Gravitational Collapse of Inhomogenous Perfect Fluid

Seema Satin,[1,2,*] Daniele Malafarina,[3,4,†] and Pankaj S. Joshi[5,‡]

[1]*Department of Mathematics, University of Pune, Ganeshkhind, 411007 Pune, India*
[2]*Department of Physics, Tamkang University (TKU), New Taipei City, 25137 Taiwan (ROC)*
[3]*Center for Field Theory and Particle Physics & Department of Physics, Fudan University, 220 Handan Road, 200433 Shanghai, China*
[4]*Department of Physics, Nazarbayev University, 53 Kabanbay Batyr avenue, 010000 Astana, Kazakhstan*
[5]*Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400005, India*

We study the complete gravitational collapse of a class of inhomogeneous perfect fluid models obtained by introducing small radial perturbations in an otherwise homogeneous matter cloud. The key feature that we assume for the perturbation profile is that of a mass profile that is separable in radial and temporal coordinates. The known models of dust and homogeneous perfect fluid collapse can be obtained from this choice of the mass profile as special cases. This choice is very general and physically well motivated and we show that this class of collapse models can lead to the formation of a naked singularity as the final state.

PACS numbers: 04.20.Dw,04.20.Jb,04.70.Bw
Keywords: Gravitational collapse, black holes, naked singularities

I. INTRODUCTION

One of the issues that is central to black hole physics today is the realistic description of the dynamical formation of black holes. The basic paradigm for black hole formation from gravitational collapse is based on the Oppenheimer-Snyder-Datt (OSD) [1] model which describes the complete collapse of a spherical cloud of non interacting (i.e. dust) particles with an homogeneous density profile. In the OSD model the central shell-focusing singularity that forms at the end of collapse is always hidden behind a trapping horizon thus showing that a Schwarzschild black hole is indeed the final state of collapse. Nevertheless the OSD model is too simplistic, as it assumes pressureless matter and homogeneity. Therefore it is legitimate to wonder how general is the result obtained for homogeneous dust. This question was at the heart of the original formulation of the Cosmic Censorship Conjeture [2], which stated that every natural collapse process must lead to the formation of a singularity that is covered by a horizon at all times, and has been a very fruitful area of research for many years.

Over the past decades there has been substantial analysis of the properties of spherically symmetric dust collapse both in the homogeneous (OSD) case as well as in the inhomogeneous case (the Lemaître-Tolman-Bondi (LTB) collapse model) [3]. These studies show that the behaviour of the free functions describing the mass and velocity distributions of the particles in the cloud are the crucial elements that determine the final outcome of collapse (see for example [4]). Several studies aimed at clarifying the role of pressures in collapse models have also been done [5], in order to determine the end states, resulting in a covered singularity or otherwise (see [6] and references therein). The key result that emerges from the many models studied over the years is that there are essentially two possible scenarios to describe relativistic gravitational collapse:

1. In the black hole case the horizon forms at an outer radius before the formation of the singularity thus leaving it hidden from far away observers.

2. In the naked singularity case the singularity and the horizon form at the center at the same time thus leaving the possibility for geodesics to escape the singularity and reach far away observers.

The naked singularity in these models must be understood as a region where the fluid model and possibly the relativistic description break down, therefore they just underline the limits of the classical theory and offer a possible window onto new phenomena.

Nevertheless today is still not clear how different kind of pressures affect the final outcome of collapse. Also how realistic and generic are the known solutions that present naked singularities is still a matter of debate. One very fruitful approach is based on the study of small perturbations in the initial data of known collapse models. The key issue is to understand and work out the stability and genericity aspects of black holes or naked singularities as they may occur in different models, by varying the initial

*Electronic address: sesatin@mail.tku.edu.tw
†Electronic address: daniele.malafarina@nu.edu.kz
‡Electronic address: psj@tifr.res.in

parameters and observing the effect of the same on the final outcome of collapse. For example starting with the homogeneous OSD model and incorporating small inhomogeneities in the initial density one can see that the space of final outcomes is equally divided between black holes and naked singularities [7]. It is seen that the introduction of small pressure perturbations to dust models, can drastically change the outcome of the collapse from covered to naked singularity or vice versa and that both the black hole and the naked singularity final outcomes are in some sense stable in these cases, with the exception of some special cases. From this perspective, once suitable definitions of genericity and stability are provided in this context, one can see that naked singularities must be considered a stable and generic outcome of spherically symmetric collapse models just as much as black holes [8].

We consider here endless gravitational collapse of a massive star, under the influence of its own gravitational field. The collapse initiates from regular initial conditions, where there is no trapping of light and no singularity. As matter source we consider a perfect fluid where small inhomogeneities are introduced in the initial pressure profile and where no equation of state is required. Still the fluid source is required to satisfy standard energy conditions. The end states resulting from such a collapse of a massive cloud, has been an active area of research, aiming to give precise mathematical formulation for the Cosmic Censorship Conjecture, which still remains to be established.

The paper is organized as follows: In section II we review the general setting for relativistic collapse in spherical symmetry. In section III we study the solution for the system of Einstein's equations under certain physically reasonable assumptions for the fluid source. Finally in section IV we briefly discuss the implications of these result for possible astrophysical observations. Throughout the paper we make use of natural units where $c = G = 1$ and we absorb the factor 8π appearing in Einstein's equations into the definition of energy density and pressure.

II. THE EINSTEIN'S EQUATIONS FOR SPHERICALLY COLLAPSING CLOUD

The most general spherically symmetric metric describing a collapsing cloud in co-moving coordinates r and t is given by

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R(t,r)^2 d\Omega \ . \tag{1}$$

The energy momentum tensor is

$$T_t^t = -\rho \ , \ T_r^r = p_r \ , \ T_\theta^\theta = T_\phi^\phi = p_\theta \ ,$$

where ρ is the energy density and p_θ and p_r are radial and tangential pressures. In the case of a perfect fluid source we have $p_\theta = p_r = p$ and Einstein's equations in this case take the form

$$p = -\frac{\dot{F}}{R^2\dot{R}} \ , \tag{2}$$

$$\rho = \frac{F'}{R^2R'} \ , \tag{3}$$

$$\nu' = -\frac{p'}{\rho+p} \ , \tag{4}$$

$$2\dot{R}' = R'\frac{\dot{G}}{G} + \dot{R}\frac{H'}{H} \ , \tag{5}$$

$$F = R(1-G+H) \ , \tag{6}$$

where H and G are given by $H = e^{-2\nu}\dot{R}^2$, $G = e^{-2\psi}R'^2$ and F is the Misner-Sharp mass of the system [9].

There are thus, five equations in the six unknowns ρ, p, F, ν, ψ and v, meaning that one of the functions is free to choose. However the system becomes closed after prescription of an equation of state for the matter content that relates density and pressure. In the following analysis we proceed without assuming any equation of state. This is due to the fact that at present we do not know the properties of matter under strong gravitational fields even though we know that the classical description of fluids, as given by barotropic or polytropic equations of state, will not remain unchanged towards the formation of the singularity. Therefore it makes sense to consider matter fields that have a physically reasonable behaviour in the weak field while not being subject to the constraint of a fixed equation of state.

There are several assumptions that need to me made in order for the system to be physically viable. We shall discuss them briefly here (refer to [6] for a more detailed treatment). We shall consider an initial time $t_i = 0$ for which the matter functions ρ and p are regular and present no cusps at the center. This implies that the Misner-Sharp mass takes the form $F(r,t) = r^3M(r,t)$ with M a suitably regular function. Furthermore we shall consider only collapse scenarios where the weak energy conditions, given by $\rho \geq 0$ and $\rho + p \geq 0$, are satisfied throughout the whole evolution. The system has a scaling degree of freedom that can be fixed by choosing the area-radius function R at the initial time. We shall set $R(r,t_i)=r$ thus introducing an adimensional scale factor $v(r,t)$ that describes the rate of collapse for which

$$R(r,t)=rv(r,t)\ . \tag{7}$$

For a homogeneous perfect fluid we have $p\equiv p(t)$ and $\rho\equiv\rho(t)$. This implies a Misner Sharp mass of the form $F(r,t)=r^3M(t)$ (fulfilling all the regularity conditions), thus making M depend only on $'t'$ and $v\equiv v(t)$. Using the definition of G and H we can rewrite equation (6) as

$$\dot{G}=2\frac{\nu'}{R'}r\dot{v}G\ , \tag{8}$$

which can be integrated once we define a function $A(r,t)$ such that

$$\dot{A}:=\frac{\nu'}{R'}r\dot{v}\ . \tag{9}$$

Then we get

$$G=b(r)e^{2A}\ , \tag{10}$$

where b is a free function coming from the integration and related to the initial velocity of the infalling particles, usually referred to as the velocity profile. Then regularity requires that near the center b has the form

$$b(r)=1+r^2b_0(r)\ . \tag{11}$$

In the following we will consider the marginally bound case by imposing $b_0(r)=0$. Matching with a Schwarzschild or generalized Vaidya exterior can always be performed at the co-moving boundary radius r_b which corresponds to a shrinking area-radius $R_b(t)=R(r_b,t)$, while matching to a Vaidya solution can be done if one considers an evolving boundary $r_b(t)$ [10].

III. INTRODUCING INHOMOGENEITIES

We shall now proceed to integrate the system of Einstein's equations for some inhomogeneous perfect fluid models. Given the absence of an equation of state we shall consider the mass profile M as the free function of the system. Then we can consider inhomogeneities in the density and pressure profiles by introducing radial inhomogeneities in the mass profile $M(t)\to M(t,r)$. This in turn leads to $v(t)\to v(r,t)$. Note that given the monotonic behaviour of v close to the center and close to the formation of the singularity we can consider a change of coordinates from the co-moving frame $\{r,t\}$ to the area-radius frame given by $\{r,v\}$ and consider v as a new coordinate to label time. This implies that for any function $X(r,t)$ we can consider $X(r,v)=X(r,t(r,v))$ so that the radial derivatives in the old coordinates become $X'=X_{,r}+X_{,v}v'$, where $X_{,r}$ is the radial derivative in the new coordinates and v' has now to be understood as a function of r and v.

Introducing inhomogeneities in the pressure profile then would result in p and ρ having a radial dependence given by

$$p(v,r)\ =\ p_0(v)+p_1(v)r+\frac{1}{2}p_2(v)r^2\ , \tag{12}$$

$$\rho(v,r)\ =\ \rho_0(v)+\rho_1(v)r+\frac{1}{2}\rho_2(v)r^2\ , \tag{13}$$

where the forms of $p_i(v)$ and $\rho_i(v)$ depend upon the specific choice of the mass function M. For simplicity we shall now choose the Misner-Sharp mass F in such a way that M be separable in r and v so that has it has the following form

$$M(r,v)=m(v)[1+\epsilon(r)]\ , \tag{14}$$

where $\epsilon(r)$ can be seen as the radial perturbation to the mass profile and will be assumed to be 'small' with respect to m, in order to study the departure of the system from the well known homogeneous solutions. We also assume that M given above be at least $\mathcal{C}^2$ in r and at least $\mathcal{C}^1$ in v. We see then that in order for the regularity conditions to be satisfied, the following is required: Expanding M in powers of r, near $r=0$, and choosing the mass function up to second order in r we must have

$$M(r,v)=M_0(v)+M_1(v)r+\frac{1}{2}M_2(v)r^2\ . \tag{15}$$

Also, since we do not want the initial density and pressure to have cusps at the origin, regularity of the initial data requires that M_1 vanishes at $r=0$. Thus $M_1(v)=m(v)\epsilon'(0)=0$, which gives $\epsilon'(0)=0$. We also assume $\epsilon(0)=0$ for consistency, so that at the center the cloud the system behaves like the homogeneous case described by $m(v)$. This is only a further gauge fixing that does not restrict the generality of the model. Finally we have to require $|M_2| << M_0$, from which we obtain $\epsilon''(0) << 1$.

Given the continuity required for M we see that the form of $m(v)$ can be taken as

$$m(v) = m_0 + m_1 v \ . \tag{16}$$

Given the above definitions once we expand pressure and density near $r = 0$ we obtain

$$p(v, r) = -\frac{m(v)_{,v}}{v^2} - \frac{1}{2}\frac{m(v)_{,v}}{v^2}\epsilon''(0)r^2 \ , \tag{17}$$

$$\rho(v, r) = \frac{3m(v)}{v^3} + \frac{1}{2}\frac{(5m(v)\epsilon''(0))}{v^2}r^2 \ . \tag{18}$$

Since we assume, for a realistic matter model, that the density decreases away from the center, it is clear that we must then requite $\epsilon''(0) < 0$.

By simplifying equation (6), using the form of G given by equation (10) we can obtain the equation of motion for the system as

$$\dot{v} = -e^{\nu}\sqrt{\frac{M}{v} + \frac{be^{2A} - 1}{r^2}} \ , \tag{19}$$

which, once solved for a given choice of the free functions M and b, allows to solve the system of Einstein's equations completely. As said before, in the following we assume the marginally bound velocity profile given by $b(r) = 1$. To evaluate the solution of the above equation we need the explicit expressions for ν and A in terms of the only free function left. From equations (5) and (9) we get

$$\nu(r, v) = \int_0^r \frac{M_{,vr}\, v + (M_{,vv}\, v - 2M_{,v})v'}{(3M + rM_{,r} - M_{,v}\, v)v} R' d\tilde{r} \ , \tag{20}$$

$$A(v, r) = \int_v^1 \frac{M_{,vr}\, v + (M_{,vv}\, v - 2M, v)v'}{(3M + rM_{,r} - vM_{,v})v} r dv \ . \tag{21}$$

Given the expansion for M we obtain the corresponding expansion for $A(v, r)$ near $r \approx 0$ as $A(v, r) \approx A_0(v) + A_1(v)r + A_2(v)r^2 + A_3(v)r^3 + A_4(v)r^4 + ...$ from which we see that using the form of the mass profile given by equation (15) implies $A_0 = A_1 = A_3 = 0$ and

$$A_2(v) = \int_v^1 \frac{2M_{2,v}}{(3M_0 - M_{0,v})} dv = \frac{2}{3}\frac{m_1\epsilon''(0)}{m_0}(1 - v) \ . \tag{22}$$

Then we can invert equation (19) to obtain $t(r, v)$ as

$$t(r, v) = t_i + \int_v^1 \frac{e^{-\nu}\sqrt{v}}{\sqrt{M + 2A_2 v + 2r^2 A_4 v}} dv \ . \tag{23}$$

Given the regularity of the functions involved the solution $t(r, v)$ is in general at least $\mathcal{C}^2$ near the singularity and therefore can be expanded as

$$t(v, r) = t(0, v) + \chi_1(v)r + \chi_2(v)r^2 + o(r^3) \ , \tag{24}$$

where $\chi_1 = \frac{dt}{dr}|_{r=0}$ and $\chi_2 = \frac{1}{2}\frac{d^2t}{dr^2}|_{r=0}$. The singularity curve $t_s(r)$, representing the time at which the shell labelled by r becomes singular, can be written as

$$t_s(r) = t(r, 0) = t_i + \int_0^1 \frac{e^{-\nu}\sqrt{v}}{\sqrt{M + 2A_2 v + 2r^2 A_4 v}} dv \ , \tag{25}$$

and according to equation (24) it can be expanded as

$$t_s(r) = t_0 + r\chi_1(0) + r^2\chi_2(0) + o(r^3) \ . \tag{26}$$

To obtain the expressions for χ_1 and χ_2 we must derive $t(r, v)$ with respect to r. Then from

$$\frac{dt}{dr} = \int_v^1 \left[\frac{\nu' e^{-\nu}\sqrt{v}}{\sqrt{M + 2A_2 v + 2r^2 A_4 v}} - \frac{1}{2}\frac{e^{-\nu}(M_{,r} + 4rA_4 v)\sqrt{v}}{(M + 2A_2 v + 2r^2 A_4 v)^{3/2}}\right] dv \ , \tag{27}$$

evaluated at $r = 0$ we obtain the following expression for χ_1

$$\chi_1(v) = -\frac{1}{2}\int_v^1 \frac{M_1\sqrt{v}}{(M_0 + 2A_2 v)^{3/2}}dv \ . \tag{28}$$

For simplicity we shall assume that the fourth order term of the expansion of $A(v, r)$ be negligible and thus we will take $A_4 = 0$. Then similarly we obtain the expression for χ_2 as

$$\chi_2(v) = \int_v^1 \left[\frac{3}{8}\frac{M_1^2}{(M_0 + 2A_2 v)^{5/2}} - \frac{\nu''}{\sqrt{M_0 + 2A_2 v}} - \frac{1}{2}\frac{M_2 + 2A_2^2 v}{(M_0 + 2A_2 v)^{3/2}}\right]\sqrt{v}dv \ , \tag{29}$$

Since $M_1 = 0$ we finally get $\chi_1(0) = 0$. Therefore the first non vanishing coefficient in the expansion of the singularity curve is the second order term $\chi_2(0)$. Then χ_2 is given by equation (29) evaluated for $v = 0$ and gives

$$\chi_2(0) = -\frac{1}{2}\int_0^1 \frac{2\nu''(M_0 + 2A_2 v) + M_2 + A_2^2 v}{(M_0 + 2A_2 v)^{3/2}}\sqrt{v}dv \ . \tag{30}$$

As it was shown in [12] it is the value of $\chi_2(0)$ that determines the nature of the singularity and its local visibility. Positivity of $\chi_2(0)$ implies that the singularity curve is increasing in the co-moving time t and thus the singularity forms at first at the shell $r = 0$. Positivity of $\chi_2(0)$ is also the necessary and sufficient condition for the apparent horizon to be increasing in t and it is possible to show that this is a necessary and sufficient condition also for null geodesics to escape the central singularity that forms at $t_s(0)$. Now by using the expression for M_0 and M_2 in terms of m and ϵ as given by $M_0 = m(v) = (m_0 + m_1 v)$ and $M_2 = m(v)\epsilon''(0)$ we can obtain the expressions for ν'' and A_2, up to first order in m_1/m_0, from equations (20) and (21) as

$$\nu'' = \frac{1}{3}\frac{m_1}{m_0}\epsilon''(0)v \ , \tag{31}$$

$$A_2 = \frac{2}{3}\frac{m_1}{m_0}\epsilon''(0)(1 - v) \ , \tag{32}$$

from which we get

$$\chi_2(0) = \int_0^1 \frac{-\frac{1}{3}\frac{m_1}{m_0}\epsilon''(0)v^{3/2}[m_0(1 + \frac{m_1}{m_0}v) + \frac{4}{3}\frac{m_1}{m_0}\epsilon''(0)v(1 - v)] - \frac{1}{2}[m_0(1 + \frac{m_1}{m_0}v)\epsilon''(0)]v^{1/2}}{m_0^{3/2}[1 + \frac{m_1}{m_0}v + \frac{4}{3}\frac{m_1}{m_0}\frac{\epsilon''(0)}{m_0}v(1 - v)]^{3/2}}dv \ . \tag{33}$$

Keeping terms up to order m_1/m_0 and neglecting higher order in the same, the expression reads

$$\chi_2(0) = -\int_0^1 \frac{\epsilon''(0)}{m_0^{1/2}}\left[\frac{v^{1/2}}{2} + \frac{m_1}{m_0}\left(\frac{7}{12}v^{3/2} - \frac{\epsilon''(0)}{m_0}(v^{3/2} - v^{5/2})\right)\right]dv \ , \tag{34}$$

which, after performing the integration gives

$$\chi_2(0) = -\frac{\epsilon''(0)}{m_0^{1/2}}\left(\frac{1}{3} + \frac{m_1}{m_0}\left(\frac{7}{30} - \frac{4}{35}\frac{\epsilon''(0)}{m_0}\right)\right) \ . \tag{35}$$

Since we expect $|\epsilon''(0)|/m_0 < 1$ we can ignore the last term in the above due to smallness of the multiplying factor and write

$$\chi_2(0) = -\frac{\epsilon''(0)}{3m_0^{1/2}}\left(1 + \frac{7}{10}\frac{m_1}{m_0}\right) \ . \tag{36}$$

As we have discussed earlier, physically reasonable density profiles require that $\epsilon''(0) < 0$, then the sign of χ_2 is decided by the quantity in brackets. It is clear that for small departures from the homogeneous perfect fluid model we must have $|m_1/m_0| < 1$, regardless of the sign of m_1 in the above. It follows that the quantity in brackets is always positive, which in turn implies that $\chi_2(0)$ is always positive. We then conclude that for scenarios described by small deviations from the homogeneous perfect fluid model as described above collapse results in the formation of a locally naked singularity.

By looking at the final expression (36) we can see how the above model can be related to the well known homogeneous perfect fluid and dust models.

1. We can retrieve the OSD collapse model in the case when m_1 and $\epsilon''(0)$ (and all the higher derivatives of $\epsilon(r)$ at $r = 0$) vanish. In this case the mass function reduces to $M = m_0$ which implies that $\chi_2(0) = 0$ and we obtain a simultaneous singularity resulting in a black hole final state.

2. In the same manner we can obtain the homogeneous perfect fluid model by imposing that $\epsilon''(0)$ and all higher derivatives of the same at $r = 0$ are zero. In this case M becomes a function of t only, through $v(t)$ and again the singularity is simultaneous thus resulting in the formation of a black hole.

3. To obtain the inhomogeneous dust collapse described by the LTB model we must impose $m_1 = 0$. Then the mass profile becomes a function of r only given by $M = m_0(1 + \epsilon(r))$ and we get $\chi_2(0) = -\epsilon''(0)/(3\sqrt{m_0})$ which, for $\epsilon''(0) < 0$ leads to the formation of a naked singularity.

Thus from the above framework we recover the widely studied cases of spherical collapse that lead to the formation of a black hole, namely homogeneous dust and homogeneous perfect fluid collapse, as well as the inhomogeneous dust model leading to a naked singularity. The important result of the above analysis is that we immediately see how the addition of a small pressure perturbation to a known collapse model can change the outcome of collapse from black hole to a locally naked singularity.

IV. CONCLUSION

In the present paper we have investigated how the presence of small pressure perturbations in the known collapse scenarios of dust and homogeneous perfect fluid affect the final outcome of collapse. We have chosen a very general class of physically valid mass profiles given by a separable mass function. By using such a form for M it is easy to see how the introduction of inhomogeneities in the dust and homogenous perfect fluid model changes the behaviour of collapse of a spherically symmetric cloud. Our calculations show that inhomogeneous perfect fluids can collapse to a naked singularity and that the homogeneous case is somehow ‘special’ in the fact that it leads to a simultaneous singularity, a result that agrees with what was previously found in [7].

This analisys suggests that towards the final stages of collapse a star could reach a stage with arbitrarily high densities at the center before the formation of the trapping horizon. This might have important astrophysical consequences as the regime where ‘quantum gravity’ effect become important might be causally connected with the outside universe thus changing drastically the classical picture for collapse (see for example [13] for some studies on how quantum effects might change the final stages of collapse). In recent years a lot of attention has been devoted to the theoretical study of observational features of naked singularities (see for example [14]) in the hope that future observations will be able to provide much needed experimental evidence on whether black holes are the only necessary outcome of collapse or if other possibilities, as allowed by theoretical models such as the ones discussed here, do occur in nature.

Our result support the idea that gravitational collapse to black holes as described in the OSD model might not be the most general paradigm to describe the final moments of the life of a star. Different density and pressure profiles contribute to the occurrence of naked singularities within theoretical models. Such singularities, that signal a breakdown of the classical relativistic description, might in turn indicate the existence of an observable window into the physics that dominates at small scales when the gravitational field is large.

[1] J.R Oppenheimer and H.Snyder, Phys. Rev. **56**, 455 (1939); S. Datt, Zs. f. Phys. **108**, 314 (1938).

[2] R. Penrose, Riv. Nuovo Cimento **1** 252 (1969).

[3] R. C. Tolman, Proc. Natl. Acad. Sci. USA, **20**, 410 (1934); H. Bondi, Mon. Not. Astron. Soc., **107**, 343 (1947); G. Lemaìtre, Ann. Soc. Sci. Bruxelles I, A **53**, 51 (1933).

[4] P. S. Joshi, N. Dadhich and R. Maartens, Phys. Rev. D **65**, 101501(R) (2002); P. S. Joshi and I. H. Dwivedi, Commun. Math. Phys. **146**, 333 (1992); P. S. Joshi and I. H. Dwivedi, Phys. Rev. D **47**, 5357 (1993); P. S. Joshi and I. H. Dwivedi, Commun. Math. Physics, **166**, 117 (1994); B. Waugh and K. Lake, Phys. Rev. D **38**, 1315 (1988); R. P. A. C. Newman, Class. Quantum Grav. **3**, 527 (1986); D. Christodoulou, Commun. Math. Phys. **93**, 171 (1984); D. M. Eardley and L. Smarr, Phys. Rev. D **19**, 2239 (1979).

[5] A. Ori and T. Piran, Phys. Rev. Lett. **59**, 2137 (1987); A. Ori and T. Piran, Phys. Rev. D **42**, 1068 (1990); G. Magli, Class. Quantum Grav. **14**, 1937 (1997); G. Magli, Class. Quantum Grav. **15**, 3215 (1998); T. Harada, Phys. Rev. D **58**, 104015 (1998); T. Harada, K. Nakao and H. Iguchi, Class. Quantum Grav. **16**, 2785 (1999); J. F. Villas da Rocha, Anzhong Wang, Class. Quantum Grav. **17**, 2589 (2000); T. Harada and H. Maeda, Phys. Rev. D **63**, 084022 (2001); P. S. Joshi and R. Goswami, Class. Quantum Grav. **19**, 5229 (2002): S. M. C. V. Goncalves, S. Jhingan, G. Magli, Phys.Rev. D **65**, 064011 (2002); R. Goswami and P. S. Joshi, Class. Quantum Grav. **19**, 5229 (2002); R. Giambò, F. Giannoni, G. Magli and P. Piccione, Comm. Math. Phys. **235**, 545 (2003); R. Giambò, F. Giannoni, G. Magli, P. Piccione, Gen. Rel. Grav. **36**, 1279 (2004); P. S. Joshi and R. Goswami, Class. Quantum Grav. **21**, 3645 (2004).

[6] P. S. Joshi and D. Malafarina, Int. J. Mod. Phys. D **20** (14), 2641 (2011).

[7] P. S. Joshi and D. Malafarina, Phys Rev D **83**, 024009 (2011); P. S. Joshi and D. Malafarina, Gen. Rel. Grav. **45** (2), 305 (2013).

[8] P. S. Joshi, D. Malafarina and R. V. Saraykar, Int. J. Mod. Phys. D **21** (8), 12500 (2012); P. S. Joshi and R. V. Saraykar, arXiv1207.3469 [gr-qc].

[9] Misner, C., Sharp, D.: Phys. Rev. 136, B571 (1964).

[10] W. Israel, Nuovo Cimento B **44**, 1 (1966); Nuovo Cimento B **48**, 463 (1966); F. Fayos, X. Jaen, E. Llanta and J. M. M. Senovilla, Phys. Rev. D **45**, 2732 (1992); F. Fayos, J. M. M. Senovilla and R. Torres, Phys. Rev. D **54**, 4862 (1996): F. Fayos, M. Mercè-Prats, J. M. M. Senovilla, Class. Quantum Grav. **12**, 2565 (1995); P. S. Joshi and I. H. Dwivedi, Class. Quant. Grav. **16**, 41 (1999).

[11] P. Yodzis, H.-J. Seifert and H. Muller zum Hagen, Commun. Math. Phys. **34**, 135 (1973); C. Hellaby and K. Lake, Astrophysical Journal, **290**, 381 (1985); C. Hellaby and K. Lake, Astrophysical Journal, **300**, 461 (1986); P. S. Joshi and R. V. Saraykar, Int. J. of Mod. Phys. D **22** (05), 1350027 (2013).

[12] P. S. Joshi and I.H. Dwivedi, Class. Quantum Grav. **16**, 41 (1999); R. Goswami and P. S. Joshi, Phys. Rev. D **69**, 104002 (2004).

[13] M. Bojowald, R. Goswami, R. Maartens and P. Singh, Phys. Rev. Lett. **95**, 091302 (2005); C. Barcelo, S. Liberati, S. Sonego and M. Visser, Phys. Rev. D **77**, 044032 (2008); R. Goswami, P. S. Joshi and P. Singh, Phys. Rev. Lett. **96**, 031302 (2006); C. Bambi, D. Malafarina and L. Modesto, Phys.Rev. D **88**, 044009 (2013); C. Bambi, D. Malafarina and L. Modesto, Eur. Phys. J. C **74**, 2767 (2014); R. Torres, Phys. Lett. B **733**, 21 (2014).

[14] P. S. Joshi, D. Malafarina and R. Narayan, Class. Quantum Grav. **28**, 235018 (2011); P. S. Joshi, D. Malafarina and R. Narayan, Class. Quantum Grav. **31**, 015002 (2014); C. Bambi and D. Malafarina, Phys. Rev. D **88**, 064022 (2013); Z. Li and C. Bambi, JCAP **1303**, 031 (2013); C. Bambi, Phys. Rev. D **87**, 023007 (2013); L. Kong, D. Malafarina and C. Bambi, Eur. Phys. J. C **74**, 2983 (2014).