FROBENIUS SINGULARITIES OF ALGEBRAIC SETS OF MATRICES

Graduation project

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Abstract

When one studies certain rings, it is natural to classify them according to certain properties. This project focuses on the study of properties of commutative rings associated with algebraic sets. In particular, we consider the algebraic set of pairs of square matrices whose commutator has a zero diagonal. We prove that it is irreducible and F-regular for matrices of all sizes and when the matrix entries are from a field of positive prime characteristic. In addition, we provide a proof of its F-purity and find a system of parameters on it. Moreover, we state several conjectures associated to this topic.

1 Introduction

1.1 Basic Definitions

Before we proceed to the setup and the goals of this project in details, we are introducing definitions. It is assumed that readers are familiar with basic abstract algebra notions such as *rings, fields, ideals* etc.
We state some more advanced notions from abstract algebra which mostly appear in assumptions of theorems and other definitions. Next, we introduce key concepts from commutative algebra. Throughout this project, the definitions come from [1], [2], and [3].

Definition 1.1. A commutative ring R with 1 is called *Noetherian* if every ideal of R is finitely generated.

From now on, all rings in this project are assumed to be commutative Noetherian with 1.

Definition 1.2. Let R be a ring. An ideal P is called a *prime ideal* if $P \neq R$ and whenever the product ab of two elements $a, b \in R$ is an element of P, then at least one of a and b is an element of P.

Remark 1. The set of prime ideals in R has a minimal element.

Definition 1.3. A prime ideal P is called a *minimal prime ideal* over/of an ideal I if it is minimal among all prime ideals containing I.

Definition 1.4. Let I be an ideal of a ring R and define $\mathrm{Rad(I)} := \{r \in R | r^n \in I \text{ for some } n \in \mathbb{N}\}$ called the *radical* of I.
An ideal I of R is called a *radical ideal* if $\mathrm{Rad}(I) = I$.

Definition 1.5. An ideal $\mathfrak{m}$ in a ring R is called a *maximal ideal* if $\mathfrak{m} \neq R$ and the only ideals containing $\mathfrak{m}$ are $\mathfrak{m}$ and R.

Definition 1.6. A ring R is *local*, if it has a unique maximal ideal.

Definition 1.7. If I and J are ideals of a ring R, their *ideal quotient* $(I : J)$ is the set $(I : J) := \{r \in R | \ rJ \subset I\}$. $(I : J)$ is an ideal itself.

Definition 1.8. If I is an ideal of a ring R, then $I^{[p]} := (\{a_\alpha^p\}_{\alpha \in A})$, where a_α is a generator of I.

Definition 1.9. A ring S is called a *graded ring* if it is the direct sum of additive subgroups: $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$ such that $S_i S_j \subset S_{i+j}$ for all $i, j \geq 0$. The elements of S_k are said to be *homogeneous of degree* k, and S_k is called the *homogeneous component of* S *of degree* k.

Remark 2. Polynomial rings are graded rings by degree, that is, each of S_i in the previous definition consists of homogeneous polynomials of degree i.

Definition 1.10. An ideal I of a polynomial ring R is *homogeneous* if I is generated by finitely many homogeneous polynomials.

Definition 1.11. A ring R is a *reduced ring* if it has no non-zero elements with square zero, i.e, $x^2 = 0 \to x = 0$.

Remark 3. A quotient ring R/I is reduced if and only if I is a radical ideal.

Definition 1.12. For any ring R the *Krull dimension* n of R is the maximum possible length of a chain $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n$ of distinct prime ideals in R.

Definition 1.13. A *regular local ring* is a local ring having the property that the minimal number of generators of its maximal ideal is equal to its Krull dimension.

Definition 1.14. Let R be a ring, let $S \subset R$ be a multiplicative closed set/system. Define an equivalence relation $\sim$ on $R \times S$ by setting $(r_1, s_1) \sim (r_2, s_2)$, if there exists $t \in S$ such that $t(r_1 s_2 - r_2 s_1) = 0$. $S^{-1}R$ (or R_S) is then defined as a set of the equivalence classes in $R \times S$.

The next notions are central in this project.

Definition 1.15. Let R be a ring. A sequence of elements $a_1, ..., a_r$ is a *regular sequence* if a_i is nonzero divisor in $R/(a_1, ..., a_{i-1})$, that is, if $a_i \neq 0$ in $R/(a_1, ..., a_{i-1})$ and $a_i b = 0$ for $b \in R/(a_1, ..., a_{i-1})$, then $b = 0$.

Definition 1.16. A graded ring R is a *complete intersection* if $R \simeq S/I$, where S is a regular local ring, I is a *complete intersection ideal*, i.e., I is generated by a regular sequence.

Definition 1.17. Let R be a ring with prime characteristic $p > 0$, then the *Frobenius endomorphism* is defined as $F(r) = r^p$ for all $r \in R$.

Remark 4. We can define $R^{1/p}$ under this endomorphism.

Definition 1.18. Let R be a reduced ring of characteristic $p > 0$. R is said to be *F-pure*, if the map $M \to M \otimes R^{1/p}$ is injective for every R-module M.

Definition 1.19. Let R be a ring of characteristic p. For an ideal I of R, we define the ideal I^* by $I^* = \{x \in R \,|$ there exists $c \in R^0$ such that $cx^q \in I^{[q]}$ for every $q = p^e \gg 0$ (much greater than 0), $e \in \mathbb{N}\}$. We call I^* the *tight closure* of I. We say that I is *tightly closed* if $I^* = I$.
R is *weakly F-regular* if every ideal of R is tightly closed. R is *F-regular* if $W^{-1}R$ is weakly F-regular for every multiplicative system W.

Remark 5. $R^0 = \{x \in R \,|\, x$ is not contained in any minimal prime ideal of $R\}$.

Remark 6. In our setup, weakly and general F-regularities are equivalent. See Corollary (4.7) and Theorem (5.5) in [4].

1.2 Setup and Notations

Let k be an algebraically closed field of positive prime characteristic p. Let $X = (x_{i,j})_{1 \leq i,j \leq n}$, $Y = (y_{i,j})_{1 \leq i,j \leq n}$ be $n \times n$ matrices of indeterminates over k. Let $C = XY - YX$. We are interested in studying the ideal, generated by the diagonal entries of commutator matrix C denoted as I, and the corresponding quotient ring $k[X,Y]/I$, denoted as R. The notations are fixed throughout the project.

Remark 7. The ideal I has generating elements $c_{i,i} = \sum_{k=1}^{n}(x_{i,k}y_{k,i} - y_{i,k}x_{k,i})$ for $i \in \{1, ..., n\}$. Since the $tr(C) = 0$, we can remove one generating element from the set, and still retain the same ideal.

1.3 Motivation and Goals

H. Young [5] (Theorem 5.3.1) proved that R is a complete intersection for matrices of all sizes and all characteristics of the field.

In the same paper, it was proved that in such rings with dimensions of matrices 2 and 3, I is a prime ideal (Theorem 5.3.3). It was conjectured that it is true for all dimension (Item 7 in Section 5.4).

Then, we want to know whether

1. R is F-pure, in which case I is radical.
2. R is F-regular, in which case I is prime.

In our approach, we use certain criteria and known F-regularity results of other rings. In addition, we implemented software Macaulay2 [6] to back up our findings.

2 Main results

2.1 F-purity

For Statement 1, we apply Fedder's criterion:

Theorem 2.1. ***Fedder's criterion*** *[7] (Theorem 1.12). Let $(S, \mathfrak{m})$ be a polynomial ring over a field of characteristic $p > 0$ with the homogeneous maximal ideal $\mathfrak{m}$, generated by all variables, and let J be a homogeneous ideal in S. Then S/J is F-pure if and only if $(J^{[p]} : J) \not\subset \mathfrak{m}^{[p]}$.*

Theorem 2.2. ***F-purity****. Let k, X, Y, C, I, R be as defined in the notations. Then R is F-pure.*

Proof. Our ring $k[X][Y]$ is a polynomial ring, satisfying the assumption of the previous theorem. $\mathfrak{m} := (\{x_{i,j}\}_{1\leq i,j\leq n}) + (\{y_{i,j}\}_{1\leq i,j\leq n})$. As it was mentioned, we can remove one element from the generating set of I. Let us remove $c_{1,1}$. Thus, $I = (\{c_{i,i}\}_2^n)$. Moreover, I is a complete intersection, meaning $(I^{[p]} : I) = I^{[p]} + (\Pi)$ [2], Corollary 2.3.10, where $\Pi = \prod_{i=2}^n c_{i,i}^{p-1}$. So it is sufficient to find a monomial term in Π that does not belong to $\mathfrak{m}^{[p]}$. We claim that Π has a monomial term with nonzero coefficients modulo p. To see this, we apply the binomial theorem. Each factor $c_{i,i}^{p-1}$ in Π will have unique terms with indices containing 1, i.e. $x_{i,1}y_{1,i}$ and $x_{1,i}y_{i,1}$, as we removed the $c_{1,1}$. Multiplying all the factors, we obtain the unique monomials $\prod_{i=2}^n (x_{i,1}y_{1,i})^{p-1}$ and $\prod_{i=2}^n (x_{1,i}y_{i,1})^{p-1}$ that are not $\mathfrak{m}^{[p]}$. Hence, $k[X,Y]/I$ is F-pure by Fedder's criterion. □

Remark 8. We can generalize previous method for i, meaning we can remove $c_{i,i}$ and get the corresponding monomials $\prod_{j\neq i}^n (x_{j,i}y_{i,j})^{p-1}$ and $\prod_{j\neq i}^n (x_{i,j}y_{j,i})^{p-1}$

Corollary 2.3. *The ideal I is a radical ideal.*

Proof. Since R is F-pure, R is reduced. □

2.2 F-regularity

Now our target is to show that Statement 2 is true. We have developed two approaches to prove this. However, before showing them, we need more background.

The next theorem is key in this project, as it introduces notion of deformation.

Theorem 2.4. ***Deformation*** *[4], [2], (Corollary 4.7(c), Proposition 3.1.20). Let S be a polynomial ring of characteristic p, and suppose S is a complete intersection. Let x be a nonzero divisor in S. Then if $S/(x)$ is F-regular, then S is F-regular, in other words, F-regularity deforms for complete intersections.*

Since we want to use deformations, we also need information how elements can be eliminated appropriately. The next notion is useful in this aspect.

Definition 2.1. Let S be a graded ring of Krull dimension d with a homogeneous maximal ideal $\mathfrak{m}$. A sequence of elements $f_1, \ldots, f_d$ in S is called a *system of parameters* (s.o.p for short) if $\mathrm{Rad}(f_1, \ldots, f_d) = \mathfrak{m}$, equivalently, $S/(f_1, \ldots, f_d)$ has Krull dimension 0.

We can eliminate any element in a s.o.p. to deform a ring.

Theorem 2.5. *A system of parameters in R is given by*

1. *All the entries in X except for $\{x_{1,i}\}_{i=2}^{n}$,*
2. *All the entries in Y except for $\{y_{i,1}\}_{i=2}^{n}$,*
3. *Associated remaining entries, $\{x_{1,i} - y_{i,1}\}_{i=2}^{n}$.*

Remark 9. As in the previous theorem, we could pick not only elements with indices $1, i$ and $i, 1$, but also others.

Proof. We start with determining the number of elements in the s.o.p. By complete intersection property, for $n \times n$ matrices, we have $\dim R = 2n^2 - n + 1$, since $\dim k[X, Y] = 2n^2$ and I is generated by $n - 1$ elements, as the $tr(C) = 0$. Thus, our s.o.p must have $2n^2 - n + 1$ elements.

We add the cardinalities of the sets defined in the hypothesis respectively: $(n^2 - n + 1) + (n^2 - n + 1) + (n - 1) = 2n^2 - n + 1$. Now we check the dimension of the respective quotient ring:

$$\frac{R}{(s.o.p)} \equiv \frac{k[x_{1,2}, ..., x_{1,n}]}{(x_{1,2}^2, ..., x_{1,n}^2)}.$$

The last ring has dim $= 0$, as if P is a prime ideal containing $(x_{1,2}^2, ..., x_{1,n}^2)$, it must contain its radical, thus $(x_{1,2}, ..., x_{1,n}) \subset P$, but $(x_{1,2}, ..., x_{1,n})$ is maximal ideal $\mathfrak{m}$, hence $P = (x_{1,2}, ..., x_{1,n})$.

□

For the second approach, we need more definitions.

Definition 2.2. A *test element* is an element $c \in S^0$ that works in all tight closure tests, i.e. $cJ^* \subset J$ for all ideals $J \subset S$.

Remark 10. If f is a test element for S, then S_f is F-regular, [3].

Definition 2.3. The *Jacobian matrix* $\mathfrak{J}$ of an ideal $L = (\{l_i\}_{i=1}^{k})$ of a ring $S = K[x_1, x_2, ..., x_N]$, where K is a field, is defined in the following way: $\mathfrak{J}_{i,j} := \frac{\partial l_i}{\partial x_j}$. A *Jacobian ideal* J is an ideal generated by minors of size $(N - d)$ of $\mathfrak{J}$, where $d = \dim S/L$.

Remark 11. In our setup, minors are of maximal size, since $d = 2n^2 - n + 1$, $N = 2n^2$, and $k = n - 1$.

The next theorem is crucial to our second approach. In addition, we will also make use of the Corollary 2.3.

Theorem 2.6. ***Glassbrenner's criterion*** *[8] (Theorem 3.1). Let $S = K[X_1, ..., X_d]$ be a polynomial ring over an algebraically closed field K of prime characteristic p. Let J be a homogeneous radical ideal in S.*
Let s be a homogeneous element of S not in any minimal prime ideals of J for which $(S/J)_s$ is strongly F-regular. Let $\mathfrak{m}$ be $(X_1, ..., X_d)$. The following are equivalent

1. *S/J is F-regular.*
2. *J is prime and $J = \cap_{e\geq 1}(\mathfrak{m}^{[p^e]} : (J^{[p^e]} : J))$.*
3. *There exists a positive integer e such that $s(J^{[p^e]} : J) \not\subset \mathfrak{m}^{[p^e]}$.*

Theorem 2.7. ***F-regularity****. Let k, X, Y, C, I, R be as defined in the notations. Then R is F-regular.*

Proof. **Approach 1**
We use deformation of F-regularity for complete intersection rings stated in Theorem 2.4. First, we have to eliminate variables appropriately in X and Y. We define a new set of matrices for $i \in \{2, ...n\}$

$$Z_i = \begin{bmatrix} x_{i,1} & x_{1,i} \\ y_{i,1} & y_{1,i} \end{bmatrix}$$

We eliminate variables of X and Y not in any of entries of Z_i's (looking at the s.o.p), and denote the set of them V and obtain:

$$\frac{R}{(V)} \simeq \frac{k[\{Z_i\}_{i=2}^n]}{(\{\det Z_i\}_{i=2}^n)} \simeq \bigotimes_{i=2}^{n} \frac{k[Z_i]}{(\det Z_i)},$$

where $k[Z_i]/(\det Z_i)$ is a determinantal ring, known to be F-regular, [9], Theorem 7.14. It is sufficient to show that the tensor product of F-regular rings is F-regular. This is indeed the case, by Theorem 7.45 in [4]. Thus, R is F-regular.

Approach 2
First, note that our setup satisfies assumptions of the criterion due to Corollary 2.3. For Glassbrenner's criterion, we have to find s that turns out to be a test element. A test element can be found from the Jacobian matrix of

I and computing the maximal minors ($\neq 0$) of the Jacobian matrix.

$$J(I) = \begin{bmatrix} x_{2,1} & 0 & \cdots & 0 & \cdots \\ 0 & x_{3,1} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & \cdots & x_{n,1} & \cdots \end{bmatrix}$$

From this, the candidates are $\prod_{i=2}^n x_{1,i}$, $\prod_{i=2}^n x_{i,1}$, $\prod_{i=2}^n y_{1,i}$, and $\prod_{i=2}^n y_{i,1}$. For simplicity, we chose the first one as our test element, and denote it s. We must verify that s is not in any minimal prime ideals of I.

We show for case $n = 3$. Then $s = x_{1,2}x_{1,3}$, $c_{2,2} = x_{1,2}F_1 + F_2 + A$, and $c_{3,3} = x_{1,3}G_1 + G_2 + A$, where $A = x_{2,3}y_{3,2} - x_{3,2}y_{2,3}$. We have to show that $c_{2,2}, c_{3,3}, s$ is a regular sequence. It is sufficient to show that any of the factors of s is not a zero divisor on $(c_{2,2}, c_{3,3})$. Without loss of generality, let this factor be $x_{1,2}$.

If $f, g, A \in R$ is a regular sequence on a polynomial ring R, then $f + A, g + A, A$ is a regular sequence on R as well. To see this, it is sufficient to show that $f + A, g, A$ is a regular sequence. Otherwise,

$$F(f + A) = Gg + HA \Rightarrow Ff = Gg + (H - F)A \Rightarrow F \in (g, A).$$

In our setup, it is sufficient to show that $c_{2,2}$ is not a zero divisor in $R/(c_{3,3}, x_{1,2})$, i.e. $F_2 + A, x_{1,3}G_1 + G_2 + A$ and $x_{1,2}$ form a regular sequence. We know that $F_2 + A$ and $x_{1,3}G_1 + G_2 + A$ form a regular sequence from previous observation, and adjoining element that is not in any of them preserves this property. Thus, $c_{2,2}, c_{3,3}, x_{1,2}$ is a regular sequence. The same strategy is applied on $x_{1,3}$, and we obtain the desired result: s is not zero-divisor on I.

We can generalise this result for the rest of n's. The above observation is applied to $(n - 1) + (n - 1)(n - 2)/2 = n(n - 1)/2$ distinct elements.

Now we fulfilled the assumptions of Glassbrenner's criterion, we try to prove item 3): $s(I^{[p^e]} : I) \not\subset m^{[p^e]}$. Take $e = 1$ and as in previous proof, I is complete intersection, which implies $I^{[p]} : I = I^{[p]} + (\prod_{i=2}^n c_{i,i}^{p-1})$. (For simplicity, let us call this product expansion Π). Thus, it is sufficient to show that $s\Pi \notin \mathfrak{m}^{[p]}$. As in previous theorem, there is a unique term $\prod_{i=2}^n (x_{i,1}y_{1,i})^{p-1}$. Clearly, $\prod_{i=2}^n (x_{i,1}y_{1,i})^{p-1}x_{1,i} \notin \mathfrak{m}^{[p]}$. Thus, Glassbrenner's criterion is fulfilled, and R is F-regular.

□

Corollary 2.8. *I is a prime ideal.*

Proof. We apply Glassbrenner's criterion, in particular, the equivalency of items 1 and 2. Since R is F-regular, then I must be prime. □

3 Conclusion

We propose some problems that we encountered for further research.

Problem 3.1. Prove that I is prime in 0 and mixed characteristics.

Problem 3.2. Prove F-regularity of $k[X,Y]/J$, where the ideal J is generated by the entries of the commutator's antidiagonal.

Is it possible to connect to determinental rings?
In general, a diagonal matrix is the product of antidiagonal matrices. Is there any relative correspondence in F-regularities?
Tests on [6] show that J is a complete intersection as well.

Problem 3.3. What can we say about the ideal generated by off-antidiagonal entries of commutator matrix?

The commutator matrices with only diagonal entries have been studied already [10]. No work so far has been done in this direction.

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