Finite Elements Solutions to the Black - Scholes Equation

MATH 499 Capstone Project

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Abstract

Nowadays, for the financial industry it is important to implement mathematical tools of the advanced level. World's well-known economists Fischer Black and Myron Scholes introduced the distinguished equation for option pricing in 1973. This Capstone project aims to find finite element solutions to the Black-Scholes Equation. For this, the Black-Scholes Equation is solved as a convection dominated problem through the Local Projection Stabilization, where Galerkin finite element method is applied to the parabolic equation. Hence, for the Local Projection Stabilization, the functions of L^2 – orthogonal finite element basis of arbitrary order are constructed. These functions result in a diagonal mass matrix which are useful for time discretization.

1. Introduction

Pricing of options is a prevalent topic for discussion among financial practicioners. Option is a financial contract between at least two parties or financial derivative which can be sold or bought with agreed price on the specific date in the future. There are various types of option. The most frequently used are European and American options, which differ from each other in their time of maturity. European option relates to the one that can be exercised only once on the specific date, meanwhile American option can be exercised at any point of time. Also, there are call and put options: call option gives the right for the holder to buy an asset and put option, on the contrary, allows to sell the stock. Therefore, the concept of the option is of great importance since it allows to secure the right of selling or buying a specific asset for a fixed price. Moreover, it gives an opportunity for the holder to buy a stock for the lower price in order to save money and to not overpay for it. This project focuses on the call options of European type.

Scholars Black and Scholes demonstrated how to examine the dynamics of a financial market, particularly of the options, by deriving a European Stock Purchase option with an equilibrium price. The equilibrium option prices concern the stock price, the exercise price (current price), the risk-free interest rate, time to expiration and the standard deviation of log returns (volatility). This research examines deterministic equations of Black-Scholes model with one-dimension. Further, the paper analyzes boundary conditions and solves the equation as convection dominated problem by local projection scheme for its stabilization.

2. The Black-Scholes Equation

To find the derivative of the function of a stochastic process, which is time dependent, the following Ito's lemma is used. Within stochastic framework which concerns random variables, Ito's lemma performs as chain rule in differential calculus. More detailed derivation of Ito's Lemma can be found in the book by Hilber et. al (2013).

$$dV = \frac{\partial V}{\partial S}(\sigma S dW + \mu S dt) + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt$$

The outcome is that dV has a random component involving dW and a deterministic component from dt. Stock price is driven by stochastic equation: $dS_t = \mu S_t dt + \sigma S_t dW_t$ where μ, $\sigma > 0$, in the integral expression $S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dW_s$. Here, μ characterizes the stock trend.

The solution of the expression is $S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$, which gives dP with no stochastic part dW:

$$dP = \frac{\partial V}{\partial S}\sigma S dW + (\mu S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}) - \frac{\partial V}{\partial S}(\sigma S dW + \mu S dt)$$

This is a deterministic risk-free return: $dP = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})dt$ with $P = V - S\left(\frac{\partial V}{\partial S}\right)$

dP=rPdt from investing the same value P in any other riskless asset.

Therefore, we obtain Black-Scholes equation: $rV - rS\frac{\partial V}{\partial S} = \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2$

Since Ito's lemma concerns time and random variables, it is applied widely in quantitative finance and economics. Black-Scholes Model is one of the most outstanding applications, which will be considered in detail later. But it is important to note that given variables in the equation, namely value of the option (V(S,t)), stock price (S), expiration time (t), risk-free interest rate (r) and standard deviation of log returns (σ), satisfy

$$\frac{\partial V(S,t)}{\partial t} + rS_t \frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\frac{\partial^2 V(S,t)}{\partial S^2}\sigma^2 S^2 = rV(S,t)$$

The price can be calculated by $C_0 = S_0 N(d_1) - Xe^{-rT}N(d_2)$, where

$d_1 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$ and N(x) means a cumulative distribution function for a random variable x which is normally distributed:

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-\frac{1}{2}t^2)dt$$

For simplicity, the classical Black-Scholes equation is investigated, which means that no dividends are paid. However, in the section of numerical examples dividends are taken into account, thus it implies that generalized Black-Scholes equation is used.

3. Assumptions

The Black-Scholes equation as a mathematical model plays vital role in the pricing of financial derivatives. The model has following assumptions:

a) There are no arbitrage opportunities.

b) There is frictionless market.

This implies that transaction costs including fees or taxes are not available, each party is able to access any information instantly, there are equal interest rates for lending and borrowing money, and at any time all credits and securities are available in any size. As a result, each variable is perfectly divisible and can be any real number. Moreover, the price is not affected by individual trading.

c) The price follows a geometric Brownian motion.

d) The interest rate and volatility are constant during the contract period, namely for $0 \leq t \leq T$.

e) The option is European type.

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$$

V – value of the European call option

V_t – rate of change of the value of the option at time t

V_S – rate of change of the value of the option with price S

S – stock price

r – risk-free interest rate

σ – standard deviation of log returns (volatility)

4. Boundary conditions

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Initial boundary conditions are defined as follows:

$$\begin{cases} V(0, t) = 0 \\ V(S, t) = S \text{ as } S \to \infty \\ V(S, T) = \max(S-K, 0) = \max(Ke^{x}-K, 0), \text{ where K – strike price} \end{cases}$$

Here, V (S, t) – value of derivative with price S at time t, which must satisfy a terminal condition. The terminal condition for t = T is V (S, T) = L(S), where L defines the payoff function taking into account the type of the option (Seydel 2006).

$$L(S) := \begin{cases} (S-K)^+ & \text{for a call} \\ (K-S)^+ & \text{for a put} \end{cases}$$

This condition naturally designates the definition of the option. Therefore, V (S, T) = f(S) – value of the option with price S at the final time T (time to expiration), S≥0 and 0 ≤t≤ T.

Solving initial boundary conditions, we obtain:

$$\begin{cases} \tau = \frac{\sigma^2}{2}(T-t) \rightarrow t = T - \frac{2\tau}{\sigma^2} \\ x = \ln\left(\frac{S}{K}\right) \rightarrow S = Ke^{x} \\ V(S, t) = Kv(x, \tau) \end{cases}$$

Solving the equation, the first derivatives are:

$$V_t = \frac{\partial V}{\partial t} = K\frac{\partial v}{\partial \tau} \times \frac{d\tau}{dt} = K\frac{\partial v}{\partial \tau} \times \left(-\frac{\sigma^2}{2}\right)$$

and

$$V_S = \frac{\partial V}{\partial S} = K\frac{\partial v}{\partial x} \times \frac{\partial x}{\partial S} = K\frac{\partial v}{\partial x} \times \frac{1}{S}$$

In order to do the second derivative is, it is necessary to mention that if $\frac{\partial x}{\partial S} = \frac{1}{S}$ then S $\frac{\partial S}{\partial x} = S$ because $x = \ln\left(\frac{S}{K}\right)$ → x=ln(S)-ln(K). Thus, the second derivative is:

$$V_{SS} = \frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial S}\right)\frac{\partial x}{\partial S} = \frac{\partial}{\partial x}\left(K\frac{\partial v}{\partial x} \times \frac{1}{S}\right)\frac{1}{S} = K\frac{\partial v}{\partial x} \times \left(-\frac{1}{S^2}\right) + K\frac{\partial^2 v}{\partial x^2} \times \frac{1}{S^2}$$

Since V (S, T) = max (S-K, 0) = max (Ke^x-K, 0) and V (S, T) = $Kv(x,0)$, then $v(x,0) = \max(e^x - 1,0)$.

Also, $\frac{\partial}{\partial x}\left(\frac{1}{S}\right) = \frac{\partial S}{\partial x} \times \left(-\frac{1}{S^2}\right) = -\frac{1}{S^2} \times S = -\frac{1}{S}$. Substituting all the derivatives into the Black-Scholes equation:

$$K\frac{\partial v}{\partial \tau} \times \left(-\frac{\sigma^2}{2}\right) + \frac{1}{2}\sigma^2 S^2\left(K\frac{\partial v}{\partial x} \times \left(-\frac{1}{S^2}\right) + K\frac{\partial^2 v}{\partial x^2} \times \frac{1}{S^2}\right) + rS\left(K\frac{\partial v}{\partial x} \times \frac{1}{S}\right) - rV$$

$$= K\frac{\partial v}{\partial \tau} \times \left(-\frac{\sigma^2}{2}\right) + \frac{\sigma^2}{2}\left(-K\frac{\partial v}{\partial x} + K\frac{\partial^2 v}{\partial x^2}\right) + rK\frac{\partial v}{\partial x} - rV$$

$$= K\frac{\sigma^2}{2}\left(-\frac{\partial v}{\partial \tau} - \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right) + rK\left(\frac{\partial v}{\partial x} - v\right) = 0$$

The solution of the problem will lie in such parameter as $\frac{\partial v}{\partial \tau}$ and will be the following:

$$\frac{\partial v}{\partial \tau} = \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + \frac{2r}{\sigma^2}\left(\frac{\partial v}{\partial x} - v\right) = \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} + \frac{2r}{\sigma^2} \times \frac{\partial v}{\partial x} - \frac{2r}{\sigma^2} \times v$$

$$= \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v$$

Hence,

$$\frac{\sigma^2}{2}\left(-\frac{\partial v}{\partial \tau}-\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}\right)+r\left(\frac{\partial v}{\partial x}-v\right)$$

$$=\frac{\sigma^2}{2}\left(-\frac{\partial^2 v}{\partial x^2}-\left(\frac{2r}{\sigma^2}-1\right)\frac{\partial v}{\partial x}+\frac{2r}{\sigma^2}v-\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}\right)+\ r\left(\frac{\partial v}{\partial x}-v\right)$$

$$=\frac{\sigma^2}{2}\left(\frac{\partial v}{\partial x}-\frac{2r}{\sigma^2}\frac{\partial v}{\partial x}+\frac{2r}{\sigma^2}v-\frac{\partial v}{\partial x}\right)+r\left(\frac{\partial v}{\partial x}-v\right)$$

$$=\frac{\sigma^2}{2}\left(-\frac{2r}{\sigma^2}\frac{\partial v}{\partial x}+\frac{2r}{\sigma^2}v\right)+r\left(\frac{\partial v}{\partial x}-v\right)=-r\left(\frac{\partial v}{\partial x}+v\right)+r\left(\frac{\partial v}{\partial x}-v\right)=0$$

5. Convection dominated problem

Further, in order to define the price of the European option, Zvan, Forsyth and Vetzal (1996) examined the Black-Scholes equation discretizing some features such as time taking into account the terminal and boundary conditions. They solved the problem by the perspective of convection-diffusion equation since the equation is backward linear parabolic. In their research, time t was substituted with t* = T – t which develops from expiration to the present time. After changing the variables, the equation converts like this:

$$\frac{\partial V}{\partial t*}=\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}-\ (-rS)\frac{\partial V}{\partial S}-rV$$

which has the similar form as in fluid dynamics. The term $\frac{1}{2}\ \sigma^2 S^2\frac{\partial^2 V}{\partial S^2}$ is a parabolic diffusion term, where $\frac{1}{2}\ \sigma^2 S^2$ is its magnitude. Meanwhile, $(-rS)\frac{\partial V}{\partial S}$ is the 1st order hyperbolic convective term, where $(-rS)$ is its velocity which disseminates the information. Since $(-rS)$ is a positive term, then from the S → ∞ boundary information emanates into the computational domain. Therefore, if the velocity term is larger than the diffusion term, then the equation is considered as convection dominated.

6. Local Projection Scheme

Referring to the Dr. Schieweck, Skrzypacz and Tobiska's research, in order to solve the convection dominated type of problem it is necessary to use the local projection scheme. Starting from the simple 1D model problem (boundary value problem)

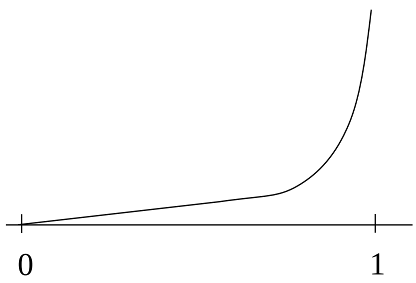
$-\varepsilon u'' + bu' = 0$ in $\Omega = (0,1) \subset \mathbb{R}^1$

with such boundary conditions as:

- u(0)=0
- u(1)=1

Singularly perturbed problem: $0 < \varepsilon \ll 1$

The exact solution to boundary value problem is: $u(x) = \frac{e^{\frac{bx}{\varepsilon}}-1}{e^{\frac{b}{\varepsilon}}-1}$



Boundary layer occurs if $0 < \frac{\varepsilon}{b} \ll 1$

To replace infinite-dimensional linear problem, it is needed to discretize by piecewise-linear conforming functions on the uniform grid.

$h = \frac{1}{M}$

Motivation for Finite Element Methods:

$$\int_{\Omega} (-\varepsilon \Delta u v + b \nabla u v) \mathrm{dx} = \int_{\Omega} 0 v \mathrm{dx} \ \forall v \in C_0^{\infty}(\Omega)$$

Then, integrating by parts:

$$\int_{\Omega} \varepsilon \nabla u \nabla v \mathrm{dx} - \int_{\partial\Omega} \varepsilon \frac{\partial \mathrm{u}}{\partial \mathrm{n}} v \mathrm{ds} + \int_{\Omega} b \nabla \mathrm{u} v \mathrm{dx} = 0$$

Thus, we obtain $\int_0^1 (\,\varepsilon u'v' + bu'v)\mathrm{dx} = 0 \qquad \forall v \in C_0^\infty(0,1)$

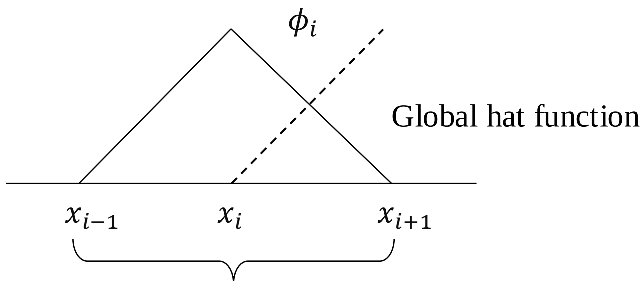
Proving the existence and uniqueness of the solution, if $u' \in L^2(\Omega)$, then LHS is well defined for all $v \in H_0^1(\Omega) = \{v \in L^2(\Omega) \colon\ v' \in L^2(\Omega), \mathrm{v}|_{\partial\Omega} = 0\ \}$

Now, it is needed to find $u \in H^1(\Omega)$, u(0)=0, u(1)=1 such that $\int_0^1 (\,\varepsilon u'v' + bu'v)\mathrm{dx} = 0$ $\forall v \in H_0^1(\Omega)$

Discretizing the function, $V_h \approx H^1(\Omega)$

$V_h = \left\{v_h \in C^0(\Omega) \quad v_h|_{(x_i, x_{i+1})} \in \mathbb{P}_1\right\}$

Thus, on each cell K=$(x_i,\ x_{i+1})$ the discrete solution should be linear polynomial



Support of ϕ_i

where the key idea of FEM is to span V_h with basis function's ϕ_i that have a local support

$u_h(x) = \sum_i u_i\, \phi(x)$ (ansatz)

$\int_0^1 \left(\,\varepsilon u_h' {\phi'}_i + bu_h'\phi_i'\right)\mathrm{dx} = 0\ \ \forall i$

$\rightarrow\ \int_0^1 \varepsilon\left(\sum_j u_j\, \phi_j'\right)\phi_i' + \mathrm{b}\left(\sum_j u_j\, \phi_j\right)'\phi_i \mathrm{dx} = 0 \qquad \forall i$

$$\rightarrow \sum_j \left(\int_0^1 \left(\varepsilon \phi_j' {\phi'}_i + b\phi_j'\phi_i\right) dx \right) u_j = 0 \qquad \forall i,$$

where $A_{ij} = \int_0^1 (\varepsilon \phi_j' {\phi'}_i + b\phi_j'\phi_i)\, dx \qquad \forall i$

To explain approximation, finite element method is introduced as a special case of Galerkin method, which minimizes error of approximation functions that project the residual. Galerkin discretization with $\mathbb{P}_1$ elements lead to the tridiagonal systems:

$$\left(-\frac{\varepsilon}{h}-\frac{b}{2}\right)u_{i-1}+\frac{2\varepsilon}{h}u_i+\left(-\frac{\varepsilon}{h}+\frac{b}{2}\right)u_{i+1}=0 \text{ for } \forall i=1,\dots,M-1$$

where $u_i = u_h(x_i),\ u_0 = 0,\ u_M = 1$

If $2\varepsilon \neq bh$, then the solution u is given by $u_i = \frac{\left(\frac{1+P_e}{1-P_e}\right)^i-1}{\left(\frac{1+P_e}{1-P_e}\right)^M-1}$, where i=1, …, M-1

where $P_e := \frac{|b|h}{2\varepsilon}$, which is a local Peclet number with such conditions as:

- $P_e > 1$ =>$\left(\frac{1+P_e}{1-P_e}\right) < 0$ => u_h is oscillatory
- $P_e < 1$ => u_h has no oscillations but $\frac{h}{2} < \frac{\varepsilon}{|b|}$ means $h \ll 1$ if $0 < \frac{\varepsilon}{|b|} \ll 1$

The mesh refinement leads to big systems which are, especially in higher dimensions, infeasible from the numerical point of view. Therefore, the following steps are needed to be taken:

1. In order to stabilize the Galerkin solution the bilinear form is to be perturbed, that is $a_h(u,v) = a(u,v) + s_h(u,v)$, where $s_h(u,v)$ is the stabilization term
2. Using different ansatz and test spaces, we take Petrov-Galerkin method.

7. Stabilization methods

One of the stabilization methods is Local Projection Stabilization (LPS), which is not consistent but easy to implement stencil as in Galerkin method.

$a_h(u,v) = a(u,v) + \sum_K \tau_K(\mathrm{K}_h(\nabla u), \mathrm{K}_h(\nabla v))_{\mathrm{K}}$, where

- $\tau_K = \gamma_0 h_K$
- $\sum_K \tau_K(K_h(\nabla u), K_h(\nabla v))_K$ is the artificial diffusion applied only to the small-scale modes of the function(s)
- $\mathrm{K}_h|_{\mathrm{K}} = \pi_h|_{\mathrm{K}} - id$; L^2 projection on some discontinuous space of $\pi_h|_K$

The idea of the Local Projection Stabilization (LPS) method is to calculate the projection of the gradient of finite element functions $v_h \in V_h$ into a discontinuous space D_h of the *large* *scale modes* and to stabilize only the remaining so-called *fine scale modes* of the function v_h for which the gradient cannot be represented by the projection space.

$$V_h = \mathbb{P}_1, D_h = \mathbb{P}_0, \; V_h \to D_h$$

$$\text{If } u_h \in \mathbb{P}_1 \to \nabla u_h \in \mathbb{P}_0 \to K_h(\nabla u_h) = \pi_h(\nabla u_h) - \nabla u_h = \nabla u_h - \nabla u_h = 0$$

Thus, in order to get stabilization of the Galerkin scheme, we need to extend finite element space $\mathbb{P}_1$. Moreover, we have to enrich this space of piecewise-linear $\mathbb{P}_1$.

Let us consider the following enrichment: $V_h = \mathbb{P}_1^+ = \mathbb{P}_2$, where $\mathbb{P}_1$ is enriched by quadratic bubbles.

$$\text{If } u_h \in \mathbb{P}_2 \to \nabla u_h \in \mathbb{P}_1 \to \mathrm{K}(\nabla u_h) = \pi_h(\nabla u_h) - \nabla u_h = \begin{cases} 0 \; \forall u_h \in \mathbb{P}_1 \\ * \; \forall u_h \in \mathbb{P}_2 - \mathbb{P}_1 \end{cases}$$

Here, π_h denotes L^2 projection: $V_h \to D_h$. Therefore, $(\pi_h v - v_1 \phi) = 0 \; \forall \phi \in D_h$ and $\pi_h \in D_h$

$$\text{Therefore, } \tau_{\mathrm{K}}(\pi_h \nabla u_h - \nabla u_h, \pi_h \nabla u_h - \nabla u_h)_{\mathrm{K}} = -\tau_{\mathrm{K}}(\pi_h \nabla u_h - \nabla u_h, \nabla u_h)_{\mathrm{K}}$$

$$= \tau_{\mathrm{K}}(\nabla u_h, \nabla u_h)_{\mathrm{K}} - \tau_{\mathrm{K}}(\pi_h \nabla u_h, \nabla u_h)_{\mathrm{K}}$$

$$= \tau_{\mathrm{K}}(\nabla u_h, \nabla u_h)_{\mathrm{K}} - \tau_{\mathrm{K}} \int_{\partial \mathrm{K}} \pi_h \nabla u_h u_h n ds = \tau_K(\nabla u_h, \nabla u_h)_K, \text{ since } u_n = 0 \text{ on } \partial \mathrm{K}$$

where $u_h \in \mathbb{P}_2 - \mathbb{P}_1$ and $\pi_h \nabla u_h \in \mathbb{P}_0$

Here, $\tau_{\mathrm{K}}(\nabla u_h, \nabla u_h)_{\mathrm{K}}$, which is an artificial diffusion was added in the region where $|\nabla u| \gg 1$

8. Main part of Local Projection Scheme

$\mathrm{K}_h: V_h(\mathrm{K}) \to D_h(\mathrm{K})$ – fluctuation operator, such that the consistency error is not too big.

$a_h(u_h, v_h) = a(u_h, v_h) + s_h(u_h, v_h) = (f, v_h) \;\; \forall v_h \in V_h$

$a_h(u, v_h) = a(u, v_h) + S_h(u, v_h) = (f, v_h) + S_h(u, v_h) \quad \forall v_h \in V_h$

where $a(u_{,}v_h) = (f_{,}v_h)$ is the weak formulation

Notice that in this case Galerkin orthogonality is violated. However, the consistency error is of the following order:

$$a(u - u_h, v_h) - s_h(u - u_h, v_h) = -S_h(u, v_h)$$

$$a_h(u - u_h, v_h) = O\left(h^{r+\frac{1}{2}}\|v_h\|\right),$$ if K_h is appropriate in case $\|\mathrm{K_h}q\|_{0,K} \leq Ch_k^r|q|_{r,\mathrm{K}}$

Special interpolation operator: $w - j_h w \perp D_h; (w - j_h w, q_h) = 0 \;\; \forall q_h \in D_h$

Applying Local Projection Scheme to non-stationary problems:

$$\partial_t u - \varepsilon\Delta u + b\nabla u + cu = f \quad in\ (0_{,}T) \times \Omega,$$

where $Au = -\varepsilon\Delta u + b\nabla u + cu$ – differential operator.

Using Finite Element Methods (FEM) in space: $M_h u_h + A_h u_h = f,$

where M_h signifies the mass matrix and A_h denotes the stiffness matrix. Discretizing in time by dG(1) results in fully discrete problem, which is necessary to solve 2×2 block system:

$$\begin{bmatrix} \frac{9}{8}M + \frac{3\tau_n}{4}A & \frac{3}{8}M \\ -\frac{9}{8}M & \frac{5}{8}M + \frac{\tau_n}{4}A \end{bmatrix} \begin{bmatrix} \underline{U^1} \\ \underline{U^2} \end{bmatrix} = \begin{bmatrix} \underline{F^1} \\ \underline{F^2} \end{bmatrix}$$

Due to L^2 orthogonality of the base functions of the new element the diagonal mass matrix M can be used in the new matrix: $\breve{A} := M^{-1}A$ which is a row scaling of A, and $R^j := M^{-1}F^j$ for j=1,2.

A reduced system for $\underline{U^2}$, which has the half dimension of the 2×2 block system, can be obtained by removing the unknown variable $\underline{U^1}$ from the block system:

$$\left\{I + \frac{2\tau_n}{3}\check{A} + \frac{{\tau_n}^2}{6}\check{A}^2\right\}\underline{U^2} = R^1 + R^2 + \frac{2\tau_n}{3}\check{A}R^2$$

9. Numerical examples

In order to ignore undesirable time discretization errors caused by dG(1), we investigate additionally the following stationary problem

$$-\varepsilon\partial_{xx}u + b\partial_x u = f \text{ in } \Omega = (0,1)$$

$$u = 0 \text{ on } \partial\Omega$$

where $\varepsilon = 10^{-8}$, $b(x) = 1$ and $f(x)$ is chosen such that

$$u(x) = x - \frac{e^{\frac{x-1}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} + e^x - 1 - x(e-1)$$

is solution for the equation above. It exhibits an exponential boundary layer at $x = 1$. We use the coarse mesh that consists of $N_{\mathrm{EL}} = 5$ equal cells and that will be uniformly refined. In Table 1, L^2 errors are presented between the exact and LPS solution for the polynomial degree $r = 1$, 2, 3. The orders of L^2 errors computed on the sub-domain $\Omega_0 =$ (0, 7/10) are optimal with respect to the polynomial degree r. The parameter γ_0 has to be chosen appropriately in oder to suppress spurious oscillations at the boundary layer, see Fig. 1. On these graphs, solid blue line signifies LPS and dashed red line denotes exact solutions for the example.

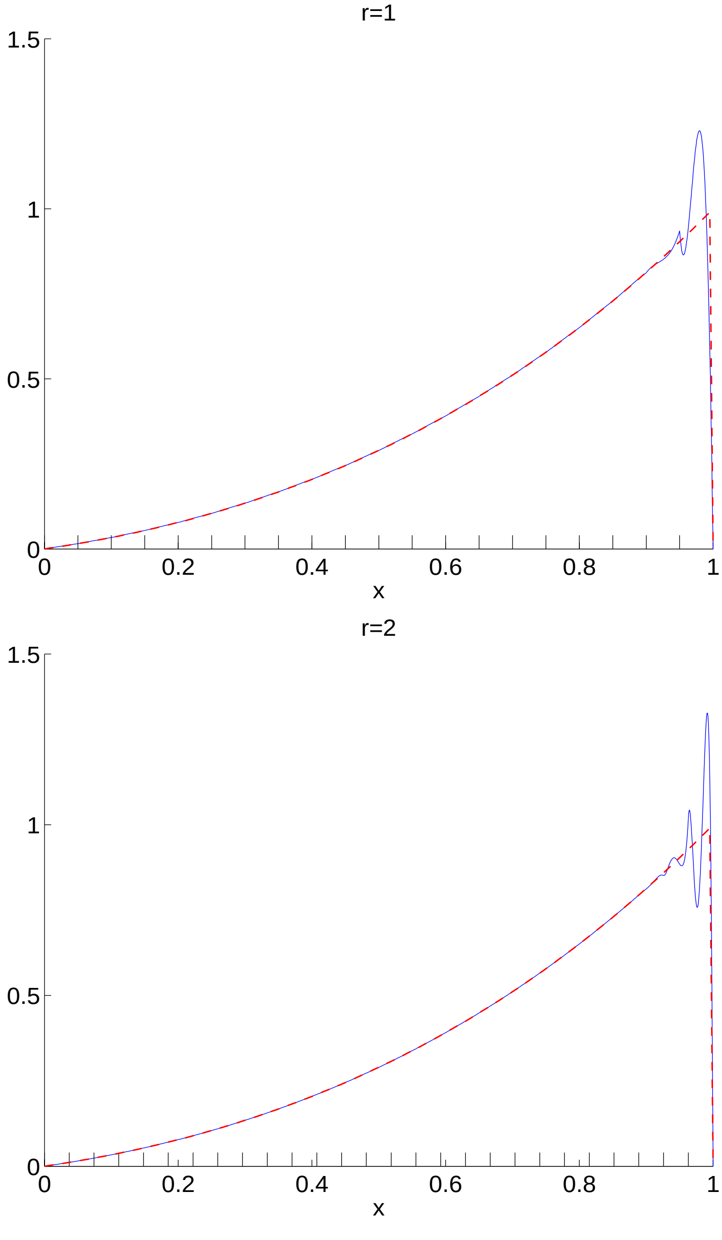
Table 1.

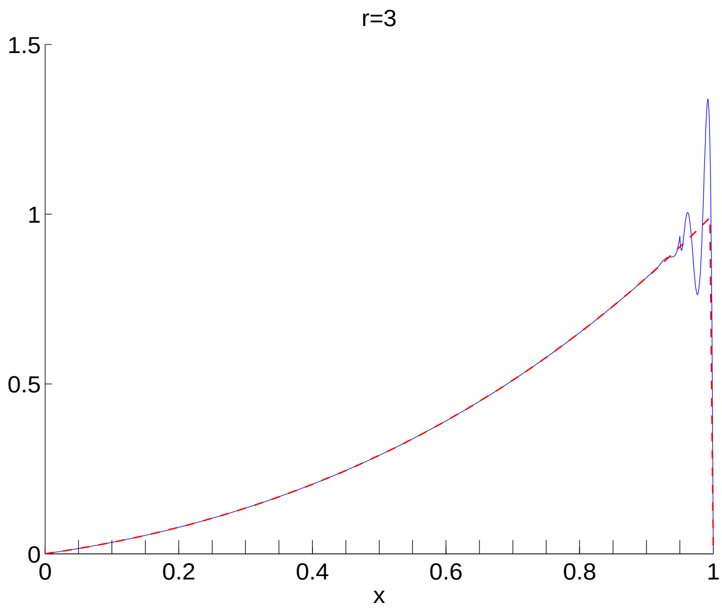
$L^2 -$ norms of the discretization errors on $\Omega_0 =$ (0, 7/10)

	r=1, $\gamma_0 = 10^{-1}$		r=2, $\gamma_0 = 6*10^{-2}$		r=3, $\gamma_0 = 5*10^{-2}$	
Level	$\|\|e_{\mathrm{h}}\|\|_{0,\Omega_0}$	order	$\|\|e_{\mathrm{h}}\|\|_{0,\Omega_0}$	order	$\|\|e_{\mathrm{h}}\|\|_{0,\Omega_0}$	order
1	5.564e-3		1.013e-04		9.442e-05	
2	1.328e-3	2.067	1.177e-05	3.105	1.189e-07	9.634
3	3.419e-4	1.957	1.506e-06	2.967	5.488e-09	3.437

4	8.667e-5	1.980	1.903e-07	2.985	3.469e-10	3.984
5	2.182e-5	1.990	2.391e-08	2.993	2.180e-11	3.992
6	5.472e-6	1.995	2.996e-09	2.996	1.367e-12	3.995
7	1.370e-6	1.998	3.750e-10	2.998	8.5115e-14	4.006

Fig. 1





The following numerical example is constructed on the basis of the generalized Black-Scholes Equation: $C_t + \frac{1}{2}\sigma^2(S,t)S^2C_{SS} + \big(r(S,t) - d(S,t)\big)SC_S - r(S,t)C = 0$, which solves the convection dominated problem.

Here, the parameters are: $C(S,t)$ - the value of European call option, S – asset price, t – time, K – exercise price, d – dividends to pay, T – maturity (expiration) date, $r(S,t) > 0$ – risk-free interest rate, $d(S,t)$ – dividend, and $\sigma(S,t) > 0$ – volatility function of the underlying asset

$r(S,t)$, $d(S,t)$, $and\ \sigma(S,t)$ are bounded on the domain and sufficiently smooth.

r, $d\ and\ \sigma$ are constant functions, which compose the classical Black-Scholes Equation.

The solution is based on such parameters as $S \in (0,8)$ and $t \in (0,8]$, which means that terminal condition for time T=8 (see Fig. 2). It can be seen from the Fig. 2 that across the time the price of the stock decreases while other parameters are constant.

Boundary conditions:

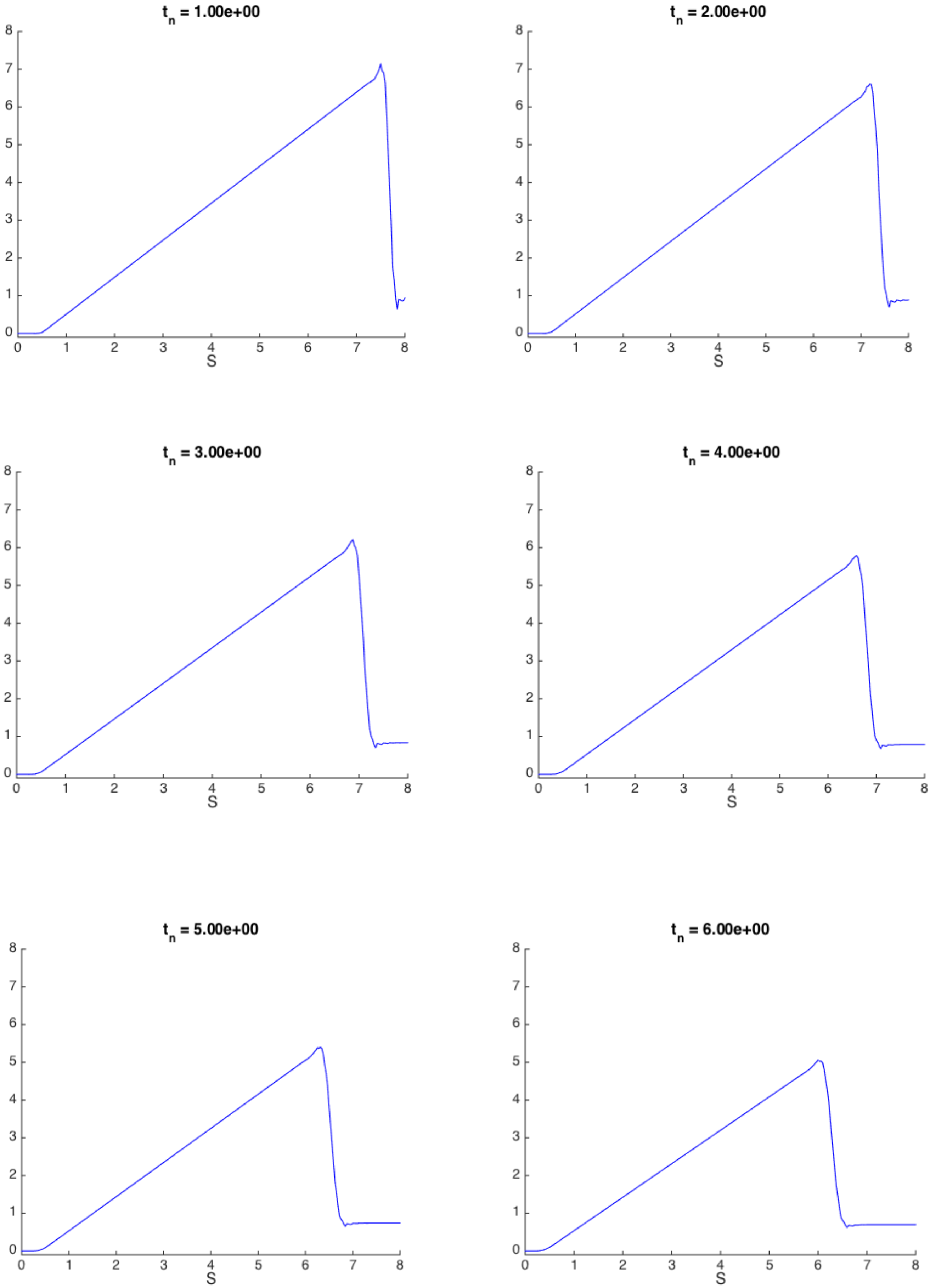
$$C(0,t) = 0$$

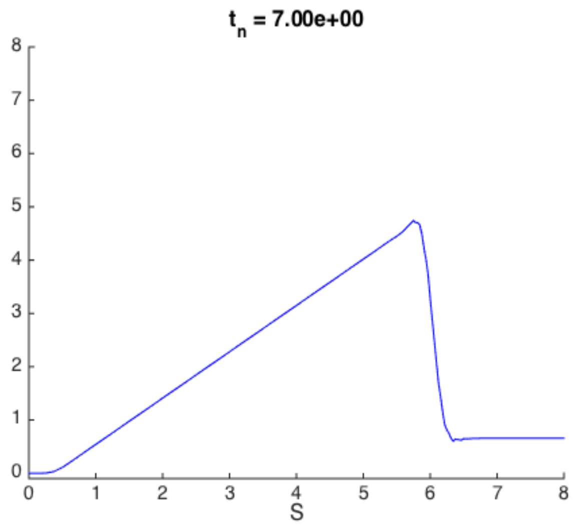
$$C(8,t) = e^{-rt}$$

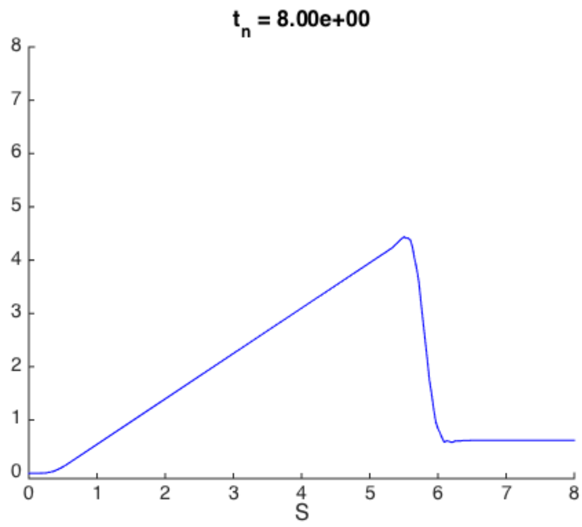
Initial conditions:

$$C(S,0) = \max(S-K,0) \text{ with } K = 0.5, \sigma = 0.001, r = 0.06 \text{ and } d = 0.02$$

Fig. 2







10. Conclusion

In this project, establishing boundary conditions Black-Scholes Equation was solved as the convection dominated problem was determined, where the velocity term was greater than the diffusion term. Thus, it can be observed that the higher price of the stock the greater value of the option. Another observation is that if the stock price does not change, value of the option decreases when the maturity date is becoming closer. Therefore, if the maturity date is in the nearest future and the value of the option is approximately stock price minus exercise price, which is about zero. Meanwhile, if the expiration date is far from today, then the exercise price will be low, and the value of the option will be approximately equal to the stock price. The presented finite elements are of arbitrary order and are L^2 – orthogonal. Moreover, using the Local Projection Stabilization they can be solved in one-dimensional problems of convection-dominated type. The functions of L^2 – orthogonal finite element basis are useful for time discretization, especially for the temporal dG(1) – discretization. In further research projects, L^2 – orthogonal basis can be constructed with the extension to two and/or three-dimensional case.

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