BERNSTEIN-WALSH INEQUALITIES IN HIGHER DIMENSIONS OVER EXPONENTIAL CURVES

SHIRALI KADYROV AND MARK LAWRENCE

Abstract. Let $\mathbf{x} = (x_1, \ldots, x_d) \in [-1,1]^d$ be linearly independent over $\mathbb{Z}$, set $K = \{(e^z, e^{x_1 z}, e^{x_2 z} \ldots, e^{x_d z}) : |z| \le 1\}$. We prove sharp estimates for the growth of a polynomial of degree n, in terms of

$$E_n(\mathbf{x}) := \sup\{\|P\|_{\Delta^{d+1}} : P \in \mathcal{P}_n(d+1), \|P\|_K \le 1\},$$

where Δ^{d+1} is the unit polydisk. For all $\mathbf{x} \in [-1,1]^d$ with linearly independent entries, we have the lower estimate

$$\log E_n(\mathbf{x}) \ge \frac{n^{d+1}}{(d-1)!(d+1)} \log n - O(n^{d+1});$$

for Diophantine $\mathbf{x}$, we have

$$\log E_n(\mathbf{x}) \le \frac{n^{d+1}}{(d-1)!(d+1)} \log n + O(n^{d+1}).$$

In particular, this estimate holds for almost all $\mathbf{x}$ with respect to Lebesgue measure. The results here generalize those of [6] for $d = 1$, without relying on estimates for best approximants of rational numbers which do not hold in the vector-valued setting.

1. Introduction

For any $\ell \in \mathbb{N}$ we let Δ^ℓ denote the unit polydisk

$$\{\mathbf{z} = (z_1, z_2, \ldots, z_\ell) \in \mathbb{C}^\ell : |z_i| \le 1, \forall i = 1, 2, \ldots, \ell\}.$$

For a given $d \in \mathbb{N}$ we consider a vector $\mathbf{x} = (x_1, \ldots, x_d) \in [-1,1]^d$ and a compact set

$$K = K(\mathbf{x}) = \{(e^z, e^{x_1 z}, e^{x_2 z} \ldots, e^{x_d z}) : |z| \le 1\}.$$

For any $n, \ell \in \mathbb{N}$ we let $\mathcal{P}_n(\ell)$ denote the subspace of polynomials $P \in \mathbb{C}[z_1, \ldots, z_\ell]$ of degree n. For any subset $D \subset \mathbb{C}^\ell$ and polynomial P we define $\|P\|_D = \{|P(\mathbf{z})| : \mathbf{z} \in D\}$. We claim that $\|\cdot\|_K$ defines a norm only if the set $\{1, x_1, x_2, \ldots, x_d\}$ is linearly independent over $\mathbb{Z}$, which is what we will assume throughout the paper. See the beginning of the next section for

2010 *Mathematics Subject Classification.* Primary: 41A17, 30D15, Secondary: 11J13, 41A17.

Key words and phrases. Bernstein-Walsh inequalities, Polynomial inequalities, Liouville vectors, Hausdorff dimension.

the justification of the claim. For any $n \in \mathbb{N}$ we let

$$E_n(\mathbf{x}) := \sup\{\|P\|_{\Delta^{d+1}} : P \in \mathcal{P}_n(d+1), \|P\|_K \le 1\}.$$

From the equivalence of the norms $\|\cdot\|_{\Delta^{d+1}}$ and $\|\cdot\|_K$ we see (c.f. [5]) for any $\mathbf{z} = (z_0, z_1, \ldots, z_d) \in \mathbb{C}^{d+1}$ that

$$|P(\mathbf{z})| \le \|P\|_K E_n(\mathbf{x}) \exp(n \log^+ \max\{|z_0|, \ldots, |z_d|\}). \tag{1}$$

Let $e_n(\mathbf{x}) = \log E_n(\mathbf{x})$. On $\mathbb{R}^d$, we fix the maximum norm $\|\cdot\|$ given by $\|\mathbf{x}\| = \max_{1\le \ell \le d} |x_\ell|$. For any $x \in \mathbb{R}$ we let $\langle x \rangle$ denote the distance from x to the nearest integer, that is, $\langle x \rangle = \min\{|x-k| : k \in \mathbb{Z}\}$. We say that a vector $\mathbf{x} \in \mathbb{R}^d$ is *Diophantine* if there exist $\mu \ge d$ and $\epsilon > 0$ such that for any $\mathbf{q} \in \mathbb{Z}^d \backslash \{\mathbf{0}\}$ we have $\langle \mathbf{q} \cdot \mathbf{x} \rangle > \epsilon \|\mathbf{q}\|^{-\mu}$. From Dirichlet's approximation theorem (see e.g. [9]) we know that there are no Diophantine vectors with $\mu < d$. For irrational $x \in \mathbb{R}$ the growth of the exponent $e_n(x)$ was studied in [6]. In particular, when $d = 1$, it was shown in [6, Corollary 1.3] that if $x \in \mathbb{R}$ is Diophantine then the exponent $e_n(x)$ grows like $\frac{1}{2} n^2 \log n$. Our goal in this paper is to generalize this result for any $d \in \mathbb{N}$. We note that Bernstein-Walsh type inequalities on curves are much studied in the literature when $d = 1$, see e.g. [1, 4, 5] and references therein. On the other hand, as pointed out by [2] much less is known when $d > 1$ and one needs new techniques. Using the existence of exponential polynomials in $\mathcal{P}_n(d+1)$ with a zero of order at least $\deg \mathcal{P}_n - 1$ we get the following.

Theorem 1.1. *For any $\mathbf{x} \in \mathbb{R}^d$ with $\{1, x_1, \ldots, x_d\}$ linearly independent over $\mathbb{Z}$ we have*

$$e_n(\mathbf{x}) \ge \frac{n^{d+1}}{(d-1)!(d+1)} \log n - O(n^{d+1}),$$

where the implied constant depends on $\mathbf{x}$ and d only.

In [2] it was proved that for general exponential curves the exponent $e_n(x)$ is at most $n^{3(d+1)}$. However, in our situation we show that the upper estimate for the exponent $e_n(x)$ can be improved and this exponent is sharp for generic x.

Theorem 1.2. *If $\mathbf{x} \in [-1,1]^d$ is Diophantine then for any $n \in \mathbb{N}$ we have*

$$e_n(\mathbf{x}) \le \frac{n^{d+1}}{(d-1)!(d+1)} \log n + O(n^{d+1}), \tag{2}$$

where the implied constant depends on $\mathbf{x}$ and d only. In particular, (2) *holds for a.e. $\mathbf{x} \in [-1,1]^d$.*

To prove their result Coman and Poletsky make use of the well developed theory of continued fractions in $\mathbb{R}$. As there is no good analogue of continued fractions theory in higher dimensions we will consider a different approach.

We say that a vector $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with $\{x_1, \ldots, x_d\}$ linearly independent over $\mathbb{Q}$ is *Liouville* if it is not Diophantine, that is, for any $n \in \mathbb{N}$

there exists $\mathbf{q} \in \mathbb{Z}^d \backslash \{\mathbf{0}\}$ such that $\langle \mathbf{q} \cdot \mathbf{x} \rangle < \|\mathbf{q}\|^{-n}$. Let $\mathcal{L}_d$ denote the set of Liouville vectors in $\mathbb{R}^d$. Let $W_d(\alpha)$ denote the set of vectors $\mathbf{x} \in \mathbb{R}^d$ such that there are infinitely many integer vectors $\mathbf{q} \in \mathbb{Z}^d$ satisfying $\langle \mathbf{q} \cdot \mathbf{x} \rangle < \|\mathbf{q}\|^{-\alpha}$. It was proved in [3] that the Hausdorff dimension of $W_d(\alpha)$ is $(d-1) + \frac{d+1}{1+\alpha}$. Since $\mathcal{L}_d = \cap_{\alpha \geq d} W_d(\alpha)$, it follows that the Hausdorff dimension of $\mathcal{L}_d$ is at most $d-1$. In particular, $\mathcal{L}_d$ has zero Lebesgue measure which justifies the last part of Theorem 1.2.

We note that for any nonzero $\mathbf{q} \in \mathbb{Z}^d$ the set $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{q} \cdot \mathbf{x} = 0\}$ is a hyperplane in $\mathbb{R}^d$ and is contained in $\mathcal{L}_d$. Together with the above upper estimate we get that the set $\mathcal{L}_d$ of Liouville d-vectors has Hausdorff dimension $d-1$.

We now turn to discuss the exceptional set of points in $\mathbb{R}^d$ for which $e_n(\mathbf{x})$ grows faster than $Cn^{d+1} \log n$. To this end, we define the set

$$W(d) = \left\{ \mathbf{x} \in [-1,1]^d : \limsup_{\|\mathbf{q}\| \to \infty} \frac{-\log \langle \mathbf{q} \cdot \mathbf{x} \rangle}{\|\mathbf{q}\|^{d+1} \log \|\mathbf{q}\|} = \infty \right\},$$

where $\mathbf{q} \in \mathbb{Z}^d_{\geq 0} := \{(q_1, \ldots, q_d) \in \mathbb{Z}^d : q_1, \ldots, q_d \geq 0\}$.

Theorem 1.3. *For any* $\mathbf{x} \in W(d)$, $\limsup_n \frac{e_n(\mathbf{x})}{n^{d+1} \log n} = \infty$.

It is easy to see (e.g. from Theorem 1.2) that $W(d) \subset \mathcal{L}_d$ so that it has Hausdorff dimension at most $d-1$. In fact, we have

Theorem 1.4. *Hausdorff dimension of the exceptional set* $W(d)$ *is* $d-1$.

It was proved in [6] that when $d = 1$ the set of points x for which $e_n(x)$ grow faster than $\frac{1}{2} n^2 \log n$ is uncountable. For $d > 1$, since the Hausdorff dimension of $W(d)$ is positive we in particular get that $W(d)$ is uncountable. Thus, for any $d \in \mathbb{N}$ the set of points $\mathbf{x}$ for which $e_n(\mathbf{x})$ grow faster than $\frac{1}{(d-1)!(d+1)} n^{d+1} \log n$ is uncountable and has Hausdorff dimension $d-1$.

In the next section we will prove Theorem 1.2 and in § 3 we obtain Theorem 1.1, Theorem 1.3, and Theorem 1.4.

Acknowledgement. The authors are grateful to Dan Coman for useful comments in the preliminary version of the paper.

2. Upper estimate

Before beginning with the main work, we prove the fact which allows us to assert that $\|\cdot\|_K$ defines a norm only if $\{1, x_1, x_2, \ldots, x_d\}$ are linearly independent over $\mathbb{Z}$, as claimed in the introduction. This follows from the following lemma.

Lemma 2.1. *Let* $y_1, y_2, \ldots y_l$ *be distinct real numbers. Then* $e^{y_1 z}, \ldots e^{y_l z}$ *are linearly independent over* $\mathbf{C}$.

The proof of the lemma is left to the reader. We apply Lemma 2.1 to an equation $P(e^z, e^{x_1 z}, \dots e^{x_n z}) = 0$ for some polynomial P. The linear independence of the x_i's implies that exponent coeffients in the expansion will be distinct; using the lemma, we get that all the coefficients of P are 0. The claim follows.

The remaining of the section is devoted to prove Theorem 1.2. We state [6, Lemma 2.4]

Lemma 2.2. *Let $x, y \in \mathbb{Z}$ with $x \le y$ be given. For any $\alpha \in \mathbb{R}$ we have*

$$\prod_{j=x}^{y} |j-\alpha| \ge \langle \alpha \rangle \left(\frac{y-x}{2e} \right)^{y-x}.$$

Let $\mathbf{x} \in \mathbb{R}^d$ and $n \in \mathbb{N}$ be given. For any $\ell \in \{0, 1, \dots, n\}$ and $\mathbf{m} \in \mathbb{Z}^d$ with $m_1, \dots, m_d \in \{0, 1, \dots, n\}$ we define

$$\beta(\ell, \mathbf{m}) = \prod_{j_0 + j_1 + \dots + j_d \le n, (j_0, \mathbf{j}) \ne (\ell, \mathbf{m})} ((\ell - j_0) + (\mathbf{m} - \mathbf{j}) \cdot \mathbf{x}), \tag{3}$$

where each $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{Z}^d$ has nonnegative components and also $j_0 \ge 0$. We will need the following estimate.

Proposition 2.3. *If $\mathbf{x}$ is Diophantine, then there exists a constant $C_{\mathbf{x},d} > 0$ such that*

$$\log |\beta(\ell, \mathbf{m})| \ge \frac{1}{(d+1)!} n^{d+1} \log n - C_{\mathbf{x},d} n^{d+1}.$$

To obtain the proposition we need the following lemmas. We set $|\mathbf{j}| = j_1 + \dots + j_d$. Arguing inductively on d it is easy to see that

Lemma 2.4. *For any $m \in \mathbb{N}$, the set $\left\{ \mathbf{j} \in \mathbb{Z}^d : |\mathbf{j}| = m, j_1, \dots, j_d \ge 0 \right\}$ has cardinality $C(m+d-1, d-1) = \begin{pmatrix} m+d-1 \\ d-1 \end{pmatrix}$.*

Lemma 2.5. *We have*

$$\int_1^n (n-x)^{d-1} x \log x \, dx \ge \frac{1}{d(d+1)} n^{d+1} \log n - C_d n^{d+1}.$$

Proof. We claim for any $m, \ell \ge 1$ that

$$\int_1^n (n-x)^m x^\ell \log x \, dx \ge \frac{m}{\ell+1} \left[\int_1^n (n-x)^{m-1} x^{\ell+1} \log x \, dx - \frac{n^{m+\ell+1}}{\ell+1} \right].$$

We first note from integration by parts that

$$\int x^\ell \log x \, dx = \frac{x^{\ell+1}}{\ell+1} \log x - \int \frac{x^\ell}{\ell+1} dx = \frac{x^{\ell+1}}{\ell+1} \log x - \frac{x^{\ell+1}}{(\ell+1)^2} + C.$$

Now, using integration by parts again we obtain:

$$\int_1^n (n-x)^m x^\ell \log x\,dx = (n-x)^m \left(\frac{x^{\ell+1}}{\ell+1}\log x - \frac{x^{\ell+1}}{(\ell+1)^2}\right)\Big|_1^n + \int_1^n m(n-x)^{m-1}\left(\frac{x^{\ell+1}}{\ell+1}\log x - \frac{x^{\ell+1}}{(\ell+1)^2}\right) dx.$$

We note that $(n-x)^{m-1}x^{\ell+1} \le n^{m+\ell}$ for $x \in [1,n]$. Thus, simplifying we get

$$\int_1^n (n-x)^m x^\ell \log x\,dx \ge \frac{(n-1)^m}{(\ell+1)^2} + \frac{m}{\ell+1}\int_1^n \left[(n-x)^{m-1}x^{\ell+1}\log x - \frac{n^{m+\ell}}{\ell+1}\right] dx$$
$$\ge \frac{m}{\ell+1}\left[\int_1^n (n-x)^{m-1}x^{\ell+1}\log x\,dx - \frac{n^{m+\ell+1}}{\ell+1}\right].$$

To prove the lemma we iterate the claim:

$$\int_1^n (n-x)^{d-1}x\log x\,dx \ge \frac{d-1}{2}\left[\int_1^n (n-x)^{d-2}x^2\log x\,dx - \frac{n^{d+1}}{2}\right]$$
$$\ge \frac{d-1}{2}\left[\frac{d-2}{3}\left(\int_1^n (n-x)^{d-3}x^3\log x\,dx - \frac{n^{d+1}}{3}\right) - \frac{n^{d+1}}{2}\right]$$
$$\ge \cdots$$
$$\ge \frac{(d-1)!}{d!}\int_1^n x^d\log x\,dx - C'_d n^{d+1}$$
$$= \frac{1}{d(d+1)}n^{d+1}\log n - C_d n^{d+1}. \qquad \square$$

We state without proof the following

Lemma 2.6. *Let $m < n$ be integers and $f : [m,n] \to [0,\infty)$ be a continuous function with exactly one local maximum in $[m,n]$ and $f(m) = f(n) = 0$. Then, we have*

$$\left|\sum_{k=m}^n f(k) - \int_m^n f(x)\,dx\right| \le \max_{m\le x\le n} f(x).$$

Proof of Proposition 2.3. We have

$$|\beta(\ell,\mathbf{m})| \ge \prod_{|\mathbf{j}|\le n,\,\mathbf{j}\ne\mathbf{m}}\ \prod_{j_0=0}^{n-|\mathbf{j}|} |(\ell - j_0) + (\mathbf{m}-\mathbf{j})\cdot\mathbf{x}|.$$

Since $\mathbf{x}$ is Diophantine of order μ we may find some $\epsilon > 0$ such that $\langle \mathbf{q}\cdot\mathbf{x}\rangle \geq \epsilon\|\mathbf{q}\|^{-\mu}$. Using Lemma 2.2 we get

$$\begin{aligned}
|\beta(\ell,\mathbf{m})| &\geq \prod_{|\mathbf{j}|\leq n,\,\mathbf{j}\neq\mathbf{m}} \prod_{j=-\ell}^{n-|\mathbf{j}|-\ell} |j-(\mathbf{m}-\mathbf{j})\cdot\mathbf{x}| \\
&\geq \prod_{|\mathbf{j}|\leq n,\,\mathbf{j}\neq\mathbf{m}} \left(\frac{n-|\mathbf{j}|}{2e}\right)^{n-|\mathbf{j}|} \langle(\mathbf{m}-\mathbf{j})\cdot\mathbf{x}\rangle \\
&\geq \prod_{|\mathbf{j}|\leq n,\,\mathbf{j}\neq\mathbf{m}} \left(\frac{n-|\mathbf{j}|}{2e}\right)^{n-|\mathbf{j}|} \prod_{|\mathbf{j}|\leq n,\,\mathbf{j}\neq\mathbf{m}} \epsilon\|\mathbf{m}-\mathbf{j}\|^{-\mu} \\
&= \left(\prod_{k=1}^{n}\prod_{|\mathbf{j}|=n-k,\,\mathbf{j}\neq\mathbf{m}} \left(\frac{k}{2e}\right)^{k}\right)\left(\prod_{|\mathbf{j}|\leq n,\,\mathbf{j}\neq\mathbf{m}} \epsilon\|\mathbf{m}-\mathbf{j}\|^{-\mu}\right).
\end{aligned}$$

We set

$$A := \prod_{k=1}^{n}\prod_{|\mathbf{j}|=n-k,\,\mathbf{j}\neq\mathbf{m}} k^k,\ B := \prod_{k=1}^{n}\prod_{|\mathbf{j}|=n-k,\,\mathbf{j}\neq\mathbf{m}} (2e)^{-k},\ C := \prod_{|\mathbf{j}|\leq n,\,\mathbf{j}\neq\mathbf{m}} \epsilon\|\mathbf{m}-\mathbf{j}\|^{-\mu}.$$

We now estimate each of A, B, C separately. Since the set $\{\mathbf{j}\in\mathbb{Z}^d : |\mathbf{j}|\leq n\}$ has cardinality at most $(n+1)^d$ and $\|\mathbf{m}-\mathbf{j}\|\leq n$ for any $|\mathbf{j}|\leq n$ we get that

$$C = \prod_{|\mathbf{j}|\leq n,\,\mathbf{j}\neq\mathbf{m}} \epsilon\|\mathbf{m}-\mathbf{j}\|^{-\mu} \geq \prod_{|\mathbf{j}|\leq n} \epsilon n^{-\mu} \geq \epsilon^{(n+1)^d} n^{-\mu(n+1)^d} \geq \epsilon^{(2n)^d} n^{-\mu(2n)^d}.$$

Thus,

$$\log C \geq -\mu 2^d n^d \log n + 2^d n^d \log\epsilon. \tag{4}$$

Using Lemma 2.4 together with the trivial bound we get

$$\begin{aligned}
\log A &\geq \left(\sum_{k=1}^{n}\sum_{|\mathbf{j}|=n-k} k\log k\right) - n\log n \\
&= \left(\sum_{k=1}^{n}\binom{n-k+d-1}{d-1} k\log k\right) - n\log n \\
&\geq \left(\frac{1}{(d-1)!}\sum_{k=1}^{n}(n-k)^{d-1}k\log k\right) - n\log n.
\end{aligned}$$

It is easy to see that the function $f : [1,n]\to[0,\infty)$ given by $f(x) = (n-x)^{d-1}x\log x$ satisfies Lemma 2.6 for $d > 1$. Thus, when $d > 1$, Lemma 2.5

and Lemma 2.6 give

$$\log A \geq \frac{1}{(d-1)!}\left(\int_1^n (n-x)^{d-1} x \log x \, dx - \max_{1\leq x\leq n} f(x)\right) - n\log n$$
$$\geq \frac{1}{(d-1)!}\left(\frac{1}{d(d+1)} n^{d+1}\log n - C_d n^{d+1} - n^d \log n\right) - n\log n.$$

On the other hand, for $d=1$, following [6], we use the estimate (c.f. [5, Lemma 2.1]) $\sum_{k=1}^n k\log k \geq \frac{n^2\log n}{2} - \frac{n^2}{4}$ to obtain

$$\log A \geq \frac{1}{2}n^2\log n - \frac{n^2}{4} - n\log n.$$

Hence, for any $d\geq 1$ it holds

$$\log A \geq \frac{1}{(d+1)!} n^{d+1}\log n - 3C_d n^{d+1}. \tag{5}$$

As for the estimating $\log B$, we note that since $C(n-k+d-1, d-1) \leq \frac{n^{d-1}}{(d-1)!} + O(n^{d-2})$ for any $k\in[1,n]$ we get

$$\log B \geq -\sum_{k=1}^n \sum_{|\mathbf{j}|=n-k} k\log(2e) = -\sum_{k=1}^n \binom{n-k+d-1}{d-1} k\log(2e) \tag{6}$$
$$\geq -\frac{1}{(d-1)!} n^{d+1} - O(n^d),$$

where the implied constant depends d only. Thus, combining (4), (5) and (6) we arrive at

$$\log|\beta(\ell,\mathbf{m})| > \frac{1}{(d+1)!} n^{d+1}\log n - C_{d,\mu,\epsilon} n^{d+1}. \qquad \square$$

Proof of Theorem 1.2. Let $N = \dim \mathcal{P}_n - 1$, so that $N = \binom{n+d+1}{n} - 1$. Fix some $P\in\mathcal{P}_n$ with $\|P\|_K \leq 1$. Define

$$P(\mathbf{z}) = \sum_{j_0+j_1+\cdots+j_d\leq n} c(j_0,\mathbf{j}) z_0^{j_0}\cdots z_d^{j_d} \text{ and } f(z) = P(e^z, e^{x_1 z}, \ldots, e^{x_d z}),$$

where $j_0,\ldots,j_d \geq 0$. Then,

$$f(z) = \sum_{j_0+j_1+\cdots+j_d\leq n} c(j_0,\mathbf{j}) e^{(j_0+\mathbf{j}\cdot\mathbf{x})z}.$$

For any polynomial $R(\lambda) = \sum_{j=0}^m c_j\lambda^j$ we introduce the differential operator

$$D_R = R\left(\frac{d}{dz}\right) = \sum_{j=0}^m c_j \frac{d^j}{dz^j}.$$

We note that for any $a \in \mathbb{C}$ we have

$$D_R(e^{az}) \mid_{z=0} = \sum_{j=0}^{m} c_j a^j = R(a). \tag{7}$$

To estimate $c(\ell, \mathbf{m})$ we set

$$R_{\ell,\mathbf{m}}(\lambda) = \prod_{j_0+j_1+\cdots+j_d \leq n, (j_0,\mathbf{j}) \neq (\ell,\mathbf{m})} (\lambda - (j_0 + \mathbf{j} \cdot \mathbf{x})) = \sum_{t=0}^{N} a_t \lambda^t.$$

For any $\lambda \geq 0$ we have

$$\sum_{t=0}^{N} |a_t| \lambda^t \leq \prod_{j_0+j_1+\cdots+j_d \leq n, (j_0,\mathbf{j}) \neq (\ell,\mathbf{m})} (\lambda + |j_0 + \mathbf{j} \cdot \mathbf{x}|) \leq (\lambda + n)^N.$$

From (7) we note that

$$D_{R_{\ell,\mathbf{m}}}(e^{(j_0+\mathbf{j}\cdot\mathbf{x})z}) \mid_{z=0} = \begin{cases} R_{\ell,\mathbf{m}}(\ell + \mathbf{m} \cdot \mathbf{x}) & \text{if } (j_0, \mathbf{j}) = (\ell, \mathbf{m}), \\ 0 & \text{if } (j_0, \mathbf{j}) \neq (\ell, \mathbf{m}). \end{cases}$$

Thus,

$$D_{R_{\ell,\mathbf{m}}}(f(z)) \mid_{z=0} = c(\ell, \mathbf{m}) \beta(\ell, \mathbf{m})$$

where β is defined in (3).

On the other hand, using $\|P\|_K \leq 1$ and Cauchy's inequality we get

$$|f^{(t)}(0)| \leq t! \leq N^t \text{ whenever } t \leq N. \tag{8}$$

This implies that

$$\left| D_{R_{\ell,\mathbf{m}}}(f(z)) \mid_{z=0} \right| = \left| \sum_{t=0}^{N} a_t f^{(t)}(0) \right| \leq \sum_{t=0}^{N} |a_t| N^t \leq (N + n)^N.$$

Therefore,

$$\log(|c(\ell, \mathbf{m}) \beta(\ell, \mathbf{m})|) \leq N \log(N + n).$$

Using Proposition 2.3 we obtain

$$\begin{aligned} \log(|c(\ell, \mathbf{m})|) &\leq N \log(N + n) - \log |\beta(\ell, \mathbf{m})| \\ &\leq N \log(N + n) - \frac{1}{(d+1)!} n^{d+1} \log n + C_{\mathbf{x},d} n^{d+1}. \end{aligned}$$

Since $\|P\|_{\Delta^d} \leq \sum |c(j_0, \mathbf{j})| \leq (N + 1) \max |c(j_0, \mathbf{j})|$ we deduce that

$$e_n(\mathbf{x}) \leq N \log(N + n) - \frac{1}{(d+1)!} n^{d+1} \log n + C_{\mathbf{x},d} n^{d+1} + \log(N + 1).$$

Finally, using

$$N = C(n + d + 1, d + 1) - 1 = \frac{n^{d+1}}{(d+1)!} + O(n^d) \tag{9}$$

we obtain $N\log(N+n) \leq N\log N + O(N) = \frac{1}{d!}n^{d+1}\log n + O(n^{d+1})$. Hence,

$$e_n(\mathbf{x}) \leq \frac{n^{d+1}}{(d-1)!(d+1)}\log n + O(n^{d+1}). \quad \square$$

3. Lower estimate and Hausdorff dimension

We first start proving Theorem 1.1. It is essentially contained in the proof of [5, Proposition 1.3] as pointed out by D. Coman and for completeness we recall it here.

Proof of Theorem 1.1. Fix $P \in \mathcal{P}_n(d+1)$ with $ord(P(e^z, e^{x_1 z}, \ldots, e^{x_d z}), 0) \geq N$. We have $P \not\equiv 0$ implies $P(e^z, e^{x_1 z}, \ldots, e^{x_d z}) \not\equiv 0$. We let $f(z) = \frac{1}{\|P\|_K} P(e^z, e^{x_1 z}, \ldots, e^{x_d z})$ so that $\|f\|_{\Delta^1} = 1$ then $\max_{|z|=r}|f(z)| \geq r^N, r \geq 1$. From (1) we get for any $|z| = r$

$$r^N \leq E_n(\mathbf{x})\exp(n\log^+\max\{|e^z|, |e^{x_1 z}|, \ldots, |e^{x_d z}|\}) \leq E_n(\mathbf{x})e^{nC_0 r},$$

where $C_0 = \max\{1, \|\mathbf{x}\|\}$. Taking $r = N/n$ we see that

$$N\log\frac{N}{n} \leq e_n(\mathbf{x}) + C_0 N.$$

Using (9) we have

$$N\log\frac{N}{n} = \frac{n^{d+1}}{(d-1)!(d+1)}\log n + O(n^d\log n),$$

which gives

$$e_n(\mathbf{x}) \geq \frac{n^{d+1}}{(d-1)!(d+1)}\log n - O(n^{d+1}). \quad \square$$

Now we prove Thoerem 1.3 which provides us with the exceptional set of points $\mathbf{x}$ that does not satisfy Theorem 1.2.

Proof of Theorem 1.3. Let $\mathbf{x} \in W(d)$ and $(\mathbf{q}_\ell)_{\geq 1}$ be a sequence satisfying

$$C(\ell) = \frac{-\log\langle \mathbf{q}_\ell \cdot \mathbf{x}\rangle}{\|\mathbf{q}_\ell\|^{\mathbf{d+1}}\log\|\mathbf{q}_\ell\|} \to \infty \text{ as } \ell \to \infty. \tag{10}$$

For a given $\ell \geq 0$ we let $n = d\|\mathbf{q}_\ell\|$ and $p \in \mathbb{Z}$ be such that $\langle \mathbf{q}_\ell \cdot \mathbf{x}\rangle = |\mathbf{q}_\ell \cdot \mathbf{x} - p|$. Since $\|\mathbf{x}\| \leq 1$ we see that $|p| \leq d\|\mathbf{q}_\ell\|$. Then, the polynomial P given by

$$P(z_0, z_1, \ldots, z_d) = z_0^p - \prod_{\ell=1}^{d} z_\ell^{q_\ell}$$

is in $\mathcal{P}_n(d+1)$. Clearly, $\|P\|_{\Delta^{d+1}} = 2$. Using $|1 - e^\xi| \leq 2|\xi|$ for $|\xi| \leq 1$ we get

$$|P(e^z, e^{x_1 z}, \ldots, e^{x_d z})| = |e^{pz}(1 - e^{(\mathbf{q}_\ell \cdot \mathbf{x} - p)z})| \leq 2e^n\langle \mathbf{q}_\ell \cdot \mathbf{x}\rangle,$$

whenever $|z| \leq 1$. Therefore,

$$E_n(\mathbf{x}) \geq \|P\|_{\Delta^{d+1}}/\|P\|_K \geq e^{-n}\frac{1}{\langle \mathbf{q}_\ell \cdot \mathbf{x}\rangle}.$$

So, using (10) we get

$$\begin{aligned} e_n(\mathbf{x}) = \log E_n(\mathbf{x}) &\geq C(\ell)\|\mathbf{q}_\ell\|^{d+1}\log\|\mathbf{q}_\ell\| - n \\ &= C(\ell)\left(\frac{n}{d}\right)^{d+1}\log\frac{n}{d} - n. \end{aligned}$$

Thus,

$$\frac{e_n(\mathbf{x})}{n^{d+1}\log n} \geq \frac{1}{d^{d+1}}C(\ell) - \frac{1}{n} \to \infty \text{ as } \ell \to \infty. \quad \square$$

It remains to give the proof of Theorem 1.4.

Proof of Theorem 1.4. We will use ubiquitous systems introduced in [8] as a method of computing Hausdorff dimension of lim-sup sets. We consider the family $\mathcal{R} = \{R(\mathbf{q}) : \mathbf{q} \in \mathbb{Z}^d_{\geq 0}\}$ where for any $\mathbf{q} \in \mathbb{Z}^d$ we set $R(\mathbf{q}) := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{q}\cdot\mathbf{x} \in \mathbb{Z}\}$. Let $\psi : \mathbb{N} \to [0,1]$ be a decreasing function converging to 0 at the infinity. Define

$$\Lambda(\mathcal{R};\psi) = \left\{\mathbf{x} \in [-1,1]^d : \operatorname{dist}(\mathbf{x}, R(\mathbf{q})) < \psi(\|\mathbf{q}\|) \text{ for infinitely many } R(\mathbf{q})\right\},$$

where $\operatorname{dist}(\mathbf{x}, S) = \inf_{\mathbf{y}\in S}\|\mathbf{x}-\mathbf{y}\|$. For any such ψ, we will prove that the Hausdorff dimension of $\Lambda(\mathcal{R};\psi)$ is at least $d-1$. Then, for $\psi(n) = n^{-n^{d+2}}$ we will show that $\Lambda(\mathcal{R};\psi) \subset W(d)$ which will finish the proof.

Let I^d denote the hypercube $[-\frac{1}{2},\frac{1}{2}]^d$ of unit length. It is well-known (see e.g.[7]) that the family $\{R(\mathbf{q}) : \mathbf{q} \in \mathbb{Z}^d\}$ is *ubiquitous with respect to* $\rho(Q) := dQ^{-1-d}\log Q$ in the sense that

$$\left| I^d \setminus \bigcup_{1\leq\|\mathbf{q}\|\leq N} B(R(\mathbf{q});\delta(N))\right| \to 0 \text{ as } N \to \infty,$$

$B(R(\mathbf{q});\delta) = \{\mathbf{x} \in \mathbb{R}^d : \operatorname{dist}(\mathbf{x}, R(\mathbf{q})) < \delta\}$. However, it is not clear if the family $\mathcal{R} = \{R(\mathbf{q}) : \mathbf{q} \in \mathbb{Z}^d_{\geq 0}\}$ is ubiquitous with respect to the same ρ. However, for our purposes we do not need to try to optimize ρ. Simply consider the constant function $\rho \equiv 1$, then for $\mathbf{q} = (0,\ldots,0,1)$ we have $I^d \subset B(R(\mathbf{q});1)$ so that $\mathcal{R}$ is ubiquitous w.r.t 1. Since $\gamma := \limsup_{Q\to\infty}\frac{\log\rho(Q)}{\log\psi(Q)} = 0$, it follows from [8, Theorem 1] that the Hausdorff dimension of $\Lambda(\mathcal{R};\psi)$ is at least $\dim\mathcal{R} + \gamma\operatorname{codim}\mathcal{R} = d-1$.

We now claim that $\Lambda(\mathcal{R};\psi) \subset W(d)$ when $\psi(n) = n^{-n^{d+2}}$. For $\mathbf{x} \in \Lambda(\mathcal{R};\psi)$ let $(\mathbf{q}_\ell)_{\geq\ell}$ denote the sequence such that $\operatorname{dist}(\mathbf{x}, R(\mathbf{q}_\ell)) < \psi(\|\mathbf{q}\|)$ and all $R(\mathbf{q}_\ell)$ are distinct. Then, for any $\mathbf{q} \in \mathbb{Z}^d$ we have $\mathbf{q}\cdot\mathbf{x} \notin \mathbb{Z}$ which means

$\{1, x_1, x_2, \ldots, x_d\}$ is linearly independent over $\mathbb{Z}$. Let $\mathbf{y} \in R(\mathbf{q}_\ell)$ be such that $\|\mathbf{x} - \mathbf{y}\| < \psi(\|\mathbf{q}\|)$. We choose $p \in \mathbb{Z}$ with $\mathbf{q}_\ell \cdot \mathbf{y} = p$. Then,

$$\langle \mathbf{q}_\ell \cdot \mathbf{x} \rangle \leq \|\mathbf{q}_\ell \cdot (\mathbf{x} - \mathbf{y}) + \mathbf{q}_\ell \cdot \mathbf{y} - p\| \leq \|\mathbf{q}_\ell\| \|\mathbf{x} - \mathbf{y}\| < \psi(\|\mathbf{q}_\ell\|).$$

Hence, $\mathbf{x} \in W(d)$ as $\frac{-\log\langle \mathbf{q}_\ell \cdot \mathbf{x} \rangle}{\|\mathbf{q}_\ell\|^{d+1} \log \|\mathbf{q}_\ell\|} \geq \|\mathbf{q}_\ell\|$ and $\|\mathbf{q}_\ell\| \to \infty$ with $\ell \to \infty$. □

References

[1] Bos, L., Brudnyi, A., Levenberg, N., Totik, V. *Tangential Markov inequalities on transcendental curves.* Constr. Approx. **19**, (2003), 339–354.

[2] Bos, L. P., Brudnyi, A., Levenberg, N. *On polynomial inequalities on exponential curves in* $\mathbb{C}^n$. Constr. Approx. **31** (2010), no.1, 139–147.

[3] J. D. Bovey and M. M. Dodson, *The Hausdorff dimension of systems of linear forms.* Acta Arith. **45** (1986), 337–358.

[4] Coman, D., Poletsky, E. A. *Measures of transcendency for entire functions.* Mich. Math. J. **51**, (2003), 575–591.

[5] ——— *Bernstein-Walsh inequalities and the exponential curve in* $\mathbb{C}^2$. Proc. Amer. Math. Soc. **131** (2003), 879–887.

[6] D. Coman and E. Poletsky, *Polynomial Estimates, Exponential Curves and Diophantine Approximation.* Math. Res. Lett. **17 (6)**, (2010), 1125–1136.

[7] M. M. Dodson, *Geometric and probabilistic ideas in metric Diophantine approximation.* Russ. Math. Surv. **48** (1993), 73–102.

[8] M. M. Dodson, B. P. Rynne, and J. A. G. Vickers *Diophantine approximation and a lower bound for Hausdorff dimension.* Mathematika **37** (1990), 59–73.

[9] W.M. Schmidt, *Diophantine approximation,* Lecture Notes in Mathematics, **785**, Springer-Verlag, Berlin, 1980.

(ML, SK) Department of Mathematics, Nazarbayev University, Astana, Kazakhstan.

E-mail address, ML: mlawrence@nu.edu.kz

E-mail address, SK: shirali.kadyrov@nu.edu.kz