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Capstone Project

Report

On the well-posedness of the Boltzmann's moment system of equations in fourth approximation

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Abstract

We study the one-dimensional non-linear non-stationary Boltzmann's moment system of equations in fourth approximation with the tools developed by Sakabekov in [4],[5] and [6]. For the third approximation system Sakabekov proves the mass conservation law (cf. Theorem 2.1 in [4]) and discusses the existence and uniqueness of the solution (cf. Theorem in [6]). We extend the analysis of the existence and uniqueness of the solution to the fourth approximation system. In particular, for the fourth approximation system we discuss the well-posed initial and boundary value problem and obtain the a-priori estimate of the solution belonging to the space of functions, continuous in time and square summable by spatial variable.

Keywords: Boltzmann equation, moment system, initial and boundary value problem, hyperbolic partial differential equations, a-priori estimate

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1 Introduction

The motion of gas particles can be described in terms of their position and velocity at any time t by the differential equations

$$\frac{d\xi_i}{dt} = X_i, \quad \frac{dx_i}{dt} = \xi_i,$$

where $x_i \in \mathbb{R}^3$ is the spatial vector of the i-th particle, $X_i \in \mathbb{R}^3$ the force per mass of a gas particle, and $\xi_i \in \mathbb{R}^3$ the velocity vector (cf. Chapter 1 of [1]). This model is however not of practical interest due to the reasons below:

1. It is necessary to know the initial values for all the particles to solve the differential equations, which in turn requires simultaneous measurement of initial values, in terms of position and velocity.

2. If one assumes that initial values can in principle be measured, a human life is insufficient time to measure all the initial values. If a human is supposed to measure the 3 components of a particle's position and velocity in one second, a life span of 65 years (or $\sim 10^9$ seconds) is not sufficient to measure initial values of 1 mole of particles (10^{23} particles and, hence, seconds).

3. The measurement of initial values is also not very accurate. One usually does not consider more than 100 decimal figures. Thus, truncation errors of order 10^{-100} have to be used. Even if one assumes that it is possible to be infinitely accurate, the measurement will still be inaccurate, as one does not take into account the influence of other particles in the universe.

4. Even if one assumes that it is possible to accurately measure the initial values, the obtained information will be useless. One usually cares about the macroscopic parameters, such as the pressure of a gas at a given temperature and given density, rather than the position and velocity of one particular particle of a gas.

5. In the end, it is impossible to imagine a computer that will solve such enormous number of differential equations.

The computational demands are far reduced when a gas is assumed to be rarefied. For a rarefied gas, consequently, one can assume that the time between collisions is much larger than the time during collisions. Under this assumption, the kinetics of a gas on a microscopic level and the evolution of a system that consists of a large number of particles moving in a three-dimensional space ($\mathbb{R}^3$) can be modelled by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \langle v, \nabla_x f \rangle = I(f, f), \tag{1}$$

where

- $f(t, x, v)$ is the particle distribution function, which represents the probability of finding a particle with velocity $v \in \mathbb{R}^3$ near a point $x \in \mathbb{R}^3$ at time t, and

- $I(f, f)$ is the collision integral, which represents the rate of change of $f(t, x, v)$ due to collisions between particles,

with $\langle v, \nabla_x f\rangle = v^T \nabla_x f = v_1 \dfrac{\partial f}{\partial x_1} + v_2 \dfrac{\partial f}{\partial x_2} + v_3 \dfrac{\partial f}{\partial x_3}$. The Boltzmann equation represents a balance between the rate of change of $f(t, x, v)$ due to convection (the left-hand side of (1)) and the rate of change of $f(t, x, v)$ due to collisions between particles (the right-hand side of (1)). The main difficulty of solving (1) lies in the distribution function, which is a function of seven independent variables: time t, three spatial coordinates x_1, x_2, x_3 and three velocity coordinates v_1, v_2, v_3.

In [5], the so-called Boltzmann's moment system was proposed to approximate (1) systematically. For one-dimensional case, i.e., the case where $x \in \mathbb{R}$, for $k \in \mathbb{N}_0$, the k-th moment approximation of (1) is given by the system of equations

$$\begin{aligned}
&\frac{\partial f_{nl}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left[l\left(\sqrt{\frac{2\left(n+l+\frac{1}{2}\right)}{(2l-1)\,(2l+1)}}f_{n,l-1}\right)\right] - \frac{1}{\alpha}\frac{\partial}{\partial x}\left[l\left(\sqrt{\frac{2(n+1)}{(2l-1)(2l+1)}}f_{n+1,l-1}\right)\right] \\
&+ \frac{1}{\alpha}\frac{\partial}{\partial x}\left[(l+1)\left(\sqrt{\frac{2\left(n+l+\frac{3}{2}\right)}{(2l+1)\,(2l+3)}}f_{n,l+1}\right)\right] - \frac{1}{\alpha}\frac{\partial}{\partial x}\left[(l+1)\left(\sqrt{\frac{2n}{(2l+1)\,(2l+3)}}f_{n-1,l+1}\right)\right] \quad = I_{nl},
\end{aligned} \tag{2}$$

where $n, l \in N_0$ such that $2n+l = 0, 1, \ldots, k$. We shall explain all variables involved in the system (2) in Section 2, where its derivation is briefly discussed. Boltzmann's moment system presents a particular interest to study, as it is an intermediate between kinetic and hydrodynamic levels of rarefied gas description.

As an example, we show the derivation of the Boltzmann's moment system in second approximation. Let $2n+l = 0, 1, 2$. Since $k = 2N = 2 \Rightarrow N = 1$, and we write the vectors $a_2 = (f_{01})^T$ and $b_2 = (f_{00}, f_{02}, f_{10})^T$ using the following specification for even approximations:

$$a_k = (f_{01}, ..., f_{0,2N-1}, f_{11}, ..., f_{1,2N-3}, f_{21}, ..., f_{2,2N-5}, ..., f_{N-1,1})^T$$

and

$$b_k = (f_{00}, ..., f_{0,2N}, f_{10}, ..., f_{1,2N-2}, f_{20}, ...f_{2,2N-4}, ..., f_{N0})^T.$$

We note that only the moments from the vecors a_2 and b_2 are legal to be in the second approximation. If $2n+l = 0 \Rightarrow n = l = 0$. Substitution of $n = l = 0$ into (2) yields

$$\frac{\partial f_{00}}{\partial t} + \frac{1}{\alpha}\frac{\partial f_{01}}{\partial x} = 0.$$

If $2n + l = 1 \Rightarrow n = 0, l = 1$. By substituting $n = 0, l = 1$ into (2) we get the equation:

$$\frac{\partial f_{01}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(f_{00} - \sqrt{\frac{2}{3}}f_{10} + \frac{2}{\sqrt{3}}f_{02}\right) = 0.$$

If $2n + l = 2 \Rightarrow n = 0, l = 2$ or $n = 1, l = 0$. Substitution of $n = 0, l = 2$ into (2) yields

$$\frac{\partial f_{02}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(\frac{2}{\sqrt{3}}f_{01}\right) = I_{02}.$$

By substituting $n = 1, l = 0$ into (2) we get the equation:

$$\frac{\partial f_{10}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(-\sqrt{\frac{2}{3}}f_{01}\right) = 0.$$

The moment system's third approximation ($k = 3$) is discussed and analysed in [4] and [6]. In particular, it has been shown the existence and the uniqueness of the solution for a short period of time (cf. Theorem 2.1 in [6]) and it has been proven the analogue of mass conservation law (cf. Theorem in [4]).

In this report, we extend the analysis of the existence and uniqueness of the solution for the moment system's fourth approximation. The main tools of the analysis are based on those used in [4], [5] and [6] for the third approximation. Our main results are the formulation of the well-posed initial and boundary value problem and the a-priori estimate of the solution belonging to a space of functions, continuous in time and square summable by spatial variable for the fourth approximation.

The report is organized as follows. In Section 2, we discuss briefly the derivation of the moment system. In Section 3, well-posedness of the associated hyperbolic system (without collisions) is discussed. A-priori estimate of the solution for the fourth approximation system is discussed in Section 4. Finally, we draw conclusions in Section 5.

2 Boltzmann's moment systems

In this section, we derive the 1D Boltzmann's moment systems. Our discussion is meant to be brief, and is based largely on [5].

As the starting point, we apply spatial coordinate transformation on (1) to the spherical coordinate. Under this transformation,

$$\langle v, \nabla_x f \rangle = |v| \left(\sin\theta \cos\phi \frac{\partial f}{\partial x_1} + \sin\theta \sin\phi \frac{\partial f}{\partial x_2} + \cos\theta \frac{\partial f}{\partial x_3} \right)$$

where θ and ϕ are respectively polar and azimuthal angles, which are given by the formulas: $\theta = \arccos\left(\frac{v_3}{|v|}\right), \phi = \arctan\left(\frac{v_2}{v_1}\right)$. By taking into consideration only the third coordinate x_3, the one-dimensional Boltzmann equation is obtained:

$$\frac{\partial f}{\partial t} + |v| \cos\theta \frac{\partial f}{\partial x} = I(f, f), \tag{3}$$

where we have dropped the subscript "3" from x_3 for notational simplicity. We note here that eigenfunctions of the linearised collision operator are orthogonal under the L^2 inner product with weight

$$f_0(\alpha|v|) = \left(\frac{\alpha^2}{2\pi} \right)^{\frac{3}{2}} \exp\left(\frac{-\alpha^2 |v|^2}{2} \right), \tag{4}$$

the so-called local Maxwell distribution, with $\alpha = \sqrt{\frac{1}{RT}}$ (R is the Boltzmann constant and T is the temperature), and have the following form [5]:

$$g_{nl}(\alpha|v|) = \gamma_{nl} \left[\frac{\alpha|v|}{\sqrt{2}} \right]^l S_n^{l+\frac{1}{2}} \left(\frac{\alpha^2 |v|^2}{2} \right) P_l(\cos\theta), \quad n, l \in N_0 \tag{5}$$

In (5), the coefficient γ_{nl} has the following form:

$$\gamma_{nl} = \left[\frac{\sqrt{\pi} n! (2l+1)}{2\Gamma(n + l + \frac{3}{2})} \right]^{\frac{1}{2}},$$

where $\Gamma(\cdot)$ is the Gamma function. Furthermore, the terms $S_n^{l+\frac{1}{2}}\left(\frac{\alpha^2|v|^2}{2}\right)$ and $P_l(\cos\theta)$ are generalized Laguerre polynomials and Legendre polynomials respectively.

By multiplying both sides of (3) by $f_0(\alpha|v|)g_{nl}(\alpha|v|)$ and integrating over R_3^v (velocity space), we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{R_3^v} f_0(\alpha|v|)g_{nl}(\alpha|v|)f(t,x,v)dv \quad &+ \quad \frac{\partial}{\partial x}\int_{R_3^v} |v|\cos(\theta) f_0\left(\alpha|v|\right)g_{nl}(\alpha|v|)f(t,x,v)dv \\ &= \quad \int_{R_3^v} f_0(\alpha|v|)g_{nl}(\alpha|v|)I(f,f)dv.\end{aligned}$$

Since the Fourier series for the distribution function $f(t,x,v)$ take the form (cf. [5])

$$f(t,x,v) = \sum_{2n+l=0}^{\infty} f_{nl}(t,x)g_{nl}(\alpha|v|),$$

we immediately get that

$$f_{nl}(t,x) = \int_{R_3^v} f_0(\alpha|v|)g_{nl}(\alpha|v|)f(t,x,v)dv$$

and

$$I_{nl}(t,x) = \int_{R_3^v} f_0(\alpha|v|)g_{nl}(\alpha|v|)I(f,f)dv.$$

The following identities for Legendre and, respectively, generalized Laguerre polynomials are useful:

$$\mu(2l+1)P_l(\mu) = (l+1)P_{l+1}(\mu) + lP_{l-1}(\mu), \tag{6}$$

and

$$\begin{aligned}\mu S_n^{\beta+1}(\mu) &= (n+\beta+1)S_n^{\beta}(\mu) - (n+1)S_{n+1}^{\beta}(\mu), \\ S_n^{\beta-1}(\mu) &= S_n^{\beta}(\mu) - S_{n-1}^{\beta}(\mu),\end{aligned} \tag{7}$$

with μ indicating the appropriate argument in the polynomials in (5). By using (6), (7), and $\Gamma(\mu+1) = \mu\Gamma(\mu)$, the so-called Boltzmann's moment system results:

$$\begin{aligned}&\frac{\partial f_{nl}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left[l\left(\sqrt{\frac{2\left(n+l+\frac{1}{2}\right)}{(2l-1)\,(2l+1)}}f_{n,l-1}\right)\right] - \frac{1}{\alpha}\frac{\partial}{\partial x}\left[l\left(\sqrt{\frac{2(n+1)}{(2l-1)(2l+1)}}f_{n+1,l-1}\right)\right] \\ &+ \frac{1}{\alpha}\frac{\partial}{\partial x}\left[(l+1)\left(\sqrt{\frac{2\left(n+l+\frac{3}{2}\right)}{(2l+1)\,(2l+3)}}f_{n,l+1}\right)\right] - \frac{1}{\alpha}\frac{\partial}{\partial x}\left[(l+1)\left(\sqrt{\frac{2n}{(2l+1)\,(2l+3)}}f_{n-1,l+1}\right)\right] \quad = I_{nl},\end{aligned}$$

where $2n+l = 0, 1, \ldots, k$ is referred to as *the quantum number*. For a fixed k, the system corresponds to the Boltzmann's moment equation in k-th approximation. In this form, the collision terms I_{nl} can be expressed as known quantities in terms of generalized Talmi and Klebsh-Gordon coefficients [3].

In this report, we focus on the fourth approximation (i.e., $2n+l = 0, 1, 2, 3, 4$), which is given by the system of equations:

$$
\begin{aligned}
\frac{\partial f_{01}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(f_{00} - \sqrt{\frac{2}{3}}f_{10} + \frac{2}{\sqrt{3}}f_{02}\right) &= 0, && (8)\\
\frac{\partial f_{03}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(\frac{3}{\sqrt{5}}f_{02} - \frac{3\sqrt{2}}{\sqrt{35}}f_{12} + \frac{4}{\sqrt{7}}f_{04}\right) &= I_{03}, && (9)\\
\frac{\partial f_{11}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(\sqrt{\frac{5}{3}}f_{10} - \frac{2}{\sqrt{3}}f_{20} + \frac{2\sqrt{7}}{\sqrt{15}}f_{12} - \frac{2\sqrt{2}}{\sqrt{15}}f_{02}\right) &= I_{11}, && (10)\\
\frac{\partial f_{00}}{\partial t} + \frac{\partial f_{01}}{\partial x} &= 0, && (11)\\
\frac{\partial f_{02}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(\frac{2}{\sqrt{3}}f_{01} - \frac{2\sqrt{2}}{\sqrt{15}}f_{11} + \frac{3}{\sqrt{5}}f_{03}\right) &= I_{02}, && (12)\\
\frac{\partial f_{04}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(\frac{4}{\sqrt{7}}f_{03}\right) &= I_{04}, && (13)\\
\frac{\partial f_{10}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(\sqrt{\frac{5}{3}}f_{11} - \sqrt{\frac{2}{3}}f_{01}\right) &= 0, && (14)\\
\frac{\partial f_{12}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(\frac{2\sqrt{7}}{\sqrt{15}}f_{11} - \frac{3\sqrt{2}}{\sqrt{35}}f_{03}\right) &= I_{12}, && (15)\\
\frac{\partial f_{20}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(\frac{-2}{\sqrt{3}}f_{11}\right) &= I_{20}. && (16)
\end{aligned}
$$

For the above system, the collision integrals are calculated and presented in [4].

3 Well-posedness of the fourth approximation

In this section, we consider the moment system of partial differential equations (8)–(16) with zero right-hand sides. First of all, we rewrite the system in a matrix-vector form as

$$
\frac{\partial U}{\partial t} + \frac{1}{\alpha}A\frac{\partial U}{\partial x} = I(U,U), \quad U, I(U,U) \in \mathbb{R}^9, \tag{17}
$$

where

$$
\begin{aligned}
U &= (f_{01},\, f_{03},\, f_{11},\, f_{00},\, f_{02},\, f_{04},\, f_{10},\, f_{12},\, f_{20})^T, && (18)\\
I(U,U) &= (0,\, I_{03},\, I_{11},\, 0,\, I_{02},\, I_{04},\, 0,\, I_{12},\, I_{20})^T, && (19)
\end{aligned}
$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & \frac{2}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{\sqrt{5}} & \frac{4}{\sqrt{7}} & 0 & -3\sqrt{\frac{2}{35}} & 0 \\ 0 & 0 & 0 & 0 & -2\sqrt{\frac{2}{15}} & 0 & \sqrt{\frac{5}{3}} & 2\sqrt{\frac{7}{15}} & \frac{-2}{\sqrt{3}} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{\sqrt{3}} & \frac{3}{\sqrt{5}} & -2\sqrt{\frac{2}{15}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{\sqrt{7}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{5}{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3\sqrt{\frac{2}{35}} & 2\sqrt{\frac{7}{15}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{9\times 9}. \tag{20}$$

By setting $I(U,U) = 0$, Equation (17) is reduced to a system of coupled, hyperbolic partial differential equations with constant coefficients A in the non-conservative form.

The symmetry of the coefficient matrix $A \in \mathbb{R}^{9\times 9}$ implies that A is orthogonally diagonalizable with real eigenvalues and a complete linearly independent set of eigenvectors. Thus, let

$$A = BDB^{-1}, \tag{21}$$

where $D \in \mathbb{R}^{9\times 9}$ is a diagonal matrix, whose entries are the eigenvalues of A, and B the matrix whose columns are eigenvectors of A. Substitution of the similarity transform (21) to (17) yields

$$\begin{aligned} \frac{\partial U}{\partial t} + \frac{1}{\alpha} A \frac{\partial U}{\partial x} = I(U,U) \quad & \Longrightarrow \quad \frac{\partial U}{\partial t} + \frac{1}{\alpha} BDB^{-1} \frac{\partial U}{\partial x} = I(U,U) \\ & \Longleftrightarrow \quad B\left(B^{-1}\frac{\partial U}{\partial t} + \frac{1}{\alpha} DB^{-1}\frac{\partial U}{\partial x}\right) = I(U,U) \\ & \Longleftrightarrow \quad B\left(\frac{\partial B^{-1}U}{\partial t} + \frac{1}{\alpha} D\frac{\partial B^{-1}U}{\partial x}\right) = I(U,U) \\ & \Longrightarrow \quad \left(\frac{\partial \Psi}{\partial t} + \frac{1}{\alpha} D\frac{\partial \Psi}{\partial x}\right) = B^{-1}I(U,U), \end{aligned} \tag{22}$$

where $\Psi = B^{-1}U$ with $I(U,U) = 0$. The system (22) corresponds to a system of hyperbolic differential equations in the conservative form. The system is decoupled, and each equation represents propagation of wave at a given wave speed.

Let A in (20) be partitioned into a 2×2 block matrix as follows:

$$A = \begin{bmatrix} 0 & A_{21}^T \\ A_{21} & 0 \end{bmatrix}, \tag{23}$$

with

$$A_{21} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{\sqrt{3}} & \frac{3}{\sqrt{5}} & -2\sqrt{\frac{2}{15}} \\ 0 & \frac{4}{\sqrt{7}} & 0 \\ -\sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{5}{3}} \\ 0 & -3\sqrt{\frac{2}{35}} & 2\sqrt{\frac{7}{15}} \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{bmatrix}. \tag{24}$$

The matrix A is similar to the so-called Golub-Kahan matrix, K, a symmetric tridiagonal matrix with zero diagonal entries, satisfying the similarity transform

$$PAP^T = K,$$

where P is a permutation matrix. Since a Golub-Kahan matrix has both $\pm\lambda$ as an eigenvalue pair, that A similar to K implies that $\pm\lambda$ are an eigenvalue pair of A.

Theorem 3.1 *Let A and correspondingly A_{21} be given as in* (23) *and* (24) *respectively. Then zero is an eigenvalue of A, with algebraic multiplicity 3.*

Proof For 0 to be an eigenvalue, we have to show that there exists a non-zero vector $w \in \mathbb{R}^9$ such that $Aw = \lambda w = 0$. Let vector w be partitioned into $w = [\tilde{w}^T \quad \hat{w}^T]^T$, where $\tilde{w} \in \mathbb{R}^3$ and $\hat{w} \in \mathbb{R}^6$. With (23) we have

$$\begin{aligned} A_{21}^T \hat{w} &= 0, \\ A_{21} \tilde{w} &= 0. \end{aligned}$$

Row 1, 3, and 4 indicate that A_{21} has three linearly independent columns. Hence, $A_{21}\tilde{w} = 0$ holds for $\tilde{w} = 0$. Since $\mathrm{rank} A_{21} = 3$, $\dim \mathcal{N}(A_{21}^T) = 3$. Thus, there exists $\hat{w} \neq 0$ such that $A_{21}^T \hat{w} = 0$ and $w = [\hat{w}^T \; 0 \; 0 \; 0]^T, \hat{w} \neq 0$ is an eigenvector associated with $\lambda = 0$. By the symmetry of A, $\lambda = 0$ is non-defective, and thus has algebraic multiplicity 3. $\square$

Theorem 3.2 *Let A and correspondingly A_{21} be given as in* (23) *and* (24)*. Then the set of eigenvalues of A is*

$$\left\{ -\sqrt{\frac{10+\sqrt{40}}{2}}, -\sqrt{\frac{10-\sqrt{40}}{2}}, -\sqrt{3}, 0, 0, 0, \sqrt{3}, \sqrt{\frac{10-\sqrt{40}}{2}}, \sqrt{\frac{10+\sqrt{40}}{2}} \right\}.$$

Proof By Theorem 3.1, 0 is in the set of eigenvalues of A with algebraic multiplicity 3. Next, recall that for a symmetric matrix, we have $\sigma_i(A) = |\lambda_i(A)| > 0$, where $\sigma_i(A)$ is the singular value of A, which is defined as the square-root of the non-zero eigenvalues of the Gram matrix $A^T A$. For (23),

$$A^T A = \begin{bmatrix} 0 & A_{21}^T \\ A_{21} & 0 \end{bmatrix}^T \begin{bmatrix} 0 & A_{21}^T \\ A_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_{21}^T \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} 0 & A_{21}^T \\ A_{21} & 0 \end{bmatrix} = \begin{bmatrix} A_{21}^T A_{21} & 0 \\ 0 & A_{21} A_{21}^T \end{bmatrix}.$$

Thus, $\{\lambda(A^TA)\} = \{\lambda(A_{21}^TA_{21})\} \cup \{\lambda(A_{21}A_{21}^T)\}$.The characteristic polynomial for the matrix $A_{21}^TA_{21}$ is given by:

$$(\lambda - 3)(-\lambda^2 + 10\lambda - 15) = 0.$$

Solving the equation for λ yields three eigenvalues:

$$\lambda_1 = \frac{10+\sqrt{40}}{2}, \lambda_2 = \frac{10+\sqrt{40}}{2}, \lambda_3 = 3.$$

Since $\pm\lambda$ is an eigenvalue pair for A, we have that matrix A has six non-zero eigenvalues, namely:

$$-\sqrt{\frac{10+\sqrt{40}}{2}}, -\sqrt{\frac{10-\sqrt{40}}{2}}, -\sqrt{3}, \sqrt{3}, \sqrt{\frac{10-\sqrt{40}}{2}}, \sqrt{\frac{10+\sqrt{40}}{2}}.$$

Hence, the set of eigenvalues of A is

$$\left\{-\sqrt{\frac{10+\sqrt{40}}{2}}, -\sqrt{\frac{10-\sqrt{40}}{2}}, -\sqrt{3}, 0, 0, 0, \sqrt{3}, \sqrt{\frac{10-\sqrt{40}}{2}}, \sqrt{\frac{10+\sqrt{40}}{2}}\right\}.$$

□

Theorem 3.3 *The moment system in* (22) *with* $I(U,U) = 0$*, defined for* $x \in \mathbb{R}$ *and* $t > 0$*, is well-posed.*

Proof (22) is an example of the Cauchy problem. By using Theorem 3.2, the symmetry of A, and Theorem 2.1.2 of [2], this problem is well-posed. □

Next, we consider the case where (22) with $I(U,U) = 0$ is solved in the finite domain $\Omega = (-a, a) \subset \mathbb{R}$, for $t > 0$. Since

$$\frac{\partial \Psi}{\partial t} + \frac{1}{\alpha} D \frac{\partial \Psi}{\partial x} = 0, \tag{25}$$

consists of 9 decoupled linear wave equations, well-posedness of the above initial- and boundary-value problem can be immediately derived from the well-posedness of each linear wave problem. The following theorem is well-known, proved in [7], and is in this regard useful.

Theorem 3.4 *The linear wave equation* $\phi_t - c\phi_u = 0$*,* $c > 0$*, with* $\phi(x,0) = \phi_0(x)$ *and* $\phi(0,t) = \phi_{BC}(t)$ *is well-posed.*

Let $\Psi = [\psi_i]_{i=1,\ldots,9} \in \mathbb{R}^9$ and rewrite (25) as

$$\frac{\partial \psi_i}{\partial t} + \frac{\lambda_i}{\alpha}\frac{\partial \psi_i}{\partial x} = 0, \quad i = 1, 2, \ldots, 9.$$

For each PDE, we set an initial condition $\psi_i(0,x) = \psi_i^0(x)$, $x \in \Omega$. Note that for $\lambda = 0$, the corresponding PDE reduces to an ordinary differential equation (in time) that requires only an initial condition.

Based on Theorem 3.4, a well-posed initial and boundary value problem for (25) can be constructed with the initial conditions:

$$\psi_i(0,x) = \psi_i^0(x), i = 1, 2, ..., 9. \tag{26}$$

Since there are three negative eigenvalues, the system has three families of characteristic curves with negative slope, i.e. three families of characteristic curves that leave the right end of the interval. Hence, three boundary conditions are set on the right end of the interval $(-a, a)$. Similarly, for equations corresponding to three positive eigenvalues, the boundary conditions are set on the left side of the interval $(-a, a)$.

In summary, we have the following initial and boundary value problem: for $i = 1, \dots, 9$,

$$\frac{\partial \psi_i}{\partial t} + \frac{\lambda_i}{\alpha}\frac{\partial \psi_i}{\partial x} = 0, \qquad \text{in } \Omega = (-a, a), \tag{27}$$

$$\psi_i(0, x) = \psi_i^0(x), \quad x \in \Omega \tag{28}$$

$$\begin{cases} \psi_i(t, a) = \psi_{i,BC}(t), & \text{if } \lambda_i < 0, \\ \psi_i(t, -a) = \psi_{i,BC}(t), & \text{if } \lambda_i > 0, \\ \text{none}, & \text{otherwise.} \end{cases} \tag{29}$$

4 A-priori estimate of the solution of the fourth approximation

In this section, we consider the fourth approximation (22) with nonzero $I(U, U)$. Due to the difficulty with more general boundary conditions, we focus only on periodic boundary conditions. Such boundary conditions allow us to obtain an a-priori estimate of the solution of the fourth approximation with the help of tools from [6].

Let $\Psi = [\psi_i]_{i=1,\dots,9} \in \mathbb{R}^9$ and $\widehat{D} = \frac{1}{\alpha}\mathrm{diag}(\lambda_i)$, where λ_i are given in Theorem 3.2. The initial and boundary value problem for the fourth approximation is given as follows:

$$\begin{aligned} &\frac{\partial \Psi}{\partial t} + \widehat{D}\frac{\partial \Psi}{\partial x} = I(\Psi, \Psi), \quad x \in [-a, a], \quad t > 0, \\ &\Psi(0, x) = \Psi^0(x), \\ &\Psi(t, a) = \Psi(t, -a). \end{aligned} \tag{30}$$

Let $L^2[-a, a]$ be the space of square integrable functions, with the inner product $\langle f, g\rangle = \int_{-a}^{a} f(x)g(x)dx$ and the norm $||f||_{L^2[-a,a]} = \sqrt{\langle f, f\rangle}$, where f and g are square integrable functions. Suppose that the initial condition $\Psi^0(x) \in L^2[-a, a]$.

We take the inner product of both sides of (30) with Ψ and integrate over the interval $[-a, a]$:

$$\int_{-a}^{a} \left\langle \left(\frac{\partial \Psi}{\partial t} + \widehat{D}\frac{\partial \Psi}{\partial x}\right), \Psi \right\rangle dx = \int_{-a}^{a} \langle I(\Psi, \Psi), \Psi\rangle\, dx. \tag{31}$$

Let us rewrite (31) as follows:

$$\int_{-a}^{a} \left\langle \frac{\partial \Psi}{\partial t}, \Psi \right\rangle + \int_{-a}^{a} \left\langle \widehat{D}\frac{\partial \Psi}{\partial x}, \Psi \right\rangle = \int_{-a}^{a} \langle I(\Psi, \Psi), \Psi\rangle\, dx.$$

Now we rewrite $\int_{-a}^{a}\left\langle \widehat{D}\frac{\partial \Psi}{\partial x},\Psi\right\rangle$ as $\int_{-a}^{a}\left\langle \frac{\partial \widehat{D}\Psi}{\partial x},\Psi\right\rangle$. We use integration by parts:

$$\int_{-a}^{a}\left\langle \frac{\partial \widehat{D}\Psi}{\partial x},\Psi\right\rangle = \int_{-a}^{a}\left\langle \frac{\partial \widehat{D}\Psi}{\partial x},\Psi\right\rangle = \left\langle \widehat{D}\Psi,\Psi\right\rangle |_{-a}^{a} - \int_{-a}^{a}\left\langle \widehat{D}\Psi,\frac{\partial \Psi}{\partial x}\right\rangle dx.$$

Hence, we have that $\int_{-a}^{a}\left\langle \widehat{D}\frac{\partial \Psi}{\partial x},\Psi\right\rangle = \frac{1}{2}\left\langle \widehat{D}\Psi,\Psi\right\rangle |_{-a}^{a}= 0$ due to periodic boundary conditions. By imposing the periodic boundary condition in (30), we get that

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a}\langle \Psi,\Psi\rangle\, dx = \int_{-a}^{a}\langle I(\Psi,\Psi),\Psi\rangle\, dx. \tag{32}$$

We now use the spherical representation of the vector Ψ, namely, $\Psi(t,x) = v(t)e(t,x)$, where $v(t) = \|\Psi(t,x)\|_{L^2[-a,a]}$ and $\|e(t,x)\|_{L^2[-a,a]} = 1$. Substitution of the spherical representation of Ψ into (32) yields

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a}\langle v(t)e(t,x),v(t)e(t,x)\rangle\, dx = \int_{-a}^{a}\langle I(v(t)e(t,x),v(t)e(t,x)),v(t)e(t,x)\rangle\,,$$

and consequently,

$$\frac{1}{2}\frac{d}{dt}v^2(t) = v^3(t)\int_{-a}^{a}\langle I(e,e),e\rangle\, dx =: v^3(t)M(t),$$

with $M(t) := \int_{-a}^{a}\langle I(e,e),e\rangle\, dx$. The above equation is an ordinary differential equation, which can be simplified further to

$$\frac{dv}{dt} = v^2(t)M(t).$$

Applying the method of separation of variables to the differential equation we get that

$$\int_0^t \frac{dv}{v^2} = \int_0^t M(t)dt.$$

After integrating from 0 to t we obtain the following:

$$-\frac{1}{v(t)} + \frac{1}{v(0)} = \int_0^t M(t)dt.$$

By using the initial condition $v(0) = \|\Psi(0,x)\|_{L^2[-a,a]} = \|\Psi^0\|_{L^2[-a,a]}$, the solution is then given by

$$v(t) = \left(\frac{1}{\|\Psi^0\|_{L^2[-a,a]}} - \int_0^t M(\tau)d\tau\right)^{-1}.$$

Let T_1 be the moment at which $\frac{1}{\|\Psi^0\|_{L^2[-a,a]}} - \int_0^{T_1} M(\tau)d\tau = 0$. Since $M(t)$ is a continuous function from the Mean Value Theorem of Integration it follows that there exists $\tilde{t} \in [0,T_1]$ such that

$$\frac{1}{\|\Psi^0\|_{L^2[-a,a]}} - \int_0^{T_1} M(\tau)d\tau = \frac{1}{\|\Psi^0\|_{L^2[-a,a]}} - M(\tilde{t})T_1.$$

Hence, we have that

$$T_1 = \frac{1}{M(\tilde{t})\|\Psi^0\|_{L^2[-a,a]}},$$

implying that

$$T_1 \sim O\left(\|\Psi^0\|_{L^2[-a,a]}\right)^{-1}.$$

Then $v(t)$ is bounded for $\forall t \in [0,T]$ with $T < T_1$ and $T_1 \sim O\left(\|\Psi^0\|_{L^2[-a,a]}\right)^{-1}$.

Since $v(t) = \|\Psi(t,x)\|_{L^2[-a,a]} = \left(\dfrac{1}{\|\Psi^0\|_{L^2[-a,a]}} - \displaystyle\int_0^t M(\tau)d\tau\right)^{-1} = \|\Psi^0\|_{L^2[-a,a]} \dfrac{1}{1 - \|\Psi^0\|_{L^2[-a,a]} \int_0^t M(\tau)d\tau}$ it follows that

$$\|\Psi\|_{C([0,T];L^2[-a,a])} \leq \max\{\|\Psi(t,x)_{L^2[-a,a]}\| : t \in [0,T]\} = \|\Psi^0\|_{L^2[-a,a]} \max\{\frac{1}{1 - \|\Psi^0\|_{L^2[-a,a]} \int_0^t M(\tau)d\tau} : t \in [0,T]\},$$

with $C\left([0,T];L^2[-a,a]\right)$ being a space of functions, continuous in time and square summable by spatial variable. We denote $\max\{\dfrac{1}{1 - \|\Psi^0\|_{L^2[-a,a]} \int_0^t M(\tau)d\tau} : t \in [0,T]\}$ by the constant C. We formulate the result as a theorem.

Theorem 4.1 *If $\Psi^0(x) \in L^2[-a,a]$, then there exists T such that the a-priori estimate $\|\Psi\|_{C([0,T];L^2[-a,a])} \leq C\|\Psi^0\|_{L^2[-a,a]}$ for the system* (30) *is true for a finite interval of time $[0,T]$, with $T_1 \sim O\left(\|\Psi^0\|_{L^2[-a,a]}\right)^{-1}$ and with the constant C not depending on Ψ.*

5 Conclusion

In this report, we presented the well-posed initial and boundary value problem and the a-priori estimate of the solution of the fourth approximation. We used symmetry of the coefficient matrix of the coupled system to use the similarity transform to a conservative form of the fourth approximation. Well-posedness of the initial value problem for the wave equation resulted in well-posedness of the decoupled system, which is a combination of nine wave equations. The consideration of domain of influence leaded to the well-posed initial and boundary value problem. In order to obtain the a-priori estimate we used the analogue of the so-called energy integral for the system of hyperbolic partial differential equations. We focused on periodic boundary conditions to obtain the a-priori estimate due to the difficulty with more general boundary conditions. A-priori estimate of the solution belonging to a space of functions continuous in time and square integrable in spatial variable is valid for a short period of time from 0 to T, where time T depends on the initial values for the decoupled fourth approximation system. Results for the fourth approximation, such as polishing the proof of mass conservation law and numerical simulations with collision term remain for the future studies.

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