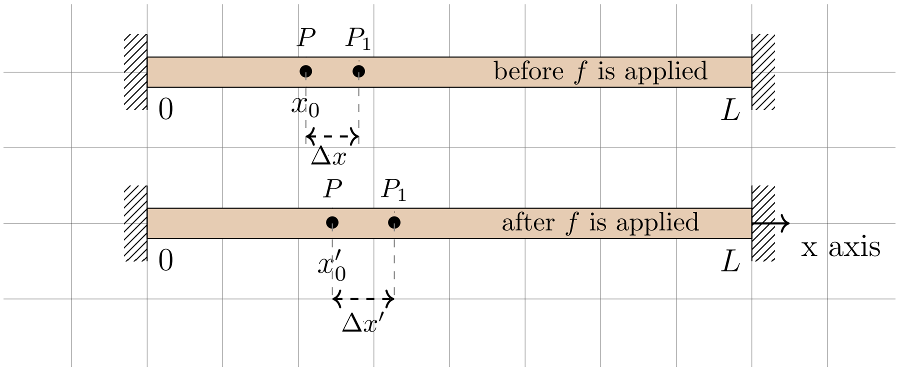

Figure 3: As P is shifted from x_0 to x_0', its neighbor P_1 is shifted from $x_0 + \Delta x$ to $x' + \Delta x'$.

The *infinitesimal strain* at P can be now defined as

$$\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta x' - \Delta x}{\Delta x}$$

Recall that $u(x) = x' - x$, then

$$\varepsilon = \frac{du}{dx} \tag{3}$$

Therefore, we obtain our constitutive relation

$$\sigma = E\frac{du}{dx}. \tag{4}$$

1.1.4 Force density f

Now we are getting closer to deriving the final equation. The last important step is to clarify what we actually mean by $f(x)$. In this case, $f(x)$ is the *force density*, i.e. the distribution of the total force per infinitesimal volume dV (which in this case is actually dx, but let's keep it general). Therefore, the units of $f(x)$ for this problem are N/m^3. We will see shortly why this is important.

1.1.5 Mathematical formulation of the problem

Consider the equilibrium state of the elastic bar after the force is applied. We would like to Newton's Second Law for each infinitesimal volume dV of the elastic bar:

$$\frac{d\sigma}{dx} + f = 0. \tag{5}$$

(Note that σ has units of N/m^2, therefore $\frac{d\sigma}{dx}$ has units N/m^3, the same as f.)

Plugging the constitutive relation (4) into equation (5), we arrive at

$$\frac{d}{dx}\left(E\frac{du}{dx}\right) + f = 0, \quad \text{or, equivalently,}$$

$$E\frac{d^2u}{dx^2} + f = 0.$$

Now, we are ready to state the mathematical formulation of the problem.

$$\boxed{\begin{array}{l}\text{Find } u : (0, L) \to \mathbb{R} \text{ that satisfies} \\ \qquad\qquad E\dfrac{d^2u}{dx^2} + f = 0, \\ \text{with the boundary conditions } u(0) = u(L) = 0.\end{array}} \tag{6}$$

We call this formulation the *strong formulation*, as the second derivative of the solution u appears explicitly in problem statement.

1.1.6 Principle of minimum potential energy

In this subsection we will reformulate our problem (6), in a different way by asking what it means physically for the elastic bar to attain deformation u caused by the external force f.

Before we jump into the main subject of this subsection, we need to first introduce the following definitions. [1]

Definition 1.1 (Strain energy) The strain energy Λ is the energy stored by a system (defined on Ω) undergoing deformation. It is given by

$$\Lambda = \frac{1}{2}\int_{\Omega} \sigma\epsilon d\Omega.$$

In our case, $\Omega = [0, L]$, so the strain energy simplifies to

$$\Lambda = \frac{1}{2}\int_{0}^{L} \sigma\epsilon dx.$$

Recalling Hooke's law (2), we arrive at

$$\Lambda = \frac{1}{2}\int_0^L E\epsilon^2 dx.$$

Now we can use constitutive relation (3) to express Λ in terms of the solution u:

$$\Lambda = \frac{1}{2}\int_0^L E(u')^2 dx \tag{7}$$

From now on, we will be using Newton's notation for derivative (e.g. u' instead of $\frac{du}{dx}$).

Definition 1.2 The work W done by the external forces f (force densities) on the body (defined on Ω) undergoing deformation is given by

$$W = \int_\Omega f v d\Omega.$$

Again, since in our 1-d case $\Omega = [0, L]$, W simplifies to

$$W = \int_0^L f(x)u(x)dx. \tag{8}$$

Definition 1.3 (Total potential energy) The total potential energy Π of a system is defined to be the difference between the total strain energy Λ and the work done by the external forces W:

$$\Pi = \Lambda - W.$$

Recalling the equations for Λ and W obtained for our problem in (7) and (8), respectively, we can express Π as

$$\Pi(u) = \frac{1}{2}\int_0^L E(u')^2 dx - \int_0^L f(x)u(x)dx. \tag{9}$$

Now we are ready to state the principle of minimum potential energy.

Definition 1.4 The *principle of minimum potential energy* states that of all admissible displacements v, the one (call it v_0) that minimizes the total potential energy Π corresponds to the equilibrium solution:

$$\Pi(v_0) \leq \Pi(v).$$

An admissible displacement is any displacement v that satisfies boundary conditions of the problem. For the purposes of this section, we shall denote V as the set of all admissible deformations.

Figure (4) illustrates some of the admissible dispalcements v for our problem.

So, the initial problem (6) is now reduced to the problem of finding $v : [0, L] \to \mathbb{R}$ satisfying B.C., which minimizes Π given by Eq. (9):

$$u = \underset{v:[0,L]\to\mathbb{R}}{\arg\min}\left(\frac{1}{2}\int_0^L E(v')^2 dx - \int_0^L f(x)v(x)dx\right).$$

How do we make sense of this? As you may have noticed, Π is a “function” that takes another function as an argument and returns a scalar. Π is what is called *functional*. For the purposes of this text, think of it as a mapping from a vector space (we can think of functions as infinite-dimensional vectors) to $\mathbb{R}$.

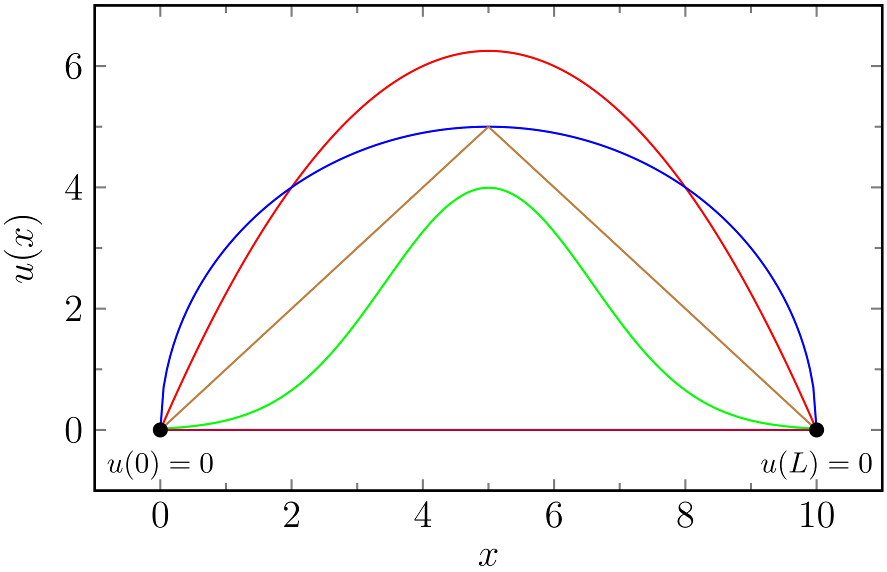

Figure 4: Some of admissible deformations for our problem (6). Note that here $L = 10$. Of all such admissible functions v, we need to find the one(s) that minimize Π.

1.1.7 Minimization of the total potential energy functional

In this section we will answer the question “How to minimize the total potential energy functional?”.

The main idea is the same as minimizing functions from calculus: we need to take the derivative and find its root(s). We shall now define the derivative of a functional.

Definition 1.5 (Directional derivative of a functional) Given functional $F : S \to \mathbb{R}$ the directional derivative of F in the direction of $v \in S$ is defined as

$$\lim_{\alpha \to 0} \frac{F(u + \alpha v) - F(u)}{\alpha} = \left[\frac{d}{d\alpha} F(u + \alpha v)\right]_{\alpha = 0}.$$

With the above definition in mind, we need to find u s.t. its directional derivative equals zero in *any direction* v.

If we assume that u is the minimizer of Π, then we can think of v as an arbitrary *perturbation* added to u, see Figure (5). As the magnitude of the perturbation gets smaller, the system gets closer and closer to the equilibrium.
Now we shall state this problem in mathematical terms using new notation.

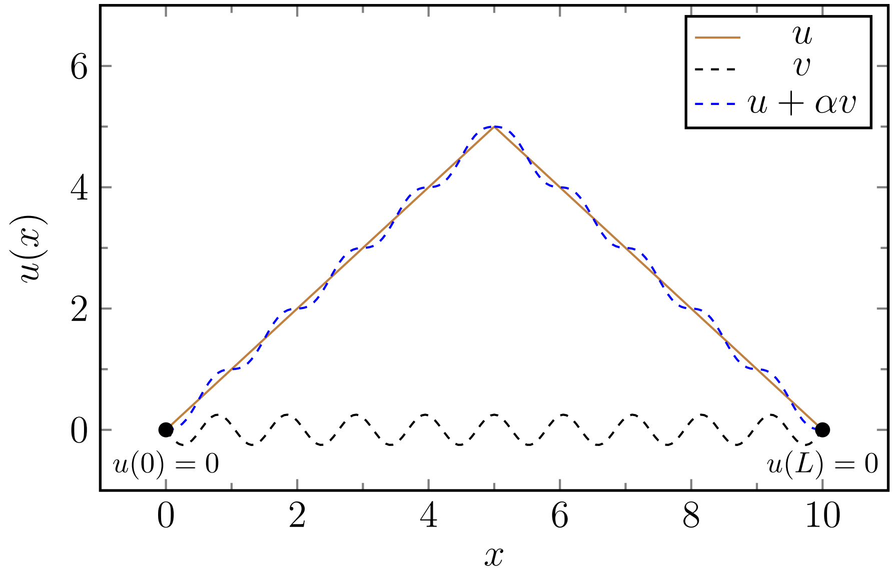

Figure 5: If we restrict $v \in V$ (Def. 1.4) to also satisfy zero Dirichlet boundary conditions, then $u + \alpha v \in V$. Even if we had nonzero Dirichlet b.c. in our problem, v would still had to satisfy zero b.c. (why?)

Find $u \in V$ that satisfies

$$\boxed{\left[\frac{d}{d\alpha}\Pi(u+\alpha v)\right]_{\alpha=0} = 0} \tag{10}$$

for all $v \in V$ which also satisfy zero Dirichlet b.c.

From now on, everything is going to be more or less straightforward, as we established all of the assumptions we needed to get to this point.

Let $v \in V$ satisfying zero b.c., then

$$\begin{aligned}
\frac{d}{d\alpha}\Pi(u+\alpha v) &= \frac{d}{d\alpha}\left(\frac{1}{2}\int_0^L E\left[(u+\alpha v)'\right]^2 dx - \int_0^L f(u+v)dx\right) \\
&= \frac{d}{d\alpha}\left(\int_0^L \frac{1}{2}E(u'+\alpha v')^2 - f(u+v)dx\right) \\
&= \int_0^L \frac{1}{2}E\frac{d}{d\alpha}\left[(u'+\alpha v')^2\right] - \frac{d}{d\alpha}[f(u+v)]dx \\
&= \int_0^L E(u'+\alpha v')\frac{d}{d\alpha}(u'+\alpha v') - f\frac{d}{d\alpha}(u+\alpha v)dx \\
&= \int_0^L E(u'+\alpha v')v' - fvdx.
\end{aligned}$$

(We assumed that we can push the derivative operator inside the definite integral, which is not always true.) By fixing $\alpha = 0$, we get

$$\left[\frac{d}{d\alpha}\Pi(u+\alpha v)\right]_{\alpha=0} = \int_0^L Eu'v' - fvdx.$$

Finally, we want to force $\left[\frac{d}{d\alpha}\Pi(u+\alpha v)\right]_{\alpha=0} = 0$, as stated in (10). This gives us

$$\int_0^L Eu'v' - fvdx,$$

or, equivalently,

$$\int_0^L Eu'v'dx = \int_0^L fvdx. \tag{11}$$

Since we let v be arbitrary, a function u that satisfies Eq. (11) is the minimizer of Π, and by the Principle of Minimum Potential Energy, it must also be the solution to the original problem (6).

(We assumed the Π does have the unique minimizer, which for this problem is true, but we are not going to show it in this text.)

1.1.8 Weak formulation and weak solution of the model problem

Let's recap what we did.

- We started with a model problem as illustrated on Figure (2). Our goal was to calculated the deformation $u(x)$ of an elastic bar of length L with fixed ends, as a result of an external force per unit length f.

- We modelled this problem using Hooke's law to arrive at the following mathematical formulation:

  Find $u : (0, L) \to \mathbb{R}$ that satisfies

  $$E\frac{d^2u}{dx^2} + f = 0,$$

  with the boundary conditions $u(0) = u(L) = 0$. E is Young's modulus of the material, f is external force.

- Using the principle of minimum potential energy, we transformed the previous (strong formulation) into the minimization problem given by

  Find $u \in V$ that satisfies

  $$\left[\frac{d}{d\alpha}\Pi(u+\alpha v)\right]_{\alpha=0} = 0$$

  for all $v \in V$ which also satisfy zero Dirichlet b.c.

Here, V is the set of all *admissible* deformations, i.e. the ones that are allowed by the nature of the experiment (mathematically, all functions, for which Π exists and which satisfy boundary conditions). Π is the total potential energy given by

$$\Pi(u) = \frac{1}{2}\int_0^L E(u')^2 dx - \int_0^L f(x)u(x)dx.$$

- Using methods of calculus of variations, we arrived at the new reformulation of the problem:

  Find $u \in V$ s.t.
  $$E\int_0^L u'v' = \int_0^L fv$$
  holds true for all $v \in V$ with zero Diriclet b.c.

The last formulation of the problem is called *weak formulation*, and the solution to weak formulation of the problem is called *weak* solution.

In the next section we will show a faster way to arrive at the weak formulation of the problem, state exactly what is the set V, and see how we can use weak formulations to obtain an approximate solution to the model problem.

1.2 Weak formulation

Let's remind ourselves with the strong formulation of the toy problem, i.e. deformation of an elastic bar.

Find $u : [0, L] \to \mathbb{R}$ s.t.

$$\begin{cases} -Eu'' = f(x), & x \in (0, L), E > 0 \\ u(0) = u(L) = 0. \end{cases} \tag{12}$$

We shall now present a simpler way to arrive at the weak formulation.

Suppose that $v : [0, L] \to \mathbb{R}$ is a *sufficiently* regular function. Now we will do the following. We multiply both sides of the equation by v, and integrate both sides from 0 to L.

$$\textstyle\int_0^L -Eu''(x)v(x)dx = \int_0^L f(x)v(x)dx. \tag{13}$$

(We call u a *trial* function, and v a *test function*, and we say that we "test" u with v)

Note that we can integrate the left hand side by parts:

$$\int_0^L -Eu''(x)v(x)dx = -E\left[u'(x)v(x)\right]\Big|_0^L + \int_0^L Eu'(x)v'(x)dx \tag{14}$$

Plugging (14) into (13) yields

$$\int_0^L Eu'(x)v'(x)dx = \int_0^L f(x)v(x)dx + Eu'(L)v(L) - Eu'(0)v(0). \tag{15}$$

Let's impose further condition on v that $v(0) = v(L) = 0$, then

$$\int_0^L Eu'(x)v'(x)dx = \int_0^L f(x)v(x)dx. \tag{16}$$

Now that we have obtained this equality, we would like to know two things:

- When does $\int_0^L Eu'(x)v'(x)dx$ make sense?
- What about $\int_0^L f(x)v(x)dx$?

1.2.1 Deriving appropriate spaces

Now we shall address the questions raised in the previous section.

We first consider the integral $\int_0^L Eu'(x)v'(x)dx$. We want to first make sure that

$$\left|\int_0^L Eu'(x)v'(x)dx\right| < \infty.$$

For that, recall (if that's the case) one of the Cauchy-Schwartz inequalities:

$$\left|\int_0^L f(x)g(x)dx\right| \leq \left(\int_0^L [f(x)]^2 dx\right)^{\frac{1}{2}} \left(\int_0^L [g(x)]^2 dx\right)^{\frac{1}{2}}. \tag{17}$$

Now we infer that

$$\left|\int_0^L Eu'(x)v'(x)dx\right| \leq \left(\int_0^L [Eu'(x)]^2 dx\right)^{\frac{1}{2}} \left(\int_0^L [v'(x)]^2 dx\right)^{\frac{1}{2}}.$$

We can see that if u' and v' are *square-integrable*, then the integral on the left-hand side exists.

We now consider the integral $\int_0^L f(x)v(x)dx$. Again, applying (17) yields

$$\left|\int_0^L f(x)v(x)dx\right| \leq \left(\int_0^L [f(x)]^2 dx\right)^{\frac{1}{2}} \left(\int_0^L [v(x)]^2 dx\right)^{\frac{1}{2}}.$$

Now we further infer that f needs to be square-integrable as well.

1.2.2 Note on function spaces

Before we jump into further discussion of the weak formulation, we would like to introduce the appropriate notation for certain *function spaces*. One can think of a function space $FS(\Omega)$ as of the set of all functions from Ω to $\mathbb{R}$ that satisfy certain properties.

We shall now present some of the important for our discussion function spaces.

- $C^n(\Omega)$ is the space of all functions from Ω to $\mathbb{R}$, all of whose derivatives up to n-th order are continuous. $C^0(\Omega)$ denotes the space of all continuous functions $f : \Omega \to \mathbb{R}$. $C^\infty(\Omega)$ denotes the space of all infinitely-many times differentiable functions from Ω to $\mathbb{R}$.

- $L^p(\Omega)$ (called *Lebesgue spaces*) are the spaces of all functions $f : \Omega \to \mathbb{R}$ s.t.

  $$\left(\int_\Omega |f|^p d\Omega\right)^{\frac{1}{p}} < \infty.$$

  For the purposes of this text, we will use the L^2-space, also known as the space of all (Lebesgue) square-integrable functions, a lot. It is defined as

  $$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{R} \middle| \int_\Omega f^2 d\Omega < \infty \right\}.$$

- $H^p(\Omega)$ are called *Sobolev spaces*, and defined as

  $$H^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \middle| f, f', ..., f^{(p)} \in L^2(\Omega) \right\}.$$

Note that some spaces can be *subspaces* of others, i.e. subsets which are closed under addition and scalar multiplication. For instance, $H^2 \subset H^1$, or $H^1 \subset L^2$. This will be crucial when we discuss Galerkin's method.

1.2.3 The weak formulation

Now, we present the weak formulation of the problem using appropriate spaces.

Let $H_0^1 = \left\{ v : [0, L] \to \mathbb{R} \middle| v \in H^1([0, L]) \text{ and } v(0) = v(L) = 0 \right\}$.

$$\boxed{\begin{array}{l} \text{Find } u \in H_0^1 \text{ s.t.} \\ \displaystyle\int_0^L E u'(x) v'(x) dx = \int_0^L f(x) v(x) dx, \quad \forall v \in H_0^1. \end{array}} \tag{18}$$

Note that v is from the same space as u, which happened because (12) also has zero Dirichlet b.c. It is not the case in general. We will discuss other cases in section 1.2.7.

1.2.4 An important remark on weak solutions

Now that we have the weak formulation (18), you may wonder why do we bother doing all these transformations to our original problem (12), when can just solve it explicitly the way it is. Well, the latter is definitely true for this simple problem, however it will no longer be true, once we make it more complicated. For instance, if we had

$$\begin{cases} -Eu'' = \delta\left(x - \frac{L}{2}\right), & x \in (0, L) \\ u(0) = u(L) = 0. \end{cases}$$

where $\delta(x)$ is the Dirac-delta function (or rather distibution). In the more "classical" sense, we would expect the solution to be two-times continuously differentiable (i.e., $\in C^2([0, L))$. However, for this particular problem, the solution is given by a piece-wise linear function:

$$u(x) = \begin{cases} \frac{1}{2E}x, & x \in \left[0, \frac{L}{2}\right) \\ \frac{L}{2E} - \frac{1}{2E}x, & x \in \left[\frac{L}{2}, L\right], \end{cases}$$

whose first derivative does not exist at $x = \frac{L}{2}$, and the second derivative can only be modelled by a Dirac-delta distribution (not even a function). So, what is presented above is a not a solution in the classical sense, but rather in a "weak" sense. Consider its weak formulation:

Find $u \in H_0^1$ s.t.

$$\int_0^L Eu'(x)v'(x)dx = v\left(\frac{L}{2}\right), \quad \forall v \in H_0^1 \cap C^0\left(\left\{\frac{L}{2}\right\}\right).$$

What you can notice is that in the weak formulation we only require u to be from the function space H_0^1, i.e. functions on $[0, L]$ whose first derivative is square integrable. This is a very relaxed, or weak condition compared to C^2, which classical solutions require. We do not even require the first derivative to be continuous, only square-integrable.

This is one of the reasons why weak formulations are so important for complicated problems: they allow us to relax the conditions on the solution, thus increasing the space of functions wherein we look for the solution.

1.2.5 Bilinear forms

Recall the weak formulation (18). In order to work with weak formulation in a more systematic and consice way, we introduce two shortcuts for this problem:

- $a(u, v) := \int_0^L Eu'(x)v'(x)dx$, where $a(\cdot, \cdot)$ is a *bilinear form*, and
- $(f, v) := \int_0^L f(x)v(x)dx$, which is an inner product between f and v.

Definition 1.6 (Bilinear form) A bilinear form is a mapping from the cross product of a vector (or function) space to $\mathbb{R}$, which is also linear in the first and the second argument separately.

Let's make the basic properties of bilinear forms more explicit. Let V be a vector space. Let $u, v, w \in V$, and $\alpha \in \mathbb{R}$. Let $a : V \times V \to \mathbb{R}$ be a bilinear form. Then, the following holds true for all u, v, w, α:

- $a(u + w, v) = a(u, v) + a(w, v)$

- $a(u, v+w) = a(u,v) + a(u,w)$
- $a(\alpha u, v) = a(u, \alpha v) = \alpha a(u,v)$

Bilinear forms can be symmetric, $a(u,v) = a(v,u)$, which is the case in our problem:

$$a(u,v) = \int_0^L Eu'(x)v'(x)dx = \int_0^L Ev'(x)u'(x)dx = a(v,u).$$

1.2.6 Relation between three formulations

Let's recall the derivation of the weak formulation obtained by minimizing the total potential energy functional (10). Let's now rewrite it using the new notation.

$$\boxed{\begin{aligned}&\text{Find } u \in H_0^1 \text{ s.t.}\\ &\qquad u = \arg\min_{v \in H_0^1} \Pi(v)\\ &\qquad\quad = \arg\min_{v \in H_0^1} \left[\frac{1}{2}a(v,v) - (f,v)\right].\end{aligned}} \tag{19}$$

What we want to do now is we want to establish the relation between the weak formulation (V), the energy minimization problem (M), and the strong formulation (D).

Theorem 1.7 (Relation between formulations) For our problem, the relation between (D), (V) and (M) is

$$(\text{D}) \implies (\text{V}) \iff (\text{M}).$$

Proof: It suffices to prove (**1**) (D) $\implies$ (V), (**2**) (V) $\implies$ (M), and (**3**) (M) $\implies$ (V).

(**1**): We actually already did it, refer to (12) - (18).

(**2**): Assume that $u \in H_0^1$ is the solution to the weak formulation, that is

$$a(u,v) = (f,v), \quad \forall v \in H_0^1.$$

Let $v \in H_0^1$, let $w = v - u \in H_0^1$. Then

$$\begin{aligned}\Pi(v) &= \Pi(w+u)\\ &= \frac{1}{2}a(w+u, w+u) - (f, w+u)\\ &= \frac{1}{2}\left[a(w, w+u) + a(u, w+u)\right] - (f,w) - (f,u)\\ &= \frac{1}{2}\left[a(w,w) + a(w,u) + a(u,w) + a(u,u)\right] - (f,w) - (f,u)\\ &= \left[\frac{1}{2}a(u,u) - (f,u)\right] + \left[\frac{1}{2}a(u,w) + \frac{1}{2}a(v,u) - (f,w)\right] + \frac{1}{2}a(w,w)\\ &= \underbrace{\left[\frac{1}{2}a(u,u) - (f,u)\right]}_{\Pi(u)} + \underbrace{\left[a(u,w) - (f,w)\right]}_{=0} + \underbrace{\frac{1}{2}a(w,w)}_{\geq 0}\\ &\geq \Pi(u).\end{aligned}$$

Thus, $\Pi(u) \leq \Pi(v)$ for any $v \in H_0^1$, which means that u is a minimizer of $\Pi(v)$.

(**3**): Even though we have already showed informally that the minimization problem reduces to the weak formulation in 1.1.7, we shall now do it in a more rigorous way.

Assume that $u \in H_0^1$ is a minimizer of $\Pi(v)$. Then consider a perturbation of u (recall Diagram 5):

$$\Pi(u) \leq \underbrace{\Pi(u + \alpha v)}_{\in H_0^1}, \quad \forall v \in H_0^1, \forall \alpha \in \mathbb{R}. \tag{20}$$

Consider

$$\begin{aligned} g(\alpha) &= \Pi(u + \alpha v) \\ &= \frac{1}{2} a(u + \alpha v, u + \alpha v) - (f, u + \alpha v) \\ &= \frac{1}{2}[a(u,u) + a(u, \alpha v) + a(\alpha v, u) + a(\alpha v, \alpha v)] - (f, u) - (f, \alpha v) \\ &= \frac{1}{2}[a(u,v) + \alpha a(u,v) + \alpha a(v,u) + \alpha^2 a(v,v)] - (f,u) - \alpha(f,v) \\ &= \frac{1}{2} a(u,u) + \alpha a(u,v) + \frac{1}{2}\alpha^2 a(v,v) - (f,u) - \alpha(f,v). \end{aligned}$$

If we let $\alpha = 0$, then we get $\Pi(u)$, thus $g(\alpha)$ attains a minimum at $\alpha = 0$. In other words, we need to compute $g'(0) = 0$.

$$\begin{aligned} g'(0) &= \frac{d}{d\alpha}\left[\frac{1}{2} a(u,u) + \alpha a(u,v) + \frac{1}{2}\alpha^2 a(v,v) - (f,u) - \alpha(f,v)\right]_{\alpha=0} \\ &= [a(u,v) + \alpha a(v,v) - (f,v)]_{\alpha=0} \\ &= a(u,v) - (f,v). \end{aligned}$$

Now, if we enforce $g'(0) = 0$, we obtain $a(u,v) - (f,v) = 0$, which is our weak formulation.

Q.E.D.

Thus, in this section we established the following relation:

$$\text{(D)} \implies \text{(V)} \iff \text{(M)}.$$

One may also show that if $u \in C^2([0,L])$, and (D) $\impliedby$ (V). [2]

1.2.7 Boundary conditions

So far we have only considered the simplest case of boundary conditions given in (6). In this section, we will briefly discuss what happens to the weak formulation with different b.c. What we want to know is how to incorporate boundary conditions into our weak formulation. In general, there are two ways to do that: 1) by imposing additional conditions on the *test space* V ($v \in V$), 2) by plugging b.c. into Eq. (15) directly.

Before we jump into the discussion, let's first introduce a new bit of notation: let U denote the *trial space*, i.e. $u \in U$.

Consider a new problem called a *pure Dirichlet problem*: Find $u : [0, L] \to \mathbb{R}$ s.t.

$$\begin{cases} -Eu'' = f(x), & x \in (0, L), E > 0 \\ u(0) = u_0, \quad u_0 \in \mathbb{R} \\ u(L) = u_L, \quad u_L \in \mathbb{R}. \end{cases} \tag{21}$$

As was discussed earlier, one necessary condition for u and v is that they are from $H^1([0, L])$. Taking into account the boundary conditions in (21), we need to further restrict the trial space:

$$U = \{u \in H^1([0, L]); u(0) = u_0, u(L) = u_L\}.$$

Now, if we "test" u with v, we obtain (as in Equation 15):

$$\int_0^L Eu'(x)v'(x)dx = \int_0^L f(x)v(x)dx + Eu'(L)v(L) - Eu'(0)v(0).$$

In this situation, we have to incorporate the Dirichlet b.c. into the test space by explicitly imposing that $v(0) = v(L) = 0$ (this assumption is essential, hence Diriclet b.c. are also called "*essential*"), thus eliminating the last two terms (which are called *boundary contribution*). Thus, the test space is given by

$$V = \{v \in H^1([0, L]); v(0) = v(L) = 0\}.$$

A good intuition why these additional assumptions on v are necessary can be seen in Figure (5). In general, we want $u + \alpha v \in U$ for the energy minimization method to work properly (see Eq. 20).

Let's now consider a pure Newmann problem

$$\begin{cases} -Eu'' = f(x), & x \in (0, L), E > 0 \\ u'(0) = u'_0, \quad u'_0 \in \mathbb{R} \\ u'(L) = u'_L, \quad u'_L \in \mathbb{R}. \end{cases} \tag{22}$$

Let for now $u, v \in H^1([0, L])$ without any further assumptions. Let's test u with v:

$$\int_0^L Eu'(x)v'(x)dx = \int_0^L f(x)v(x)dx + Eu'(L)v(L) - Eu'(0)v(0).$$

You may see that here we can actually incorporate the Newmann b.c. directly or "naturally" (hence Newmann b.c. are also called "*natural*" b.c.) into the formulation:

$$\int_0^L Eu'(x)v'(x)dx = \int_0^L f(x)v(x)dx + Eu'_L v(L) - Eu'_0 v(0). \tag{23}$$

Since we have already incorporated Newmann b.c. into the formulation, we no longer need to impose it onto the spaces U and V, hence $u + \alpha v \in U$ still holds. The justification comes from the fact that total potential energy includes the phenomena corresponding to the Newmann b.c. naturally in its definition. One can derive the formula for Π for this problem from (23).

Let's now summarize the results in the final table:

B.C.	Variational name	Proper name	How to treat?
$u(x) = u_x$	essential	Dirichlet	Impose $u(x) = u_x$ on U, $v(x) = 0$ on V
$u'(x) = u'_x$	natural	Newmann	Plug u'_x directly into Eq. (15)

Table 1: How to incorporate different boundary conditions into the weak formulation, $x = 0$ or L. [3]

1.2.8 Ritz-Galerkin method

If we restrict ourselves to a *finite* subspace $S \subset U$ rather than the entire space, will the solution to the weak formulation in this subspace be the best approximation of the true weak solution among all the functions in S?

Let's go back to our weak formulation (18). Recall that $U = V = \{u \in H^1([0, L]) : u(0) = u(L) = 0\}$. Let $S \subset U$ be a finite-dimensional subspace.

What we are trying to solve now is

$$\boxed{\begin{array}{l}\text{Find } u_S \in S \text{ s.t.} \\ \\ \quad a(u_S, v_S) = (f, v_S), \qquad \forall v_S \in S.\end{array}} \tag{24}$$

Since S is finite dimensional space, we can express S as $\text{span}\{\phi_i\}_{i=1}^N$ where $\phi_i : [0, L] \to \mathbb{R}$. Here, ϕ_i are called *basis functions*. We can express $u_S(x) = \sum_{i=1}^N u_i \phi_i(x)$, $v_S = \sum_{j=1}^N v_j \phi_j(x)$ (u_i, v_j are scalars) [3]. Plugging u_S and v_S into (24), we get

$$\int_0^L E \frac{d}{dx}\left(\sum_{i=1}^N u_i \phi_i(x)\right) \frac{d}{dx}\left(\sum_{j=1}^N v_j \phi_j(x)\right) dx = \int_0^L f(x)\left(\sum_{j=1}^N v_j \phi_j(x)\right) dx.$$

$$\int_0^L E \left(\sum_{i=1}^N u_i \phi'_i(x)\right)\left(\sum_{j=1}^N v_j \phi'_j(x)\right) dx = \int_0^L f(x)\left(\sum_{j=1}^N v_j \phi_j(x)\right) dx.$$

This must hold for any $v_S \in S$. But since S is finite and a vector space, it is equivalent to saying that the above equation must hold for all coefficients v_j. Let's expand the sum:

$$E\int_0^L \sum_{j=1}^N \sum_{i=1}^N u_i\phi_i'(x)v_j\phi_j'(x)dx = \int_0^L \sum_{j=1}^N v_j f(x)\phi_j(x)dx$$

$$E\int_0^L \sum_{j=1}^N v_j \sum_{i=1}^N u_i\phi_i'(x)\phi_j'(x)dx = \int_0^L \sum_{j=1}^N v_j \left[f(x)\phi_j(x)\right]dx$$

$$E\int_0^L \sum_{j=1}^N v_j \left[\sum_{i=1}^N \phi_i'(x)\phi_j'(x)\right] u_i dx = \int_0^L \sum_{j=1}^N v_j \left[f(x)\phi_j(x)\right]dx$$

Moving everything to the left-hand side:

$$\int_0^L \sum_{j=1}^N v_j \left(\left[\sum_{i=1}^N E\phi_i'(x)\phi_j'(x)\right] u_i - \left[f(x)\phi_j(x)\right]\right) dx = 0$$

$$\sum_{j=1}^N v_j \int_0^L \left[\sum_{i=1}^N E\phi_i'(x)\phi_j'(x)\right] u_i - f(x)\phi_j(x)dx = 0$$

Consequently,

$$\sum_{j=1}^N v_j \left[\sum_{i=1}^N \left(\int_0^L E\phi_i'(x)\phi_j'(x)dx\right) u_i - \int_0^L f(x)\phi_j(x)dx\right] = 0$$

must be satisfied for any $v_j \in \mathbb{R}$. If we let $K_{ij} = \int_0^L E\phi_i'(x)\phi_j'(x)dx$, $F_j = \int_0^L f(x)\phi_j(x)dx$, then

$$\sum_{j=1}^N v_j \left[\sum_{i=1}^N K_{ij}u_i - F_j\right] = 0, \quad \forall v_j. \tag{25}$$

Here, $\mathbf{K} = [K_{ij}] \in \mathbb{R}^{N\times N}$ is called *stiffness matrix*, and $\mathbf{F} = [F_j] \in \mathbb{R}^N$ is called *load vector*.

One can notice that by linearity of S, Eq. (25) is equivalent to

$$v_j \left[\sum_{i=1}^N K_{ij}u_i - F_j\right] = 0, \quad \forall v_j \in \mathbb{R}.$$

Consequently, we arrive at

$$\sum_{i=1}^N K_{ij}u_i - F_j = 0. \tag{26}$$

If we let $\mathbf{U} = [u_i] \in \mathbb{R}^N$, then Eq. (26) can be expressed in a more compact form as

$$\mathbf{KU} = \mathbf{F}. \tag{27}$$

What we need now is to show that we can solve (27) and that the solution $\mathbf{U}$ is unique.

Theorem 1.8 (K is s.p.d.) The stiffness matrix $\mathbf{K}$ is symmetric positive definite.

Proof: $\mathbf{K}$ is s.p.d. $\iff \forall \mathbf{x} \in \mathbb{R}^N \setminus \{\mathbf{0}\}, \mathbf{x}^T\mathbf{K}\mathbf{x} > 0$. Let $\mathbf{x} \in \mathbb{R}^N$, then

$$
\begin{aligned}
\mathbf{x}^T\mathbf{K}\mathbf{x} &= \begin{bmatrix} x_1 & \cdots & x_N \end{bmatrix} \begin{bmatrix} \int_0^L E\phi_1'(x)\phi_1'(x)dx & \cdots & \int_0^L E\phi_1'(x)\phi_N'(x)dx \\ \vdots & \ddots & \vdots \\ \int_0^L E\phi_N'(x)\phi_1'(x)dx & \cdots & \int_0^L E\phi_N'(x)\phi_N'(x)dx \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \\
&= \begin{bmatrix} x_1 & \cdots & x_N \end{bmatrix} \begin{bmatrix} \sum_{i=1}^N \int_0^L Ex_i\phi_1'(x)\phi_i'(x)dx \\ \vdots \\ \sum_{i=1}^N \int_0^L Ex_i\phi_N'(x)\phi_i'(x)dx \end{bmatrix} \\
&= \begin{bmatrix} x_1 & \cdots & x_N \end{bmatrix} \begin{bmatrix} E\int_0^L \phi_1'(x) \left[\sum_{i=1}^N x_i\phi_i'(x)\right] dx \\ \vdots \\ E\int_0^L \phi_N'(x) \left[\sum_{i=1}^N x_i\phi_i'(x)\right] dx \end{bmatrix} \\
&= E\int_0^L \left[\sum_{j=1}^N x_j\phi_j'(x)\right] \left[\sum_{i=1}^N x_i\phi_i'(x)\right] dx \\
&= \int_0^L E(v')^2 dx \geq 0.
\end{aligned}
$$

We need to show that $\mathbf{x}^T\mathbf{K}\mathbf{x} = 0 \iff \mathbf{x} = \mathbf{0}$. ($\Longleftarrow$) is trivial, so we only show ($\Longrightarrow$).

($\Longrightarrow$) $\int_0^L E(v')^2dx = 0 \implies v' = 0 \implies v$ is const., but since $v \in S \subset U \implies v(0) = 0 \implies v = 0 \iff \sum_{j=1}^N x_j\phi_j(x) = 0$. Since $\{\phi_j\}$ forms a basis for S, we conclude that $\mathbf{x} = 0$.

Therefore, $\forall \mathbf{x} \in \mathbb{R}^N \setminus \{\mathbf{0}\}, \mathbf{x}^T\mathbf{K}\mathbf{x} > 0$, or, equivalently, $\mathbf{K}$ is s.p.d.

Q.E.D.

Since $\mathbf{K}$ is s.p.d., it must be invertible. If we assume that $f \in L^2([0, L])$, then $\mathbf{U}$ is unique, and, consequently, $u_S \in S$ is unique for Dirichlet b.c. [2]

1.2.9 Galerkin orthogonality

So far we have shown that under certain assumption, we can always find a unique solution u_S in a finite-dimensional subspace of U. Still, this does not imply that it is the "best" solution in S, and by "best" we mean the closest approximation to u w.r.t. some norm.

Definition 1.9 (Norm) Let V be a vector space. Then norm on V is defined by a function $\|\cdot\| : V \to \mathbb{R}^{nonneg}$ that satisfies

1. $\|x\| = 0 \iff x = 0$

2. $\|cx\| = |c|\|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

$\forall x, y \in V, c \in \mathbb{R}$. In addition, $(V, \|\cdot\|)$ is called normed space, or normed vector space, or normed linear space. [4]

One can verify that $\|v\| := \sqrt{a(v,v)}$ defines a norm on H_0^1. Since this norm is not unique for H_0^1, we shall denote it as $\|\cdot\|_{H_0^1}$ to avoid ambiguity.

Before we state the Galerkin orthogonality theorem, the following *Schwarz' inequality* will be of great use:

$$a(u,v) \leq \|u\|_{H_0^1}\|v\|_{H_0^1}, \quad \forall u, v \in H_0^1. \tag{28}$$

Theorem 1.10 (Galerkin orthogonality) If $u \in H_0^1$ is the solution to (18), and $u_S \in S$ is the solution to (24), then $\|u_S - u\|_{H_0^1} \leq \|v_S - u\|_{H_0^1}, \forall v_S \in H_0^1$.

Proof: Suppose $u \in H_0^1$ solves (18), then $a(u,v) = (f,v)$, $\forall v \in H_0^1$. Since $S \subset H_0^1$, we can test u against $v_S \in S$:

$$a(u, v_S) = (f, v_S), \quad \forall v_S \in S. \tag{29}$$

Suppose $u_S \in S$ solves (24), then

$$a(u_S, v_S) = (f, v_S),\ \forall v_S \in S. \tag{30}$$

If we subtract (30) from (29), we obtain

$$a(u - u_S, v_S) = 0,\ \forall v_S \in S. \tag{31}$$

$$\begin{aligned}
\|u - u_S\|_{H_0^1}^2 &= a(u - u_S, u - u_S) \\
&= a(u - u_S, u - v_S + v_S - u_S) \\
&= a(u - u_S, u - v_S) + a(u - u_S, v_S - u_S) \\
&\quad \text{we use the fact that } a(u - u_S, w_S) = 0 \text{ where } w_S = v_S - u_S \in S \text{ by (31),} \\
&= a(u - u_S, u - v_S) \\
&\leq \|u - u_S\|_{H_0^1}\|u - v_S\|_{H_0^1}, \quad \text{by Ineq. (28)}
\end{aligned}$$

Consequently, we obtain

$$\|u - u_S\|_{H_0^1} \leq \|u - v_S\|_{H_0^1},\ \forall v_S \in S.$$

Thus, $u_S \in S$ is in fact the best approximation to $u \in H_0^1$ w.r.t. H_0^1-norm. [3]

Q.E.D.

Equation 31 was the key in the proof. It is called *Galerkin orthogonality*. It says that the error $\|u - u_S\|_{H_0^1}$ is "orthogonal" to the finite-dimensional subspace S.

So far we have shown that $\mathbf{K}$ is invertible, that $u_S \in S$ exists and unique (assuming $f \in L^2([0, L])$), and that u_S is the best approximation to u (w.r.t. H_0^1-norm) from all S.

By choosing different S and it basis, we will get different matrices $\mathbf{K}$, and all of them are guaranteed to be invertible, which is great news. Now, we need to switch our framework, and think about how to choose S and its basis so that our computations become the least expensive.

1.3 Finite element method in 1d

Now we can finally talk about the central topic of this project, which is Finite Element Method, or simply FEM. In essence, it is a special case of Ritz-Galerkin's method, where we choose a special kind of finite-dimensional subspace S with a special kind of basis functions $\phi_i(x)$, which make the stiffness matrix $\mathbf{K}$ as sparse as possible.

1.3.1 Motivation

Why does it matter whether $\mathbf{K}$ is sparse or not? Recall from the previous section, that we have reduced our weak formulation given by Eq. (18) to a linear system given by Eq. (27). Thus, by fixing the finite-dimensional subspace S and its basis $\{\phi_i\}_{i=1}^N$, we need to solve $\mathbf{KU} = \mathbf{F}$ for $\mathbf{U}$.

The best known algorithms for solving a general linear system of n equations have computational complexity of approximately $\mathcal{O}(n^{2.373})$ [5, 6]. Both of these algorithms are based on Coppersmith–Winograd algorithm, whose complexity is approximately $\mathcal{O}(n^{2.375})$ [7]. Even though they have the best complexities, it comes with a very large constant. Hence they are only efficient for extremely large systems.

In practice, Strassen's algorithm is used for moderately large systems, whose complexity is $\mathcal{O}(n^{\log_2 7}) \approx \mathcal{O}(n^{2.8074})$ [8], and Gaussian elimination with $\mathcal{O}(n^3)$ is used for the smaller ones.

The point is that if the matrix $\mathbf{K}$ is large and dense, then our computation of the solution vector $\mathbf{U}$ may become infeasible. This may not be a problem for linear 1d problems, but it will surely be if we go to higher dimensions and also have non-linearities. But if we find a way to make $\mathbf{K}$ sparse (meaning most of the entries are zero), then we can employ very fast algorithms like Thomas algorithm for tridiagonal systems which is only $\mathcal{O}(n)$ [9].

1.3.2 Discretization of the domain

First, we shall discretize the domain. Let $\Omega = [0, L]$. Let $0 = x_0 < ...x_j < ... < x_{NE} = L$ be the partition of Ω. Let $\Omega_e = [x_{e-1}, x_e], e = 1, ..., NE$. We shall call Ω_e an *element* of Ω. Thus, we have NE number of elements, and each element has two nodes. It will also be

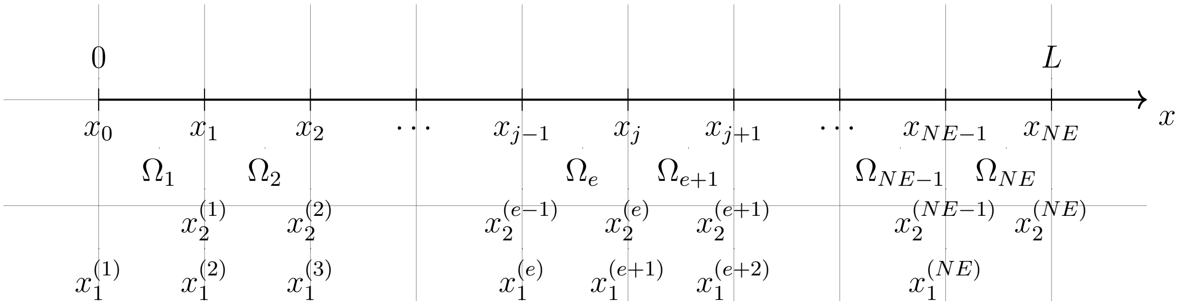

Figure 6: Discretization of the domain.

very useful to define a *local node numbering.* Let $x_i^{(e)} = x_{e+i-2}$, then $\Omega_e = [x_1^{(e)}, x_2^{(e)}]$. See Figure (6).

The last step is to define the step size $h_j = x_j - x_{j-1}, j = 1, ..., NE$. For the purposes of this toy problem, let's assume that we use an equidistant mesh, i.e. $h_1 = h_2 = ... = h_{NE} = h = \frac{L}{NE}$.

1.3.3 Basis functions

Now, we define the piecewise linear basis functions $\phi_i(x), i = 1, ..., NE-1$ as

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h} & \text{if } x_{j-1} \leq x \leq x_j \\ \frac{x_{j+1} - x}{h} & \text{if } x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

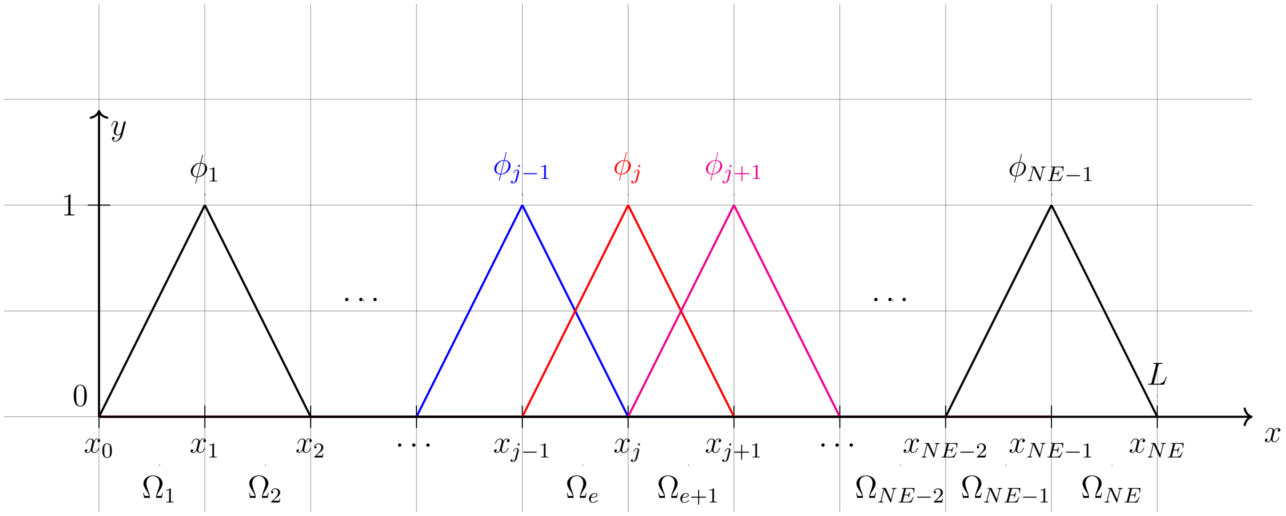

Figure 7: Piecewise linear basis functions, also known as "hat" functions.

Figure (7) illustrates the basis functions $\phi_i(x)$. As you may or may not recognize, these functions actually form a basis for the piecewise linear Lagrange interpolating polynomials, i.e. their linear combination can produce any piecewise linear polynomials on Ω (assuming the start and end points are fixed to 0).

1.3.4 Global stiffness matrix

A remarkable property of these functions is that they are zero mostly everywhere. This results in $\phi_i(x)\phi_j(x) = 0$ for all i, j with $|i - j| > 1$. In other words, the product is only nonzero if ϕ_i and ϕ_j are adjacent. Same applies to their derivatives which are given by

$$\phi_j'(x) = \begin{cases} \frac{1}{h} & \text{if } x_{j-1} \leq x \leq x_j \\ -\frac{1}{h} & \text{if } x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

Now consider the stiffness matrix $\mathbf{K}$:

$$K_{ij} = \int_0^L \phi_i'(x)\phi_j'(x)dx.$$

As was mentioned earlier, $\phi_i'(x)\phi_j'(x) = 0$ unless $|i - j| \leq 1$. Thus, there are only three cases to consider: K_{ii}, $K_{i,i+1}$, $K_{i+1,i}$. Actually, since we know that $\mathbf{K}$ is symmetric, we infer that $K_{i,i+1} = K_{i+1,i}$. Let's now compute $\mathbf{K}$:

$$\begin{aligned} K_{ii} &= \int_0^L [\phi_i'(x)]^2 dx = \int_{x_{i-1}}^{x_i} \left[\frac{1}{h}\right]^2 dx + \int_{x_i}^{x_{i+1}} \left[-\frac{1}{h}\right]^2 dx = \frac{2}{h}. \\ K_{i+1,i} = K_{i,i+1} &= \int_0^L \phi_i'(x)\phi_{i+1}'(x)dx = \int_{x_i}^{x_{i+1}} \frac{1}{h}\left(-\frac{1}{h}\right) dx = -\frac{1}{h}. \end{aligned} \tag{34}$$

From Eq. (34) we can now construct $\mathbf{K}$.

$$\mathbf{K} = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \tag{35}$$

As expected, this matrix is tridiagonal, and produces a system that can be solved in linear time (w.r.t. to NE).

Thus, the main idea of finite elements is to choose such basis functions that are zero everywhere but a small subdomain. In mathematical terms, such basis functions are said to have a *compact support*, meaning that they are zero outside of a compact set.

1.3.5 Stiffness matrix assembly

Even though we have obtained a simple closed formula for the stiffness matrix **K**, we had to integrate piecewise functions, which is not very straightforward (nor efficient) in computational sense. To illustrate the last big idea behind FEM, we need to switch our conceptual framework a little bit. Instead of thinking about global basis functions, let's now think about each element Ω_e (See Figure 8), and all the basis functions that are associated with it.

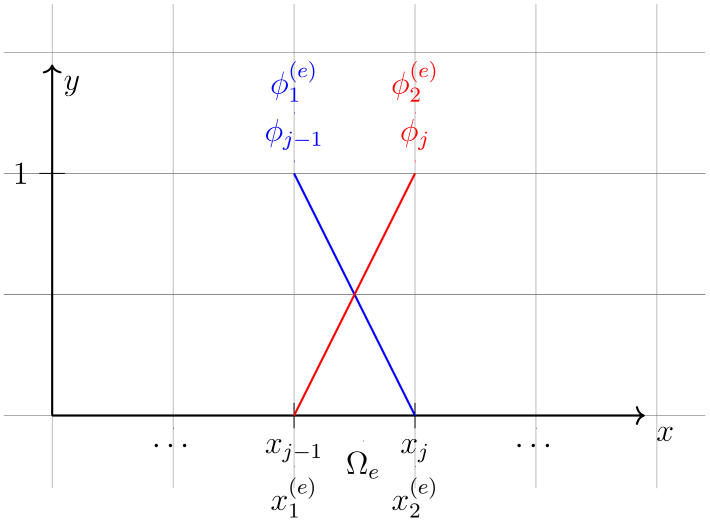

Figure 8: A closer look at element Ω_e.

Recall that $\Omega_1, ..., \Omega_e, ...\Omega_{NE}$ is the partition of $\Omega = [0, L]$. Then

$$K_{ij} = \int_\Omega \phi_i'(x)\phi_j'(x)dx = \sum_{e=1}^{NE} \int_{\Omega_e} \phi_i'(x)\phi_j'(x)dx. \tag{36}$$

We know that $\int_{\Omega_e} \phi_i'(x)\phi_j'(x)dx$ is nonzero if and only if x_i, x_j are both nodes of Ω_e. If we simply plug each term K_{ij} from Eq. (36) into **K** as they are, and then remove all terms which give zero, then we are left with

$$\mathbf{K} = \begin{bmatrix} \int_{\Omega_1} \phi_1'\phi_1' & \int_{\Omega_1} \phi_1'\phi_2' & 0 & 0 & 0 & \cdots \\ \int_{\Omega_1} \phi_2'\phi_1' & \int_{\Omega_1} \phi_2'\phi_2' + \int_{\Omega_2} \phi_2'\phi_2' & \int_{\Omega_2} \phi_2'\phi_3' & 0 & 0 & \cdots \\ 0 & \int_{\Omega_2} \phi_3'\phi_2' & \int_{\Omega_2} \phi_3'\phi_3' + \int_{\Omega_3} \phi_3'\phi_3' & \int_{\Omega_3} \phi_3'\phi_4' & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{37}$$

By taking a closer look, we can see a blocking pattern, which is illustrated on Figure (9). Let $\mathbf{K}^{(e)}$ be a *local stiffness matrix* defined by

$$K_{ij}^{(e)} = \int_{\Omega_e} \frac{d}{dx}\left[\phi_i^{(e)}(x)\right] \frac{d}{dx}\left[\phi_j^{(e)}(x)\right] dx \tag{38}$$

where $i, j \in \{1, 2\}$ is *local* numbering, corresponding to the numbering of $x_j^{(e)}$ Fig. (7). Thus, each $\mathbf{K}^{(e)}$ is a 2-by-2 matrix, which can be computed by integrating the product of two *continuous* polynomials over the element Ω_e.

Having computed each $\mathbf{K}^{(e)}$, we can "assemble" the *global stiffness matrix* $\mathbf{K}$. We first initialize $\mathbf{K}$ as a zero matrix. Then we loop over the elements and add the values of each $\mathbf{K}^{(e)}$ to its cooresponding location (See Figure 9).

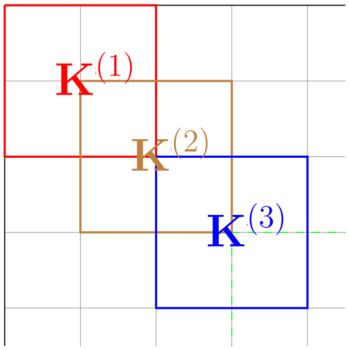

Figure 9: Global stiffness matrix layout. The global stiffness matrix $\mathbf{K}$ can be obtained by "assembling" local stiffness matrices $\mathbf{K}^{(e)}$.

The same idea applies to the load vector $\mathbf{F}$, we can compute it by assembling *local* load vectors $\mathbf{F}^{(e)}$.

At first sight, this may sound like we over-complicated our problem again. In reality, these procedures make FEM much easier to implement.

1.3.6 Shape functions and the master element

In the previous section we reduced our problem of computing the global stiffness matrix $\mathbf{K}$ (which involved integrating piecewise functions) to a problem of computing small 2-by-2 matrices, which involve integrating continuous functions. In this section we will introduce another trick that allows us to further reduce the number of integrations.

Recall Eq. (38). To compute $\mathbf{K}^{(e)}$, we need to integrate $\phi_i'\phi_j'$ over the element Ω_e:

$$K_{ij}^{(e)} = \int_{\Omega_e} \frac{d}{dx}\left[\phi_i^{(e)}(x)\right] \frac{d}{dx}\left[\phi_j^{(e)}(x)\right] dx$$

The last improvement is that we can actually perform integration over the same reference element, if we do a simple change of variable:

$$x = x_j + \xi(x_{j+1} - x_j) = x_j + \xi h, \quad \xi \in [0, 1].$$

$$dx = hd\xi \implies \frac{dx}{d\xi} = h \implies \frac{d\xi}{dx} = \frac{1}{h}.$$

If we apply this change of variable to $K_{ij}^{(e)}$, then we obtain

$$
\begin{aligned}
K_{ij}^{(e)} &= \int_{\Omega_e} \frac{d}{dx}\left[\phi_i^{(e)}(x)\right]\frac{d}{dx}\left[\phi_j^{(e)}(x)\right]dx \\
&= \int_0^1 \frac{d}{d\xi}\frac{d\xi}{dx}\left[\phi_i^{(e)}(x(\xi))\right]\frac{d}{d\xi}\frac{d\xi}{dx}\left[\phi_j^{(e)}(x(\xi))\right]hd\xi \\
&= \frac{1}{h}\int_0^1 \frac{d}{d\xi}\left[\phi_i^{(e)}(x(\xi))\right]\frac{d}{d\xi}\left[\phi_j^{(e)}(x(\xi))\right]d\xi
\end{aligned}
$$

Now we need to figure our what are $\phi_i^{(e)}(x(\xi))$ and $\phi_j^{(e)}(x(\xi))$. This is actually very simple if you refer to Figure (7).

$$
\begin{aligned}
\phi_i^{(e)}(x(\xi)) &= \begin{cases} \frac{x-x_1^{(e)}}{h}, & i=1 \\ \frac{x_2^{(e)}-x}{h}, & i=2 \end{cases} \\
&= \begin{cases} \frac{x_1^{(e)}+\xi h-x_1^{(e)}}{h}, & i=1 \\ \frac{x_2^{(e)}-(x_1^{(e)}+\xi h)}{h}, & i=2 \end{cases} \\
&= \begin{cases} \frac{\xi h}{h}, & i=1 \\ \frac{h-\xi h}{h}, & i=2 \end{cases} \\
&= \begin{cases} \xi, & i=1 \\ 1-\xi, & i=2 \end{cases}
\end{aligned}
$$

This gives us two very simple linear functions defined on $[0,1]$. We call them *shape functions* and denote them by $N_i(\xi)$. We call the reference element $[0,1]$ the *master element*, and denote it by Ω_M.

Going back to $K_{ij}^{(e)}$:

$$
\begin{aligned}
K_{ij}^{(e)} &= \frac{1}{h}\int_0^1 \frac{d}{d\xi}\left[\phi_i^{(e)}(x(\xi))\right]\frac{d}{d\xi}\left[\phi_j^{(e)}(x(\xi))\right]d\xi \\
&= \frac{1}{h}\int_{\Omega_M} \frac{d}{d\xi}N_i(\xi)\frac{d}{d\xi}N_j(\xi)d\xi
\end{aligned}
$$

As a result, it is sufficient to perform only one integration in the entire procedure of calculating $\mathbf{K}$, as once $\int_{\Omega_M}\frac{d}{d\xi}N_i(\xi)\frac{d}{d\xi}N_j(\xi)d\xi$ is computed, $K_{ij}^{(e)}$ depends only on the step size h_e (which we chose to be uniform for this toy problem).

Numerical integration can be performed using appropriate Gauss-Legendre quadrature rule. For our problem, we can use the cheapest midpoint rule to compute $\mathbf{K}$.

1.3.7 A note on boundary conditions

If take a closer look at Figure (7), you may wonder why don't we have ϕ_0 and ϕ_{NE}. The answer has to do with the boundary conditions. Since in our toy problem (Eq. 6), both b.c. are zero Dirichlet, we must have that

$$u_S(x) = \sum_{i=1}^{NE-1} u_i \phi_i(x)$$

also satisfies zero Dirichlet b.c. If we include ϕ_0 or ϕ_{NE} in our basis, then we are not guaranteed that their linear combination will satisfy zero Dirichlet b.c., which are enforced on H_0^1 and consequently on S as well.

If we have, for instance, a non-zero Dirichlet b.c. at $x = 0$, then we must include ϕ_0 in our basis. Same goes for the other boundary.

If we have a Neumann b.c. at one end, let's say at $x = L$, then we must include ϕ_{NE} in our basis by the same logic. Not only that, but we must also be able to recover $u'(0)$ if it is unknown. Let's elaborate on that.

Suppose that $u'(L) = \tilde{u}'_L \in \mathbb{R}$, and $u(0) = \tilde{u}_0 \in \mathbb{R}$ are the new boundary conditions of our toy problem. Then our weak formulation is given by

$$\int_0^L Eu'(x)v'(x)dx = \int_0^L f(x)v(x)dx + E\tilde{u}'_L v(L), \quad \text{recall Eq. (22).} \tag{39}$$

As explained earlier, our basis functions will contain ϕ_0 and ϕ_{NE}:

$$\phi_0(x) = \begin{cases} \frac{x_1 - x}{h} & \text{if } x \in \Omega_1 \\ 0 & \text{otherwise.} \end{cases}, \quad \phi_{NE}(x) = \begin{cases} \frac{x - x_{NE-1}}{h} & \text{if } x \in \Omega_{NE} \\ 0 & \text{otherwise.} \end{cases},$$

and $\{\phi_i(x)\}_{i=1}^{NE-1}$ is the same as in Eq. (32). Thus, u_S is given by

$$u_S(x) = \sum_{i=0}^{NE} u_i \phi_i(x).$$

This means that $\mathbf{U} \in \mathbb{R}^{NE+1}$, the global stiffness matrix $\mathbf{K} \in \mathbb{R}^{(NE+1)\times(NE+1)}$, and $\mathbf{F} \in \mathbb{R}^{NE+1}$. Let us also introduce a new term $\mathbf{Q} \in \mathbb{R}^{NE+1}$ to address the boundary term(s) in the weak formulation (Eq. 39). It is given by

$$\mathbf{Q} = \begin{bmatrix} Eu'_0 \\ 0 \\ \vdots \\ 0 \\ Eu'_L \end{bmatrix}, \quad \text{where } u'_0 \text{ and } u'_L \text{ are unknowns.}$$

Thus, our linear system $\mathbf{KU} = \mathbf{F} + \mathbf{Q}$ takes the following form:

$$\begin{bmatrix} K_{0,0} & \cdots & K_{0,NE} \\ \vdots & \ddots & \vdots \\ K_{NE,0} & \cdots & K_{NE,NE} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{NE-1} \\ u_{NE} \end{bmatrix} = \begin{bmatrix} f_0 \\ \vdots \\ f_{NE} \end{bmatrix} + \begin{bmatrix} Eu'_0 \\ 0 \\ \vdots \\ 0 \\ Eu'_L \end{bmatrix} \tag{40}$$

Recall that u_0 is given, and $u_0 = \tilde{u}_0$ from the Dirichlet b.c. We also know that $u'_L = \tilde{u}'_L$ from the Neumann b.c. Thus, we can plug those in directly into Eq. (40),

$$\begin{bmatrix} K_{0,0} & \cdots & K_{0,NE} \\ \vdots & \ddots & \vdots \\ K_{NE,0} & \cdots & K_{NE,NE} \end{bmatrix} \begin{bmatrix} \tilde{u}_0 \\ u_1 \\ \vdots \\ u_{NE-1} \\ u_{NE} \end{bmatrix} = \begin{bmatrix} f_0 \\ \vdots \\ f_{NE} \end{bmatrix} + \begin{bmatrix} Eu'_0 \\ 0 \\ \vdots \\ 0 \\ E\tilde{u}'_L \end{bmatrix} \tag{41}$$

From Eq. (41), we can see that our degrees of freedom (DOF, or simply unknowns) are $\{Eu'_0, u_1, u_2, ..., u_{NE}\}$. That makes $NE + 1$ unknowns in a system of $NE + 1$ equations. Thus, if we solve this system, then we obtain $u_1, ..., u_{NE}$ and u'_0 as well. We can move all the degrees of freedom into a single vector $\mathbf{U}^*$ to get the final system

$$\mathbf{K}^*\mathbf{U}^* = \mathbf{F},$$

where $\mathbf{K}^*$ is a modified version of $\mathbf{K}$ obtained by moving $\mathbf{Q}$ to the left-hand side and adjusting the first and last rows to accomodate the new solution vector $\mathbf{U}^*$.

1.3.8 Summary

We have covered the very basics of the Finite Element method. Let us summarize the key procedures that we need to do to apply it to a boundary-value problem.

1. Derive the weak formulation, see Section 1.2.
2. Discretize the domain, see Section 1.3.2.
3. Choose basis functions, see Section 1.3.3.
4. Compute local stiffness matrices $\mathbf{K}^{(e)}$, local load vectors $\mathbf{F}^{(e)}$, see Section 1.3.6.
5. Assemble the global stifness matrix $\mathbf{K}$ and the global load vector $\mathbf{F}$, see Section 1.3.5.
6. Compute the boundary contribution term $\mathbf{Q}$, see Section 1.3.7.
7. Solve the linear system $\mathbf{KU} = \mathbf{F} + \mathbf{Q}$ for the degrees of freedom.
8. Plug the solution vector $\mathbf{U}$ into $u_S(x) = \sum_i u_i\phi_i(x)$ to get the approximate solution.
9. Recover $u'(0)$ or $u'(L)$, if applicable.

The last note in this section is that we do not have to use piecewise linear polynomials to approximate $u(x)$. In fact, there is a myriad of different basis and shape functions at our disposal. We can even use basis functions of different orders in a single approximation. This gives us a great flexibility. You can read more about higher-order basis functions in [10].

2 Euler-Bernoulli beam

Now that we have given a proper introduction to Finite Element Method, we can start discussing a more complicated problem, which we will also attack with FEM. The underlying differential equation is called the Euler-Bernoulli beam equation. It models *deflection* of a beam, see Figures (10, 11).

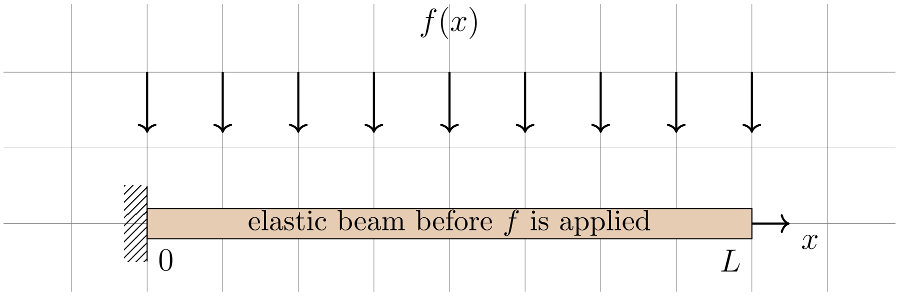

Figure 10: Euler-Bernoulli beam setup in 1d. The beam is fixed at 0. The force acts perpendicular to the axis of the beam, thus each point of the beam will *deflect* in y-direction.

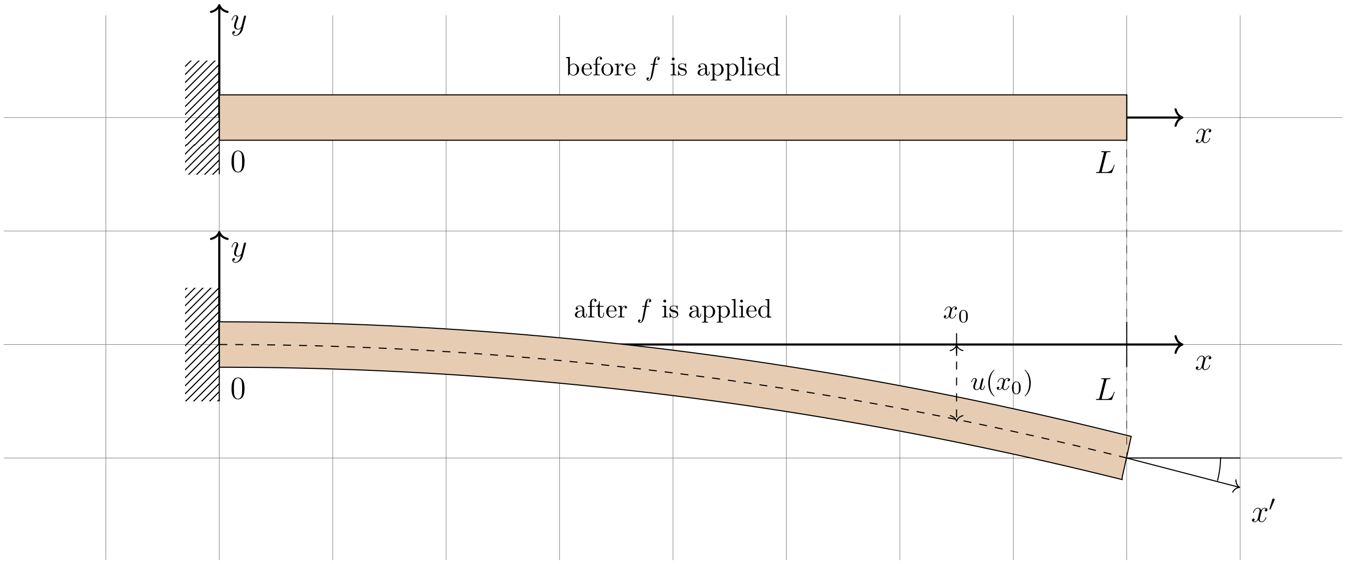

Figure 11: Euler-Bernoulli beam with its left end fixed. Given the distributed lateral force $f(x)$, we need to calculate the deflection $u(x)$ in y-direction. The cross section of the beam is plane and is perpendicular to the neutral axis (dashed, x').

This model has many applications in mechanical and structural engineering, one of which is the retaining wall (see Figure 12). These kind of walls are used to separate and hold back two ground levels. They can often be seen in parks and reserves.

The scheme of the retaining wall is illustrated on Figure (13). The wall is fixed at the bottom, and the force is acting on the wall from the right laterally (to the axis of the wall). We would like to know how much will the wall deflect while being acted upon by this force.

Figure 12: An example of a retaining wall.

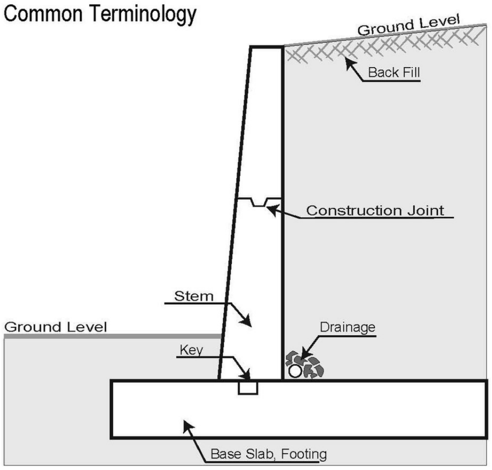

Figure 13: Retaining wall design and terminology.

2.1 Euler-Bernoulli beam equation

Euler-Bernoulli beam is in fact the simplest model of beam deflection. It makes the following assumptions about the beam:

- the deflection is very small compared to the length of the beam
- the cross-section area of the bent beam is perpendicular to the neutral axis and is a straight segment (planar).

A great derivation of the Euler-Bernoulli beam is given in Stepan Timoshenko's "Strength of materials" [11]. Here we present the equation itself.

The Euler-Bernoulli beam equation is given by

$$\frac{d^2}{dx^2}\left[EI\frac{d^2u}{dx^2}\right] = f(x),$$

where E is Young's modulus, I is the second moment of area of the beam's cross section w.r.t. to the neutral axis (see Fig. 11), u is the deflection of the beam in y-direction, and f is the lateral force (or distributed load) acting on the beam. (42)

Note that the second moment of area is a geometrical property that shows how area is distributed on the domain. In general, $I_a = \int_\Omega \rho^2 dA$, where a is some axis, dA is an element of area, and is the distance between dA and a.

In our case, we consider the cross-section area of the beam, and the axis is simply the x-axis. Suppose that the beam has thickness $2d$. Then, I can be calculated as

$$I = \int_{-d}^{d} y^2 dy = \frac{2}{3} d^3.$$

Thus, E and I are both constant in our 1d problem, and the Euler-Bernoulli beam equation can be simplified to

$$EI \frac{d^4 u}{dx^4} = f(x). \tag{43}$$

2.2 Swelling force

Now we shall talk about the distributed load f that we are going to use in this capstone. As the title of this project suggests it is the *swelling force.*

Swelling force is caused by the expansion of the soil (clay or rock), which can happen due to an increased humidity. This expansion creates pressure on the wall laterally across its height. Determining the deflection resulting from the swelling force is an important problem in the design of retaining walls.

The simplest model of the swelling pressure is due to K. Grob et al, 1972, [12], who gives a stress-strain relationship:

$$\varepsilon = -c \log \left(\frac{\sigma}{\sigma_0} \right), \tag{44}$$

where c is swelling parameter, and σ is the maximum pressure on the wall [13]. Let d be the distance from the distance from the location where swelling occurs to the retaining wall, then the strain is given by

$$\varepsilon = \frac{u(x)}{d}. \tag{45}$$

Then Eq. (44, 45) can be solved for the swelling pressure as

$$\sigma = \sigma_0 10^{-\frac{1}{cd} u(x)}. \tag{46}$$

From Eq. (46) we can obtain the formula for the distributed load:

$$f = s\sigma_0 10^{-\frac{1}{cd} u(x)} \tag{47}$$

where s is the length of the wall. Note that for the purposes of our problem, L denotes the *height* of the wall, so $s \neq L$. We need s and also the thickness of the wall in order to calculate the second moment of area I [14].

Thus, our equation for the Euler-Bernoulli beam with the swelling force is written as

$$\frac{d^4u}{dx^4} = Be^{-Cu}, \tag{48}$$

where $B = \frac{s\sigma_0}{EI}, C = \frac{\ln 10}{cd}$. Note that this is a nonlinear problem, as the distributed load depends on the deflection.

2.3 Boundary conditions

In order to find a particular solution to Eq. (48), we need four conditions. Depending on the prescribed boundary conditions, we obtain models of different, see Table 2.

Beam type	**Boundary conditions**
cantilever	$u(0) = u'(0) = u''(L) = u'''(L) = 0$
simply supported	$u(0) = u(L) = u''(0) = u''(L) = 0$
fixed	$u(0) = u(L) = u''(0) = u''(L) = 0$

Table 2: Different beam variations with the corresponding boundary conditions. Here we use the prime-notations for the derivatives w.r.t. x [15].

Referring to the design of a retaining wall (Figure 13), we can see that the *cantilever beam* model describes the retaining wall subjected to a lateral force best.

Thus, we can now present the strong formulation of the *steady-state* Euler-Bernoulli beam:

Find $u : (0, L) \to \mathbb{R}$ that satisfies

$$\frac{d^4u}{dx^4} = Be^{-Cu}, \tag{49}$$

with the boundary conditions $u(0) = u'(0) = u''(L) = u'''(L) = 0$. Coefficients B, C are given in Eq. (48).

2.4 Dynamic Euler-Bernoulli beam

In the previous section we have derived the BVP for the lateral deflection of the retaining wall subjected to a swelling pressure, Eq. (49). The solution to Eq. (49) gives us the deflection $u(x)$ at *steady state*, i.e. if we let the system go to equilibrium. What we might also want to know is how exactly will the beam go to the equilibrium state as a function of time. This leads us to consider the *dynamic* Euler-Bernoulli beam equation, given by

$$\mu\frac{\partial^2 u}{\partial t^2} + EI\frac{\partial^4 u}{\partial x^4} = f(x, u), \tag{50}$$

where μ denotes mass per unit length of the beam [16]. If we plug the swelling force (Eq. 47) into Eq. (50), then we get

$$\mu\frac{\partial^2 u}{\partial t^2} + EI\frac{\partial^4 u}{\partial x^4} = s\sigma_0 10^{-\frac{1}{cd}u(x)}, \tag{51}$$

for the meaning of E, I, s, σ_0, c, and d, refer to Eq. (44)-(48). If we let $A = \frac{EI}{\mu}$, $B = \frac{s\sigma_0}{\mu}$, $C = \frac{\ln 10}{cd}$, then arrive at the strong formulation of the dynamic Euler-Bernoulli beam that describes the vibration in lateral direction of the retaining wall subjected to swelling pressure:

Find $u : (0, L) \times \mathbb{R}^{nonneg} \to \mathbb{R}$ that satisfies

$$\frac{\partial^2 u}{\partial t^2} + A\frac{\partial^4 u}{\partial x^4} = Be^{-Cu} \tag{52}$$

with the boundary conditions $u(0) = u_x(0) = u_{xx}(L) = u_{xxx}(L) = 0$ and initial conditions $u(x, 0) = u_0(x)$ and $u_t(x, 0) = u_1(x)$.

2.5 What this capstone is about

In this capstone project, the Finite Element solution to the steady-state Euler-Bernoulli beam problem (Eq. 49) will be presented with each step annotated, and the working code will be shared. In addition, a FEM scheme for solving the dynamic Euler-Bernoulli beam (Eq. 52) will be provided. It is important to note that, while the steady-state Euler-Bernoulli beam with swelling force is a relatively well-studied problem, its dynamic variation is still a novel problem.

3 Steady-state Euler beam in 1d

We shall first attack the steady state version of the 1d Euler-Bernoulli beam. The techniques are similar to the ones we used to solve the toy problem, however there are slight modifications, as we will see.

3.1 Strong formulation

Let $\Omega = (0, L)$. Find $u : \Omega \to \mathbb{R}$, s.t.

$$\begin{cases} \frac{d^4}{dx^4}u(x) = Be^{-Cu(x)}, \quad x \in \Omega = (0, L), B, C \in \mathbb{R}^+, \\ u(0) = \frac{d}{dx}u(0) = 0, \\ \frac{d^2}{dx^2}u(L) = \frac{d^3}{dx^3}u(L) = 0. \end{cases} \tag{53}$$

3.2 Weak formulation

Let $v : \Omega \to \mathbb{R}$ be some sufficiently regular function. We shall use the prime notation for the derivative w.r.t. x in this section.

$$\int_0^L u''''(x)v(x)dx = \int_0^L Be^{-Cu(x)}v(x)dx.$$

We integrate the left-hand side by parts:

$$\begin{aligned} \int_0^L u''''(x)v(x)dx &= [u'''(x)v(x)]_0^L - \int_0^L u'''(x)v'(x)dx \\ &= [u'''(L)v(L)] - [u'''(0)v(0)] - \int_0^L u'''(x)v'(x)dx \\ &\quad \text{we assume that } v(0) = 0 \\ &= -\int_0^L u'''(x)v'(x)dx. \end{aligned} \tag{54}$$

We can apply integration by parts again:

$$\begin{aligned} \int_0^L -u'''(x)v'(x)dx &= [-u''(x)v'(x)]_0^L - \int_0^L -u''(x)v''(x)dx \\ &= [u''(L)v'(L) - u''(0)v'(0)] + \int_0^L u''(x)v''(x)dx \\ &\quad \text{we assume that } v'(0) = 0 \\ &= \int_0^L u''(x)v''(x)dx. \end{aligned} \tag{55}$$

In the end, we have

$$\int_0^L u''(x)v''(x)\ dx = \int_0^L Be^{-Cu(x)}v(x)dx. \tag{56}$$

Following the same steps as in Section 1.2.1, we can see that the necessary conditions on u and v are

$$\int_0^L [u''(x)]^2 dx < \infty, \text{ and } \int_0^L [v''(x)]^2 dx < \infty, \text{ respectively.}$$

Thus, $u, v \in H^2(\Omega)$ (see Section 1.2.2). If we let $H_0^2 = \{v \in H^2(\Omega) : v(0) = v'(0) = 0\}$, then weak formulation is given by

$$\boxed{\begin{array}{l}\text{Find } u \in H_0^2 \text{ s.t.}\\[1em] \displaystyle\int_0^L u''(x)v''(x)dx = \int_0^L Be^{-Cu(x)}v(x)dx, \quad \forall v \in H_0^2.\end{array}} \tag{57}$$

3.3 Note on boundary conditions

In Section 1.3.7, we introduced the notions of "essential" and "natural" boundary conditions. For the second-order problems, essential b.c. is equivalent to Dirichlet b.c., and natural is equivalent to Neumann b.c. (see Table 1). In this case, however, boundary conditions involving both u and u' are classified as "essential", while b.c. involving u'' and u''' are "natural".

As we have already seen in Eq. (54, 55) we imposed $v(0)$ and $v'(0)$ onto H^2 thus making them essential b.c., while $u''(L)$ and $u'''(L)$ appeared explicitly in the weak formulation (Eq. 54 and 55, though we cancelled them out immediately) thus making them natural.

3.4 Basis and shape functions

For they toy problem we used piecewise linear basis functions. They will not work for this problem, because they are not in $H^2(\Omega)$ (their second derivative is not square-integrable, as it is the Dirac-delta distribution). In fact, it can be shown that $H^2(\Omega)$ implies $C^1(\Omega)$, thus our basis functions must be continuously differentiable.

If you recall interpolation theory, we need to use the basis for splines. The best candidate for this is Hermite cubic splines. Given the values of some function f at points a, b (call it v_1, v_2) and the values f' at a, b (call it m_1, m_2), we need to interpolate a cubic polynomial through a, b. Thus, the degree of freedom at each inner node is 2 instead of 1 (see Figure 15), as it was in the first section.

To derive Hermite cubic shape functions, we first choose the master element, which in our case is $\Omega_M = [0, 1]$. Then given v_1, v_2, m_1, m_2, we shall find polynomials $N_i(\xi) \in P_3(\Omega_M)$ such that

$$N(\xi) = N_1(\xi)v_1 + N_2(\xi)m_1 + N_3(\xi)v_2 + N_4(\xi)m_2.$$

By letting $N(\xi) = \sum_{i=0}^{3} a_i x_i$, we first obtain the coefficients a_i in terms of v_1, v_2, m_1, m_2 by solving a system of algebraic equations. From that we derive $N_i(\xi)$.

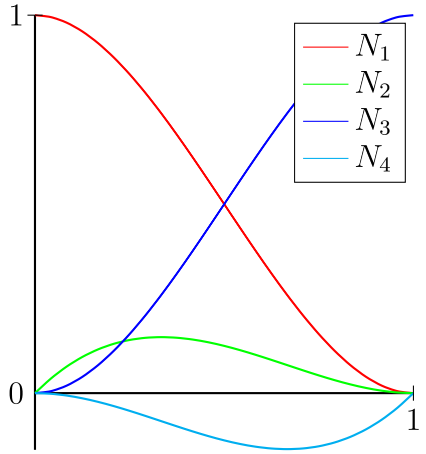

Figure 14: Hermite cubic shape functions.

The Hermite cubic basis functions on Ω_M (see Figure 14) are given by

$$N_1(\xi) = 2\xi^3 - 3\xi^2 + 1, \quad N_2(\xi) = \xi^3 - 2\xi^2 + \xi,$$
$$N_3(\xi) = -2\xi^3 + 3\xi^2, \quad N_4(\xi) = \xi^3 - \xi^2.$$

Given our weak formulation (Eq. 57), we also need to know the second derivatives of the shape functions.

$$\frac{d^2}{d\xi^2}N_1(\xi) = 12\xi - 6, \quad \frac{d^2}{d\xi^2}N_2(\xi) = 6\xi - 4,$$
$$\frac{d^2}{d\xi^2}N_3(\xi) = -12\xi + 6, \quad \frac{d^2}{d\xi^2}N_4(\xi) = 6\xi - 2.$$

If we let $0 = x_0 < ... < x_n = L$ be a partition of Ω for some n, then Hermite cubic elements allow us to compute both $u(x_i)$ and $u'(x_i)$ at each node.

3.5 Discretization of the domain

Now that we have chosen basis functions, we can now split our domain Ω into elements. Let NE denote the number of elements. Let $0 = x_0 < x_1 < ... < x_{NE} = L$ be the partition of Ω. Let $x_1^{(e)}$ and $x_2^{(e)}$ denote the coordinates of the e-th element, $e = 1, ..., NE$ (See Figure 15). Let $l^{(e)} = x_2^{(e)} - x_1^{(e)}$ denote the length of the e-th element.
Let $\Omega_M = [0, 1]$ denote the domain of the master element, and Ω_e denote the domain of the element e.

At each node $x_i, i \in \{0, ..., NE\}$ we want to find $u(x_i)$ and $u'(x_i)$.

Let $u_{2i} = u(x_i)$ and $u_{2i+1} = u'(x_i), i \in \{0, ..., NE\}$. These are our unknowns with *global DOF numbering.*

Let $u_j^{(e)} = u_{2e+j-3}, \ e \in \{1, ..., NE\}, j \in \{1, 2, 3, 4\}$. Note that $u_j^{(e)}$ is our *local numbering* of DOF, wherein e denotes the element number, and j denotes the degree of freedom of this element. Since each element e has two nodes, it means that it has up to 4 (local) degrees of freedom.

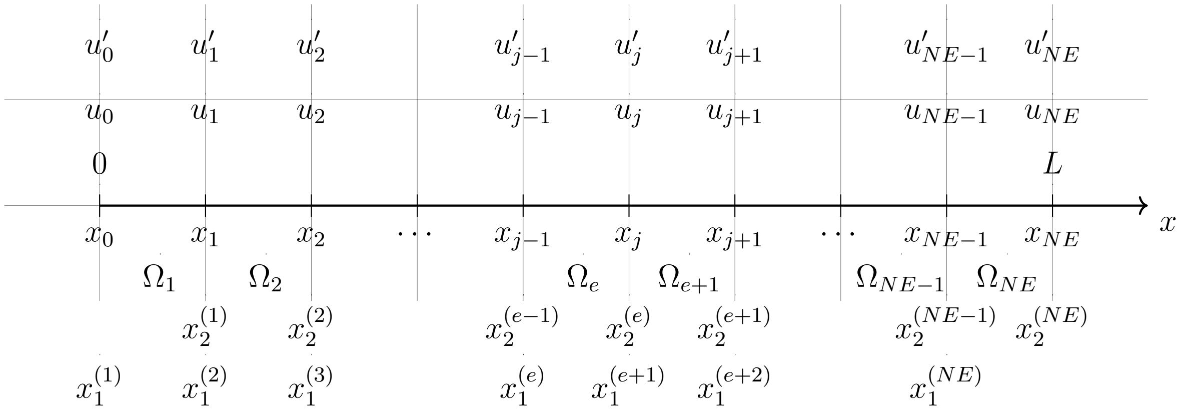

Figure 15: Discretization of the domain.

3.5.1 Change of variable

The transformation from local coordinates to the global ones is given by $x(\xi) = x_1^{(e)} + \left(x_2^{(e)} - x_1^{(e)}\right)\xi$. Thus, $\frac{dx}{d\xi} = l^{(e)}$, the same transformation that we had with the toy problem.

3.6 Local element matrices

By Galerkin method, let

$$u(x) \approx u_S(x) = \sum_{i=0}^{2NE+1} u_i \phi_i(x).$$

Then, the local stiffness matrices $\mathbf{K}^{(e)}$ can be computed using

$$\begin{aligned} K_{ij}^{(e)} &= \int_{\Omega_e} \frac{d^2}{dx^2} N_i^{(e)} \frac{d^2}{dx^2} N_j^{(e)} dx \\ &= \int_{\Omega_M} \frac{d^2}{d\xi^2}\frac{d\xi^2}{dx^2} N_i \frac{d^2}{d\xi^2}\frac{d\xi^2}{dx^2} N_j \left[l^{(e)} d\xi\right] \qquad \text{(change of variable)}, \\ &= \frac{1}{\left[l^{(e)}\right]^3} \int_{\Omega_M} \frac{d^2}{d\xi^2} N_i \frac{d^2}{d\xi^2} N_j \; d\xi, \end{aligned}$$

and the load vector by

$$F_j^{(e)} = \int_{\Omega_e} B e^{-C \sum_{i=0}^{2NE+1} u_i \phi_i(x)} N_j^{(e)} dx$$

3.6.1 Dealing with the exponent

In order to simplify the load vector, we use Taylor series for e^x

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \mathcal{O}(x^5).$$

Let $exp(x) = \sum_{n=0}^{4} \frac{x^n}{n!}$, then

$$F_j^{(e)} = \int_{\Omega_e} Bexp\left(-C \sum_{i=0}^{2NE+1} u_i \phi_i(x)\right) N_j \; dx. \tag{58}$$

3.6.2 Gauss quadrature

Note that $N_i(\xi) \in \mathbb{P}_3$ and $exp(x) \in \mathbb{P}_4$. Therefore, $Bexp\left(-C \sum_{i=0}^{2NE+1} u_i \phi_i(x)\right) \in \mathbb{P}_{12}$.

Consequently, the force term (Eq. 58) is an integral over a polynomial of degree 15 (we multiply by additionally N_j), which means that 8-point Gauss-Legendre quadrature rule will suffice to get the exact result (up to the precision of the IEEE-754 `double`).

3.7 Assembling global matrices

Since each node has 2 degrees of freedom, the overlap between two adjacent elements accordingly has 2 degrees of freedom as well. We address this by modifying the procedure of assembling the global system, see Figure (16).

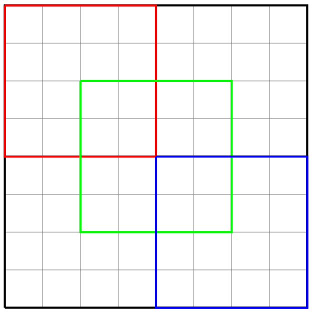

Figure 16: Global Element matrix layout for 3 Hermite cubic elements. Each small square represents a 4 × 4 matrix, and the big black square has 8 × 8 size.

3.8 The global system

The final system is given by

$$\mathbf{KU} = \mathbf{F} + \mathbf{Q}, \tag{59}$$

where $\mathbf{Q}$ is the boundary contribution given by

$$\begin{bmatrix} u_{xx}(0) \\ u_{xxx}(0) \\ 0 \\ \vdots \\ 0 \\ u_{xx}(L) \\ u_{xxx}(L) \end{bmatrix} = \begin{bmatrix} u_{xx}(0) \\ u_{xxx}(0) \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

as prescribed in the problem statement (Eq. 49).

Putting $\mathbf{Q}$ on the left hand side and modifying the terms accordingly, we obtain

$$\mathbf{K}^*\mathbf{U}^* = \mathbf{F}, \tag{60}$$

where $\mathbf{U}^*$ is given by

$$\begin{bmatrix} u_{xx}(0) \\ u_{xxx}(0) \\ u_2 \\ u_3 \\ \vdots \\ u_{2NE+1} \end{bmatrix}$$

3.9 Numerical results

3.9.1 Fixed-point iteration

To obtain the steady-state solution, we need to solve Eq. (60) by the means of the fixed-point iteration.

$$\mathbf{U}^*_{n+1} = [\mathbf{K}^*]^{-1}\mathbf{F}(\mathbf{U}^*_n)$$

with $\mathbf{U}^*_0 = \mathbf{0}$.

3.9.2 Results

The numerical solver was implemented using Python 3. Given $(B, C) = (1.47 \times 10^{-8}, 1.68 \times 10^{-1})$,

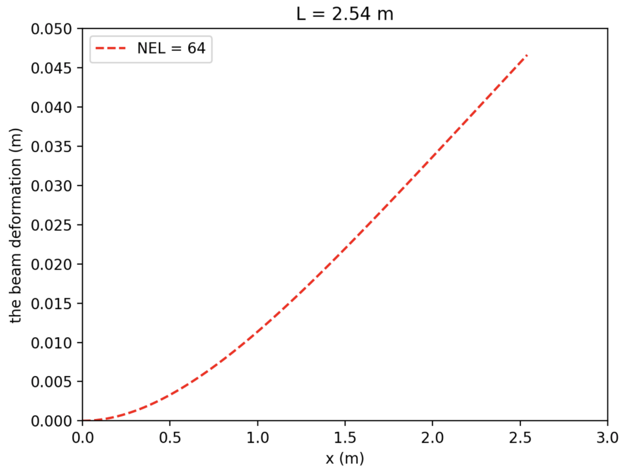

Figure 17: My FEM solution

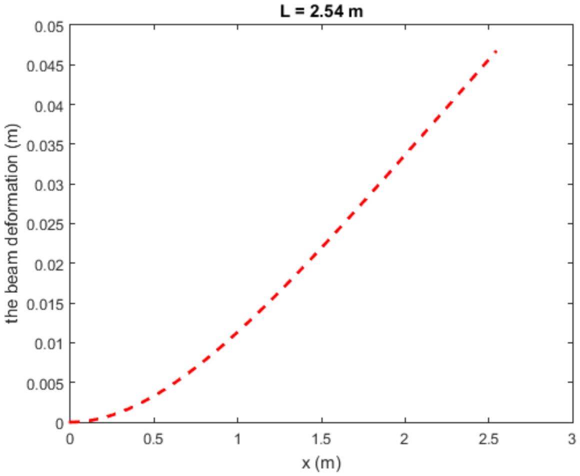

Figure 18: Reference solution [14].

4 Dynamic Euler beam in 1d

From Section 2.4 we obtained the following strong formulation of the problem:

Find $u : [0, L] \times \mathbb{R}^{nonneg} \to \mathbb{R}$ s.t.

$$\begin{cases} u_{tt} + Au_{xxxx} = Be^{-Cu}, & x \in \Omega = (0, L), \quad t \in \mathbb{R}^+, \\ & A, B, C \text{ are positive constants,} \\ u(x, 0) = u_0(x), & x \in \Omega, \\ u_t(x, 0) = u_1(x), & x \in \Omega, \\ \frac{\partial^i}{\partial x^i} u(0, t) = 0, & t \in \mathbb{R}^+, i \in \{0, 1\}, \\ \frac{\partial^j}{\partial x^j} u(L, t) = 0, & t \in \mathbb{R}^+, j \in \{2, 3\}. \end{cases}$$

4.1 Weak formulation and discretization

4.1.1 Function spaces

Deriving the weak formulation for a time-dependent problem is slightly different from the steady-state ones. For that, we need to first introduce new function spaces in addition to the ones defined in Section 1.2.2.

- $H^{-1}(0, L)$ is the dual space of $H_0^2(\Omega)$. A dual space of some vector space V is the space of all bounded linear functionals from V to $\mathbb{R}$.
- $L^p(0, T; H^q(\Omega))$, a Bochner space. The formal definition of the Boncher space is beyond the scope of this project. The intuition is that it allows us to consider a function of space and time as a collection of functions in space only, parametrized by time. We will need this space in order to derive the weak formulation.

4.1.2 Weak formulation

Let $v : \Omega \to \mathbb{R}$ be a sufficiently regular function. Using the results from Eq. (54-57), we infer that

$$\int_\Omega u_{tt} v + Au_{xx} v_{xx} dx = \int_\Omega Be^{-Cu} v dx.$$

Let $H_0^2 = \{v(x) \in H^2(\Omega) : v(0) = v_x(0) = 0\}$. Then the weak formulation is given by

$$\boxed{\begin{array}{l} \text{Find } u(x,t) \in L^2(0, T; H_0^2(\Omega)) \text{ with } u_t \in L^2(0, T; L^2(\Omega)),\ u_{tt} \in L^2(0, T; H^{-1}(\Omega)) \text{ s.t.} \\ \displaystyle\int_0^L u_{tt} v + Au_{xx} v_{xx}\ dx = \int_0^L Be^{-Cu} v dx, \quad \forall v \in H_0^2(\Omega). \quad [17] \end{array}} \tag{61}$$

4.2 Discretization

We use the same domain discretization as given in Figure (15), we shall also use the same Hermite cubic shape functions. There are two key differences in the dynamic case:

- Discretization of u. $u(x,t) \approx u_S = \sum_{i=1}^{N} u_i(t)\phi_i(x) = \sum_{e=1}^{NE}\left[\chi^{(e)}(x)\sum_{i=1}^{4} u_i^{(e)} N_i^{(e)}(x)\right]$, where $\chi^{(e)}(x)$ is the indicator function. Notice here, that the coefficients before global basis functions are no longer constant, but functions of time. $v(x)$, on the other hand, is discretized in the same way as it was in the steady-state case.

- The final system. Since we have an additional term in the weak formulation (Eq. 61), our system of equations will take a different form. Let us elaborate on this one.

Let Ω_e be the domain of the e-th element. Let $u_i^{(e)} = [u^{(e)}]$, $N_i^{(e)}(x) = [N^{(e)}]$, $u_j = [u]$,

$$\int_{\Omega_e} u_{tt}v + Au_{xx}v_{xx}dx = \int_{\Omega_e} Be^{-Cu}vdx.$$

By Galerkin method, we get

$$\int_{\Omega_e} \left[N^{(e)}\right]^T \left[N^{(e)}\right]\left[\frac{\partial^2 u^{(e)}}{\partial t^2}\right] + A\left[\frac{d^2N^{(e)}}{dx^2}\right]^T\left[\frac{d^2N^{(e)}}{dx^2}\right]\left[u^{(e)}\right]dx =$$

$$= \int_{\Omega_e} Be^{-C\sum_{e=1}^{NE}\left[\chi^{(e)}(x)\sum_{i=1}^{4} u_i^{(e)} N_i^{(e)}(x)\right]}[N^{(e)}]dx.$$

We know already know that

$$K_{ij}^{(e)} = \frac{1}{\left[l^{(e)}\right]^3}\int_{\Omega_M} A\frac{d^2}{d\xi^2}N_i\frac{d^2}{d\xi^2}N_j\ d\xi,$$

and

$$F_j^{(e)} = \int_{\Omega_e} Bexp\left(-C\sum_{i=0}^{2NE+1} u_i\phi_i(x)\right)N_j\ dx.$$

If we let

$$\begin{aligned} M_{ij}^{(e)} &= \int_{\Omega_e} N_i^{(e)}N_j^{(e)}dx \\ &= l^{(e)}\int_{\Omega_M} N_iN_j\ d\xi, \end{aligned}$$

Then our system of equations would take the form of

$$\mathbf{M}\mathbf{U}_{tt} + \mathbf{K}\mathbf{U} = \mathbf{F} + A\mathbf{Q}, \tag{62}$$

By moving $\mathbf{Q}$ to the left hand side, we arrive at the system

$$\mathbf{M}^*\mathbf{U}^*_{tt} + \mathbf{K}^*\mathbf{U}^* = \mathbf{F}, \tag{63}$$

where $\mathbf{U}^* = [u_{xx}(0), u_{xxx}(0), u_2, ..., u_{2NE+1}]^T$.

Now the question is, how do we solve for $\mathbf{U}(t)$?

4.3 Newmark-β method

So, how do we attack the dynamic case? Recall our system

$$\mathbf{M}^*\mathbf{U}^*_{tt} + \mathbf{K}^*\mathbf{U}^* = \mathbf{F}.$$

Let's first clean up the notation a little bit: let $\mathbf{M} = \mathbf{M}^*, \ddot{\mathbf{u}} = \mathbf{U}^*_{tt}, \mathbf{K} = \mathbf{K}^*, \mathbf{u} = \mathbf{U}^*$, and $\mathbf{f} = \mathbf{F}$. Thus,

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}.$$

A well-established method for solving such systems is called the Newmark method [18, 19].

Let Δt be the temporal step size. Let $\mathbf{u}_j = \mathbf{u}(j\Delta t)$. This method assumes that $\ddot{\mathbf{u}}$ is linear with time, see Figure (19).

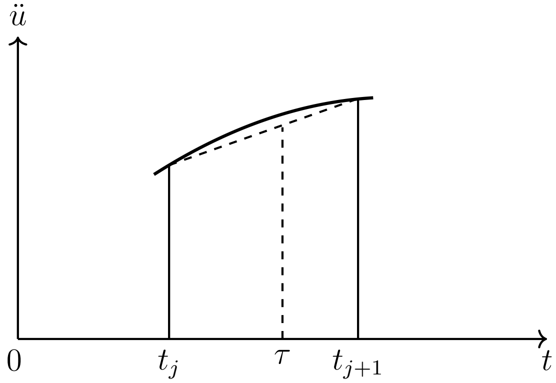

Figure 19: Linear acceleration approximation [19].

Thus, on the time interval (t_j, t_{j+1}),

$$\ddot{\mathbf{u}} = \ddot{\mathbf{u}}_j + \frac{1}{\Delta t}(\ddot{\mathbf{u}}_{j+1} - \ddot{\mathbf{u}}_j)\tau, \quad 0 \leq \tau \leq \Delta t.$$

Integrating over time gives

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}_j + \ddot{\mathbf{u}}_j\tau + \frac{1}{2\Delta t}(\ddot{\mathbf{u}}_{j+1} - \ddot{\mathbf{u}}_j)\tau^2 \tag{64}$$

since $\dot{\mathbf{u}} = \dot{\mathbf{u}}_j$ when $\tau = 0$.

Integrating again,

$$\mathbf{u} = \mathbf{u}_j + \dot{\mathbf{u}}_j\tau + \frac{1}{2}\ddot{\mathbf{u}}_j\tau^2 + \frac{1}{6\Delta t}(\ddot{\mathbf{u}}_{j+1} - \ddot{\mathbf{u}}_j)\tau^3 \tag{65}$$

since $\mathbf{u} = \mathbf{u}_j$ when $\tau = 0$.

Evaluating Eq. (64) and (65) at $\tau = \Delta t$ gives

$$\dot{\mathbf{u}}_{j+1} = \dot{\mathbf{u}}_j + \frac{\Delta t}{2}(\ddot{\mathbf{u}}_j + \ddot{\mathbf{u}}_{j+1})$$

and

$$\mathbf{u}_{j+1} = \mathbf{u}_j + \dot{\mathbf{u}}_j \Delta t + \frac{(\Delta t)^2}{6}(2\ddot{\mathbf{u}}_j + \ddot{\mathbf{u}}_{j+1}).$$

In Newmark method, new parameters γ and β are introduced to transforms the last two equations into the following

$$\dot{\mathbf{u}}_{j+1} = \dot{\mathbf{u}}_j + \Delta t((1-\gamma)\ddot{\mathbf{u}}_j + \gamma\ddot{\mathbf{u}}_{j+1}) \tag{66}$$

and

$$\mathbf{u}_{j+1} = \mathbf{u}_j + \dot{\mathbf{u}}_j \Delta t + (\Delta t)^2 \left(\left(\frac{1}{2} - \beta \right) \ddot{\mathbf{u}}_j + \beta \ddot{\mathbf{u}}_{j+1} \right). \tag{67}$$

If we use Eq. (67) to solve for $\ddot{\mathbf{u}}_{j+1}$, we obtain

$$\ddot{\mathbf{u}}_{j+1} = \frac{1}{\beta(\Delta t^2)}(\mathbf{u}_{j+1} - \mathbf{u}_j) - \frac{1}{\beta\Delta}\dot{\mathbf{u}}_j - \left(\frac{1}{2\beta} - 1 \right) \ddot{\mathbf{u}}_j. \tag{68}$$

Plugging the above equation into Eq. (66),

$$\dot{\mathbf{u}}_{j+1} = \frac{\gamma}{\beta\Delta t}(\mathbf{u}_{j+1} - \mathbf{u}_j) + \left(1 - \frac{\gamma}{\beta} \right) \dot{\mathbf{u}}_j + \Delta t \left(1 - \frac{\gamma}{2\beta} \right) \ddot{\mathbf{u}}_j \tag{69}$$

Now we want to plug Eq. (68) into our system $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}$, we get

$$\left(\frac{1}{\beta(\Delta t)^2}\mathbf{M} + \mathbf{K} \right) \mathbf{u}_{j+1} = \mathbf{f}_{j+1} + \frac{1}{\beta(\Delta t)^2}\mathbf{M}\mathbf{u}_j + \frac{1}{\beta\Delta t}\mathbf{M}\dot{\mathbf{u}}_j + \left(\frac{1}{2\beta} - 1 \right) \mathbf{M}\ddot{\mathbf{u}}_j. \tag{70}$$

As you can see, everything on the right hand side is explicit, except the force term. In most application, f depends only on x, so this method becomes explicit. In our case, however, $\mathbf{f}$ does depend on $\mathbf{u}$, and thus this method still requires fixed-point iteration.

One question that may arise is how do we calculate $\ddot{\mathbf{u}}_0$ for the very first iteration. And the answer is, we simply plug the initial condition into $\mathbf{M}\ddot{\mathbf{u}}_0 + \mathbf{K}\mathbf{u}_0 = \mathbf{f}(\mathbf{u}_0)$ and solve for $\ddot{\mathbf{u}}_0$. Then, everything else follows from the application of Eq. (68, 69 and 70).

5 Conclusion

This capstone project gave an extensive introduction to finite element method, and gave a detailed case study of a real-life engineering application. There is still room for improvement though, the next steps would be to implement the dynamic beam solver, and to try to attack the 2d version of this problem.

A Implementation

All the codes can be found in a github repository [20].

A.1 Polynomial tools

I wrote a small library that allows to do manipulations with basis functions. Given an $n \times n$ matrix, it can generate an array of n polynomials of degree $n-1$ (they are stored as functions using functional programming in Python). It can also find its derivatives using linear algebra (polynomial derivative matrix). Lastly, it can integrate a polynomial using Gauss-Legendre quadrature.

The code is very short, so I decided to include it.

```python
import numpy as np
import functools as ft

def get_monomial_basis(size):
    return np.array([(lambda pow: lambda x: x**pow)(pow) for pow in range(size)])

def get_basis_functions(coefficient_matrix):
    size = len(coefficient_matrix)
    monomial_basis = get_monomial_basis(size)
    basis = [ft.reduce( (lambda monom1, monom2: lambda x: monom1(x) + monom2(x)), [(lambda monom, scalar: lambda x: monom(x)*scalar)(monomial_basis[j], coefficient_matrix[i,j]) for j in range(size)]) for i in range(size)]
    return np.array(basis).ravel()

def get_basis_derivatives(coefficient_matrix, order):
    der_mat = get_derivative_matrix(len(coefficient_matrix), order)
    return get_basis_functions(coefficient_matrix@der_mat)

def get_derivative_matrix(basis_order, derivative_order):
    derivative_matrix = np.zeros((basis_order, basis_order))
    for i in range(derivative_order, basis_order):
        derivative_matrix[i,i-derivative_order] = ft.reduce(lambda xx,yy: xx*yy, range(i,i-derivative_order,-1))
    return derivative_matrix

def gauss(fun, a, b, gauss_points_num):
    x, w = np.polynomial.legendre.leggauss(gauss_points_num)
    return ft.reduce(lambda xx,yy: xx+yy, map(lambda xx,ww: (b-a)/2*ww*fun((b-a)/2*xx + (a+b)/2), x,w))
```

Listing 1: Polynomial tools

References

[1] J. N. Reddy, *An Introduction to Continuum Mechanics*, 2nd ed. Cambridge University Press, 2013.

[2] P. Skrzypacz, "Lecture notes on Applied FEM," April 2020.

[3] S. Brenner and R. Scott, *The Mathematical Theory of Finite Element Methods*, ser. Texts in Applied Mathematics. Springer New York, 2007. [Online]. Available: https://books.google.kz/books?id=kGFW2uWYU0YC

[4] E. Lee, "Lecture note on Real Analysis 2," April 2020.

[5] F. L. Gall, "Powers of tensors and fast matrix multiplication," 2014.

[6] V. V. Williams, "Multiplying matrices in o(n 2:373) time," 2014.

[7] D. Coppersmith and S. Winograd, "Matrix multiplication via arithmetic progressions," *Journal of Symbolic Computation*, vol. 9, no. 3, pp. 251 – 280, 1990, computational algebraic complexity editorial. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0747717108800132

[8] V. STRASSEN, "Gaussian elimination is not optimal." *Numerische Mathematik*, vol. 13, pp. 354–356, 1969. [Online]. Available: http://eudml.org/doc/131927

[9] N. Higham, *Accuracy and Stability of Numerical Algorithms: Second Edition*, ser. Other Titles in Applied Mathematics. Society for Industrial and Applied Mathematics, 2002. [Online]. Available: https://books.google.kz/books?id=7J52J4GrsJkC

[10] H. P. Langtangen, "Introduction to finite element methods," http://hplgit.github.io/INF5620/doc/pub/sphinx-fem/._main_fem003.html, accessed: 2020-05-10.

[11] S. Timoshenko, *Strength of Materials*, ser. Strength of Materials. Van Nostrand, 1955, no. pt. 1, pp. 134–137. [Online]. Available: https://books.google.kz/books?id=kDMIAQAAIAAJ

[12] K. Grob and H. J. Jaeggi, "Methodik der kapillar-gas-chromatographie. hinweise zur vollen ausnützung hochwertiger säulen," *Chromatographia*, vol. 5, no. 7, pp. 382–391, Jul 1972. [Online]. Available: https://doi.org/10.1007/BF02270890

[13] Y. Abou Rjeily and M. Khouri, "Longitudinal stress analysis of buried pipes under expansive soils," *International Journal of Science and Research (IJSR)*, vol. Volume 3, pp. 2592–2599, 11 2014.

[14] D. Wei, L. Yu, D. Zhang, W. Ko, and J. Kim, "Numerical analysis for retaining walls subjected to swelling pressure," 2016, 2016 International Conference on Architecture, Structure and Civil Engineering ; Conference date: 26-03-2016 Through 27-03-2016. [Online]. Available: https://architexturez.net/pst/az-cf-173127-1446189391

[15] F. Ching, *A Visual Dictionary of Architecture*. Wiley, 1995. [Online]. Available: https://books.google.kz/books?id=l1oYAAAAIAAJ

[16] R. Naz and F. M. Mahomed, "Dynamic euler-bernoulli beam equation: Classification and reductions," *Mathematical Problems in Engineering*, vol. 2015, p. 520491, Aug 2015. [Online]. Available: https://doi.org/10.1155/2015/520491

[17] D. Wei, L. Zhapsarbayeva, and Z. Kazbek, "Mathematical analysis of the euler-bernoulli beam subject to swelling pressure," *Journal of Mathematics, Mechanics and Computer Science*, vol. 97, pp. 25–37, 01 2018.

[18] J. Lehtinen, "Time-domain numerical solution of the wave equation," 03 2003.

[19] M. Petyt, *Introduction to Finite Element Vibration Analysis*, 2nd ed. Cambridge University Press, 2010.

[20] M. Shakipov, "Finite element method for the euler-bernoulli equation," 2020. [Online]. Available: https://github.com/chromomons/fem_euler_bernoulli_beam