

# Fujita-type results and lifespan of solutions for the higher-order semilinear parabolic equations



NAZARBAYEV  
UNIVERSITY

A Thesis Presented by

Nurdaulet Tobakhanov

Submitted to the School of Sciences and Humanities  
In Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY IN MATHEMATICS

Nazarbayev University

Astana, Kazakhstan

April 2026

# Supervisory Committee

- Lead Supervisor:** Durvudkhan Suragan  
Professor, Department of Mathematics  
Nazarbayev University  
email: durvudkhan.suragan@nu.edu.kz
- Co-Supervisor:** Berikbol Torebek  
Professor, Department of Nonlinear Analysis  
Institute of Mathematics and Mathematical Modeling  
torebek@math.kz
- External Supervisor:** Bessem Samet  
Full Professor, Department of Mathematics  
King Saud University  
email: bsamet@ksu.edu.sa

# Declaration

I, Nurdaulet Tobakhanov, certify that, with the exception of properly referenced quotes and citations, this thesis titled "*Fujita-type results and lifespan of solution for the higher-order semilinear parabolic equations*" is entirely my original work. No part of this thesis has been submitted for any other degree at Nazarbayev University or any other institution, nor does it contain material previously submitted for a degree.

Signed: \_\_\_\_\_

Date:

*Dedicated to my late father, Nurlan Azhikhanov, who made me who I am, and who made this work possible. You are the silent force behind every page of this work.*

# Acknowledgements

First and foremost, I thank my supervisor, Professor Durvudkhan Suragan, for his trust in my potential and for providing the professional environment. His backing allowed me to focus fully on my doctoral studies.

I am deeply grateful to my co-supervisor, Professor Berikbol Torebek. It was his encouragement that convinced me to pursue this PhD when I was considering leaving science. His extensive guidance, feedback, and support were essential to the completion of this thesis.

My time as a PhD student was made much better by the friendship and teamwork of my colleagues. I extend my sincere appreciation to the members of the RTM research group: Samat Mambetov (Ahmet Yasawi University), Meiirkhan Borikhanov (SDU University), and Nurbol Koshkarbayev (Institute of Mathematics and Mathematical Modeling). I am equally thankful for the vibrant environment shared with my friends and colleagues from C4.336, including Mukhtar Karazym and Aisha Tursynkozha (Astana IT University), Rakhymzhan Kazbek (Umeå University), Khumoyun Jabbarkhanov, Kathiresan Sivakumar and Meiirbek Slyamkhan (Nazarbayev University), and Azizbek Mamanazarov (V.I. Romanovskiy Institute of Mathematics), and Priyank Oza (BITS Pilani Dubai Campus).

Finally, I want to express my deepest thanks to my family. I am especially grateful to my mother, Bagila Moldabekova, for her constant love, patience, and support throughout this long journey. I am also very thankful to Nurgeldi, Sandugash, Nurzada, Nurlibek, and Ailin.

This thesis is supported by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP26195417).

# Abstract

This thesis investigates the critical behavior, finite-time blow-up, and sharp lifespan estimates for solutions to semilinear higher-order parabolic equations. The research provides a comprehensive analysis across three primary analytical frameworks:

- **Exterior Biharmonic Problems without Forcing:** We examine the semilinear biharmonic heat equation  $u_t + \Delta^2 u = |u|^p$  in an exterior domain  $D^c$  under six distinct homogeneous boundary configurations. By constructing specialized trial functions involving harmonic and biharmonic kernels, we prove the nonexistence of global weak solutions in the subcritical and critical regimes. The Fujita critical exponent for these exterior problems is identified as  $p_{Fuj} = 1 + 4/N$ .
- **Influence of Inhomogeneous Forcing term:** We further analyze the exterior biharmonic problem subject to a spatial forcing term  $f(x)$ . We demonstrate that the presence of forcing fundamentally shifts the critical threshold to  $p_{Crit}$ , which is defined as  $\infty$  for  $N \leq 4$  and  $N/(N-4)$  for  $N \geq 5$ . In sub-critical dimensions, no global-in-time weak solutions exist for any  $p > 1$ . In the supercritical regime, we identify a second critical exponent in the sense of Lee and Ni, denoted as  $\omega_{Crit} = 4p/(p-1)$ . This threshold is governed by the spatial decay rate of the forcing term  $f(x)$  at infinity. We prove that slow decay results in blow-up, whereas sufficiently fast decay allows for the existence of global stationary solutions.
- **Polyharmonic Cauchy Problem and Lifespan Estimates:** For the higher-order parabolic Cauchy problem  $u_t + (-\Delta)^m u = |u|^p$  in  $\mathbb{R}^N$ , we establish sharp asymptotic estimates for the maximal lifespan  $T_\epsilon$  of mild solutions. Utilizing the test function method combined with Faà di Bruno's formula to manage higher-order spatial derivatives, we prove that the lifespan follows a power-law  $T_\epsilon \simeq \epsilon^{-(\frac{1}{p-1} - \frac{N}{2m})^{-1}}$  in the subcritical regime and exhibits exponential behavior  $T_\epsilon \simeq \exp(\epsilon^{-(p-1)})$  at the critical threshold  $p_{Fuj} = 1 + 2m/N$ .

**Keywords:** Polyharmonic heat equation, Semilinear biharmonic heat equation, Finite-time blow-up, Lifespan estimates, Fujita critical exponent, Second critical exponent, Exterior domain, Forcing term.

# Аңдатпа

Бұл диссертациялық жұмыста жартылай сызықты жоғары ретті параболалық теңдеулер шешімдерінің кризистік әрекеті, ақырлы уақытта қирауы (blow-up) және өмір сүру ұзақтығының дәл бағалаулары зерттеледі. Зерттеу жұмысы үш негізгі аналитикалық бағыт бойынша кешенді талдауды ұсынады:

- **Бигармониялық жылу теңдеу есебі:** Біз алты түрлі біртекті шекаралық шарттар қойылған  $D^c$  сыртқы облысында  $u_t + \Delta^2 u = |u|^p$  жартылай сызықты бигармониялық жылу теңдеуін қарастырамыз. Гармониялық және бигармониялық ядролардан құрылған арнайы тест функцияларын пайдалану арқылы біз субкризистік және кризистік режимдерде глобалды әлсіз шешімдердің жоқтығын дәлелдейміз. Осы сыртқы есептер үшін Фуджита кризистік көрсеткіші  $p_{Fuj} = 1 + 4/N$  ретінде анықталды.
- **Сыртқы әсері бар бигармониялық жылу теңдеуі:** Біз кеңістіктік  $f(x)$  сыртқы әсері бар бигармониялық есепті одан әрі талдаймыз. Сыртқы әсері бар болуы кризистік шекті  $p_{Crit}$  мәніне түбегейлі өзгертетінін көрсетеміз; мұнда  $N \leq 4$  болса,  $p_{Crit} = \infty$ , ал  $N \geq 5$  болса,  $p_{Crit} = N/(N - 4)$ . Субкризистік өлшемдерде кез келген  $p > 1$  үшін жаһандық уақыт бойынша әлсіз шешімдер болмайды. Суперкризистік режимде Ли және Ни мағынасындағы  $\omega_{Crit} = 4p/(p - 1)$  деп белгіленетін екінші кризистік көрсеткішті анықтаймыз. Бұл шекті мән  $f(x)$  сыртқы әсерінің шексіздіктегі өшу жылдамдығымен анықталады.
- **Полигармониялық Коши есебі және өмір сүру ұзақтығын бағалау:**  $\mathbb{R}^N$  кеңістігіндегі  $u_t + (-\Delta)^m u = |u|^p$  жоғары ретті параболалық Коши есебі үшін интегралдық (mild) шешімдердің максималды өмір сүру ұзақтығы  $T_\epsilon$  үшін дәл асимптотикалық бағалауларды белгілейміз. Жоғары ретті кеңістіктік туындыларды басқару үшін Фаа ди Бруно формуласымен біріктірілген тест функциялары әдісін қолдана отырып, өмір сүру ұзақтығы субкризистік режимде  $T_\epsilon \simeq \epsilon^{-(\frac{1}{p-1} - \frac{N}{2m})^{-1}}$  дәрежелік заңдылыққа бағынатынын және  $p_{Fuj} = 1 + 2m/N$  кризистік шегінде экспоненциалды сипатқа ие болатынын дәлелдейміз.

**Түйінді сөздер:** Полигармониялық жылу теңдеуі, Жартылай сызықты бигармониялық жылу теңдеуі, Ақырлы уақыттағы қирау, Өмір сүру ұзақтығын бағалау, Фуджита кризистік көрсеткіші, Екінші кризистік көрсеткіш, Сыртқы облыс.

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# List of Symbols

|                      |                                                                                                                                                     |
|----------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------|
| $\Delta$             | Laplace operator                                                                                                                                    |
| $\mathbb{N}$         | set of all natural numbers                                                                                                                          |
| $\nabla$             | standard gradient                                                                                                                                   |
| $\operatorname{div}$ | divergence operator                                                                                                                                 |
| $C_0^\infty(\Omega)$ | class of all smooth functions $u : \Omega \rightarrow \mathbb{R}$ with compact support in $\Omega$                                                  |
| $L_{loc}^1(\Omega)$  | space of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that $\int_K  f  dx < \infty$ for all compact subsets $K \subset \Omega$ |
| $L^\infty(\Omega)$   | space of all essentially bounded measurable functions $u : \Omega \rightarrow \mathbb{R}$                                                           |
| $L^p(\Omega)$        | space of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that $\int_\Omega  u ^p dx < \infty$                                     |
| $\epsilon$           | Small positive parameter                                                                                                                            |
| $\omega_{Crit}$      | Second critical exponent (Lee-Ni type)                                                                                                              |
| $\psi_R$             | Cut-off test function                                                                                                                               |
| $\Delta^2$           | Biharmonic operator                                                                                                                                 |
| $\partial_n$         | Normal derivative                                                                                                                                   |
| $\operatorname{div}$ | Divergence operator                                                                                                                                 |
| $f(x)$               | Forcing term                                                                                                                                        |
| $p$                  | Power of the nonlinearity ( $p > 1$ )                                                                                                               |
| $p_{Crit}$           | Critical exponent (varies by dimension)                                                                                                             |
| $p_{Fuj}$            | Fujita critical exponent ( $1 + 4/N$ )                                                                                                              |
| $T_\epsilon$         | Lifespan of the mild solution                                                                                                                       |
| $u(x, t)$            | Solution to the biharmonic heat equation                                                                                                            |
| $u_0(x)$             | Initial data                                                                                                                                        |
| $\mathbb{N}$         | Set of all natural numbers                                                                                                                          |

|                      |                                                            |
|----------------------|------------------------------------------------------------|
| $\mathbb{R}^N$       | $N$ -dimensional Euclidean space                           |
| $\mathbb{R}_+$       | Set of all nonnegative real numbers                        |
| $C_0^\infty(\Omega)$ | Space of smooth functions with compact support in $\Omega$ |
| $D$                  | Closed unit ball, $\overline{B(0, 1)}$                     |
| $D^c$                | Exterior domain (complement of $D$ )                       |
| $Q$                  | Space-time cylinder $[0, T) \times D^c$                    |

# Chapter 1

## Introduction

### 1.1 Historical background

In his seminal 1966 paper, Hiroshi Fujita [Fuj66] pioneered the study of critical behavior in nonlinear evolution equations by investigating the competition between linear diffusion and nonlinear reaction in the whole Euclidean space.

$$\begin{cases} u_t - \Delta u = u^p, & \text{in } \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1.1)$$

He established a sharp critical threshold, now known as the Fujita critical exponent,  $p_F = 1 + \frac{2}{N}$ . He proved the following:

- (i) If  $1 < p < p_F$  (the subcritical case), there is no global non-negative solution to the problem (1.1.1) other than the trivial one. In this regime, the nonlinearity dominates diffusion, causing the solution to blow up in finite time for *any* non-trivial non-negative initial data  $u_0$ , regardless of how small it is.
- (ii) If  $p > p_F$  (the supercritical case), global solutions exist provided the initial data is sufficiently small and decays rapidly enough at infinity. Specifically, Fujita demonstrated that if the initial data  $u_0(x)$  is bounded by a small Gaussian heat kernel, such that

$$0 \leq u_0(x) \leq \delta(4\pi\gamma)^{-N/2} \exp(-|x|^2/4\gamma)$$

for some small  $\delta > 0$  and  $\gamma > 0$ , then the diffusion is strong enough to prevent blow-up, yielding a global classical solution.

However, Fujita's original work left the critical case  $p = p_F$  unresolved, as his theorems initially addressed only the strictly subcritical and supercritical regimes. Consequently, determining whether the critical exponent itself belongs to the blow-up or global existence regime remained an open challenge. This gap at the threshold was subsequently bridged by Hayakawa [Hay73] for dimensions  $N = 1, 2$ , and by Sugitani [Sug75] and Kobayashi et al. [KST77] for  $N \geq 3$ . Their findings confirmed that the critical case  $p = p_F$  indeed belongs to the blow-up regime. Subsequently, Weissler [Wei81] significantly advanced the theory by analyzing the Cauchy problem in the framework of Lebesgue spaces  $L^q(\mathbb{R}^N)$ , rather than relying on the pointwise Gaussian bounds used by Fujita. He investigated the existence and

long-time behavior of solutions as curves in  $L^q$  spaces. Weissler established that in the subcritical and critical regimes, all non-negative solutions belonging to certain  $L^q$  classes blow up in finite time. Furthermore, for the supercritical case, he proved that global solutions exist provided the initial data is sufficiently small in the  $L^q$  norm, where  $q = \frac{N(p-1)}{2}$ . This result was pivotal as it identified the scale-invariant norm required for global existence, broadening the class of admissible initial data beyond the Gaussian decay originally proposed by Fujita. Furthermore, Zhang [Zha02] extended the classical Fujita blow-up theory to include sign-changing initial data. Zhang proved that if the initial data satisfies  $\int_{\mathbb{R}^N} u_0(x) dx > 0$ , then the critical exponent remains the classical Fujita exponent. Specifically, he demonstrated that for  $1 < p \leq p_F$ , no global solutions exist under this integral condition, even if the solution takes negative values. Additionally, Zhang investigated the problem in conical domains with a potential  $V(x) \sim |x|^\sigma$ , successfully resolving an open question regarding the critical case. He proved that the critical exponent  $p^* = 1 + (2 + \sigma)/(N + m)$  belongs to the blow-up regime. Complementing these findings, Pinsky [Pin05] investigated the delicate borderline case where the initial data has vanishing total mass. He established that the non-existence of global solutions persists in the subcritical and critical regimes  $p \leq p_F$ , even for sign-changing initial data satisfying the zero-mass condition:

$$\int_{\mathbb{R}^N} u_0(x) dx = 0, \quad u_0 \not\equiv 0.$$

In the supercritical regime  $p > p_F$ , the global existence of solutions depends heavily on the asymptotic behavior of the initial data at infinity. Lee and Ni [LN92] made a breakthrough by introducing the concept of a second critical exponent related to the spatial decay of the initial value. By considering initial data bounded by a polynomial decay

$$u_0(x) \sim |x|^{-\omega} \text{ as } |x| \rightarrow \infty,$$

they identified the threshold  $\omega_c = 2/(p-1)$ . They established that solutions generally fail to exist globally when the decay rate satisfies  $\omega < \frac{2}{p-1}$ . Conversely, in the regime where  $\omega \geq \frac{2}{p-1}$ , they proved that a global solution exists, provided the initial data  $u_0$  is sufficiently small.

Having established the conditions under which solutions blow up, the focus naturally shifts to determining when this phenomenon occurs. Specifically, we are interested in estimating the lifespan of the solution, defined as the maximal time interval  $[0, T_\varepsilon)$  during which the solution exists. Lee and Ni [LN92] derived precise asymptotic estimates for this lifespan when  $m = 1$  in (1.1.6). They established that if the initial data decays sufficiently fast at infinity and is scaled by a small parameter  $\varepsilon$ , the lifespan satisfies the following sharp asymptotic estimates:

$$T_\varepsilon \simeq \begin{cases} \varepsilon^{-\left(\frac{1}{p-1} - \frac{N}{2}\right)^{-1}}, & 1 < p < p_F, \\ \exp(\varepsilon^{-(p-1)}), & p = p_F. \end{cases}$$

Following the extensive study of the Cauchy problem in the whole space  $\mathbb{R}^N$ , attention shifted towards understanding how domain geometry and boundary conditions influence the blow-up phenomenon. A natural and significant extension was the investigation of the semi-linear heat equation in exterior domains. Notably, Bandle and Levine [BL89] investigated

the problem under Dirichlet boundary conditions:

$$\begin{cases} u_t - \Delta u = u^p, & \text{in } D^c \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } D^c, \\ u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \end{cases} \quad (1.1.2)$$

where  $D^c$  represents the exterior domain. They stated that problem (1.1.2) admits no global positive solutions in the subcritical regime  $1 < p < p_F$ , whereas global solutions do exist in the supercritical regime  $p > p_F$ . The critical case, where  $p = p_F$ , was subsequently addressed by Suzuki [Suz96] for dimensions  $N \geq 3$  and by Ikeda and Sobajima [IS19b] for the two-dimensional case  $N = 2$ , confirming blow-up behavior at the threshold as well. Moreover, the two-dimensional exterior case exhibits a distinct behavior compared to the whole space. Notably, if  $u_0 \in H_0^1(D^c) \times L^2(D^c)$  and  $u_0 \log |x| \in L^1(D^c)$ , then the lifespan for (1.1.2) satisfies the upper bound

$$\text{LifeSpan}(u) \leq \begin{cases} C\varepsilon^{-\frac{p-1}{2-p}} (\log \varepsilon^{-1})^{p-1} & \text{if } 1 < p < 2, \\ \exp \exp(C\varepsilon^{-1}) & \text{if } p = 2. \end{cases}$$

The double exponential type phenomenon is attributed to the recurrence property of Brownian motion in  $\mathbb{R}^2$ , prompting further investigation into whether this recurrence effect similarly induces a unique blow-up rate in the inhomogeneous setting. Moreover, higher dimensional  $N \geq 3$  lifespan observed in [IS19b]. They found that the upper lifespan of solutions to problem (1.1.2) is same as the case of whole space  $\mathbb{R}^N$  for  $N \geq 3$ .

Subsequently, Levine and Zhang [LZ00] investigated the problem under Neumann boundary conditions to determine if the nature of the boundary interaction shifts the critical exponent. They considered the following initial-boundary value problem:

$$\begin{cases} u_t - \Delta u = u^p, & \text{in } D^c \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } D^c, \\ \frac{\partial u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty), \end{cases} \quad (1.1.3)$$

where  $\frac{\partial}{\partial n}$  denotes the outward unit normal vector on  $\partial D$ . They established that the critical exponent remains the Fujita exponent, indicating that the Neumann boundary does not alter the critical threshold compared to the whole space or the Dirichlet case. Specifically, they proved that if  $1 < p \leq p_F$ , every non-trivial non-negative solution blows up in finite time, whereas if  $p > p_F$ , global solutions exist for sufficiently small initial data.

In addition to Dirichlet and Neumann conditions, Rault [Rau11] examined the problem under Robin boundary conditions:

$$\begin{cases} u_t - \Delta u = u^p, & \text{in } D^c \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } D^c, \\ \frac{\partial u(x, t)}{\partial n} + \alpha u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \end{cases} \quad (1.1.4)$$

where  $\alpha > 0$  is a constant. Utilizing a truncation argument combined with a comparison principle, he established that the critical exponent remains the same as the classical

case. This result further confirms that the Fujita critical exponent is robust with respect to standard homogeneous boundary conditions in exterior domains.

A distinct and significant shift in critical behavior occurs when the equation includes a spatial forcing term. Bandle, Levine, and Zhang [BLZ00] analyzed the inhomogeneous semilinear heat equation in exterior domains, considering the problem:

$$\begin{cases} u_t - \Delta u = |u|^p + f(x), & \text{in } D^c \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } D^c, \\ u(x, t) = 0, & \text{in } \partial D. \end{cases} \quad (1.1.5)$$

Their investigation yielded the following classification regarding global existence and blow-up:

- (i) If  $p < \frac{N}{N-2}$  and the forcing term satisfies  $\int_{D^c} f(x) dx > 0$ , then Problem (1.1.5) admits no global solutions.
- (ii) If  $p > \frac{N}{N-2}$  and  $N \geq 3$ , then for sufficiently small forcing  $f(x)$  and initial data  $u_0(x)$ , global solutions exist.
- (iii) For any  $p > 1$ , there exists a negative forcing term  $f(x)$  such that the equation admits a global solution.

This study highlights a fundamental phenomenon: the presence of the spatial forcing term  $f(x)$  alters the critical threshold. Consequently, the critical exponent shifts from the classical Fujita exponent  $1 + \frac{2}{N}$  to the exponent  $\frac{N}{N-2}$ .

A natural generalization of the classical second-order theory involves higher-order diffusion operators. Now, consider the following higher-order problem

$$\begin{cases} u_t + (-\Delta)^m u = |u|^p, & \text{in } \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1.6)$$

where  $m > 1$  is an integer. Assuming bounded integrable initial data, they established that the critical exponent is given by  $p_F = 1 + \frac{2m}{N}$ . Specifically, they proved that if  $1 < p \leq 1 + \frac{2m}{N}$ , solutions blow up in finite time, whereas if  $p > 1 + \frac{2m}{N}$ , global solutions exist for sufficiently small initial data. Complementing this, Galaktionov and Pohozaev [GP02] proved that in the supercritical regime  $p > p_{Fuj}$ , the problem admits globally existing solutions, provided the initial data is sufficiently small. Subsequent research focused on the specific behavior of the biharmonic operator and the influence of the spatial decay of initial data. Caristi and Mitidieri [CM03] demonstrated the existence of global solutions for  $p > 1 + \frac{4}{N}$  even for slowly decaying initial data satisfying  $0 \leq u_0(x) \leq C(1 + |x|)^{-\beta}$ , provided that  $\beta > \frac{4}{p-1}$ . Gazzola and Grunau [GG07] further extended these global existence results to the equation with the nonlinearity  $|u|^{p-1}u$  (for  $m = 2$ ), establishing the precise decay threshold  $\beta = \frac{4}{p-1}$ . For additional results in this direction, we refer the reader to [FGG08; Gaz13; GG09]. Most recently, the influence of a spatial forcing term on the higher-order problem was investigated by Majdoub [Maj23]. He analyzed the problem (1.1.6) with an additive source term  $f(x)$  and determined that the presence of the forcing term shifts the critical threshold to the new exponent  $p = \frac{N}{N-2m}$ . It is also worth mentioning that Caristi and Mitidieri [CM03],

and subsequently Sun [Sun10], investigated the higher-order semilinear parabolic equation (1.1.6) subject to slowly decaying initial data. They proved that any nontrivial solution blows up in finite time and established an explicit upper bound for the lifespan

$$T_\varepsilon \leq C\varepsilon^{-\left(\frac{1}{p-1}-\frac{\kappa}{2m}\right)^{-1}}.$$

This analysis was conducted for initial data satisfying the lower bound

$$u_0(x) \geq \begin{cases} \delta, & |x| \leq \varepsilon_0, \\ C_0|x|^{-\kappa}, & |x| > \varepsilon_0, \end{cases} \quad (1.1.7)$$

where  $\delta > 0$  and  $\varepsilon_0 > 0$  are constants, and the decay rate parameter  $\kappa$  falls within the specific ranges:

$$\kappa \in \begin{cases} (N, \frac{2m}{p-1}), & 1 < p < p_F, \\ (0, N), & p = p_F. \end{cases}$$

These results reveal that the spatial decay of the initial data fundamentally influences the blow-up time, indicating that a slower rate of decay leads to a faster blow-up.

It is important to highlight that numerous papers have investigated several extensions of the nonexistence of the higher-order semilinear parabolic equations in  $\mathbb{R}^N$  [Ego+04; GP02; Phi15] and evolution equations with higher-order time-derivatives [JKS19; Lap03a; Lap03b] in exterior domains.

While compiling an exhaustive list of contributions to this vast field is nearly impossible, we have highlighted the developments most pertinent to this thesis. For a comprehensive overview of studies on Fujita exponents prior to 1990, we refer the reader to the survey paper by Levine [Lev90], while for subsequent developments, we refer to the survey by Deng and Levine [DL00] and the comprehensive monograph by Quittner and Souplet [QS07].

### 1.1.1 Organization of the manuscript

The present thesis is structured into six chapters, organized as follows:

Chapter 1 provides an introduction to the research topic. It presents the historical background of Fujita-type blow-up problems, outlines the scope of the study, and details the organization of the thesis.

Chapter 2 investigates the initial-boundary value problem for the semilinear biharmonic heat equation in an exterior domain  $D^c = \mathbb{R}^N \setminus \overline{B(0, 1)}$  with a zero forcing term. The problem is analyzed under six distinct homogeneous boundary conditions, including Navier, Dirichlet, and Neumann-type conditions. We construct specific test functions involving harmonic and biharmonic functions to prove the non-existence of global weak solutions in the subcritical and critical Fujita regimes and establish upper bounds for the lifespan of solutions.

Chapter 3 extends the analysis to the exterior biharmonic heat equation with a spatial forcing term  $f(x)$ . We examine how the asymptotic behavior of the forcing term influences the critical exponents. Specifically, we identify a second critical exponent in the sense of Lee and Ni, which serves as a threshold for the existence of stationary global solutions. The chapter distinguishes between necessary conditions for blow-up and sufficient conditions for global existence based on the decay rate of  $f(x)$ .

Chapter 4 focuses on the Cauchy problem for the higher-order semilinear parabolic equation  $u_t + (-\Delta)^m u = |u|^p$  in the whole Euclidean space  $\mathbb{R}^N$ . We define mild solutions and derive sharp asymptotic estimates for the lifespan  $T_\epsilon$ . This includes establishing a lower bound using a contraction mapping principle and an upper bound employing the test function method adapted via Faà di Bruno's formula.

Chapter 5 summarizes the main conclusions and contributions of the thesis.

Chapter 6 provides supplementary information, including essential inequalities and technical lemmas used throughout the proofs.

# Chapter 2

## Exterior biharmonic heat equation

Chapters 2 and 3 are adapted from the following article:

[**TT26b**] N. N. Tobakhanov and B. T. Torebek. On the critical behavior for the semilinear biharmonic heat equation with forcing term in exterior domain. *J. Differential Equations*, 451, Paper No. 113758, 53, 2026

In Chapter 2, we investigate the following fourth-order parabolic problem in an exterior domain:

$$\begin{cases} u_t + \Delta^2 u = |u|^p, & \text{in } D^c \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } D^c. \end{cases} \quad (2.0.1)$$

where  $D = \overline{B(0, 1)}$  denotes the closed unit ball in  $\mathbb{R}^N$ , and  $D^c := \mathbb{R}^N \setminus D$  represents its exterior. Our analysis focuses on the interplay between the biharmonic operator  $\Delta^2$  and the power-law nonlinearity  $|u|^p$  under various boundary constraints. We investigate problem (2.0.1) under six distinct sets of boundary conditions, all evaluated on  $\partial D \times (0, \infty)$ . For convenience, we assign a Roman numeral to each configuration:

(I) Navier boundary conditions:

$$\begin{cases} u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \\ \Delta u(x, t) = 0, & \text{in } \partial D \times (0, \infty); \end{cases} \quad (2.0.2)$$

(II) Dirichlet boundary conditions:

$$\begin{cases} u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \\ \frac{\partial u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty); \end{cases} \quad (2.0.3)$$

(III) Dirichlet-Navier boundary conditions:

$$\begin{cases} u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \\ \frac{\partial \Delta u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty); \end{cases} \quad (2.0.4)$$

(IV) Kuttler-Sigillito boundary conditions:

$$\begin{cases} \frac{\partial u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty), \\ \frac{\partial \Delta u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty); \end{cases} \quad (2.0.5)$$

⟨V⟩ Neumann-Navier boundary conditions:

$$\begin{cases} \Delta u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \\ \frac{\partial \Delta u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty); \end{cases} \quad (2.0.6)$$

⟨VI⟩ Neumann boundary conditions:

$$\begin{cases} \frac{\partial u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty), \\ \Delta u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \end{cases} \quad (2.0.7)$$

where  $n$  denotes the outward unit normal vector on  $\partial D$ . For convenience, we label the boundary conditions using Roman numerals. For instance, Problem I refers to the semilinear equation (2.0.1) equipped with the Navier boundary conditions, and so forth.

Before presenting our main results, we must first establish several preliminary concepts, with a particular focus on the properties of the test functions. In each of the following subsections, we construct specific test functions to the boundary conditions of the respective problem. Moreover, we establish test function estimates that are essential for our main proofs.

**Definition 2.0.1.** Let  $Q := [0, T] \times D^c$ . We say that a function  $u \in L^p_{\text{loc}}(\overline{Q})$  is a weak solution to Problems I–VI if there exists  $T > 0$  such that the integral identity

$$\int_Q |u|^p \varphi_j \, dx \, dt + \int_{D^c} u_0(x) \varphi_j(x, 0) \, dx = - \int_Q u \partial_t \varphi_j \, dx \, dt + \int_Q u \Delta^2 \varphi_j \, dx \, dt$$

holds for every  $j \in \{1, 2, 3, 4, 5, 6\}$  and for all test functions  $\varphi_j$  satisfying:

- (i)  $\varphi_j \in C^{1,4}_{t,x}(\overline{Q}) \cap C(Q)$  and  $\varphi_j \geq 0$ ;
- (ii)  $\varphi_j(t, \cdot) \equiv 0$  for all  $t \geq T$ ;
- (iii)  $\varphi_j(\cdot, x) \equiv 0$  for all  $|x| \geq R$  (where  $R > 1$ ).

Properties (i) through (iii) are universal for all test functions. A subsequent property, which we will denote as (iv), will identify the weak solutions for each of the respective boundary value problems I–VI.

| (iv)        |                                                                                  |            |                                                                                                              |
|-------------|----------------------------------------------------------------------------------|------------|--------------------------------------------------------------------------------------------------------------|
| Problem I   | $\varphi_1 = \Delta \varphi_1 = 0$ on $\partial D$ ;                             | Problem IV | $\frac{\partial \varphi_4}{\partial n} = \frac{\partial \Delta \varphi_4}{\partial n} = 0$ on $\partial D$ ; |
| Problem II  | $\varphi_2 = \frac{\partial \varphi_2}{\partial n} = 0$ on $\partial D$ ;        | Problem V  | $\Delta \varphi_5 = \frac{\partial \Delta \varphi_5}{\partial n} = 0$ on $\partial D$ ;                      |
| Problem III | $\frac{\partial \varphi_3}{\partial n} = \Delta \varphi_3 = 0$ on $\partial D$ ; | Problem VI | $\varphi_6 = \frac{\partial \Delta \varphi_6}{\partial n} = 0$ on $\partial D$ .                             |

## 2.1 Main results

**Theorem 2.1.1.** Assume  $u_0 \in L^1_{\text{loc}}(D^c)$ . If  $\int_{D^c} u_0(x) \, dx \geq 0$  with  $u_0 \not\equiv 0$ , then problems III–V associated with (2.0.1) admit no global weak solution for

$$1 < p \leq p_{\text{Fuj}} := 1 + \frac{4}{N}.$$

**Remark 2.1.2.** Theorem 2.1.1 assumes that  $\int_{D^c} u_0(x) dx \geq 0$ ,  $u_0 \not\equiv 0$ . This means that the initial data  $u_0$  that satisfy  $\int_{D^c} u_0(x) dx = 0$ , but  $u_0 \not\equiv 0$  is also included in this class of functions.

**Remark 2.1.3.** Theorem 2.1.1 addresses the subcritical and critical cases, but does not contain results for the supercritical regime  $p > 1 + \frac{4}{N}$ . However, we note that the exponent  $p_{\text{Fuj}} := 1 + \frac{4}{N}$  coincides with the known Fujita critical exponent for the biharmonic heat equation given in the whole space  $\mathbb{R}^N$  (see [GP02; CM03; GG07; Ego+04]):

$$u_t + \Delta^2 u = |u|^p.$$

Consequently, we conjecture that if

$$p > p_{\text{Fuj}} = 1 + \frac{4}{N},$$

then problems I–VI admit global-in-time solutions for sufficiently small initial data  $u_0$ . Proving this supercritical case relies on analyzing the properties of the Green's functions associated with Problems I–VI, an extensive study that requires separate consideration.

By following the standard theory for local existence, it can be verified that there exists a unique local-in-time nonnegative solution to (2.0.1). Accordingly, we define the upper bound for the lifespan  $T_\varepsilon$  of a local solution  $u_\varepsilon$  as follows:

$$T_\varepsilon := \sup \{T > 0 : u_\varepsilon \text{ is a weak solution to problems I–VI in } [0, T) \times D^c\}.$$

With this definition, we present the following assertion regarding the relationship between the lifespan and the small parameter  $\varepsilon$ .

**Theorem 2.1.4.** *Let  $u_\varepsilon$  be the unique local weak solution to (2.0.1) subject to boundary conditions III–V. Suppose  $1 < p \leq p_{\text{Fuj}}$ ,  $N \geq 3$ , and  $u_0 \in L^1_{\text{loc}}(D^c)$  satisfies:*

$$\int_{D^c} u_0(x) dx \geq 0, \quad u_0 \not\equiv 0.$$

*Then, there exists a threshold  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0]$ , the following upper bounds hold:*

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-\left(\frac{1}{p-1} - \frac{N}{4}\right)^{-1}} & \text{if } p \in (1, p_{\text{Fuj}}), \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = p_{\text{Fuj}}. \end{cases}$$

**Remark 2.1.5.** At present, lifespan estimates for local solutions in the one- and two-dimensional cases remain undetermined. This limitation arises because the test function method employed in this work relies on specific kernel properties that do not scale appropriately for  $N < 3$ .

**Theorem 2.1.6.** *Let  $H(x)$  be defined as in (2.1.6), and assume  $u_0 \in L^1_{\text{loc}}(D^c)$ . If*

$$\int_{D^c} u_0(x) H(x) dx \geq 0, \quad \text{with } u_0 \not\equiv 0,$$

*then problem I and problem VI associated with (2.0.1) admit no global weak solution for*

$$1 < p \leq p_{\text{Fuj}} := 1 + \frac{4}{N}.$$

Moreover for  $N = 2$ ,  $\text{LifeSpan}(u)$  has the following upper bound:

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-\frac{2(p-1)}{3-p}} (\log \varepsilon^{-1})^{p-1} & \text{if } 1 < p < 3, \\ \exp \exp (\varepsilon^{-2}) & \text{if } p = 3. \end{cases}$$

**Remark 2.1.7.** The lifespan of the higher domain  $N \geq 3$  case is the same as the previous theorem 4.1.2.

**Theorem 2.1.8.** Let  $N \geq 3$ , and  $B(x)$  be defined as in (2.1.20), and assume  $u_0 \in L^1_{\text{loc}}(D^c)$ . If

$$\int_{D^c} u_0(x) B(x) dx \geq 0, \quad \text{with } u_0 \not\equiv 0,$$

then problem II associated with (2.0.1) admits no global weak solution for

$$1 < p \leq p_{\text{Fuj}} := 1 + \frac{4}{N}.$$

Moreover for  $N = 4$ ,  $\text{LifeSpan}(u)$  has the following upper bound:

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-\frac{p-1}{2-p}} (\log \varepsilon^{-1})^{-1} & \text{if } 1 < p < 2, \\ \exp \exp (\varepsilon^{p-1}) & \text{if } p = 2. \end{cases}$$

**Remark 2.1.9.** The lifespan of the higher domain  $N = 3$  and  $N \geq 5$  cases are the same as theorem 4.1.2. However, the case when  $N = 2$  is left open. The reason is that our method is not proper to show the nonexistence of a critical case.

## 2.1.1 Properties of test functions

### Properties of $\varphi_1(x, t)$ .

Let us construct the test function  $\varphi_1$  for Problem I by separating the temporal and spatial variables. We define

$$\varphi_1(t, x) = \eta(t)\psi(x), \quad (t, x) \in Q. \quad (2.1.1)$$

For a sufficiently large time  $T > 0$ , the temporal component is given by

$$\eta(t) = \zeta^\ell \left( \frac{t}{T} \right), \quad t \geq 0, \quad (2.1.2)$$

where  $\ell \geq \frac{4p}{p-1}$ , and  $\zeta \in C^\infty([0, \infty))$  is a smooth cut-off function satisfying

$$\zeta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \text{decreasing}, & \text{if } \frac{1}{2} < s < 1, \\ 0, & \text{if } s \geq 1. \end{cases} \quad (2.1.3)$$

Similarly, for a sufficiently large radius  $R > 1$ , the spatial component is defined as

$$\psi(x) = H(x) \xi^\ell \left( \frac{|x|}{R} \right), \quad x \in D^c, \quad (2.1.4)$$

where  $\xi \in C^\infty([0, \infty))$  is a smooth spatial cut-off function such that

$$\xi(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq 1, \\ \text{decreasing}, & \text{if } 1 < s < 2, \\ 0, & \text{if } s \geq 2, \end{cases} \quad (2.1.5)$$

and  $H(x)$  is a harmonic function in  $D^c$  given explicitly by

$$H(x) = \begin{cases} \ln|x|, & \text{if } N = 2, \\ 1 - |x|^{2-N}, & \text{if } N \geq 3. \end{cases} \quad (2.1.6)$$

**Lemma 2.1.10.** The function  $H(x)$  satisfies the following properties:

- (i)  $H(x) \geq 0$ ;
- (ii)  $\Delta H(x) = \Delta^2 H(x) = 0$  in  $D^c$ ;
- (iii)  $H(x) = \Delta H(x) = \frac{\partial(\Delta H(x))}{\partial \nu} = 0$  on  $\partial D$ .

*Proof.* Property (i) is clear from the explicit definition of  $H(x)$  given in (2.1.6), as  $|x| \geq 1$  in the exterior domain  $D^c$ . Property (ii) follows immediately from the fact that  $H(x)$  is a harmonic function, which trivially implies it is also biharmonic. Finally, property (iii) holds because evaluating (2.1.6) at  $|x| = 1$  yields  $H(x) = 0$  on  $\partial D$ . Furthermore, since  $\Delta H(x) \equiv 0$  everywhere in  $D^c$ , its normal derivative on the boundary  $\partial D$  is zero.  $\square$

Moreover, the definition of the harmonic function in (2.1.6) requires us to treat the cases  $N = 2$  and  $N \geq 3$  separately.

### The case $N = 2$

**Lemma 2.1.11.** For  $N = 2$  and  $R < |x| < 2R$ , the following estimates hold for sufficiently large  $R$ :

$$H(x) \leq C \ln R \quad \text{and} \quad |\nabla H(x)| \leq CR^{-1}.$$

*Proof.* For  $N = 2$ , the harmonic function defined in (2.1.6) is  $H(x) = \ln|x|$ . Within the annulus  $R < |x| < 2R$ , we can estimate  $H(x)$  as follows:

$$H(x) \leq \ln(2R) = \ln R + \ln 2 \leq 2 \ln R = C \ln R.$$

By direct computation, the gradient is given by

$$\nabla H(x) = \frac{x}{|x|^2}.$$

Taking the norm and applying the lower bound  $|x| > R$ , we obtain

$$|\nabla H(x)| = \frac{1}{|x|} < R^{-1},$$

which yields the desired gradient bound with  $C = 1$ . This completes the proof.  $\square$

**Lemma 2.1.12.** For  $N = 2$ , assume  $R > 0$  and  $T > 0$  are sufficiently large. Then, the following estimate holds:

$$\int_Q \varphi_1^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_1(t, x)|^{p'} dx dt \leq CT^{1-p'} R^2 \ln R.$$

*Proof.* Using the structure of the test function defined in (2.1.1) and the fact that  $p' - \frac{1}{p-1} = 1$ , we can deduce that

$$\varphi_1^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_1(t, x)|^{p'} = \psi(x) \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{p'}.$$

This relation allows us to decouple as

$$\int_Q \varphi_1^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_1(t, x)|^{p'} dx dt = \left( \int_{D^c} \psi(x) dx \right) \left( \int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{p'} dt \right).$$

For the spatial integral, using the definition of  $\psi(x)$  from (2.1.4) and applying the bound from Lemma 2.1.11, we have

$$\begin{aligned} \int_{D^c} \psi(x) dx &= \int_{1 < |x| < 2R} H(x) \xi^\ell \left( \frac{|x|}{R} \right) dx \\ &\leq C \ln R \int_{1 < |x| < 2R} \xi^\ell \left( \frac{|x|}{R} \right) dx. \end{aligned} \quad (2.1.7)$$

Passing to polar coordinates, we obtain

$$\int_{D^c} \psi(x) dx \leq C \ln R \int_1^{2R} r dr \leq CR^2 \ln R. \quad (2.1.8)$$

For the temporal integral, the properties of the cut-off function  $\zeta$  given in (2.1.2) and (2.1.3) yield

$$\begin{aligned} \int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{p'} dt &= T^{-p'} \int_{\frac{T}{2}}^T \zeta^{-\frac{\ell}{p-1}} \left( \frac{t}{T} \right) \zeta^{(\ell-1)p'} \left( \frac{t}{T} \right) \left| \zeta' \left( \frac{t}{T} \right) \right|^{p'} dt \\ &\leq CT^{-p'} \int_{\frac{T}{2}}^T \zeta^{\ell-p'} \left( \frac{t}{T} \right) dt \\ &= CT^{1-p'} \int_{\frac{1}{2}}^1 \zeta^{\ell-p'}(s) ds \\ &\leq CT^{1-p'}, \end{aligned} \quad (2.1.9)$$

where we used the substitution  $s = t/T$  and the fact that  $\ell \geq p'$ . Finally, combining the spatial bound (2.1.8) and the temporal bound (2.1.9) yields the desired result. This completes the proof.  $\square$

The subsequent estimate arises from the standard calculation. The function  $\xi$  is radial, which means the estimate can be directly calculated by changing to the polar coordinates. The power of  $R$  provides a relation of the derivatives to the cut-off function.

**Lemma 2.1.13.** For  $R < |x| < 2R$ , the following estimates hold:

$$\left| \nabla \left( \Delta \xi^\ell \left( \frac{|x|}{R} \right) \right) \right| \leq CR^{-3} \xi^{\ell-4} \left( \frac{|x|}{R} \right),$$

and

$$\left| \Delta^2 \xi^\ell \left( \frac{|x|}{R} \right) \right| \leq CR^{-4} \xi^{\ell-4} \left( \frac{|x|}{R} \right).$$

Furthermore, applying the product expansion for the biharmonic operator to (2.1.4), we obtain

$$\begin{aligned}\Delta^2\psi(x) &= \xi^\ell \left( \frac{|x|}{R} \right) \Delta^2 H(x) + 4\nabla(\Delta H(x)) \cdot \nabla \xi^\ell \left( \frac{|x|}{R} \right) \\ &\quad + 6\Delta H(x) \Delta \xi^\ell \left( \frac{|x|}{R} \right) + 4\nabla H(x) \cdot \nabla \left( \Delta \xi^\ell \left( \frac{|x|}{R} \right) \right) \\ &\quad + H(x) \Delta^2 \xi^\ell \left( \frac{|x|}{R} \right).\end{aligned}\tag{2.1.10}$$

Since  $H(x)$  is a harmonic function, we know from Lemma 2.1.10 that  $\Delta H(x) = \Delta^2 H(x) = 0$ , and consequently  $\nabla(\Delta H(x)) = 0$ . This significantly simplifies the expansion to

$$\Delta^2\psi(x) = H(x) \Delta^2 \xi^\ell \left( \frac{|x|}{R} \right) + 4\nabla H(x) \cdot \nabla \left( \Delta \xi^\ell \left( \frac{|x|}{R} \right) \right).$$

Applying the bounds derived in Lemmas 2.1.11 and 2.1.13, we can estimate this expression for sufficiently large  $R < |x| < 2R$  as follows:

$$\begin{aligned}|\Delta^2\psi(x)| &\leq C (R^{-4} \ln R + R^{-4}) \xi^{\ell-4} \left( \frac{|x|}{R} \right) \\ &\leq CR^{-4} \ln R \xi^{\ell-4} \left( \frac{|x|}{R} \right).\end{aligned}\tag{2.1.11}$$

**Lemma 2.1.14.** For  $N = 2$ , assume  $R > 0$  and  $T > 0$  are sufficiently large. Then, the following estimate holds:

$$\int_Q \varphi_1^{-\frac{1}{p-1}}(x, t) |\Delta^2 \varphi_1(x, t)|^{p'} dx dt \leq CT^1 R^{-\frac{2(p+1)}{p-1}} (\ln R)^{\frac{p}{p-1}}.$$

*Proof.* Using the separable structure of the test function defined in (2.1.1), we deduce

$$\varphi_1^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_1(t, x)|^{p'} = \eta(t) \psi^{-\frac{1}{p-1}}(x) |\Delta^2 \psi(x)|^{p'}.$$

This relation allows us to decouple as

$$\int_Q \varphi_1^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_1(t, x)|^{p'} dx dt = \left( \int_0^T \eta(t) dt \right) \left( \int_{D^c} \psi^{-\frac{1}{p-1}}(x) |\Delta^2 \psi(x)|^{p'} dx \right).$$

By the definition of the temporal cut-off function in (2.1.2), we evaluate the first integral as

$$\int_0^T \eta(t) dt = \int_0^T \zeta^\ell \left( \frac{t}{T} \right) dt = CT \int_0^1 \zeta^\ell(s) ds = CT.\tag{2.1.12}$$

Moreover, using Lemma 2.1.10 and the bound (2.1.11), we can estimate

$$\begin{aligned}\int_{D^c} \psi^{-\frac{1}{p-1}}(x) |\Delta^2 \psi(x)|^{p'} dx &= \int_{R < |x| < 2R} \left[ H(x) \xi^\ell \left( \frac{|x|}{R} \right) \right]^{-\frac{1}{p-1}} |\Delta^2 \psi(x)|^{p'} dx \\ &\leq CR^{-4p'} (\ln R)^{\frac{p}{p-1}} \int_{R < |x| < 2R} \xi^{\ell-4p'} \left( \frac{|x|}{R} \right) dx \\ &\leq CR^{-\frac{2(p+1)}{p-1}} (\ln R)^{\frac{p}{p-1}},\end{aligned}\tag{2.1.13}$$

where we used the substitution  $r = |x|$ . Combining (2.1.12) and (2.1.13) yields the desired result. This completes the proof.  $\square$

### The Case When $N \geq 3$

**Lemma 2.1.15.** For  $N \geq 3$  and  $R < |x| < 2R$ , the following estimates hold:

$$H(x) \leq C \quad \text{and} \quad |\nabla H(x)| \leq CR^{1-N}.$$

*Proof.* For  $N \geq 3$ , the harmonic function defined in (2.1.6) is  $H(x) = 1 - |x|^{2-N}$ . Since the exponent  $2 - N$  is negative, the term  $|x|^{2-N}$  is strictly positive for all  $|x| > R > 0$ . Therefore, we immediately obtain the upper bound

$$H(x) < 1 \leq C.$$

By direct computation, the gradient is given by

$$\nabla H(x) = -(2 - N)|x|^{1-N} \frac{x}{|x|} = (N - 2) \frac{x}{|x|^N}.$$

Taking the norm and applying the lower bound  $|x| > R$ , we deduce

$$|\nabla H(x)| = \frac{N - 2}{|x|^{N-1}} < (N - 2)R^{1-N} \leq CR^{1-N},$$

which establishes the desired gradient bound. This completes the proof.  $\square$

**Lemma 2.1.16.** For  $N \geq 3$ , assume  $R > 0$  and  $T > 0$  are sufficiently large. Then, the following estimate holds:

$$\int_Q \varphi_1(x, t)^{-\frac{1}{p-1}} |\partial_t \varphi_1(x, t)|^{p'} dx dt \leq CT^{1-p'} R^N.$$

*Proof.* Using the separable structure of the test function defined in (2.1.1), we first decouple the integral into spatial and temporal components:

$$\int_Q \varphi_1^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_1(t, x)|^{p'} dx dt = \left( \int_{D^c} \psi(x) dx \right) \left( \int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{p'} dt \right).$$

We evaluate these two components separately. Using the definition of  $\psi$  in (2.1.4) and the upper bound  $H(x) \leq C$  established in Lemma 2.1.15, we can bound the spatial integral. Passing to polar coordinates, we obtain

$$\int_{D^c} \psi(x) dx \leq C \int_{1 < |x| < 2R} \xi^\ell \left( \frac{|x|}{R} \right) dx \leq C \int_1^{2R} r^{N-1} dr \leq CR^N. \quad (2.1.14)$$

The temporal component is evaluated exactly as in the  $N = 2$  case. As previously derived in (2.1.9), this integral satisfies

$$\int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{p'} dt \leq CT^{1-p'}. \quad (2.1.15)$$

Multiplying the spatial bound (2.1.14) by the temporal bound (2.1.15) directly yields the desired result, completing the proof.  $\square$

Since  $H(x)$  is a harmonic function, we know from Lemma 2.1.10 that  $\Delta H(x) = \Delta^2 H(x) = 0$ , and consequently  $\nabla(\Delta H(x)) = 0$ . This significantly simplifies the expansion to

$$\Delta^2 \psi(x) = H(x) \Delta^2 \xi^\ell \left( \frac{|x|}{R} \right) + 4 \nabla H(x) \cdot \nabla \left( \Delta \xi^\ell \left( \frac{|x|}{R} \right) \right).$$

Applying the bounds derived in Lemmas 2.1.13 and 2.1.15, we derive

$$\begin{aligned} |\Delta^2 \psi(x)| &\leq C (R^{-4} + R^{1-N} R^{-3}) \xi^{\ell-4} \left( \frac{|x|}{R} \right) \\ &\leq C R^{-4} \xi^{\ell-4} \left( \frac{|x|}{R} \right), \end{aligned} \quad (2.1.16)$$

for  $R < |x| < 2R$ .

**Lemma 2.1.17.** For  $N \geq 3$ , assume  $R > 0$  and  $T > 0$  are sufficiently large. Then, the following estimate holds:

$$\int_Q \varphi_1^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_1(t, x)|^{p'} dx dt \leq C T R^{N-4p'}.$$

*Proof.* Using the separable structure of the test function defined in (2.1.1), we first factor the integral into temporal and spatial components:

$$\int_Q \varphi_1^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_1(t, x)|^{p'} dx dt = \left( \int_0^T \eta(t) dt \right) \left( \int_{D^c} \psi^{-\frac{1}{p-1}}(x) |\Delta^2 \psi(x)|^{p'} dx \right).$$

As previously established in (2.1.12), the temporal integral evaluates directly to

$$\int_0^T \eta(t) dt = C T. \quad (2.1.17)$$

Using Lemma 2.1.15 and the bound (2.1.16), we estimate the spatial component. Passing to polar coordinates, we obtain

$$\begin{aligned} \int_{D^c} \psi^{-\frac{1}{p-1}}(x) |\Delta^2 \psi(x)|^{p'} dx &\leq C R^{-4p'} \int_{R < |x| < 2R} \xi^{\ell-4p'} \left( \frac{|x|}{R} \right) dx \\ &\leq C R^{-4p'} \int_R^{2R} r^{N-1} dr \\ &\leq C R^{N-4p'}. \end{aligned} \quad (2.1.18)$$

Combining the temporal evaluation (2.1.17) and the spatial bound (2.1.18) yields the desired result. This completes the proof.  $\square$

### Properties of $\varphi_2(x, t)$ .

We introduce a second class of trial functions, denoted by  $\varphi_2(t, x)$ , which are defined as

$$\varphi_2(t, x) = \eta(t) \phi(x) = \zeta^\ell \left( \frac{t}{T} \right) B(x) \xi^\ell \left( \frac{|x|}{R} \right), \quad (t, x) \in Q. \quad (2.1.19)$$

In this formulation,  $\eta(t)$  is the temporal cut-off function defined in (2.1.2),  $\xi$  is the spatial cut-off function defined in (2.1.5), and  $B(x)$  is a biharmonic function in  $D^c$  given by

$$B(x) = \begin{cases} |x|^2 \ln |x| - |x|^2 + \ln |x| + 1 & \text{if } N = 2, \\ |x|^{-1} + |x| - 2 & \text{if } N = 3, \\ 2 \ln |x| + |x|^{-2} - 1 & \text{if } N = 4, \\ |x|^{2-N} - 1 + \frac{N-2}{N-4} (1 - |x|^{4-N}) & \text{if } N \geq 5. \end{cases} \quad (2.1.20)$$

**Lemma 2.1.18.** The function  $B(x)$  satisfies the following properties:

- (i)  $B(x) \geq 0$  in  $D^c$ ;
- (ii)  $\Delta^2 B(x) = 0$  in  $D^c$ ;
- (iii)  $B(x) = \frac{\partial B(x)}{\partial n} = 0$  on  $\partial D$ .

The asymptotic behavior of these test functions depends on the dimension  $N$ . In particular, the cases  $N = 2$  and  $N = 4$  involve logarithmic singularities that require a separate treatment from the cases where  $N = 3$  or  $N \geq 5$ . Consequently, we shall analyze the properties of the biharmonic function (2.1.20) and the resulting integral estimates by distinguishing these dimensional regimes.

### The case when $N = 2$ .

**Lemma 2.1.19.** For  $R < |x| < 2R$ , the following estimates hold:

$$\begin{aligned} B(x) &\leq CR^2 \ln R, \\ |\nabla B(x)| &\leq CR \ln R, \\ |\Delta B(x)| &\leq C \ln R, \end{aligned}$$

and

$$|\nabla \cdot \Delta B(x)| \leq CR^{-1}.$$

*Proof.* Let  $r = |x|$ . From the definition of  $B(x)$  in (2.1.20) for  $N = 2$ , we have

$$B(x) = r^2 \ln r - r^2 + \ln r + 1 \leq CR^2 \ln R,$$

for  $R < r < 2R$ . Differentiating with respect to  $r$  yields the radial derivative

$$h'(r) = 2r \ln r - r + r^{-1}.$$

Since  $\nabla B(x) = h'(r) \frac{x}{r}$ , it follows that  $|\nabla B(x)| = |h'(r)|$ . To compute the Laplacian, we find the second radial derivative:

$$h''(r) = 2 \ln r + 1 - r^{-2}.$$

Using the expression for the Laplacian in radial coordinates for  $N = 2$ , we obtain

$$\Delta B(x) = h''(r) + \frac{1}{r} h'(r) = (2 \ln r + 1 - r^{-2}) + (2 \ln r - 1 + r^{-2}) = 4 \ln r.$$

Thus, we immediately have  $|\Delta B(x)| \leq C \ln R$ . Finally, the gradient of the Laplacian is given by

$$\nabla(\Delta B(x)) = \nabla(4 \ln |x|) = \frac{4x}{|x|^2},$$

which leads to the bound

$$|\nabla(\Delta B(x))| = \frac{4}{|x|} \leq CR^{-1}.$$

This completes the proof.  $\square$

**Lemma 2.1.20.** For  $R > 0$  and  $T > 0$  sufficiently large, the following estimate holds:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_2(t, x)|^{p'} dx dt \leq CT^{1-p'} R^4 \ln R.$$

*Proof.* Based on the separable structure of the test function  $\varphi_2$  defined in (2.1.19), we factor the integral into its spatial and temporal components as follows:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_2(t, x)|^{p'} dx dt = \left( \int_{D^c} \phi(x) dx \right) \left( \int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{p'} dt \right).$$

To estimate the spatial integral, we utilize the bound from Lemma 2.1.19. For  $\ell > \frac{4p}{p-1}$  and  $R$  sufficiently large, we obtain

$$\begin{aligned} \int_{D^c} \phi(x) dx &= \int_{1 < |x| < 2R} B(x) \xi^\ell \left( \frac{|x|}{R} \right) dx \\ &\leq CR^2 \ln R \int_{1 < |x| < 2R} \xi^\ell \left( \frac{|x|}{R} \right) dx. \end{aligned} \tag{2.1.21}$$

By passing to polar coordinates and evaluating the resulting integral, we find

$$\int_{D^c} \phi(x) dx \leq CR^2 \ln R \int_1^{2R} r dr \leq CR^4 \ln R.$$

The temporal integral is evaluated in the same manner as (2.1.9). Combining these spatial and temporal estimates completes the proof.  $\square$

By combining the estimates for the biharmonic kernel  $B(x)$  from Lemma 2.1.19 with the derivative bounds for the cut-off function established in Lemma 2.1.13, we arrive at the following combined estimates for the spatial component.

**Lemma 2.1.21.** For  $R < |x| < 2R$ , the following estimates hold:

$$\begin{aligned} \left| B(x) \Delta^2 \xi^\ell \left( \frac{|x|}{R} \right) \right| &\leq CR^{-2} \ln R \xi^{\ell-4} \left( \frac{|x|}{R} \right), \\ \left| \nabla B(x) \cdot \nabla \left( \Delta \xi^\ell \left( \frac{|x|}{R} \right) \right) \right| &\leq CR^{-2} \ln R \xi^{\ell-4} \left( \frac{|x|}{R} \right), \\ \left| \Delta B(x) \Delta \xi^\ell \left( \frac{|x|}{R} \right) \right| &\leq CR^{-2} \ln R \xi^{\ell-4} \left( \frac{|x|}{R} \right), \\ \left| \nabla(\Delta B(x)) \cdot \nabla \xi^\ell \left( \frac{|x|}{R} \right) \right| &\leq CR^{-2} \xi^{\ell-4} \left( \frac{|x|}{R} \right), \end{aligned}$$

where  $\cdot$  denotes the standard inner product in  $\mathbb{R}^N$ .

**Lemma 2.1.22.** For  $N = 2$ , let  $R > 0$  and  $T > 0$  be sufficiently large. Then, the following estimate holds:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_2(t, x)|^{p'} dx dt \leqslant CTR^{-\frac{2}{p-1}} (\ln R)^{\frac{p}{p-1}}.$$

*Proof.* By utilizing the separable structure of the test function  $\varphi_2$ , the integral factors into temporal and spatial components:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_2(t, x)|^{p'} dx dt = \left( \int_0^T \eta(t) dt \right) \left( \int_{D^c} \phi^{-\frac{1}{p-1}}(x) |\Delta^2 \phi(x)|^{p'} dx \right).$$

Applying the product expansion formula for the biharmonic operator in (5.0.2) and incorporating the estimates from Lemma 2.1.21, we find that for  $R < |x| < 2R$ :

$$|\Delta^2 \phi(x)| \leqslant CR^{-2} \ln R \xi^{\ell-4} \left( \frac{|x|}{R} \right). \quad (2.1.22)$$

Furthermore, the lower bound for the biharmonic function  $B(x)$  in the region  $R < |x| < 2R$  is provided by Lemma 2.1.19. Substituting these into the spatial integral and passing to polar coordinates, we obtain

$$\begin{aligned} \int_{D^c} \phi^{-\frac{1}{p-1}}(x) |\Delta^2 \phi(x)|^{p'} dx &= \int_{R < |x| < 2R} \left[ B(x) \xi^\ell \left( \frac{|x|}{R} \right) \right]^{-\frac{1}{p-1}} |\Delta^2 \phi(x)|^{p'} dx \\ &\leqslant CR^{-2p'} (\ln R)^{p'} (R^2 \ln R)^{-\frac{1}{p-1}} \int_R^{2R} r dr \\ &\leqslant CR^{-\frac{2p}{p-1}} (\ln R)^{\frac{p}{p-1}} \int_R^{2R} r dr \\ &\leqslant CR^{-\frac{2}{p-1}} (\ln R)^{\frac{p}{p-1}}. \end{aligned} \quad (2.1.23)$$

Finally, by combining the temporal evaluation from (2.1.12) with the spatial estimate in (2.1.23), we arrive at the desired outcome.  $\square$

### The case when $N = 4$

**Lemma 2.1.23.** For  $R < |x| < 2R$ , the following estimates hold:

$$\begin{aligned} B(x) &\leqslant C \ln R, \\ |\nabla B(x)| &\leqslant CR^{-1}, \\ |\Delta B(x)| &\leqslant CR^{-2}, \\ |\nabla(\Delta B(x))| &\leqslant CR^{-3}. \end{aligned}$$

*Proof.* Let  $r = |x|$ . From the definition of  $B(x)$  in (2.1.20) for  $N = 4$ , we have

$$B(x) = 2 \ln r + r^{-2} - 1 \leqslant C \ln R,$$

for  $R < r < 2R$ . Differentiating with respect to  $r$  gives the radial derivative

$$h'(r) = \frac{2}{r} - \frac{2}{r^3} = \frac{2}{r} (1 - r^{-2}).$$

Since  $\nabla B(x) = h'(r)\frac{x}{r}$ , it follows that  $|\nabla B(x)| = |h'(r)|$ . For large  $r$ , the term  $2r^{-1}$  dominates, yielding  $|\nabla B(x)| \leq CR^{-1}$ .

To compute the Laplacian in four dimensions, we find the second radial derivative:

$$h''(r) = -\frac{2}{r^2} + \frac{6}{r^4}.$$

Substituting these into the radial Laplacian formula for  $N = 4$ , we obtain

$$\Delta B(x) = h''(r) + \frac{3}{r}h'(r) = \left(-\frac{2}{r^2} + \frac{6}{r^4}\right) + \frac{3}{r}\left(\frac{2}{r} - \frac{2}{r^3}\right) = \frac{4}{r^2}.$$

This yields the estimate  $|\Delta B(x)| \leq CR^{-2}$ . Finally, the gradient of the Laplacian is computed as

$$\nabla(\Delta B(x)) = \nabla\left(\frac{4}{|x|^2}\right) = -\frac{8x}{|x|^4},$$

which leads to the bound

$$|\nabla(\Delta B(x))| = \frac{8}{r^3} \leq CR^{-3}.$$

This completes the proof.  $\square$

**Lemma 2.1.24.** For  $N = 4$ , let  $R > 0$  and  $T > 0$  be sufficiently large. Then, the following estimate holds:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_2(t, x)|^{p'} dx dt \leq CT^{1-p'} R^4 \ln R.$$

*Proof.* By the separable structure of the trial function  $\varphi_2$  defined in (2.1.19), the integral factors into independent temporal and spatial components:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_2(t, x)|^{p'} dx dt = \left(\int_{D^c} \phi(x) dx\right) \left(\int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{p'} dt\right).$$

To estimate the spatial integral for the  $N = 4$  case, we apply the bound  $B(x) \leq C \ln R$  from Lemma 2.1.23. For  $\ell > \frac{4p}{p-1}$  and  $R$  sufficiently large, we have

$$\begin{aligned} \int_{D^c} \phi(x) dx &= \int_{1 < |x| < 2R} B(x) \xi^\ell \left(\frac{|x|}{R}\right) dx \\ &\leq C \ln R \int_{1 < |x| < 2R} \xi^\ell \left(\frac{|x|}{R}\right) dx. \end{aligned} \tag{2.1.24}$$

By transforming the integral into polar coordinates

$$\int_{D^c} \phi(x) dx \leq C \ln R \int_1^{2R} r^3 dr \leq CR^4 \ln R.$$

The temporal integral is evaluated according to the standard cut-off derivative bound derived in (2.1.9). Combining these two estimates provides the desired result and completes the proof.  $\square$

By noting the estimates for the biharmonic function  $B(x)$  from Lemma 2.1.23 with the derivative bounds for the cut-off function established in Lemma 2.1.13, we obtain the following spatial estimates for  $N = 4$ .

**Lemma 2.1.25.** For  $R < |x| < 2R$ , the following estimates hold:

$$\begin{aligned} \left| B(x) \Delta^2 \xi^\ell \left( \frac{|x|}{R} \right) \right| &\leq CR^{-4} \ln R \xi^{\ell-4} \left( \frac{|x|}{R} \right), \\ \left| \nabla B(x) \cdot \nabla \left( \Delta \xi^\ell \left( \frac{|x|}{R} \right) \right) \right| &\leq CR^{-4} \xi^{\ell-4} \left( \frac{|x|}{R} \right), \\ \left| \Delta B(x) \Delta \xi^\ell \left( \frac{|x|}{R} \right) \right| &\leq CR^{-4} \xi^{\ell-4} \left( \frac{|x|}{R} \right), \\ \left| \nabla(\Delta B(x)) \cdot \nabla \xi^\ell \left( \frac{|x|}{R} \right) \right| &\leq CR^{-4} \xi^{\ell-4} \left( \frac{|x|}{R} \right), \end{aligned}$$

where  $\cdot$  denotes the standard inner product in  $\mathbb{R}^N$ .

**Lemma 2.1.26.** For  $N = 4$ , let  $R > 0$  and  $T > 0$  be sufficiently large. Then, the following estimate holds:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_2(t, x)|^{p'} dx dt \leq CT R^{-\frac{4}{p-1}} (\ln R)^{\frac{p}{p-1}}.$$

*Proof.* By the separable structure of the test function  $\varphi_2$ , we get

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_2(t, x)|^{p'} dx dt = \left( \int_0^T \eta(t) dt \right) \left( \int_{D^c} \phi^{-\frac{1}{p-1}}(x) |\Delta^2 \phi(x)|^{p'} dx \right).$$

Using the biharmonic product expansion from (5.0.2) and the derivative estimates for  $B(x)$  and  $\xi^\ell$  provided in Lemma 2.1.25, we find that for  $R < |x| < 2R$ :

$$|\Delta^2 \phi(x)| \leq CR^{-4} \ln R \xi^{\ell-4} \left( \frac{|x|}{R} \right). \quad (2.1.25)$$

Next, we estimate the spatial integral. Recalling the Lemma 2.1.23 and utilizing the bound (2.1.25), we obtain

$$\begin{aligned} \int_{D^c} \phi^{-\frac{1}{p-1}}(x) |\Delta^2 \phi(x)|^{p'} dx &= \int_{R < |x| < 2R} \left[ B(x) \xi^\ell \left( \frac{|x|}{R} \right) \right]^{-\frac{1}{p-1}} |\Delta^2 \phi(x)|^{p'} dx \\ &\leq CR^{-4p'} (\ln R)^{p'} (\ln R)^{-\frac{1}{p-1}} \int_{R < |x| < 2R} \xi^{\ell-4p'} \left( \frac{|x|}{R} \right) dx. \end{aligned} \quad (2.1.26)$$

By passing to polar coordinates in  $\mathbb{R}^4$ , we deduce

$$\int_{D^c} \phi^{-\frac{1}{p-1}}(x) |\Delta^2 \phi(x)|^{p'} dx \leq CR^{-\frac{4p}{p-1}} (\ln R)^{\frac{p}{p-1}} \int_R^{2R} r^3 dr \leq CR^{-\frac{4}{p-1}} (\ln R)^{\frac{p}{p-1}}.$$

Combining the temporal estimate  $\int_0^T \eta(t) dt \leq CT$  with this spatial bound yields the required result.  $\square$

**The case when  $N = 3$  and  $N \geq 5$ .**

**Lemma 2.1.27.** For  $R < |x| < 2R$ , the following estimates hold:

$$\begin{aligned} |B(x)| &\leq C, \\ |\nabla B(x)| &\leq CR^{3-N}, \\ |\Delta B(x)| &\leq CR^{2-N}, \\ |\nabla(\Delta B(x))| &\leq CR^{1-N}. \end{aligned}$$

*Proof.* From the definition of the biharmonic function in (2.1.20), we have

$$B(x) = |x|^{2-N} - 1 + \frac{N-2}{N-4} (1 - |x|^{4-N}).$$

Since  $N \geq 3$ , the terms  $|x|^{2-N}$  and  $|x|^{4-N}$  remain bounded for  $|x| > R > 1$ , which directly implies  $|B(x)| \leq C$ . To derive the remaining estimates, we define the radial profile  $h(r)$  as

$$h(r) = r^{2-N} - 1 + \frac{N-2}{N-4} (1 - r^{4-N}).$$

Differentiating with respect to  $r$  yields the first and second radial derivatives:

$$\begin{aligned} h'(r) &= (2-N)r^{1-N} + (N-2)r^{3-N}, \\ h''(r) &= (2-N)(1-N)r^{-N} + (N-2)(3-N)r^{2-N}. \end{aligned}$$

In the region  $R < r < 2R$ , the term  $r^{3-N}$  dominates the first derivative, leading to  $|\nabla B(x)| = |h'(r)| \leq CR^{3-N}$ . Substituting these derivatives into the radial Laplacian formula for  $N$  dimensions, we find

$$\begin{aligned} \Delta B(x) &= h''(r) + \frac{N-1}{r} h'(r) \\ &= [(2-N)(1-N) + (N-1)(2-N)] r^{-N} \\ &\quad + [(N-2)(3-N) + (N-1)(N-2)] r^{2-N} \\ &= 2(N-2)r^{2-N}. \end{aligned}$$

This confirms the estimate  $|\Delta B(x)| \leq CR^{2-N}$ . Finally, by taking the gradient of the Laplacian, we obtain

$$\nabla(\Delta B(x)) = \nabla(2(N-2)|x|^{2-N}) = 2(N-2)(2-N) \frac{x}{|x|^N},$$

which results in the bound

$$|\nabla(\Delta B(x))| = \frac{2(N-2)|2-N|}{r^{N-1}} \leq CR^{1-N}.$$

This completes the proof.  $\square$

**Lemma 2.1.28.** For  $N = 3$  and  $N \geq 5$ , let  $R > 0$  and  $T > 0$  be sufficiently large. Then, the following estimate holds:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_2(t, x)|^{p'} dx dt \leq CT^{1-p'} R^N.$$

*Proof.* By the separable structure of the test function  $\varphi_2$  defined in (2.1.19), the integral factors into independent temporal and spatial components:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_2(t, x)|^{p'} dx dt = \left( \int_{D^c} \phi(x) dx \right) \left( \int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{p'} dt \right).$$

To estimate the spatial component, we recall Lemma 2.1.27

$$\begin{aligned} \int_{D^c} \phi(x) dx &= \int_{1 < |x| < 2R} B(x) \xi^\ell \left( \frac{|x|}{R} \right) dx \\ &\leq C \int_{1 < |x| < 2R} \xi^\ell \left( \frac{|x|}{R} \right) dx. \end{aligned} \quad (2.1.27)$$

By transforming the integral into polar coordinates, we get

$$\int_{D^c} \phi(x) dx \leq C \int_1^{2R} r^{N-1} dr \leq CR^N.$$

The temporal integral is evaluated as in (2.1.9). Combining these spatial and temporal estimates yields the stated result directly.  $\square$

By combining the derivative bounds for the cut-off function from Lemma 2.1.13 with the estimates for the biharmonic function  $B(x)$  and its derivatives from Lemma 2.1.27, we obtain the following spatial estimates for the cases  $N = 3$  and  $N \geq 5$ .

**Lemma 2.1.29.** For  $R < |x| < 2R$ , the following estimates hold:

$$\begin{aligned} \left| B(x) \Delta^2 \xi^\ell \left( \frac{|x|}{R} \right) \right| &\leq CR^{-4} \xi^{\ell-4} \left( \frac{|x|}{R} \right), \\ \left| \nabla B(x) \cdot \nabla \left( \Delta \xi^\ell \left( \frac{|x|}{R} \right) \right) \right| &\leq CR^{-N} \xi^{\ell-4} \left( \frac{|x|}{R} \right), \\ \left| \Delta B(x) \Delta \xi^\ell \left( \frac{|x|}{R} \right) \right| &\leq CR^{-N} \xi^{\ell-4} \left( \frac{|x|}{R} \right), \\ \left| \nabla(\Delta B(x)) \cdot \nabla \xi^\ell \left( \frac{|x|}{R} \right) \right| &\leq CR^{-N} \xi^{\ell-4} \left( \frac{|x|}{R} \right), \end{aligned}$$

where  $\cdot$  denotes the standard inner product in  $\mathbb{R}^N$ .

**Lemma 2.1.30.** For  $N = 3$  and  $N \geq 5$ , let  $R > 0$  and  $T > 0$  be sufficiently large. Then, the following estimate holds:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_2(t, x)|^{p'} dx dt \leq CTR^{N-4p'}.$$

*Proof.* By the separable structure of the trial function  $\varphi_2$ , we get

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_2(t, x)|^{p'} dx dt = \left( \int_0^T \eta(t) dt \right) \left( \int_{D^c} \phi^{-\frac{1}{p-1}}(x) |\Delta^2 \phi(x)|^{p'} dx \right).$$

Applying the expansion for the biharmonic operator in (5.0.2) and incorporating the spatial estimates from Lemma 2.1.29, we observe that for  $R < |x| < 2R$ :

$$\begin{aligned} |\Delta^2 \phi(x)| &\leq C(R^{-4} + R^{-N}) \xi^{\ell-4} \left( \frac{|x|}{R} \right) \\ &\leq CR^{-4} \xi^{\ell-4} \left( \frac{|x|}{R} \right). \end{aligned}$$

Since  $|B(x)| \leq C$ , the spatial integral is bounded as follows:

$$\begin{aligned} \int_{D^c} \phi^{-\frac{1}{p-1}}(x) |\Delta^2 \phi(x)|^{p'} dx &= \int_{R < |x| < 2R} \left[ B(x) \xi^\ell \left( \frac{|x|}{R} \right) \right]^{-\frac{1}{p-1}} |\Delta^2 \phi(x)|^{p'} dx \\ &\leq CR^{-4p'} \int_{R < |x| < 2R} \xi^{\ell-4p'} \left( \frac{|x|}{R} \right) dx. \end{aligned} \quad (2.1.28)$$

By transforming the integral into polar coordinates, we obtain

$$\int_{D^c} \phi^{-\frac{1}{p-1}}(x) |\Delta^2 \phi(x)|^{p'} dx \leq CR^{-4p'} \int_R^{2R} r^{N-1} dr \leq CR^{N-4p'}.$$

Recalling the temporal estimate from (2.1.12) and combining it with the spatial bound above, we arrive at the desired result.  $\square$

### Properties of the Test Function $\varphi_3(t, x)$ .

We consider a standard class of trial functions defined by

$$\varphi_3(t, x) = \eta(t) \chi(x) = \zeta^\ell \left( \frac{t}{T} \right) \xi^\ell \left( \frac{|x|}{R} \right), \quad (t, x) \in Q, \quad (2.1.29)$$

where  $R$  and  $T$  are sufficiently large constants. Here,  $\eta(t)$  is the temporal cut-off function defined in (2.1.2), and  $\xi$  is the smooth spatial cut-off profile defined in (2.1.5).

**Lemma 2.1.31.** For  $R > 0$  and  $T > 0$  sufficiently large, the following estimate holds:

$$\int_Q \varphi_3^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_3(t, x)|^{p'} dx dt \leq CT^{1-p'} R^N.$$

*Proof.* By the separable definition of  $\varphi_3$  in (2.1.29), we get

$$\int_Q \varphi_3^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_3(t, x)|^{p'} dx dt = \left( \int_{D^c} \chi(x) dx \right) \left( \int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{p'} dt \right).$$

From the properties of the spatial cut-off function in (2.1.5), the spatial integral over the exterior domain  $D^c$  is bounded by

$$\int_{D^c} \chi(x) dx = \int_{1 < |x| < 2R} \xi^\ell \left( \frac{|x|}{R} \right) dx \leq C \int_1^{2R} r^{N-1} dr \leq CR^N, \quad (2.1.30)$$

where we have employed polar coordinates. Combining this with the temporal derivative estimate from (2.1.9) leads directly to the desired result.  $\square$

**Lemma 2.1.32.** For  $R > 0$  and  $T > 0$  sufficiently large, the following estimate holds:

$$\int_Q \varphi_3^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_3(t, x)|^{p'} dx dt \leq CTR^{N-4p'}.$$

*Proof.* Based on the definition of the test function  $\varphi_3$  in (2.1.29), the space-time integral factors into its temporal and spatial components:

$$\int_Q \varphi_3^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_3(t, x)|^{p'} dx dt = \left( \int_0^T \eta(t) dt \right) \left( \int_{D^c} \chi^{-\frac{1}{p-1}}(x) |\Delta^2 \chi(x)|^{p'} dx \right).$$

Utilizing the derivative estimates for the spatial cut-off function from Lemma 2.1.13, we have the following bound for the biharmonic operator in the region  $R < |x| < 2R$ :

$$|\Delta^2 \chi(x)| \leq CR^{-4} \xi^{\ell-4} \left( \frac{|x|}{R} \right).$$

Applying the temporal estimate  $\int_0^T \eta(t) dt \leq CT$  and substituting the spatial bound into the remaining integral, we obtain

$$\begin{aligned} \int_Q \varphi_3^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_3(t, x)|^{p'} dx dt &\leq CTR^{-4p'} \int_{R < |x| < 2R} \xi^{\ell-4p'} \left( \frac{|x|}{R} \right) dx \\ &\leq CTR^{-4p'} \int_R^{2R} r^{N-1} dr \\ &\leq CTR^{N-4p'}, \end{aligned}$$

where we have employed polar coordinates to evaluate the spatial volume. This completes the proof.  $\square$

### Properties of the test function $\psi$ .

Let us consider the following family of space-time cut-off functions:

$$\psi_R(x, t) := \left[ \phi \left( \frac{(|x| - 1)^4 + t}{R} \right) \right]^{4p'}, \quad \psi_R^*(x, t) := \left[ \phi^* \left( \frac{(|x| - 1)^4 + t}{R} \right) \right]^{4p'}, \quad (2.1.31)$$

for all  $(x, t) \in D^c \times [0, \infty)$ . Here, the profile function  $\phi \in C^\infty([0, \infty))$  is a non-increasing smooth function satisfying:

$$\phi(s) = \begin{cases} 1 & \text{if } s \in [0, 1/2], \\ \searrow & \text{if } s \in (1/2, 1), \\ 0 & \text{if } s \in [1, \infty), \end{cases} \quad \text{and} \quad \phi^*(s) = \begin{cases} 0 & \text{if } s \in [0, 1/2], \\ \phi(s) & \text{if } s \in [1/2, \infty). \end{cases}$$

Based on these profiles, we define our composite test function as:

$$\psi(x, t) := A(x) \psi_R(x, t), \quad \text{for all } (x, t) \in D^c \times [0, \infty), \quad (2.1.32)$$

where  $A(x)$  corresponds to the stationary biharmonic profile defined in (3.1.1).

**Lemma 2.1.33.** Let  $\psi_R$  and  $\psi_R^*$  be the cut-off functions defined in (2.1.31). For a positive constant  $C$ , the following estimates hold:

$$\begin{aligned} |\partial_t \psi_R(x, t)| &\leq CR^{-1} [\psi_R^*(x, t)]^{\frac{1}{p}}, \\ |\Delta^2 \psi_R(x, t)| &\leq CR^{-1} [\psi_R^*(x, t)]^{\frac{1}{p}}. \end{aligned}$$

*Proof.* Let  $s_R(x, t) = \frac{(|x|-1)^4+t}{R}$ . For the first inequality, a direct application of the chain rule yields:

$$|\partial_t \psi_R(x, t)| = 4p'R^{-1} [\phi(s_R(x, t))]^{4p'-1} |\phi'(s_R(x, t))|.$$

Since the derivative  $\phi'(s)$  is supported only on  $s \in [1/2, 1]$ , we can replace  $\phi$  with  $\phi^*$ . Furthermore, noting that  $4p' - 1 = \frac{4p'}{p} + 3$ , and because  $\phi$  is bounded, we have:

$$\begin{aligned} |\partial_t \psi_R(x, t)| &\leq CR^{-1} [\phi^*(s_R(x, t))]^{\frac{4p'}{p}} [\phi(s_R(x, t))]^3 |\phi'(s_R(x, t))| \\ &\leq CR^{-1} [\psi_R^*(x, t)]^{\frac{1}{p}}. \end{aligned}$$

The second assertion regarding the biharmonic operator  $\Delta^2$  follows similarly. By differentiating the radial profile, each spatial derivative  $\partial_{x_i}$  pulls out a factor of  $R^{-1/4}$  from the argument  $(|x| - 1)^4/R$ . Consequently, the fourth-order operator  $\Delta^2$  produces a factor of  $(R^{-1/4})^4 = R^{-1}$ . Combining this with the same power-counting argument for the profile function  $\phi$  completes the proof. Detailed proof will be in Appendix.  $\square$

### 2.1.2 Proof of Theorem 2.1.1

(i) **Subcritical case  $p < p_{Fuj}$ .**

We establish the non-existence of global weak solutions by employing the method of contradiction. Suppose  $u(x, t)$  is a global weak solution to Problems III, IV, or V. According to the weak formulation provided in Definition 2.0.1, the following integral identity holds for any valid trial function:

$$\int_Q |u|^p \varphi_3 \, dx \, dt + \int_{D^c} u_0(x) \varphi_3(x, 0) \, dx = - \int_Q u \partial_t \varphi_3 \, dx \, dt + \int_Q u \Delta^2 \varphi_3 \, dx \, dt.$$

Applying Young's inequality with  $\epsilon$  to the terms on the right-hand side, we obtain:

$$\begin{aligned} &\int_Q u(x, t) \partial_t \varphi_j(x, t) \, dx \, dt \\ &\leq \epsilon_1 \int_Q |u(x, t)|^p \varphi_j(x, t) \, dx \, dt + c(\epsilon_1) \int_Q \varphi_j^{-\frac{1}{p-1}}(x, t) |\partial_t \varphi_j(x, t)|^{p'} \, dx \, dt, \end{aligned}$$

and

$$\begin{aligned} &\int_Q u(x, t) \Delta^2 \varphi_j(x, t) \, dx \, dt \\ &\leq \epsilon_2 \int_Q |u(x, t)|^p \varphi_j(x, t) \, dx \, dt + c(\epsilon_2) \int_Q \varphi_j^{-\frac{1}{p-1}}(x, t) |\Delta^2 \varphi_j(x, t)|^{p'} \, dx \, dt. \end{aligned}$$

Applying Young's inequality with  $\epsilon = \frac{1}{4}$  to the terms on the right-hand side, we derive the following upper bound:

$$\begin{aligned} &\frac{1}{2} \int_Q |u|^p \varphi_3 \, dx \, dt + \int_{D^c} u_0(x) \varphi_3(x, 0) \, dx \\ &\leq C \int_Q \varphi_3^{-\frac{1}{p-1}} |\Delta^2 \varphi_3|^{p'} \, dx \, dt + C \int_Q \varphi_3^{-\frac{1}{p-1}} |\partial_t \varphi_3|^{p'} \, dx \, dt. \end{aligned} \tag{2.1.33}$$

By invoking the estimates for the temporal and spatial integrals from Lemma 2.1.31 and Lemma 2.1.32, respectively, we obtain:

$$\frac{1}{2} \int_Q |u|^p \varphi_3 dx dt + \int_{D^c} u_0(x) \varphi_3(x, 0) dx \leq C \left( T R^{N-4p'} + T^{1-p'} R^N \right).$$

By choosing the scaling relation  $T = R^4$ , the inequality reduces to

$$\frac{1}{2} \int_0^{R^4} \int_{D^c} |u(x, t)|^p \varphi_3 dx dt + \int_{D^c} u_0(x) \varphi_3(x, 0) dx \leq C R^{N - \frac{4}{p-1}}. \quad (2.1.34)$$

In the subcritical range  $p < 1 + \frac{4}{N}$ , the exponent  $N - \frac{4}{p-1}$  is strictly negative. Taking the limit as  $R \rightarrow \infty$ , the right-hand side of (2.1.34) vanishes. By applying the Monotone Convergence Theorem to the space-time integral and the Lebesgue Dominated Convergence Theorem to the initial data integral, we find:

$$\frac{1}{2} \int_0^\infty \int_{D^c} |u(x, t)|^p dx dt + \int_{D^c} u_0(x) dx \leq 0.$$

Since  $|u|^p$  is non-negative and  $u_0(x)$  satisfies  $\int_{D^c} u_0(x) dx > 0$  by assumption, this result provides the required contradiction. Consequently, no global weak solution exists for the specified range of  $p$ .

**(i) Critical case  $p = p_{Fuj}$ .** In the critical regime where  $p = p_{Fuj}$ , the estimate derived in (2.1.34) implies the following bound:

$$\int_0^{R^4} \int_{1 < |x| < 2R} |u(x, t)|^p \varphi_3(x, t) dx dt + \int_{1 < |x| < 2R} u_0(x) \varphi_3(x, 0) dx \leq C.$$

Given the non-negativity assumption on the initial data,  $\int_{D^c} u_0(x) dx \geq 0$ , we may pass to the limit as  $R \rightarrow \infty$  in the aforementioned inequality to obtain:

$$\int_0^\infty \int_{D^c} |u|^p dx dt \leq C.$$

which implies that  $u \in L^p(Q)$ .

By the weak formulation provided in Definition 2.0.1, the following integral identity holds:

$$\int_Q |u|^p \varphi_3 dx dt + \int_{D^c} u_0(x) \varphi_3(x, 0) dx = - \int_Q u \partial_t \varphi_3 dx dt + \int_Q u \Delta^2 \varphi_3 dx dt.$$

Utilizing the scaling  $T = R^4$  and the specific support properties of the test function defined in (2.1.29), we can isolate the regions where the derivatives of the cut-off functions are non-trivial. This leads to the following refinement:

$$\int_0^{R^4} \int_{D^c} |u|^p \varphi_3 dx dt + \int_{D^c} u_0(x) \varphi_3(x, 0) dx = - \int_{\frac{R^4}{2}}^{R^4} \int_{D^c} u \partial_t \varphi_3 dx dt + \int_0^{R^4} \int_{R < |x| < 2R} u \Delta^2 \varphi_3 dx dt.$$

Applying Hölder's inequality in combination with the estimates from Lemma 2.1.31 and Lemma 2.1.32, and specifically considering the critical exponent  $p = 1 + \frac{4}{N}$ , we obtain

$$\begin{aligned} \int_{\frac{R^4}{2}}^{R^4} \int_{D^c} u \partial_t \varphi_3 dx dt &\leq \left( \int_{\frac{R^4}{2}}^{R^4} \int_{D^c} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}} \left( \int_{\frac{R^4}{2}}^{R^4} \int_{D^c} \varphi_3^{-\frac{1}{p-1}} |\partial_t \varphi_3|^{p'} dx dt \right)^{\frac{1}{p'}} \\ &= C \left( \int_{\frac{R^4}{2}}^{R^4} \int_{D^c} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}}, \end{aligned}$$

and

$$\begin{aligned} \int_0^{R^4} \int_{D^c} u \Delta^2 \varphi_3 dx dt &\leq \left( \int_0^{R^4} \int_{R < |x| < 2R} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}} \left( \int_0^{R^4} \int_{R < |x| < 2R} \varphi_3^{-\frac{1}{p-1}} |\Delta^2 \varphi_3|^{p'} dx dt \right)^{\frac{1}{p'}} \\ &= C \left( \int_0^{R^4} \int_{R < |x| < 2R} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}}. \end{aligned}$$

By combining these estimates, we establish the fundamental inequality for the critical case:

$$\begin{aligned} \int_0^{R^4} \int_{D^c} |u|^p \varphi_3 dx dt + \int_{D^c} u_0(x) \varphi_3(x, 0) dx \\ \leq C \left( \int_{\frac{R^4}{2}}^{R^4} \int_{D^c} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}} + C \left( \int_0^{R^4} \int_{R < |x| < 2R} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}}. \end{aligned} \quad (2.1.35)$$

Since  $u \in L_{loc}^p([0, \infty) \times D^c)$ , the integral property implies

$$\begin{aligned} \lim_{R \rightarrow \infty} \left( \int_{\frac{R^4}{2}}^{R^4} \int_{D^c} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}} &= \lim_{R \rightarrow \infty} \left( \int_0^{R^4} \int_{D^c} |u|^p \varphi_3 dx dt - \int_0^{\frac{R^4}{2}} \int_{D^c} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}} \\ &= \lim_{R \rightarrow \infty} \left( \int_0^{\infty} \int_{D^c} |u|^p \varphi_3 dx dt - \int_0^{\infty} \int_{D^c} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned}
& \lim_{R \rightarrow \infty} \left( \int_0^{R^4} \int_{R < |x| < 2R} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}} \\
&= \lim_{R \rightarrow \infty} \left( \int_0^{R^4} \int_{0 < |x| < 2R} |u|^p \varphi_3 dx dt - \int_0^{R^4} \int_{0 < |x| < R} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}} \\
&= \lim_{R \rightarrow \infty} \left( \int_0^{\infty} \int_{0 < |x| < \infty} |u|^p \varphi_3 dx dt - \int_0^{\infty} \int_{0 < |x| < \infty} |u|^p \varphi_3 dx dt \right)^{\frac{1}{p}} \\
&= 0.
\end{aligned}$$

Hence, passing to the lim  $R \rightarrow \infty$  in (2.1.35), we get

$$\begin{aligned}
& \lim_{R \rightarrow \infty} \left( \int_0^{R^4} \int_{D^c} |u|^p \varphi_3 dx dt + \int_{D^c} u_0(x) \varphi_3(x, 0) dx \right) \\
&= \int_0^{\infty} \int_{D^c} |u|^p dx dt + \int_{D^c} u_0(x) dx \\
&\leq 0,
\end{aligned}$$

This inequality implies that  $\int_{D^c} u_0(x) dx \leq 0$ , which directly contradicts our initial assumption that  $\int_{D^c} u_0(x) dx \geq 0$  for non-trivial, non-negative initial data. This contradiction ensures that no global weak solution exists, thereby completing the proof of the theorem.

**Remark 2.1.34.** Note that the nonexistence of the global solution in the critical case provided above will be similar to the Navier boundary when  $N \geq 3$  and for the Dirichlet boundary when  $N = 3$  and  $N \geq 5$ .

### 2.1.3 Proof of Theorem 2.1.4

In this subsection, we derive the upper bound for the lifespan of solutions to Problems III–V. As the test function method applies to all considered boundary conditions, it is sufficient to demonstrate the proof for a single case without loss of generality.

Assume that  $R_0 := 2r_0^2 < \text{LifeSpan}(u_\varepsilon)$ . According to the definition of a weak solution 2.0.1, we satisfy the following integral inequality:

$$\begin{aligned}
& \int_0^T \int_{D^c} |u(t, x)|^p \psi(t, x) dx dt + \varepsilon \int_{D^c} u_0(x) \psi(0, x) dx \\
&\leq \int_0^T \int_{D^c} |u(t, x)| (|\partial_t \psi(t, x)| + |\Delta^2 \psi(t, x)|) dx dt.
\end{aligned}$$

From lemma 4.1.5 and noting that  $A(x)$  is independent of  $t$ , it follows that

$$\begin{aligned}
& \left| \partial_t (A(x)\psi_R(t)) - \Delta^2 (A(x)\psi_R(t, x)) \right| \\
&= \left| \partial_t (A(x)\psi_R(t, x)) - 4\nabla A(x)\nabla\Delta\psi_R(t, x) - A(x)\Delta^2\psi_R(t, x) \right| \\
&\leq \frac{2C}{R}A(x) [\psi_R^*(t, x)]^{\frac{1}{p}} + \frac{C}{R}A(x) [\psi_R^*(t, x)]^{\frac{1}{p}} + \frac{C}{R}A(x) [\psi_R^*(t, x)]^{\frac{1}{p}} \\
&\leq \frac{C}{R}A(x) [\psi_R^*(t)]^{\frac{1}{p}},
\end{aligned}$$

making some algebraic improvements and noting  $u_0(x) \in L^1_{loc}(D^c)$

$$\begin{aligned}
c_0\varepsilon + \int_0^T \int_{D^c} |u(t, x)|^p A(x)\psi_R(t, x) dx dt \\
\leq CR^{-1} \int_0^T \int_{D^c} |u(t, x)| A(x) [\psi_R^*(t, x)]^{\frac{1}{p}} dx dt \\
\leq CR^{-1+\frac{N+4}{4p}} \left( \int_0^T \int_{D^c} |u(t, x)|^p A(x)\psi_R^*(t, x) dx dt \right)^{\frac{1}{p}},
\end{aligned}$$

which is

$$\begin{aligned}
c_0\varepsilon + \int_0^T \int_{D^c} |u(t, x)|^p A(x)\psi_R(t, x) dx dt \\
\leq CR^{-\frac{1}{p}(\frac{1}{p-1}-\frac{N}{4})} \left( \int_0^T \int_{D^c} |u(t, x)|^p A(x)\psi_R^*(t, x) dx dt \right)^{\frac{1}{p}},
\end{aligned}$$

using lemma 5.0.3 with  $w(x, t) = |u(x, t)|^p A(x)$  and  $\theta = \frac{1}{p-1} - \frac{N}{4}$ , we have the desired results.

**Remark 2.1.35.** Note that the upper lifespan of solutions provided above contains a Navier boundary when  $N \geq 3$  and the Dirichlet boundary when  $N = 3$  and  $N \geq 5$ .

### 2.1.4 Proof of Theorem 2.1.6

The theorem is proved using the method of contradiction. Suppose that  $u$  is a global weak solution to Problem I.

$$\begin{aligned}
& \int_Q |u(x, t)|^p \varphi_j(x, t) dx dt + \int_{D^c} u_0(x) \varphi_j(x, 0) dx \\
&= - \int_Q u(x, t) \partial_t \varphi_j(x, t) dx dt + \int_Q u(x, t) \Delta^2 \varphi_j(x, t) dx dt.
\end{aligned}$$

Applying Young's inequality with  $\epsilon$  to the terms on the right-hand side, we obtain:

$$\begin{aligned}
& \int_Q u(x, t) \partial_t \varphi_j(x, t) dx dt \\
&\leq \epsilon_1 \int_Q |u(x, t)|^p \varphi_j(x, t) dx dt + c(\epsilon_1) \int_Q \varphi_j^{-\frac{1}{p-1}}(x, t) |\partial_t \varphi_j(x, t)|^{p'} dx dt,
\end{aligned}$$

and

$$\begin{aligned} & \int_Q u(x, t) \Delta^2 \varphi_j(x, t) dx dt \\ & \leq \epsilon_2 \int_Q |u(x, t)|^p \varphi_j(x, t) dx dt + c(\epsilon_2) \int_Q \varphi_j^{-\frac{1}{p-1}}(x, t) |\Delta^2 \varphi_j(x, t)|^{p'} dx dt. \end{aligned}$$

By choosing  $\epsilon_1 = \epsilon_2 = \frac{1}{2}$  and absorbing the terms into the left-hand side, we arrive at the fundamental inequality:

$$\begin{aligned} & \int_{D^c} u_0(x) \varphi_j(x, 0) dx \\ & \leq \int_Q \varphi_j^{-\frac{1}{p-1}}(x, t) |\Delta^2 \varphi_j(x, t)|^{p'} dx dt + \int_Q \varphi_j^{-\frac{1}{p-1}}(x, t) |\partial_t \varphi_j(x, t)|^{p'} dx dt. \end{aligned} \quad (2.1.36)$$

**(i) The case  $N = 2$ .** Based on the construction of the trial functions, it is observed that  $\varphi_1$  satisfies the boundary requirements specified in Definition 2.0.1 for both Problem I and Problem VI. Consequently, establishing the result for Problem I is sufficient. By invoking the estimates for the temporal and spatial integrals derived in Lemma 2.1.20 and Lemma 2.1.22, respectively, we can evaluate the right-hand side of the fundamental inequality (2.1.36) for  $N = 2$ :

$$\int_{D^c} u_0(x) \varphi_1(x, 0) dx \leq C \left( T^1 R^{2-4p'} (\ln R)^{\frac{p}{p-1}} + T^{1-p'} R^2 \ln R \right).$$

To balance the growth rates, we choose the scaling  $T = R^4$ . Recalling that  $p' = \frac{p}{p-1}$ , the inequality can be rewritten as:

$$\int_{D^c} u_0(x) \varphi_1(x, 0) dx \leq C T^1 R^{2-4p'} (\ln R)^{\frac{p}{p-1}},$$

Selecting  $T = R^4$ , we arrive at

$$\int_{D^c} u_0(x) \varphi_1(x, 0) dx \leq C R^{2-\frac{4}{p-1}} (\ln R)^{\frac{p}{p-1}}.$$

When  $1 < p < 1 + \frac{4}{N}$ , we get

$$2 - \frac{4}{p-1} < 0,$$

then passing to the limit  $R \rightarrow \infty$  in (2.1.37), we arrive at

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < 2R} u_0(x) \varphi_1(x, 0) dx = \int_{D^c} u_0(x) H(x) dx \leq 0.$$

This result contradicts our initial assumption that  $\int_{D^c} u_0(x) H(x) dx > 0$ . Therefore, the proof of part (i) for  $N = 2$  is complete.

Now, we consider a two-dimensional critical case for the Navier boundary condition. We use the same method as in [IS19a]. From the definition of weak solution and Lemma 5.0.8, we get

$$\begin{aligned} & \int_Q |u|^p \psi(x, t) dx dt + \varepsilon \int_{D^c} u_0(x) \psi(x, 0) dx \leq \frac{C}{R} \iint_{P(R)} |u| H(x) [\psi_R^*(t)]^{\frac{1}{p}} dx dt \\ & \leq \frac{C}{|\lambda| R} \left( \iint_{P(R)} H(x) dx dt \right)^{\frac{1}{p'}} \left( \iint_{P(R)} |u(t)|^p H(x) \psi_R^*(t) dx dt \right)^{\frac{1}{p}}. \end{aligned}$$

Therefore noting that

$$\iint_{P(R)} H(x) dx dt \leq \pi R (R^{\frac{1}{4}} + 1)^2 \log(R^{\frac{1}{4}} + 1),$$

and since  $\int_{D^c} u_0(x) \psi(x, 0) dx > 0$ , we deduce

$$c_0 \varepsilon + \iint_{P(R)} |u(t)|^p H(x) \psi_R(t) dx dt \leq C R^{\frac{1}{2} - \frac{3}{2p}} (\log R)^{\frac{1}{p'}} \left( \iint_{P(R)} |u(t)|^p H(x) \psi_R^*(t) dx dt \right)^{\frac{1}{p}}$$

or

$$c_0 \varepsilon + \iint_{P(R)} |u(t)|^p H(x) \psi_R(t) dx dt \leq C R^{-\left(\frac{3-p}{2(p-1)}\right) \frac{1}{p'}} (\log R)^{\frac{1}{p'}} \left( \iint_{P(R)} |u(t)|^p H(x) \psi_R^*(t) dx dt \right)^{\frac{1}{p}}$$

for some positive constant  $C > 0$ . Applying Lemma 5.0.6 with  $w = |u|^p H(x)$ , we have the desired estimate for  $\text{LifeSpan}(u)$ . The proof is complete.

**(ii) The  $N \geq 3$  case.** In this regime, we employ the test function defined in (2.1.1), where the spatial component incorporates the harmonic kernel  $H(x)$  for  $N \geq 3$ . By reiterating the procedure from the previous case and applying the fundamental inequality (2.1.36) alongside the estimates from Lemma 2.1.28 and Lemma 2.1.30, we deduce

$$\int_{D^c} u_0(x) \varphi_1(x, 0) dx \leq C \left( T^1 R^{N-4p'} + T^{1-p'} R^N \right).$$

By setting the scaling relationship  $T = R^4$  and recalling that  $p' = \frac{p}{p-1}$ , the right-hand side simplifies as follows:

$$\int_0^{R^4} \int_{1 < |x| < 2R} |u(x, t)|^p \varphi_1(x, t) dx dt + \int_{1 < |x| < 2R} u_0(x) \varphi_1(x, 0) dx \leq C R^{N - \frac{4}{p-1}}, \quad (2.1.37)$$

when  $1 < p < 1 + \frac{4}{N}$ , we get

$$N - \frac{4}{p-1} < 0,$$

then passing to the limit  $R \rightarrow \infty$  in (2.1.37), by Dominated Convergence Theorem, we arrive at

$$\begin{aligned} & \lim_{R \rightarrow \infty} \left( \int_0^{R^4} \int_{1 < |x| < 2R} |u(x, t)|^p \varphi_1(x, t) dx dt + \int_{1 < |x| < 2R} u_0(x) \varphi_1(x, 0) dx \right) \\ &= \int_0^\infty \int_{D^c} |u(x, t)|^p H(x) dx dt + \int_{D^c} u_0(x) H(x) dx \\ &\leq 0. \end{aligned}$$

Since both terms on the left-hand side are non-negative, this inequality implies that  $\int_{D^c} u_0(x) H(x) dx \leq 0$ , which contradicts our initial assumption that  $\int_{D^c} u_0(x) H(x) dx > 0$ . Furthermore, it implies that  $u \equiv 0$ , which is inconsistent with the non-trivial initial data. This completes the proof for the  $N \geq 3$  case.

### 2.1.5 Proof of Theorem 2.1.8

**N=4 case.** We follow the same methodology as established for the Navier boundary conditions. Starting from the fundamental integral inequality derived from the weak formulation and Young's inequality, we have

$$\int_{D^c} u_0(x) \varphi_2(x, 0) dx \leq \int_Q \varphi_2^{-\frac{1}{p-1}} |\Delta^2 \varphi_2|^{p'} dx dt + \int_Q \varphi_2^{-\frac{1}{p-1}} |\partial_t \varphi_2|^{p'} dx dt.$$

By invoking the estimates for the temporal and spatial integrals from Lemma 2.1.24 and Lemma 2.1.26, respectively, we obtain:

$$\int_{D^c} u_0(x) \varphi_2(x, 0) dx \leq CT^{1-p'} R^4 \ln R + CTR^{-\frac{4}{p-1}} (\ln R)^{\frac{p}{p-1}}. \quad (2.1.38)$$

Setting the scaling relationship  $T = R^4$ , we get

$$\int_{D^c} u_0(x) \varphi_2(x, 0) dx \leq CR^{N-\frac{4}{p-1}} (\ln R)^{\frac{p}{p-1}}. \quad (2.1.39)$$

For  $N = 4$ , the subcritical condition  $p < 1 + \frac{4}{N}$  implies  $p < 2$ , which means the exponent  $4 - \frac{4}{p-1}$  is strictly negative. Consequently, the power-law decay of  $R$  dominates the logarithmic growth of  $(\ln R)^{\frac{p}{p-1}}$ . Taking the limit as  $R \rightarrow \infty$ , the right-hand side of (2.1.39) vanishes:

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < 2R} u_0(x) \varphi_2(x, 0) dx = \int_{D^c} u_0(x) B(x) dx \leq 0.$$

This result contradicts the initial assumption

$$\int_{D^c} u_0(x) B(x) dx \geq 0.$$

This proves the subcritical case of the theorem for  $N = 4$ . From the definition of weak solution and Lemma 5.0.7, we get

$$\begin{aligned} \int_Q |u|^p \psi(x, t) dx dt + \varepsilon \int_{D^c} u_0(x) \psi(x, 0) dx &\leq \frac{C}{R} \iint_{P(R)} |u| B(x) [\psi_R^*(t)]^{\frac{1}{p}} dx dt \\ &\leq \frac{C}{|\lambda| R} \left( \iint_{P(R)} B(x) dx dt \right)^{\frac{1}{p'}} \left( \iint_{P(R)} |u(t)|^p B(x) \psi_R^*(t) dx dt \right)^{\frac{1}{p}}. \end{aligned}$$

Therefore noting that

$$\iint_{P(R)} B(x) dx dt \leq \pi R (R^{\frac{1}{4}} + 1)^2 \log(R^{\frac{1}{4}} + 1),$$

and since  $\int_{D^c} u_0(x) \psi(x, 0) dx > 0$ , we deduce

$$c_0 \varepsilon + \iint_{P(R)} |u(t)|^p B(x) \psi_R(t) dx dt \leq CR^{\frac{1}{2} - \frac{3}{2p}} (\log R)^{\frac{1}{p'}} \left( \iint_{P(R)} |u(t)|^p B(x) \psi_R^*(t) dx dt \right)^{\frac{1}{p}}$$

or

$$c_0\varepsilon + \iint_{P(R)} |u(t)|^p B(x) \psi_R(t) dx dt \leqslant C R^{-\left(\frac{3-p}{2(p-1)}\right)\frac{1}{p'}} (\log R)^{\frac{1}{p'}} \left( \iint_{P(R)} |u(t)|^p B(x) \psi_R^*(t) dx dt \right)^{\frac{1}{p}}$$

for some positive constant  $C > 0$ . Applying Lemma 5.0.6 with  $w = |u|^p B(x)$ , we have the desired estimate for  $\text{LifeSpan}(u)$ . Thus, the proof of non-existence for  $N = 4$  is complete.

**The Case  $N = 3$  and  $N \geqslant 5$ .** Following the methodology established for the Navier boundary conditions, we utilize the weak formulation and Young's inequality to derive the following integral bound:

$$\int_{D^c} u_0(x) \varphi_2(x, 0) dx \leqslant C \int_Q \varphi_2^{-\frac{1}{p-1}} |\Delta^2 \varphi_2|^{p'} dx dt + C \int_Q \varphi_2^{-\frac{1}{p-1}} |\partial_t \varphi_2|^{p'} dx dt. \quad (2.1.40)$$

By substituting the estimates for the temporal and spatial integrals from Lemma 2.1.28 and Lemma 2.1.30, we obtain:

$$\int_{D^c} u_0(x) \varphi_2(x, 0) dx \leqslant C \left( T R^{N-4p'} + T^{1-p'} R^N \right). \quad (2.1.41)$$

Choosing the scaling  $T = R^4$ , the right-hand side simplifies to a single power of  $R$ :

$$\int_{D^c} u_0(x) \varphi_2(x, 0) dx \leqslant C R^{N-\frac{4}{p-1}}. \quad (2.1.42)$$

Given the subcritical condition  $1 < p < 1 + \frac{4}{N}$ , the exponent  $N - \frac{4}{p-1}$  is strictly negative. Consequently, as  $R \rightarrow \infty$ , the right-hand side vanishes. Applying the Dominated Convergence Theorem to the left-hand side, we find:

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < 2R} u_0(x) \varphi_2(x, 0) dx = \int_{D^c} u_0(x) B(x) dx \leqslant 0,$$

where  $B(x)$  is the stationary biharmonic profile. This result contradicts our assumption  $\int_{D^c} u_0(x) B(x) dx > 0$ , thereby establishing the non-existence of global weak solutions. The proof of Theorem 2.1.8 for problem II is thus complete.

# Chapter 3

## Exterior biharmonic heat equation with forcing term

In Chapter 3, we consider the following model

$$\begin{cases} u_t + \Delta^2 u = |u|^p + f(x), & \text{in } D^c \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } D^c, \end{cases} \quad (3.0.1)$$

where  $D = \overline{B(0, 1)}$  denotes the closed unit ball in  $\mathbb{R}^N$ , and  $D^c := \mathbb{R}^N \setminus D$  represents its exterior. We investigate problem (3.0.1) under six distinct sets of boundary conditions, all evaluated on  $\partial D \times (0, \infty)$ . For convenience, we assign a Roman numeral to each configuration:

(I) Navier boundary conditions:

$$\begin{cases} u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \\ \Delta u(x, t) = 0, & \text{in } \partial D \times (0, \infty); \end{cases} \quad (3.0.2)$$

(II) Dirichlet boundary conditions:

$$\begin{cases} u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \\ \frac{\partial u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty); \end{cases} \quad (3.0.3)$$

(III) Dirichlet-Navier boundary conditions:

$$\begin{cases} u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \\ \frac{\partial \Delta u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty); \end{cases} \quad (3.0.4)$$

(IV) Kuttler-Sigillito boundary conditions:

$$\begin{cases} \frac{\partial u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty), \\ \frac{\partial \Delta u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty); \end{cases} \quad (3.0.5)$$

⟨V⟩ Neumann-Navier boundary conditions:

$$\begin{cases} \Delta u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \\ \frac{\partial \Delta u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty); \end{cases} \quad (3.0.6)$$

⟨VI⟩ Neumann boundary conditions:

$$\begin{cases} \frac{\partial u(x, t)}{\partial n} = 0, & \text{in } \partial D \times (0, \infty), \\ \Delta u(x, t) = 0, & \text{in } \partial D \times (0, \infty), \end{cases} \quad (3.0.7)$$

where  $n$  denotes the outward unit normal vector on  $\partial D$ . For convenience, we label the boundary conditions using Roman numerals. For instance, Problem I refers to the semilinear equation (3.0.1) equipped with the Navier boundary conditions, and so forth. We denote the Fujita critical exponent by  $p_{Crit}$ , which is defined as:

$$p_{Crit} := \begin{cases} \infty, & N = 2, 3, 4, \\ \frac{N}{N-4}, & N \geq 5. \end{cases} \quad (3.0.8)$$

In accordance with established terminology, the regimes corresponding to  $p < p_{Crit}$ ,  $p = p_{Crit}$ , and  $p > p_{Crit}$  are referred to as the Fujita subcritical, critical, and supercritical cases, respectively.

## 3.1 Main results

This section is dedicated to establishing the proofs for the primary theorems presented in this work. We begin by examining the criteria for the existence and non-existence of solutions within the first critical regime. Subsequently, we extend our analysis to the second critical case, where the corresponding existence results are proven.

The local existence of solutions to Problems I–VI is a direct consequence of the results presented in [Paz83]. Therein, the author establishes the existence of a unique local classical solution for a general class of abstract parabolic problems, of which our models are specific instances.

Consequently, we take the local existence as a given and focus our attention on the long-term global behavior and blow-up phenomena associated with solutions to Problems I–VI.

**Definition 3.1.1.** A function  $u$  is said to be a **weak solution** to Problems I–VI if there exists a time  $0 < T \leq \infty$  such that  $u \in L_{loc}^p(\overline{Q})$  and the following integral identity holds:

$$\begin{aligned} \int_Q |u|^p \varphi_j \, dx \, dt + \int_Q f(x) \varphi_j(x, t) \, dx \, dt + \int_{D^c} u_0(x) \varphi_j(x, 0) \, dx \\ = - \int_Q u \partial_t \varphi_j \, dx \, dt + \int_Q u \Delta^2 \varphi_j \, dx \, dt, \end{aligned}$$

where  $Q := [0, T] \times D^c$ . This identity must hold for every non-negative test function  $\varphi_j \in C_{t,x}^{1,4}(\overline{Q})$  satisfying:

- (i)  $\varphi_j(t, \cdot) \equiv 0$  for all  $t \geq T$ ;
- (ii)  $\varphi_j(\cdot, x) \equiv 0$  for all  $|x| \geq R$  where  $R > 1$ .

While properties (i) and (ii) remain consistent across all cases. (iii) — will differentiate the weak solutions for each of the six boundary conditions.

| (iii)       |                                                                                |            |                                                                                                           |
|-------------|--------------------------------------------------------------------------------|------------|-----------------------------------------------------------------------------------------------------------|
| Problem I   | $\varphi_1 = \Delta\varphi_1 = 0$ on $\partial D$ ;                            | Problem IV | $\frac{\partial\varphi_4}{\partial n} = \frac{\partial\Delta\varphi_4}{\partial n} = 0$ on $\partial D$ ; |
| Problem II  | $\varphi_2 = \frac{\partial\varphi_2}{\partial n} = 0$ on $\partial D$ ;       | Problem V  | $\Delta\varphi_5 = \frac{\partial\Delta\varphi_5}{\partial n} = 0$ on $\partial D$ ;                      |
| Problem III | $\frac{\partial\varphi_3}{\partial n} = \Delta\varphi_3 = 0$ on $\partial D$ ; | Problem VI | $\varphi_6 = \frac{\partial\Delta\varphi_6}{\partial n} = 0$ on $\partial D$ .                            |

Furthermore, if the existence of the weak solution  $u$  can be established for any arbitrary choice of  $T > 0$ , then  $u$  is referred to as a global weak solution to the problem. Below, we formulate our first primary result regarding the global behavior of solutions.

**Theorem 3.1.2.** *Suppose that  $u_0 \in L^1_{loc}(D^c)$  and  $f \in L^1_{loc}(D^c) \cap L^\infty(D^c)$  satisfy the following positivity condition:*

$$\int_{D^c} A(x)f(x) dx > 0,$$

where the stationary profile  $A(x)$  is defined according to the specific boundary conditions as follows:

$$A(x) := \begin{cases} \text{The harmonic function (2.1.6),} & \text{for Problems I and VI,} \\ \text{The biharmonic function (2.1.20),} & \text{for Problem II,} \\ 1, & \text{for Problems III–V.} \end{cases} \quad (3.1.1)$$

Then, the following assertions hold:

- (i) **Sub-critical Dimensions:** If  $N \in \{2, 3, 4\}$ , then for all  $p > 1$ , Problems I–VI admit no global-in-time weak solution.
- (ii) **Critical and Sub-critical Regimes:** If  $N \geq 5$  and  $1 < p \leq p_{\text{Crit}} := \frac{N}{N-4}$ , then Problems I–VI admit no global-in-time weak solution.
- (iii) **Super-critical Regime:** If  $N \geq 5$  and  $p > p_{\text{Crit}}$ , Problems I–VI admit global (stationary) solutions for sufficiently small initial data  $u_0 > 0$  and source term  $f > 0$ .

Let us consider the problems I–VI for the following semilinear biharmonic Poisson equation:

$$\Delta^2 u(x) = |u(x)|^p + f(x), \quad \text{in } D^c. \quad (3.1.2)$$

Then the following statement holds.

**Corollary 3.1.3.** Assume that  $f \in L^1_{loc}(D^c) \cap L^\infty(D^c)$  and  $f \geq 0$ . Then:

- (i) If  $N \in \{2, 3, 4\}$ , then for all  $p > 1$ , Problems I–VI for the equation (3.1.2) admit no weak solutions.

- (ii) If  $N \geq 5$  and  $1 < p \leq p_{Crit} := \frac{N}{N-4}$ , then Problems I–VI for the equation (3.1.2) admit no weak solutions.
- (iii) If  $N \geq 5$  and  $p > p_{Crit}$ , Problems I–VI for the equation (3.1.2) admit positive solutions.

**Remark 3.1.4.** An analysis of the results presented in Theorem 3.1.2 and Corollary 3.1.3 allows us to draw the following conclusions:

- (a) The critical exponent

$$p_{Crit} := \begin{cases} \infty, & N = 1, 2, 3, 4, \\ \frac{N}{N-4}, & N \geq 5 \end{cases}$$

for the exterior Problems I–VI coincides with the critical exponent for the corresponding problem in  $\mathbb{R}^N$  obtained by Majdoub in [Maj23]. Additionally, the parabolic equation (3.0.1) and the biharmonic Poisson equation (3.1.2) share the same critical threshold  $p_{Crit}$ . It is noteworthy that the source term  $f(x)$  does not shift this threshold, and the results of Corollary 3.1.3 remain valid for  $f(x) = 0$ .

- (b) Theorem 3.1.2 part (iii) establishes global existence specifically for stationary positive solutions. Currently, information regarding global non-stationary solutions remains unavailable. This is attributed to technical complexities in analyzing the Green's functions for Problems I–VI, which need dedicated studies.
- (c) Of particular interest is the critical behavior when the forcing term satisfies  $\int_{D^c} f(x)A(x)dx = 0$  with  $f \not\equiv 0$ . Our current approach does not encompass this case, leaving it as an open question. A analogous problem was posed by Bandle et al. in [BL89] for the classical heat equation and remains unresolved to date.
- (d) Unlike the case  $D^c = \mathbb{R}^N$ , where the assumption  $\int_{\mathbb{R}^N} f(x)dx > 0$  is standard, the exterior domain problem requires the weighted condition  $\int_{D^c} f(x)A(x)dx > 0$ . Here, the weight  $A(x)$  is the stationary profile given by (3.1.1).

In part (iii) of Theorem 3.1.2, it was established that for  $p > p_{Crit}$ , a global solution exists only for sufficiently small  $u_0$  and  $f$ . Therefore, it is natural to consider this regime separately and investigate the second critical exponent in the sense of Lee–Ni. To facilitate this analysis, for  $\omega < N$ , we define the following sets:

$$\mathcal{I}_\omega^+ = \{f \in C(D^c) : f \geq 0 \text{ and } f(x) \geq C|x|^{-\omega} \text{ for sufficiently large } |x|\},$$

and

$$\mathcal{I}_\omega^- = \{f \in C(D^c) : f > 0 \text{ and } f(x) \leq C|x|^{-\omega} \text{ for sufficiently large } |x|\},$$

where  $C > 0$  is a positive constant.

We are now ready to formulate the second main result of this work.

**Theorem 3.1.5** (Second Critical Exponent). *Suppose  $p > p_{Crit}$ ,  $u_0 \in L_{loc}^1(D^c)$ , and  $f \not\equiv 0$ .*

- (i) *If  $f \in \mathcal{I}_\omega^+$  and  $\omega < 0$ , Problems I–VI admit no global positive weak solutions.*
- (ii) *If  $f \in \mathcal{I}_\omega^+$  and  $0 \leq \omega < \omega_{Crit} := \frac{4p}{p-1}$ , Problems I–VI admit no global positive weak solutions.*

(iii) If  $\omega_{Crit} \leq \omega < N$ , Problems I–VI admit positive global classical solutions for sufficiently small  $f \in \mathcal{I}_\omega^-$  and  $u_0 > 0$ .

**Remark 3.1.6.** We highlight two significant consequences of Theorem 3.1.5:

- (a) Given the supercritical assumption  $p > p_{Crit} = \frac{N}{N-4}$  for  $N \geq 5$ , the set of decay exponents  $\omega$  satisfying  $\frac{4p}{p-1} \leq \omega < N$  is guaranteed to be non-empty. This confirms that the fast-decay regime defined in part (iii) is mathematically well-posed.
- (b) The second critical exponent  $\omega_{Crit} := \frac{4p}{p-1}$  for Problems I–VI is determined by the asymptotic behavior of the forcing term  $f$ , rather than the initial data  $u_0$ . While the magnitude of  $u_0$  affects global existence, the threshold for blow-up versus global-in-time solutions is governed by the spatial decay rate of the external source.

## 3.2 Properties of test functions

### Test Functions with Logarithmic Arguments $\varphi_1(t, x)$ .

Let us introduce a modified class of test functions utilizing logarithmic spatial arguments. For  $(t, x) \in Q$ , we define

$$\varphi_1(t, x) = \eta(t)\psi(x) = \zeta^\ell \left( \frac{t}{T} \right) H(x) \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln(\sqrt{R})} \right), \quad (t, x) \in Q, \quad (3.2.1)$$

where  $H(x)$  is the harmonic function defined in (2.1.6),  $\eta(t)$  is the temporal cut-off function given by (2.1.2), and  $\mathcal{F} : \mathbb{R} \rightarrow [0, 1]$  is a smooth spatial cut-off function satisfying

$$\mathcal{F}(s) = \begin{cases} 1, & \text{if } -1 \leq s \leq 0, \\ \text{decreasing}, & \text{if } 0 < s < 1, \\ 0, & \text{if } s \geq 1. \end{cases} \quad (3.2.2)$$

**Lemma 3.2.1.** For  $N \geq 5$  and  $p = \frac{N}{N-4}$ , assume  $R > 0$  and  $T > 0$  are sufficiently large. Then, the following estimate holds:

$$\int_Q \varphi_1^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_1(t, x)|^{p'} dx dt \leq CT^{1-\frac{N}{4}} R^N. \quad (3.2.3)$$

*Proof.* Using the separable structure of the test function defined in (3.2.1), we first factor the integral into its spatial and temporal components. Noting that for  $p = \frac{N}{N-4}$ , the conjugate exponent is  $p' = \frac{N}{4}$ , we obtain

$$\int_Q \varphi_1^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_1(t, x)|^{p'} dx dt = \left( \int_{D^c} \psi(x) dx \right) \left( \int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{\frac{N}{4}} dt \right).$$

Applying Lemma 2.1.15 and noting the support of the logarithmic cut-off function  $\mathcal{F}$ , we

can bound the spatial integral. Passing to polar coordinates, we find

$$\begin{aligned}
\int_{D^c} \psi(x) dx &= \int_{1 < |x| < R} H(x) \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln(\sqrt{R})} \right) dx \\
&\leq C \int_{1 < |x| < R} \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln(\sqrt{R})} \right) dx \\
&\leq C \int_1^R r^{N-1} dr \\
&\leq CR^N.
\end{aligned} \tag{3.2.4}$$

As derived previously in (2.1.9), the temporal integral containing the derivative of the cut-off function evaluates to

$$\int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{\frac{N}{4}} dt \leq CT^{1-\frac{N}{4}}. \tag{3.2.5}$$

Multiplying the spatial bound (3.2.4) by the temporal bound (3.2.5) directly yields the desired estimate (3.2.3). This completes the proof.  $\square$

The proof of Lemma 3.2.2 relies on straightforward direct computations exploiting the radial symmetry of the composite function, and is therefore omitted.

**Lemma 3.2.2.** Let  $\mathcal{F}^\ell$  be defined as in (3.2.1). Then the following estimates hold:

$$\left| \nabla \cdot \Delta \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) \right| \leq C \frac{1}{|x|^3 \ln \sqrt{R}} \mathcal{F}^{\ell-4} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right),$$

and

$$\left| \Delta^2 \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) \right| \leq C \frac{1}{|x|^4 \ln \sqrt{R}} \mathcal{F}^{\ell-4} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right).$$

Using the harmonic properties of  $H(x)$  and the expansion formula (5.0.2), we deduce

$$\Delta^2 \psi(x) = H(x) \Delta^2 \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) + 4 \nabla H(x) \cdot \nabla \left( \Delta \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) \right).$$

Applying the bounds derived in Lemma 3.2.2, alongside the gradient bound  $|\nabla H(x)| \leq C|x|^{1-N}$  from Lemma 2.1.15, we can estimate

$$\begin{aligned}
|\Delta^2 \psi(x)| &\leq C \left( \frac{1}{|x|^4 \ln \sqrt{R}} + \frac{|x|^{1-N}}{|x|^3 \ln \sqrt{R}} \right) \mathcal{F}^{\ell-4} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) \\
&\leq \frac{C}{|x|^4 \ln \sqrt{R}} \mathcal{F}^{\ell-4} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right),
\end{aligned} \tag{3.2.6}$$

where the second inequality holds because  $N \geq 5$  and  $|x| \geq 1$ .

**Lemma 3.2.3.** For  $N \geq 5$  and  $p = \frac{N}{N-4}$ , assume  $R > 0$  and  $T > 0$  are sufficiently large. Then, the following estimate holds:

$$\int_Q \varphi_1^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_1(t, x)|^{p'} dx dt \leq CT(\ln R)^{-\frac{N}{4}+1}.$$

*Proof.* By the separable structure of the test function defined in (3.2.1), the integral factors into temporal and spatial components. Specifically, we have

$$\int_Q \varphi_1^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_1(t, x)|^{p'} dx dt = \left( \int_0^T \eta(t) dt \right) \left( \int_{D^c} \psi^{-\frac{1}{p-1}}(x) |\Delta^2 \psi(x)|^{p'} dx \right).$$

The temporal integral is evaluated directly as in (2.1.12). To estimate the spatial component, we utilize the bound for  $|\Delta^2 \psi(x)|$  from (3.2.6). Recalling that  $p' = \frac{N}{4}$  when  $p = \frac{N}{N-4}$ , and restricting our focus to the support of the derivatives, it follows that

$$\begin{aligned} \int_{D^c} \psi^{-\frac{1}{p-1}}(x) |\Delta^2 \psi(x)|^{p'} dx &\leq C \int_{\sqrt{R} < |x| < R} \mathcal{F}^{\ell-4p'} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) (|x|^{-4} (\ln \sqrt{R})^{-1})^{\frac{N}{4}} dx \\ &\leq C(\ln R)^{-\frac{N}{4}} \int_{\sqrt{R} < |x| < R} |x|^{-N} dx. \end{aligned}$$

By passing to polar coordinates, the remaining integral evaluates to

$$\int_{\sqrt{R}}^R r^{-N} r^{N-1} dr = \ln R - \ln \sqrt{R} = \frac{1}{2} \ln R.$$

Combining these temporal and spatial bounds leads to the desired estimate, which completes the proof.  $\square$

### Test Functions with Logarithmic Arguments $\varphi_2(t, x)$ .

We introduce a class of test functions for the critical case, defined as

$$\varphi_2(t, x) = \eta(t)\phi(x) = \zeta^\ell \left( \frac{t}{T} \right) B(x) \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right), \quad (t, x) \in Q, \quad (3.2.7)$$

where  $B(x)$  is the biharmonic function defined in (2.1.20),  $\eta(t)$  is the temporal cut-off function from (2.1.2), and  $\mathcal{F}$  is the smooth logarithmic profile defined in (3.2.2).

**Lemma 3.2.4.** For  $N \geq 5$  and  $p = \frac{N}{N-4}$ , the following estimate holds:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_2(t, x)|^{p'} dx dt \leq CT^{1-\frac{N}{4}} R^N. \quad (3.2.8)$$

*Proof.* Based on the separable structure of the test function in (3.2.7) and noting that  $p' = \frac{N}{4}$  for the critical exponent, the integral factors into temporal and spatial components:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_2(t, x)|^{\frac{N}{4}} dx dt = \left( \int_{D^c} \phi(x) dx \right) \left( \int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{\frac{N}{4}} dt \right).$$

To estimate the spatial integral, we observe from Lemma 2.1.27 that  $B(x)$  is bounded for  $N \geq 5$  in the exterior domain  $D^c$ . Consequently, we have

$$\begin{aligned} \int_{D^c} \phi(x) dx &= \int_{1 < |x| < R} B(x) \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) dx \\ &\leq C \int_{1 < |x| < R} \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) dx. \end{aligned} \quad (3.2.9)$$

Transforming the integral into polar coordinates and evaluating the  $N$ -dimensional volume integration yields

$$\int_{D^c} \phi(x) dx \leq C \int_1^R r^{N-1} dr \leq CR^N.$$

The temporal integral is bounded as (2.1.9). Combining these spatial and temporal bounds yields the result in (3.2.8) directly.  $\square$

The proof of the following lemma is established by synthesizing the estimates for the biharmonic function from Lemma 2.1.27 with the logarithmic derivative bounds from Lemma 3.2.2. The introduction of the biharmonic kernel  $B(x)$  slightly modifies the resulting decay rates.

**Lemma 3.2.5.** The following estimates hold:

$$\begin{aligned} \left| \nabla(\Delta B(x)) \cdot \nabla \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) \right| &\leq \frac{C|x|^{-N}}{\ln \sqrt{R}} \mathcal{F}^{\ell-4} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right), \\ \left| \nabla B(x) \cdot \nabla \left( \Delta \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) \right) \right| &\leq \frac{C|x|^{-N}}{\ln \sqrt{R}} \mathcal{F}^{\ell-4} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right), \\ \left| B(x) \Delta^2 \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) \right| &\leq \frac{C|x|^{-4}}{\ln \sqrt{R}} B(x) \mathcal{F}^{\ell-4} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right), \\ \left| \Delta B(x) \Delta \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) \right| &\leq \frac{C|x|^{-N}}{\ln \sqrt{R}} \mathcal{F}^{\ell-4} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right), \end{aligned}$$

where  $\cdot$  denotes the standard inner product in  $\mathbb{R}^N$ .

**Lemma 3.2.6.** For  $N \geq 5$  and  $p = \frac{N}{N-4}$ , the following integral estimate holds:

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_2(t, x)|^{p'} dx dt \leq CTR^2 (\ln R)^{-\frac{N}{4}}. \quad (3.2.10)$$

*Proof.* Utilizing the properties of the test function  $\varphi_2$  as defined in (3.2.7), we get

$$\int_Q \varphi_2^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_2(t, x)|^{p'} dx dt = \left( \int_0^T \eta(t) dt \right) \left( \int_{D^c} \phi^{-\frac{1}{p-1}}(x) |\Delta^2 \phi(x)|^{p'} dx \right).$$

Applying the biharmonic expansion (5.0.2) and the derivative bounds from Lemma 3.2.5, we establish the following estimate for the spatial operator

$$|\Delta^2 \phi(x)| \leq \frac{C}{\ln \sqrt{R}} (|x|^{-4} + |x|^{-N}) \mathcal{F}^{\ell-4} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right).$$

Given that  $p' = \frac{N}{4}$  in the critical case and using the spatial behavior of the biharmonic kernel  $B(x)$ , the spatial integral (3.2.11) can be bounded as:

$$\begin{aligned} \int_{D^c} \phi^{-\frac{1}{p-1}}(x) |\Delta^2 \phi(x)|^{p'} dx &= \int_{\sqrt{R} < |x| < R} B(x)^{-\frac{1}{p-1}} \mathcal{F}^{-\frac{\ell}{p-1}} |\Delta^2 \phi(x)|^{p'} dx \\ &\leq C (\ln \sqrt{R})^{-\frac{N}{4}} \int_{\sqrt{R} < |x| < R} |x|^{2-N} \mathcal{F}^{\ell-N} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) dx. \end{aligned} \quad (3.2.11)$$

By transforming the integral into polar coordinates, we obtain

$$\int_{D^c} \phi^{-\frac{1}{p-1}}(x) |\Delta^2 \phi(x)|^{p'} dx \leq C (\ln \sqrt{R})^{-\frac{N}{4}} \int_{\sqrt{R}}^R r^{2-N} r^{N-1} dr = C (\ln R)^{-\frac{N}{4}} \int_{\sqrt{R}}^R r dr.$$

Evaluating the integral leads to the bound  $CR^2 (\ln R)^{-\frac{N}{4}}$ . Finally, combining this with the temporal estimate  $\int_0^T \eta(t) dt \leq CT$  from (2.1.12) yields the desired result in (3.2.10).  $\square$

### The Case of Logarithmic Arguments $\varphi_3(t, x)$ .

For the critical regime, we introduce a third class of trial functions defined by

$$\varphi_3(t, x) = \eta(t) \chi(x) = \zeta^\ell \left( \frac{t}{T} \right) \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right), \quad (t, x) \in Q. \quad (3.2.12)$$

In this configuration,  $\eta(t)$  represents the temporal cut-off function defined in (2.1.2), while  $\mathcal{F} : \mathbb{R} \rightarrow [0, 1]$  is the smooth logarithmic profile introduced in (3.2.2). These functions are particularly effective for capturing the precise blow-up rates.

**Lemma 3.2.7.** Let  $N \geq 5$  and  $p = \frac{N}{N-4}$ . For  $R > 0$  and  $T > 0$  sufficiently large, the following estimate holds:

$$\int_Q \varphi_3^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_3(t, x)|^{p'} dx dt \leq CT^{1-\frac{N}{4}} R^N.$$

*Proof.* By the separable construction of the trial function  $\varphi_3$ , we get

$$\int_Q \varphi_3^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_3(t, x)|^{p'} dx dt = \left( \int_{D^c} \chi(x) dx \right) \left( \int_0^T \eta^{-\frac{1}{p-1}}(t) |\partial_t \eta(t)|^{p'} dt \right).$$

To estimate the spatial integral, we consider the support of the logarithmic cut-off function in  $D^c$ . For  $R$  sufficiently large, we have

$$\begin{aligned} \int_{D^c} \chi(x) dx &= \int_{1 < |x| < R} \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) dx \\ &\leq C \int_1^R r^{N-1} dr \leq CR^N, \end{aligned} \quad (3.2.13)$$

where we have employed polar coordinates for the  $N$ -dimensional volume integration. Given that  $p' = N/4$  in the critical case, the temporal derivative estimate from (2.1.9) yields a bound of order  $CT^{1-N/4}$ . Combining these spatial and temporal estimates leads directly to

$$\int_Q \varphi_3^{-\frac{1}{p-1}}(t, x) |\partial_t \varphi_3(t, x)|^{p'} dx dt \leq CT^{1-\frac{N}{4}} R^N,$$

which completes the proof.  $\square$

**Lemma 3.2.8.** Let  $N \geq 5$  and  $p = \frac{N}{N-4}$ . The following integral estimate holds:

$$\int_Q \varphi_3^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_3(t, x)|^{p'} dx dt \leq CT(\ln \sqrt{R})^{1-\frac{N}{4}}.$$

*Proof.* By factoring the space-time integral according to the definition of  $\varphi_3$  in (3.2.12), we obtain:

$$\int_Q \varphi_3^{-\frac{1}{p-1}}(t, x) |\Delta^2 \varphi_3(t, x)|^{p'} dx dt = \left( \int_0^T \eta(t) dt \right) \left( \int_{D^c} \chi^{-\frac{1}{p-1}}(x) |\Delta^2 \chi(x)|^{p'} dx \right).$$

The temporal integral is bounded (2.1.12). For the spatial component, we use the logarithmic-derivative bounds from Lemma 3.2.2. In the region  $\sqrt{R} < |x| < R$ , the bi-harmonic operator acting on the logarithmic profile satisfies

$$|\Delta^2 \chi(x)| \leq \frac{C|x|^{-4}}{\ln \sqrt{R}} \mathcal{F}^{\ell-4} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right).$$

Substituting this into the spatial integral and setting  $p' = N/4$ , we have

$$\int_{D^c} \chi^{-\frac{1}{p-1}} |\Delta^2 \chi|^{p'} dx \leq C(\ln \sqrt{R})^{-N/4} \int_{\sqrt{R} < |x| < R} |x|^{-N} \mathcal{F}^{\ell-N} \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) dx.$$

By changing to polar coordinates and noting that the volume element is  $r^{N-1} dr$ , the integral simplifies to:

$$\begin{aligned} \int_{\sqrt{R} < |x| < R} |x|^{-N} dx &= \int_{\sqrt{R}}^R r^{-N} r^{N-1} dr = \int_{\sqrt{R}}^R \frac{1}{r} dr \\ &= \ln R - \ln \sqrt{R} = \ln \sqrt{R}. \end{aligned}$$

Combining these results, the spatial integral scales as

$$\int_{D^c} \chi^{-\frac{1}{p-1}} |\Delta^2 \chi|^{p'} dx \leq C(\ln \sqrt{R})^{1-N/4}$$

Multiplying by the temporal bound  $CT$  completes the proof.  $\square$

**Lemma 3.2.9.** For sufficiently large  $R > 0$ , the following integral evaluations hold:

$$\int_Q f(x) \varphi_j(x, t) dx dt = CT \int_{1 < |x| < 2R} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx,$$

alternatively, under logarithmic scaling:

$$\int_Q f(x) \varphi_j(x, t) dx dt = CT \int_{1 < |x| < R} f(x) A(x) \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln(\sqrt{R})} \right) dx,$$

where  $A(x)$  corresponds to the stationary profile defined in (3.1.1).

*Proof.* Applying the identity for the temporal component  $\int_0^T \eta(t) dt = CT$ , we derive:

$$\begin{aligned} \int_Q f(x) \varphi_j(x, t) dx dt &= \int_0^T \eta(t) dt \int_{D^c} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx \\ &= CT \int_{1 < |x| < 2R} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx. \end{aligned}$$

A similar calculation yields the second integral evaluation, taking into account the logarithmic support of the test function  $\mathcal{F}^\ell$ .  $\square$

### 3.2.1 Proof of Theorem 3.1.2 to the problem I and problem VI

The theorem is proved using the method of contradiction. Suppose that  $u$  is a global weak solution to Problem I.

By definition 3.1.1, we have:

$$\begin{aligned} \int_Q |u(x, t)|^p \varphi_j(x, t) dx dt + \int_Q f(x) \varphi_j(x, t) dx dt + \int_{D^c} u_0(x) \varphi_j(x, 0) dx \\ = - \int_Q u(x, t) \partial_t \varphi_j(x, t) dx dt + \int_Q u(x, t) \Delta^2 \varphi_j(x, t) dx dt. \end{aligned}$$

Applying Young's inequality with  $\varepsilon$  to the terms on the right-hand side, we obtain:

$$\begin{aligned} \int_Q u(x, t) \partial_t \varphi_j(x, t) dx dt \\ \leq \varepsilon_1 \int_Q |u(x, t)|^p \varphi_j(x, t) dx dt + c(\varepsilon_1) \int_Q \varphi_j^{-\frac{1}{p-1}}(x, t) |\partial_t \varphi_j(x, t)|^{p'} dx dt, \end{aligned}$$

and

$$\begin{aligned} \int_Q u(x, t) \Delta^2 \varphi_j(x, t) dx dt \\ \leq \varepsilon_2 \int_Q |u(x, t)|^p \varphi_j(x, t) dx dt + c(\varepsilon_2) \int_Q \varphi_j^{-\frac{1}{p-1}}(x, t) |\Delta^2 \varphi_j(x, t)|^{p'} dx dt. \end{aligned}$$

By choosing  $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$  and absorbing the terms into the left-hand side, we arrive

$$\begin{aligned} \int_Q f(x) \varphi_j(x, t) dx dt + \int_{D^c} u_0(x) \varphi_j(x, 0) dx \\ \leq \int_Q \varphi_j^{-\frac{1}{p-1}}(x, t) |\Delta^2 \varphi_j(x, t)|^{p'} dx dt + \int_Q \varphi_j^{-\frac{1}{p-1}}(x, t) |\partial_t \varphi_j(x, t)|^{p'} dx dt. \end{aligned} \tag{3.2.14}$$

**(i) The case  $N = 2$ .** As previously observed, the test function defined in (2.1.1) satisfies the necessary boundary conditions in Definition 3.1.1 for both Problem I and Problem VI. Consequently, it is sufficient to demonstrate the proof for Problem I. From the integral estimates derived in Lemma 2.1.12 and Lemma 2.1.14, we have the following inequality:

$$\int_Q f(x) \varphi_1(x, t) dx dt + \int_{D^c} u_0(x) \varphi_1(x, 0) dx \leq C \left( T R^{2-4p'} (\ln R)^{\frac{p}{p-1}} + T^{1-p'} R^2 \ln R \right).$$

By setting the temporal scaling  $T = R^j$  with  $j > 4$  and applying the results of Lemma 3.2.9, we can rewrite the expression in terms of the spatial source term:

$$\begin{aligned} \int_{1 < |x| < 2R} f(x)A(x)\xi^\ell \left( \frac{|x|}{R} \right) dx &\leq CR^{-j} \int_{1 < |x| < 2R} |u_0(x)|\varphi_1(x, 0) dx \\ &\quad + CR^{2-4p'}(\ln R)^{\frac{p}{p-1}} + CR^{-jp'+2} \ln R, \end{aligned}$$

where  $A(x)$  corresponds to the harmonic profile (2.1.6) for the specific boundary configuration.

As  $R$  becomes sufficiently large, and noting that  $u_0 \in L_{loc}^1(D^c)$ , the first and third terms on the right-hand side decay rapidly due to our choice of  $j$ . Thus, the inequality simplifies to:

$$\int_{1 < |x| < 2R} f(x)A(x)\xi^\ell \left( \frac{|x|}{R} \right) dx \leq CR^{-4p'+2} \ln R.$$

Since  $p > 1$ , it is immediate that  $2 - 4p' < 0$ . By applying the dominated convergence theorem as  $R \rightarrow \infty$  and observing that:

$$\lim_{R \rightarrow \infty} A(x)\xi^\ell \left( \frac{|x|}{R} \right) = A(x)\xi^\ell(0) = A(x), \quad (3.2.15)$$

we obtain:

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < 2R} f(x)A(x)\xi^\ell \left( \frac{|x|}{R} \right) dx = \int_{D^c} f(x)A(x) dx \leq 0,$$

which directly contradicts our initial hypothesis that

$$\int_{D^c} f(x)A(x) dx > 0.$$

The proof of case (i) for  $N = 2$  is thus complete.

**(ii) The subcritical case.** In this subsection, we employ the test function defined in (2.1.1), noting that for  $N \geq 3$ , the stationary profile  $A(x)$  follows the dimensionality of the exterior domain. Following the estimation procedure established in the previous proofs and applying Lemma 2.1.16 and Lemma 2.1.17, we obtain:

$$\int_Q f(x)\varphi_1(x, t) dx dt + \int_{D^c} u_0(x)\varphi_1(x, 0) dx \leq C \left( TR^{N-4p'} + T^{1-p'} R^N \right).$$

By utilizing the temporal scaling  $T = R^j$  ( $j > 4$ ) and invoking Lemma 3.2.9, we derive the spatial inequality:

$$\begin{aligned} \int_{1 < |x| < 2R} f(x)A(x)\xi^\ell \left( \frac{|x|}{R} \right) dx &\leq CR^{-j} \int_{1 < |x| < 2R} |u_0(x)|\varphi_1(x, 0) dx \\ &\quad + C \left( R^{N-4p'} + R^{-jp'+N} \right). \end{aligned}$$

For the dimensions  $N \in \{3, 4\}$ , the exponent satisfies  $N - 4p' < 0$  for all  $p > 1$ . Similarly, for  $N \geq 5$ , the inequality  $N - 4p' < 0$  holds when  $1 < p < \frac{N}{N-4}$ .

Consequently, as  $R \rightarrow \infty$ , the right-hand side tends to zero. Applying the dominated convergence theorem to the left-hand side, we find:

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < 2R} f(x)A(x)\xi^\ell \left( \frac{|x|}{R} \right) dx = \int_{D^c} f(x)A(x) dx \leq 0,$$

which directly contradicts the assumption that  $\int_{D^c} f(x)A(x) dx > 0$ . This completes the proof for the subcritical case.

**(ii) Critical case ( $p = p_{crit}$ ).** The proof for the critical regime follows a similar methodology to the subcritical analysis, with the adjustment of employing test functions with logarithmic arguments as defined in (3.2.1). By applying the inequality (3.2.14) and incorporating the estimates from Lemma 3.2.1 and Lemma 3.2.3, we arrive

$$\begin{aligned} \int_Q f(x)\varphi_1(x, t) dx dt + \int_{D^c} u_0(x)\varphi_1(x, 0) dx \\ \leq CT \left( (\ln R)^{-\frac{N}{4}+1} + R^{\frac{N}{2}(1-\frac{N}{2})} (\ln R)^{-\frac{N}{4}} \right). \end{aligned}$$

Utilizing the scaling  $T = R^j$  and applying Lemma 3.2.9, the inequality is expressed as

$$\begin{aligned} \int_{1 < |x| < R} f(x)A(x)\mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) dx \leq C \left[ (\ln R)^{-\frac{N}{4}+1} + R^{\frac{N}{2}(1-\frac{N}{2})} (\ln R)^{-\frac{N}{4}} \right] \\ + CR^{-j} \int_{D^c} |u_0(x)|\varphi_1(x, 0) dx. \end{aligned}$$

Taking  $j > 4$  and passing to the limit as  $R \rightarrow \infty$ , we note that the right-hand side tends to zero since  $-\frac{N}{4} + 1 < 0$  for  $N \geq 5$ . Furthermore, by the dominated convergence theorem and the property, we get

$$\lim_{R \rightarrow \infty} \mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) = \mathcal{F}^\ell(-1) = 1, \quad (3.2.16)$$

which yields to

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < R} f(x)A(x)\mathcal{F}^\ell \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \sqrt{R}} \right) dx = \int_{D^c} f(x)A(x) dx \leq 0.$$

This contradicts the positivity assumption  $\int_{D^c} f(x)A(x) dx > 0$ . This concludes the proof of Theorem 3.1.2.

### 3.2.2 Proof of Theorem 3.1.2 to the problem II

The test function  $\varphi_2$  satisfies the prescribed conditions for a weak solution to Problem II, as defined in Definition 3.1.1. For this specific boundary configuration, the stationary profile  $A(x)$  is given by the biharmonic function (2.1.20).

By substituting this profile into our integral identity and proceeding with the scaling analysis for  $R$  and  $T$ , we obtain the non-existence results for the clamped Dirichlet case as stated in Theorem 3.1.2.

**(i) The case  $N = 2$ .** By Definition 3.1.1, the weak solution for Problem II satisfies the following integral identity:

$$\begin{aligned} \int_Q |u|^p \varphi_2 dx dt + \int_Q f(x)\varphi_2(x, t) dx dt + \int_{D^c} u_0(x)\varphi_2(x, 0) dx \\ = - \int_Q u \partial_t \varphi_2 dx dt + \int_Q u \Delta^2 \varphi_2 dx dt, \end{aligned}$$

where the test function  $\varphi_2$  is provided by (2.1.19). Following the standard estimation procedure, the inequality (3.2.14) for the specific case of  $\varphi_2$  is expressed as

$$\begin{aligned} & \int_Q f(x) \varphi_2(x, t) dx dt + \int_{D^c} u_0(x) \varphi_2(x, 0) dx \\ & \leq \int_Q \varphi_2^{-\frac{1}{p-1}}(x, t) |\Delta^2 \varphi_2(x, t)|^{p'} dx dt + \int_Q \varphi_2^{-\frac{1}{p-1}}(x, t) |\partial_t \varphi_2(x, t)|^{p'} dx dt. \end{aligned}$$

By synthesizing the results from Lemma 2.1.20, Lemma 2.1.22, and Lemma 3.2.9, we derive the following integral estimate:

$$\begin{aligned} \int_{1 < |x| < 2R} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx & \leq \frac{1}{CT} \int_{1 < |x| < 2R} |u_0(x)| \varphi_2(x, 0) dx \\ & + CT^{-p'} R^4 \ln R + R^{-\frac{2}{p-1}} (\ln R)^{\frac{p}{p-1}}, \end{aligned}$$

where  $A(x)$  is the biharmonic profile (2.1.20). By setting the temporal scaling parameter  $T = R^j$  for  $j > 4$ , we obtain:

$$\begin{aligned} \int_{1 < |x| < 2R} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx & \leq CR^{-j} \int_{1 < |x| < 2R} |u_0(x)| \varphi_2(x, 0) dx \\ & + CR^{4-jp'} \ln R + R^{-\frac{2}{p-1}} (\ln R)^{\frac{p}{p-1}}. \end{aligned}$$

By taking into account the limit property (3.2.15) and passing to the limit as  $R \rightarrow \infty$  in the aforementioned inequality, we arrive

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < 2R} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx = \int_{D^c} f(x) A(x) dx \leq 0,$$

which contradicts our initial assumption that

$$\int_{D^c} f(x) A(x) dx > 0.$$

This concludes the proof for the current case.

**(i) The case  $N = 4$ .** In dimension  $N = 4$ , the inequality (3.2.14) for the test function  $\varphi_2$  takes the following form

$$\begin{aligned} & \int_Q f(x) \varphi_2(x, t) dx dt + \int_{D^c} u_0(x) \varphi_2(x, 0) dx \\ & \leq \int_Q \varphi_2^{-\frac{1}{p-1}} |\Delta^2 \varphi_2|^{p'} dx dt + \int_Q \varphi_2^{-\frac{1}{p-1}} |\partial_t \varphi_2|^{p'} dx dt. \end{aligned}$$

By applying the results from Lemma 2.1.24, Lemma 2.1.26, and Lemma 3.2.9, we derive

$$\int_{D^c} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx \leq \frac{1}{CT} \int_{D^c} |u_0(x)| \varphi_2(x, 0) dx + CT^{-p'} R^4 \ln R + R^{-\frac{4}{p-1}} (\ln R)^{\frac{p}{p-1}}.$$

Upon setting  $T = R^j$  for  $j > 4$ , we obtain

$$\begin{aligned} \int_{1 < |x| < 2R} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx & \leq CR^{-j} \int_{1 < |x| < 2R} |u_0(x)| \varphi_2(x, 0) dx \\ & + CR^{4-jp'} \ln R + CR^{-\frac{4}{p-1}} (\ln R)^{\frac{p}{p-1}}. \end{aligned}$$

Finally, by accounting for (3.2.15) and passing to the limit as  $R \rightarrow \infty$ , it follows that

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < 2R} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx = \int_{D^c} f(x) A(x) dx \leq 0,$$

which contradicts the assumption  $\int_{D^c} f(x) A(x) dx > 0$ . The proof for  $N = 4$  is concluded.

**(ii) The subcritical case.** By applying the estimates from Lemma 2.1.28 and Lemma 2.1.30, we arrive at the following inequality:

$$\int_Q f(x) \varphi_2(x, t) dx dt + \int_{D^c} u_0(x) \varphi_2(x, 0) dx \leq C \left( T R^{N-4p'} + T^{1-p'} R^N \right).$$

Invoking Lemma 3.2.9 to evaluate the spatial integral of the source term, we have:

$$\begin{aligned} \int_{1 < |x| < 2R} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx &\leq \frac{1}{CT} \int_{1 < |x| < 2R} |u_0(x)| \varphi_2(x, 0) dx \\ &\quad + C \left( R^{N-\frac{4p}{p-1}} + T^{-p'} R^N \right). \end{aligned}$$

Setting  $T = R^j$  with  $j > 4$ , the inequality becomes:

$$\begin{aligned} \int_{1 < |x| < 2R} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx &\leq C R^{-j} \int_{1 < |x| < 2R} |u_0(x)| \varphi_2(x, 0) dx \\ &\quad + C \left( R^{N-\frac{4p}{p-1}} + R^{N-\frac{j p}{p-1}} \right). \end{aligned}$$

Given that  $N - \frac{4p}{p-1} < 0$  for  $N = 3$  (for all  $p > 1$ ) and for  $N \geq 5$  (whenever  $1 < p < \frac{N}{N-4}$ ), we pass to the limit as  $R \rightarrow \infty$ . This yields:

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < 2R} f(x) A(x) \xi^\ell \left( \frac{|x|}{R} \right) dx = \int_{D^c} f(x) A(x) dx \leq 0,$$

which contradicts the positivity hypothesis  $\int_{D^c} f(x) A(x) dx > 0$ . The proof for this case is now complete.

**(ii) Critical case ( $p = p_{\text{crit}}$ ).** Using the results from Lemma 3.2.4, Lemma 3.2.6, and Lemma 3.2.9, we obtain:

$$\begin{aligned} CT \int_{1 < |x| < R} f(x) A(x) \mathcal{F}^\ell \left( \frac{\ln(|x|/\sqrt{R})}{\ln \sqrt{R}} \right) dx &+ \int_{1 < |x| < R} u_0(x) \varphi_2(x, 0) dx \\ &\leq C \left( T (\ln R)^{1-\frac{N}{4}} + T^{1-\frac{N}{4}} R^N \right). \end{aligned}$$

By assigning the scaling  $T = R^j$ , the inequality can be rewritten as:

$$\begin{aligned} \int_{1 < |x| < R} f(x) A(x) \mathcal{F}^\ell \left( \frac{\ln(|x|/\sqrt{R})}{\ln \sqrt{R}} \right) dx &\leq C R^{-j} \int_{1 < |x| < R} |u_0(x)| \varphi_2(x, 0) dx \\ &\quad + C \left( (\ln R)^{1-\frac{N}{4}} + R^{-\frac{N(j-4)}{4}} \right). \end{aligned}$$

Taking  $j > 4$  and passing to the limit as  $R \rightarrow \infty$ , the right-hand side vanishes since  $1 - \frac{N}{4} < 0$  for  $N \geq 5$ . Consequently, we find:

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < R} f(x) A(x) \mathcal{F}^\ell \left( \frac{\ln(|x|/\sqrt{R})}{\ln \sqrt{R}} \right) dx = \int_{D^c} f(x) A(x) dx \leq 0,$$

which contradicts the positivity assumption  $\int_{D^c} f(x) A(x) dx > 0$ . The proof is thus finalized.

### 3.2.3 Proof of Theorem 3.1.2 to problems III-V

The test function  $\varphi_3$  satisfies the boundary and support conditions prescribed in Definition 3.1.1 for Problems III–V. We employ  $\varphi_3$  to establish the non-existence results for these three boundary configurations simultaneously. Since the stationary profile  $A(x) = 1$  for these problems, the integral estimates follow the same scaling laws as the previously analyzed cases.

**(i) The subcritical case.** By Definition 3.1.1 and the application of inequality (3.2.14) to the test function  $\varphi_3$ , we obtain

$$\begin{aligned} \int_Q f(x)\varphi_3(x, t) dx dt + \int_{D^c} u_0(x)\varphi_3(x, 0) dx \\ \leq \int_Q \varphi_3^{-\frac{1}{p-1}} |\Delta^2 \varphi_3|^{p'} dx dt + \int_Q \varphi_3^{-\frac{1}{p-1}} |\partial_t \varphi_3|^{p'} dx dt. \end{aligned}$$

From Lemma 2.1.31 and Lemma 2.1.32, the integral bound is given by

$$\int_Q f(x)\varphi_3(x, t) dx dt + \int_{D^c} u_0(x)\varphi_3(x, 0) dx \leq C \left( TR^{N-4p'} + T^{1-p'} R^N \right).$$

By invoking Lemma 3.2.9 and setting  $T = R^j$  for  $j > 4$ , we derive

$$\begin{aligned} \int_{1 < |x| < 2R} f(x) \xi^\ell \left( \frac{|x|}{R} \right) dx \leq CR^{-j} \int_{1 < |x| < 2R} |u_0(x)| \varphi_3(x, 0) dx \\ + C \left( R^{N-4p'} + R^{-jp'+N} \right). \end{aligned}$$

For large  $R$ , the inequality reduces to

$$\int_{1 < |x| < 2R} f(x) \xi^\ell \left( \frac{|x|}{R} \right) dx \leq CR^{N-4p'}.$$

Given that  $N - 4p' < 0$  for  $N \in \{2, 3, 4\}$  (for all  $p > 1$ ) and for  $N \geq 5$  (whenever  $1 < p < \frac{N}{N-4}$ ), we pass to the limit as  $R \rightarrow \infty$ . Using (3.2.15), we find:

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < 2R} f(x) \xi^\ell \left( \frac{|x|}{R} \right) dx = \int_{D^c} f(x) dx \leq 0,$$

which contradicts the positivity hypothesis  $\int_{D^c} f(x) dx > 0$ . This concludes the proof for subcritical case.

**(ii) Critical case ( $p = p_{Crit}$ ).** Using the integral inequality (3.2.14) as established in the subcritical analysis, and specific logarithmic estimates from Lemma 3.2.7 and Lemma 3.2.8, we arrive at

$$\int_Q f(x)\varphi_3(x, t) dx dt + \int_{D^c} u_0\varphi_3(0, x) dx \leq C \left( T(\ln \sqrt{R})^{1-\frac{N}{4}} + T^{1-\frac{N}{4}} R^N \right).$$

By applying Lemma 3.2.9 and adopting the scaling  $T = R^j$  ( $j > 4$ ) for the critical exponent  $p = \frac{N}{N-4}$ , the inequality becomes

$$\begin{aligned} \int_{1 < |x| < R} f(x) \mathcal{F}^\ell \left( \frac{\ln(|x|/\sqrt{R})}{\ln \sqrt{R}} \right) dx \leq C \left( (\ln \sqrt{R})^{-\frac{N-4}{4}} + R^{-\frac{N(j-4)}{4}} \right) \\ + CR^{-j} \int_{1 < |x| < R} |u_0(x)| \varphi_3(x, 0) dx. \end{aligned}$$

Upon letting  $R \rightarrow \infty$ , the right-hand side vanishes for  $N \geq 5$  since the logarithmic and power terms possess negative exponents. We thus obtain

$$\lim_{R \rightarrow \infty} \int_{1 < |x| < R} f(x) \mathcal{F}^\ell \left( \frac{\ln(|x|/\sqrt{R})}{\ln \sqrt{R}} \right) dx = \int_{D^c} f(x) dx \leq 0,$$

which provides the desired contradiction to  $\int_{D^c} f(x) dx > 0$ . The proof of Theorem 3.1.2 for the critical case is thus complete.

**(iii) The supercritical case ( $p > p_{\text{crit}}$ ).** In this regime, we establish the existence of global solutions by constructing a stationary supersolution. Let us define:

$$v(x) = \varepsilon |x|^{-m}, \quad \text{with } \varepsilon > 0 \text{ and } \frac{4}{p-1} < m < N-4. \quad (3.2.17)$$

Note that the range for  $m$  is non-empty since  $p > \frac{N}{N-4}$  for  $N \geq 5$ . A direct computation of the biharmonic operator yields:

$$\Delta^2 v(x) = \varepsilon M |x|^{-m-4}, \quad M = m(m+2)(m-N+2)(m-N+4).$$

Consequently, we have:

$$\Delta^2 v(x) - v^p(x) = \varepsilon |x|^{-m-4} (M - \varepsilon^{p-1} |x|^{-mp+m+4}).$$

Since  $-mp + m + 4 < 0$  and  $|x| \geq 1$ , it follows that  $|x|^{-mp+m+4} \leq 1$ . For sufficiently small  $\varepsilon$ , we define  $f(x) := \Delta^2 v(x) - v^p(x) > 0$ . Furthermore, evaluations on the boundary  $\partial B_1$  confirm that  $v$  satisfies the non-negativity requirements for the trace and Laplacian terms associated with Problems I–VI.

By the comparison principle,  $u(x, t) \leq v(x)$  for all  $(x, t) \in Q$ , which establishes the existence of a stationary positive supersolution. The proof of Theorem 3.1.2 is thus complete.

### 3.2.4 Proof of Theorem 3.1.5

**Cases (i)  $\omega < 0$  and (ii)  $0 \leq \omega < \frac{4p}{p-1}$ .** By applying the integral inequality (3.2.14), we obtain:

$$\int_{D^c} f(x) \xi^\ell \left( \frac{|x|}{R} \right) dx \leq CR^{-j} \int_{D^c} u_0(x) \varphi(x, 0) dx + C_1 \left( R^{N-4p'} + R^{-jp'+N} \right). \quad (3.2.18)$$

Considering the asymptotic behavior  $f(x) \geq C|x|^{-\omega}$  for  $f \in \mathcal{I}_\omega^+$ , the left-hand side of (3.2.18) is bounded from below as follows:

$$\int_{D^c} f(x) \xi^\ell \left( \frac{|x|}{R} \right) dx \geq C \int_{1 < |x| < R} |x|^{-\omega} dx \geq CR^{N-\omega}. \quad (3.2.19)$$

For sufficiently large  $R$ , substituting (3.2.19) into (3.2.18) results in:

$$CR^{N-\omega} \leq C_1 \left( R^{N-4p'} + R^{-jp'+N} \right).$$

Normalizing by  $R^{N-\omega}$ , we deduce:

$$C \leq C_1 \left( R^{\omega - \frac{4p}{p-1}} + R^{\omega - \frac{jp}{p-1}} \right).$$

In both cases,  $\omega < \frac{4p}{p-1}$ , the exponents of  $R$  are negative for  $j > 4$ . Letting  $R \rightarrow \infty$ , we arrive at the contradiction  $0 < C \leq 0$ , which completes the proof.

**Case (iii)**  $\frac{4p}{p-1} \leq \omega < N$ . To establish existence, we construct a stationary supersolution  $\hat{u}(x) = \xi|x|^{-m}$  with  $\xi > 0$ . We select the exponent  $m$  to satisfy:

$$\omega - 4 \leq m < N - 4.$$

From the biharmonic estimates established in the proof of Theorem 3.1.2, we seek to satisfy the inequality  $\Delta^2 \hat{u}(x) - \hat{u}^p(x) \geq f(x)$ . For sufficiently small  $\xi$ , this requirement is dominated by the linear part:

$$f(x) \leq C\xi|x|^{-m-4}.$$

Applying our lower bound for  $m$ , we have:

$$f(x) \leq C\xi|x|^{-(\omega-4)-4} = C\xi|x|^{-\omega}.$$

This demonstrates that  $f \in \mathcal{I}_\omega^-$  is a sufficient condition for the existence of a stationary positive supersolution. By the comparison principle, Problems I–VI admit global positive solutions in this regime. The proof of Theorem 3.1.5 is now complete.

# Chapter 4

## Higher-order parabolic equation

Chapter 4 is adapted from the following article:

[[TT26a](#)] N. N. Tobakhanov and B. T. Torebek. A note on lifespan estimates for higher-order parabolic equations. *Bull. Lond. Math. Soc.*, 2026

In this chapter, we investigate the higher-order parabolic Cauchy problem:

$$\begin{cases} u_t + (-\Delta)^m u = |u|^p, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = \varepsilon u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (4.0.1)$$

where  $m \geq 1$  is an integer,  $p > 1$ , and  $\varepsilon > 0$  is a small parameter.

This equation represents a fundamental shift in diffusion modeling, particularly in the study of thin film dynamics [[BF90](#)] and surface tension models [[PT12](#)]. Unlike second-order problems, these equations often lack a maximum principle, necessitating an analysis of the fundamental solution to establish critical exponents and blow-up conditions.

In this chapter, we refine and extend the above results by considering initial data  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  without any decay assumption. We establish both upper and lower estimates for the lifespan of solutions to problem (4.0.1), including the critical Fujita case  $p = p_{\text{Fuj}}$ . Our analysis is based on the test function method and the semigroup theory. These findings offer a more detailed understanding of the blow-up mechanism and extend several known results for the higher-order semilinear parabolic equations.

### 4.1 Main results

We begin by characterizing the solution to (4.0.1) through the integral formulation provided by Duhamel's principle. This definition is central to establishing the local-in-time existence and the subsequent blow-up analysis.

**Definition 4.1.1.** Given an initial datum  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , a function

$$u \in C([0, T_\varepsilon); L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$$

is termed a *mild solution* of (4.0.1) if it satisfies the integral equation:

$$u(t) = \varepsilon e^{-t(-\Delta)^m} u_0 + \int_0^t e^{-(t-s)(-\Delta)^m} |u(s)|^p ds, \quad (4.1.1)$$

where  $e^{-t(-\Delta)^m}$  denotes the semigroup generated by the polyharmonic operator. The *maximal lifespan*  $T_\varepsilon$  is defined as:

$$T_\varepsilon := \sup \{ T > 0 \mid \exists u \in C([0, T]; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)) \text{ satisfying (4.1.1)} \}.$$

**Theorem 4.1.2** (Global Existence and Lifespan). *Consider the initial datum  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  such that  $\int_{\mathbb{R}^N} u_0(x) dx > 0$ . The Cauchy problem (4.0.1) possesses a unique mild solution*

$$u \in C([0, T_\varepsilon]; L^1(\mathbb{R}^N)) \cap L^\infty((0, T_\varepsilon); L^\infty(\mathbb{R}^N))$$

for a maximal lifespan  $T_\varepsilon > 0$ . Moreover, as the parameter  $\varepsilon$ , the lifespan satisfies the following estimates:

$$T_\varepsilon \simeq \begin{cases} \varepsilon^{-\left(\frac{1}{p-1} - \frac{N}{2m}\right)^{-1}}, & 1 < p < 1 + \frac{2m}{N}, \\ \exp(\varepsilon^{-(p-1)}), & p = 1 + \frac{2m}{N}. \end{cases} \quad (4.1.2)$$

**Remark 4.1.3.** The results of Theorem 4.1.2 can be compared to those of Caristi et al. [CM03] and Sun [Sun10] in the following ways:

- (i) While previous works [CM03; Sun10] analyzed the lifespan of solutions to problem (4.0.1) using slowly decaying initial data as in (1.1.7), we instead utilize the condition  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ .
- (ii) Previous estimates for the lifespan  $T_\varepsilon$  often relied on an auxiliary parameter  $\kappa$ . Our approach provides a more specific bound for the subcritical regime. Additionally, for the critical case  $p = p_F$ , we identify an exponential behavior  $T_\varepsilon \simeq \exp(\varepsilon^{-(p-1)})$ , which offers a more detailed view of the solution's lifespan in this particular case.

**Remark 4.1.4.** A comparison between the estimates in (4.1.2) and the existing literature yields the following observations:

- In the case  $m = 1$ , the estimate (4.1.2) corresponds to the lifespan results for the classical semilinear heat equation developed by Lee and Ni in [LN92]. Thus, our findings include the Lee–Ni estimates as a specific case.
- We also note that an upper bound for the biharmonic case ( $m = 2$ ) in an exterior domain was recently investigated in [TT26b]. The results in (4.1.2) are consistent with those bounds when the domain  $\Omega^c$  is taken to be the whole space  $\mathbb{R}^N$ .
- To our knowledge, the estimate (4.1.2) represents a new contribution for the polyharmonic case  $m \geq 2$ , specifically for the semilinear biharmonic heat equation  $u_t + \Delta^2 u = |u|^p$ .

### 4.1.1 Preliminaries for the Test-Function Method.

To establish the upper bound for the lifespan, we first introduce the following cut-off functions, which are necessary for the integration by parts procedure:

$$\begin{aligned} \psi_R(x, t) &:= \left[ \phi \left( \frac{|x|^{2m} + t}{R} \right) \right]^l, \\ \psi_R^*(x, t) &:= \left[ \phi^* \left( \frac{|x|^{2m} + t}{R} \right) \right]^l, \end{aligned} \quad (4.1.3)$$

for all  $(x, t) \in Q$ , where  $l = 2mp'$ . The smooth function  $\phi \in C^\infty([0, \infty))$  is defined as:

$$\phi(s) = \begin{cases} 1 & \text{if } s \in [0, 1/2], \\ \text{non-increasing} & \text{if } s \in (1/2, 1), \\ 0 & \text{if } s \in [1, \infty), \end{cases}$$

and the associated function  $\phi^*(s)$  is given by:

$$\phi^*(s) = \begin{cases} 0 & \text{if } s \in [0, 1/2), \\ \phi(s) & \text{if } s \in [1/2, \infty). \end{cases}$$

By using these functions, we can effectively manage the spatial and temporal growth of the solution within the integral framework.

**Lemma 4.1.5.** Let  $\psi_R$  and  $\psi_R^*$  be the cut-off functions defined in (4.1.3). For all sufficiently large  $R > 0$ , there exists a constant  $C > 0$  such that

$$|\partial_t \psi_R(t, x)| + |\Delta^m \psi_R(t, x)| \leq CR^{-1} [\psi_R^*(t, x)]^{1/p},$$

for all  $(t, x) \in Q := (0, \infty) \times \mathbb{R}^N$ .

*Proof.* We introduce the scaled variables:

$$y = \frac{x}{R^{1/(2m)}}, \quad \tau = \frac{t}{R}, \quad s(y, \tau) := |y|^{2m} + \tau, \quad s_R(x, t) = \frac{|x|^{2m} + t}{R}.$$

The operators scale as  $\partial_t \psi_R = R^{-1} \partial_\tau \psi$  and  $\Delta_x^m \psi_R = R^{-1} \Delta_y^m \psi$ . For the time derivative, a direct calculation gives

$$|\partial_t \psi_R(t, x)| \leq CR^{-1} [\phi(s_R)]^{2mp'-1} |\phi'(s_R)| \leq CR^{-1} [\psi_R^*(t, x)]^{1/p}.$$

For the polyharmonic operator, we use Faà di Bruno's formula [Joh02] for the composite function  $\psi(s) = \phi(s)^{2mp'}$ . For  $|\alpha| = 2m$ , we have

$$|\partial_y^\alpha(\psi(s))| = \left| \sum_{k=1}^{|\alpha|} h^{(k)}(g(s)) \sum_{\dots} C_{\alpha, \gamma} \prod_{i=1}^k \partial_y^{\gamma_i}(g(s)) \right|. \quad (4.1.4)$$

Since  $\phi$  is smooth and its derivatives are bounded, we obtain

$$|\Delta_x^m \psi_R(s_R)| = R^{-1} |\Delta_y^m \psi(s)| \leq CR^{-1} \phi(s)^{2mp'-2m} \leq CR^{-1} [\psi_R^*(s_R)]^{1/p}.$$

This concludes the proof of the lemma.  $\square$

### 4.1.2 Proof of Theorem 4.1.2

The proof for the lifespan  $T_\varepsilon$  is divided into two parts, each using a different analytical strategy.

The derivation of the upper bound is based on the test-function method introduced by Ikeda and Sobajima [IS19b]. To handle the spatial derivatives of the polyharmonic operator

$(-\Delta)^m$ , we utilize Faà di Bruno's formula. This allows us to manage the higher-order terms that arise during the integration-by-parts process.

The lower bound is established using a fixed-point argument. This approach makes use of the decay and smoothing properties of the heat kernel for the polyharmonic equation. By applying these kernel properties, we can determine the time interval over which a unique mild solution is guaranteed to exist.

• *Lower estimate.* To determine the lower bound of the lifespan  $T_\varepsilon$ , we apply the contraction mapping principle. We begin by defining the following norm

$$M(t) := \sup_{\tau \in [0, t)} \left\{ (1 + \tau)^{\frac{N}{2m}} \|u(\tau)\|_{L^\infty} + \|u(\tau)\|_{L^1} \right\}.$$

Using the mild solution formulation (4.1.1), the  $L^1$  norm can be estimated as

$$\|u(t)\|_{L^1} \leq C\varepsilon \|u_0\|_{L^1} + C \int_0^t \|u(\tau)\|_{L^p}^p d\tau.$$

Applying Hölder's inequality, we bound the nonlinearity by

$$\|u(\tau)\|_{L^p}^p \leq \|u(\tau)\|_{L^1} \|u(\tau)\|_{L^\infty}^{p-1}.$$

Finally, substituting the definition of  $M(t)$  into the integral, we obtain

$$\|u(t)\|_{L^1} \leq C\varepsilon + CM(t)^p \int_0^t (1 + \tau)^{-\frac{N(p-1)}{2m}} d\tau.$$

Next, we derive the estimate for  $\|u(t)\|_{L^\infty}$ . For the short-time regime  $0 < t \leq 1$ , applying the  $L^\infty - L^\infty$  estimate to (4.1.1) results in:

$$\begin{aligned} (1 + t)^{\frac{N}{2m}} \|u(t)\|_{L^\infty} &\leq C\varepsilon \|u_0\|_{L^\infty} + C \int_0^t \|u(\tau)\|_{L^\infty}^p d\tau \\ &\leq C\varepsilon + CM(t)^p. \end{aligned}$$

For the long-time regime  $t \geq 1$ , the  $L^1 - L^\infty$  estimate leads to:

$$\begin{aligned} (1 + t)^{\frac{N}{2m}} \|u(t)\|_{L^\infty} &\leq C\varepsilon \|u_0\|_{L^1} + C(1 + t)^{\frac{N}{2m}} \int_0^{t/2} (t - \tau)^{-\frac{N}{2m}} \| |u(\tau)|^p \|_{L^1} d\tau \\ &\quad + C(1 + t)^{\frac{N}{2m}} \int_{t/2}^t \|u(\tau)\|_{L^\infty}^p d\tau \\ &\leq C\varepsilon + CM(t)^p \int_0^t (1 + \tau)^{-\frac{N(p-1)}{2m}} d\tau. \end{aligned}$$

By combining the  $L^1$  and  $L^\infty$  bounds, we arrive at the following integral inequality:

$$M(t) \leq C_0\varepsilon + C_1 M(t)^p \int_0^t (1 + \tau)^{-\frac{N(p-1)}{2m}} d\tau. \quad (4.1.5)$$

To complete the lower bound estimate, we let  $T_1$  denote the smallest time such that  $M(t) = 2C_0\varepsilon$ . Substituting this condition into the integral inequality for  $M(t)$ , we obtain:

$$\varepsilon \leq \begin{cases} C\varepsilon^p (1 + T_1)^{\left(\frac{1}{p-1} - \frac{N}{2m}\right)(p-1)} & \text{if } 1 < p < p_F, \\ C\varepsilon^p \log(T_1 + 1) & \text{if } p = p_F. \end{cases}$$

Given the smallness of  $\varepsilon$ , these inequalities lead to the following lower bounds for the lifespan  $T_\varepsilon$ :

$$T_\varepsilon > T_1 \geq \begin{cases} C\varepsilon^{-\left(\frac{1}{p-1}-\frac{N}{2m}\right)^{-1}} & \text{if } 1 < p < p_F, \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = p_F. \end{cases}$$

This concludes the proof of the lower bound for the lifespan.

• *Upper estimate.* The derivation of the upper bound follows the test-function framework developed in [IS19c]. We start with the mild solution representation (4.1.1), multiply by the test function  $\psi_R$ , and integrate over  $\mathbb{R}^N$

$$\int_{\mathbb{R}^N} u\psi_R dx = \varepsilon \int_{\mathbb{R}^N} e^{-t(-\Delta)^m} u_0 \psi_R dx + \int_{\mathbb{R}^N} \left( \int_0^t e^{-(t-s)(-\Delta)^m} |u(s)|^p ds \right) \psi_R dx.$$

Differentiating with respect to  $t$  yields two primary terms. The contribution from the initial data is

$$\varepsilon \int_{\mathbb{R}^N} \left[ -e^{-t(-\Delta)^m} u_0 (-\Delta)^m \psi_R + e^{-t(-\Delta)^m} u_0 \partial_t \psi_R \right] dx. \quad (4.1.6)$$

Similarly, the nonlinear component results in

$$\begin{aligned} \int_{\mathbb{R}^N} |u(t)|^p \psi_R dx - \int_{\mathbb{R}^N} \left( \int_0^t e^{-(t-s)(-\Delta)^m} |u(s)|^p ds \right) (-\Delta)^m \psi_R dx \\ + \int_{\mathbb{R}^N} \left( \int_0^t e^{-(t-s)(-\Delta)^m} |u(s)|^p ds \right) \partial_t \psi_R dx. \end{aligned} \quad (4.1.7)$$

Combining the results from (4.1.6) and (4.1.7) leads to

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} u\psi_R dx - \int_{\mathbb{R}^N} |u(t)|^p \psi_R dx \\ = \int_{\mathbb{R}^N} \left( e^{-t(-\Delta)^m} u_0 + \int_0^t e^{-(t-s)(-\Delta)^m} |u(s)|^p ds \right) \partial_t \psi_R dx \\ - \int_{\mathbb{R}^N} \left( e^{-t(-\Delta)^m} u_0 + \int_0^t e^{-(t-s)(-\Delta)^m} |u(s)|^p ds \right) (-\Delta)^m \psi_R dx. \end{aligned}$$

Recognizing the mild solution (4.1.1) within the right-hand side, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^N} u\psi_R dx - \int_{\mathbb{R}^N} |u(t)|^p \psi_R dx = - \int_{\mathbb{R}^N} u(-\Delta)^m \psi_R dx + \int_{\mathbb{R}^N} u \partial_t \psi_R dx.$$

Finally, by integrating over  $[0, T]$  and utilizing the property  $\psi_R(\cdot, T) = 0$ , we establish the weak identity

$$\varepsilon \int_{\mathbb{R}^N} u_0 \psi_R(x, 0) dx + \int_Q |u|^p \psi_R dx dt = \int_Q u(-\Delta)^m \psi_R dx dt - \int_Q u \partial_t \psi_R dx dt. \quad (4.1.8)$$

Following the estimates established in Lemma 4.1.5, we deduce

$$\begin{aligned} \int_Q |u|^p \psi_R dx dt + \varepsilon \int_{\mathbb{R}^N} u_0(x) \psi_R(0, x) dx &\leq \int_Q |u| (|\partial_t \psi_R| + |(-\Delta)^m \psi_R|) dx dt \\ &\leq CR^{-1} \int_Q |u| (\psi_R^*)^{\frac{1}{p}} dx dt. \end{aligned} \quad (4.1.9)$$

We then apply Hölder's inequality to the right-hand side of (4.1.9)

$$\begin{aligned} R^{-1} \int_Q |u| (\psi_R^*)^{\frac{1}{p}} dx dt &\leq R^{-1} \left( \int_Q |u|^p \psi_R^* dx dt \right)^{\frac{1}{p}} \left( \int_0^T \int_{\frac{1}{2R} < |x|^{2m} < R} dx dt \right)^{\frac{1}{p'}} \\ &\leq C R^{-\frac{p-1}{p} \left( \frac{1}{p-1} - \frac{N}{2m} \right)} \left( \int_Q |u|^p \psi_R^* dx dt \right)^{\frac{1}{p}}, \end{aligned}$$

which remains valid for  $R \in (2R_0, T)$ . Consequently, we obtain the following bound

$$\int_Q |u|^p \psi_R dx dt + \varepsilon \int_{\mathbb{R}^N} u_0(x) \psi_R(0, x) dx \leq C R^{-\frac{p-1}{p} \left( \frac{1}{p-1} - \frac{N}{2m} \right)} \left( \int_Q |u|^p \psi_R^* dx dt \right)^{\frac{1}{p}}. \quad (4.1.10)$$

To analyze the critical regime, we introduce the auxiliary quantities

$$\begin{aligned} X(r) &:= \int_Q |u(t, x)|^p \psi_r(t, x) dx dt, \\ Y(r) &:= \int_Q |u(t, x)|^p \psi_r^*(t, x) dx dt, \end{aligned}$$

and define the logarithmic average

$$W(R) := \int_0^R Y(r) \frac{dr}{r}.$$

From these definitions and the inequality (4.1.9), we deduce

$$X(R) + C\varepsilon \lesssim R^{-\frac{p-1}{p} \left( \frac{1}{p-1} - \frac{N}{2m} \right)} Y(R)^{\frac{1}{p}}, \quad (4.1.11)$$

where  $C > 0$  is a constant. Additionally, following the properties of the cut-off functions established in [GP21], we have

$$\frac{2}{\log 2} W(R) \leq X(R). \quad (4.1.12)$$

Combining the relations (4.1.11) and (4.1.12) yields the following estimate

$$\frac{2W(R)}{\log 2} + C\varepsilon \lesssim R^{-\frac{p-1}{p} \left( \frac{1}{p-1} - \frac{N}{2m} \right) + \frac{1}{p}} (W'(R))^{\frac{1}{p}}.$$

This inequality can be rearranged into a differential form

$$R^{\left( \frac{1}{p-1} - \frac{N}{2m} \right) (p-1) - 1} \leq C W'(R) \left( C\varepsilon + \frac{2W(R)}{\log 2} \right)^{-p}.$$

By identifying the right-hand side as a derivative, we obtain

$$R^{\left( \frac{1}{p-1} - \frac{N}{2m} \right) (p-1) - 1} \leq \frac{C \log 2}{2(1-p)} \frac{d}{dR} \left( C\varepsilon + \frac{2W(R)}{\log 2} \right)^{1-p}. \quad (4.1.13)$$

To finalize the proof, we integrate (4.1.13) over the interval  $[2R_0, T]$ . The integration of the left-hand side gives

$$\begin{aligned} & \int_{2R_0}^T R^{\left(\frac{1}{p-1} - \frac{N}{2m}\right)(p-1)-1} dR \\ &= \begin{cases} T^{\left(\frac{1}{p-1} - \frac{N}{2m}\right)(p-1)} - (2R_0)^{\left(\frac{1}{p-1} - \frac{N}{2m}\right)(p-1)} & \text{if } 1 < p < p_F, \\ \log\left(\frac{T}{2R_0}\right) & \text{if } p = p_F. \end{cases} \end{aligned}$$

For the right-hand side, we have

$$\frac{C \log 2}{2(1-p)} \int_{2R_0}^T \frac{d}{dR} \left( C\varepsilon + \frac{2W(R)}{\log 2} \right)^{1-p} dR \leq C\varepsilon^{1-p}.$$

In the subcritical case, assuming  $\varepsilon$  is sufficiently small, we obtain

$$T \lesssim \varepsilon^{-\left(\frac{1}{p-1} - \frac{N}{2m}\right)^{-1}}.$$

In the critical case, we find

$$T \lesssim \exp\left(\varepsilon^{-(p-1)}\right).$$

This concludes the proof of Theorem 4.1.2.

## Chapter 5: Conclusions

In this thesis, we have investigated the critical behavior, blow-up phenomena, and lifespan estimates for the semilinear biharmonic heat equation. We examined the problem in three distinct settings: the exterior problem under various boundary conditions, the exterior problem with a spatial forcing term, and the Cauchy problem in the whole space  $\mathbb{R}^N$ . Additionally, we extended our analysis to a system of higher-order semilinear evolution equations.

The main results and contributions of this thesis can be summarized as follows:

1. **The Exterior Problem and Fujita Critical Exponent.** In Chapter 2, we analyzed the problem in an exterior domain  $D^c$  under six different homogeneous boundary conditions, including Navier, Dirichlet, and their variations. We proved the non-existence of global weak solutions in the subcritical and critical regimes, identifying the Fujita critical exponent as:

$$p_{Fuj} = 1 + \frac{4}{N}$$

Using constructed trial functions involving harmonic and biharmonic functions, we demonstrated that if  $1 < p \leq p_{Fuj}$ , no global weak solutions exist for non-negative initial data. Furthermore, we provided an upper bound for the lifespan of solutions in the exterior domain for dimensions  $N \geq 3$ .

2. **The Effect of Inhomogeneous Forcing.** In Chapter 3, we investigated the influence of a spatial forcing term  $f(x)$  on the solvability of the exterior problem. We identified a second critical exponent,  $\omega_{Crit}$ , which depends on the decay rate of the forcing term at infinity. Specifically, we showed that:

- If the forcing term decays slowly ( $\omega < \frac{4p}{p-1}$ ), global solutions do not exist.
- If the forcing term decays sufficiently fast ( $\omega \geq \frac{4p}{p-1}$ ), stationary global solutions can be constructed.

This result highlights the interplay between the nonlinearity and the spatial inhomogeneity in determining the global existence of solutions.

3. **Lifespan Estimates in  $\mathbb{R}^N$ .** In Chapter 4, we studied the Cauchy problem for the higher-order semilinear parabolic equation  $u_t + (-\Delta)^m u = |u|^p$  in the whole space. We established sharp asymptotic estimates for the lifespan  $T_\epsilon$  of solutions. By employing the test function method developed by Ikeda and Sobajima, together with Faà di Bruno's formula, we derived upper bounds on the lifespan. Complementary lower bounds were obtained via a fixed-point argument using the heat kernel properties. We confirmed that the lifespan behavior in the biharmonic case ( $m = 2$ ) parallels that of the classical heat equation ( $m = 1$ ) and generalizes established results for higher-order operators.

**Future Perspectives:** While this thesis resolves several open questions regarding the biharmonic heat equation, several interesting problems remain. Notably, the lifespan estimates for the exterior problem in low dimensions ( $N = 1, 2$ ) remain an open challenge, as the specific test function construction used here is not directly applicable. Additionally, the existence of global solutions in the supercritical regime for the exterior problem with general boundary conditions warrants further investigation using Green's function estimates.

# Chapter 5

## Appendix

### ***Young's inequality.***

Let  $a, b > 0$  and  $1 < p, q < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

### ***$\epsilon$ - Young inequality.***

Let  $a, b > 0$  and  $1 < p, q < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

$$ab \leq \epsilon a^p + C(\epsilon) b^q$$

where  $C(\epsilon) = (\epsilon p)^{-q/p} q^{-1}$ .

### ***Hölder's inequality.***

Let  $1 \leq p, q \leq \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . For any functions  $u \in L^p(U), v \in L^q(U)$ , we have

$$\int_U |uv| dx \leq \|u\|_{L^p(U)} \|v\|_{L^q(U)}.$$

### ***Interpolation inequality.***

Let the exponents  $1 \leq s \leq r \leq t \leq \infty$  satisfy the interpolation relation  $\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$  for some parameter  $\theta \in [0, 1]$ . If a function  $u$  belongs to  $L^s(U) \cap L^t(U)$ , then  $u \in L^r(U)$  and the following estimate holds:

$$\|u\|_{L^r(U)} \leq \|u\|_{L^s(U)}^\theta \|u\|_{L^t(U)}^{1-\theta}.$$

### ***Faà di Bruno's Formula.***

Assuming  $f$  and  $g$  are sufficiently smooth, the  $m$ -th order derivative of the composition  $g \circ f$  admits the following expansion:

$$\frac{d^m}{dt^m} g(f(t)) = \sum \frac{m!}{\prod_{j=1}^m b_j!} g^{(k)}(f(t)) \prod_{j=1}^m \left( \frac{f^{(j)}(t)}{j!} \right)^{b_j}.$$

Here, the sum is taken over all partitions of the integer  $m$ , represented by non-negative integers  $b_1, \dots, b_m$  such that  $\sum_{j=1}^m j \cdot b_j = m$ , with  $k = \sum_{j=1}^m b_j$  representing the order of the derivative of  $g$ .

**Lemma 5.0.1** ([Pok13], Lemma 3.1). Consider an integrable function  $\omega \in L^1(\mathbb{R}^N)$  with a strictly positive total integral. One can construct a smooth, compactly supported cut-off function  $\phi \in C_c^\infty(\mathbb{R}^N)$ , taking values in  $[0, 1]$ , for which the integral remains positive:

$$\int_{\mathbb{R}^N} \omega \phi dx > 0.$$

**Proposition 5.0.2** ([MOT18], Proposition 6.1). For initial data  $f \in L^p(\mathbb{R}^n)$ , it is well known that the solution to the associated linear problem decays in the  $L^q$ -norm. Specifically, for  $1 \leq p \leq q \leq \infty$ , we have the estimate

$$\left\| e^{-t(-\Delta)^m} f \right\|_{L^q} \leq C t^{-\alpha} \|f\|_{L^p}, \quad \text{with } \alpha = \frac{n}{2m} \left( \frac{1}{p} - \frac{1}{q} \right),$$

for some positive constant  $C$ .

**Lemma 5.0.3** (Lemma 3.10 [IS19b]). Let  $\delta, C_0, R_1$  be positive constants and  $\theta \geq 0$ . Suppose  $w \in L_{\text{loc}}^1([0, T]; L^1(D^c))$  is a non-negative function. If the following integral inequality holds for every  $R \in [R_1, T]$ :

$$\delta + \int_0^T \int_{D^c} w(x, t) \psi_R(x, t) dx dt \leq C_0 R^{-\frac{\theta}{p'}} \left( \int_0^T \int_{D^c} w(x, t) \psi_R^*(x, t) dx dt \right)^{\frac{1}{p}},$$

then the time  $T$  is bounded from above. Specifically,

$$T \leq \begin{cases} \left( R_1^{(p-1)\theta} + (\log 2) C_0^p \theta \delta^{-(p-1)} \right)^{\frac{1}{(p-1)\theta}} & \text{if } \theta > 0, \\ \exp(\log R_1 + (\log 2)(p-1)^{-1} C_0^p \delta^{-(p-1)}) & \text{if } \theta = 0. \end{cases}$$

**Lemma 5.0.4** ([GP21], Lemma 3.2). Let  $g : [0, \infty) \rightarrow \mathbb{R}$  be a measurable function such that  $g(s) = 0$  for  $s \in [0, \frac{1}{2}] \cup [1, \infty)$ , and  $g$  is non-increasing on the interval  $(\frac{1}{2}, \infty)$ . Then, for any  $R > 0$  and  $A > 0$ , the following logarithmic bound holds:

$$\int_0^R g\left(\frac{A}{r^2}\right) \frac{dr}{r} \leq \frac{\log 2}{2} g\left(\frac{A}{R^2}\right).$$

**Lemma 5.0.5** ([Fin22], Lemma 14). Let  $m \geq 1, \ell > 2m$ , and  $\phi$  is

$$\phi(x) = \phi(|x|) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Then, the following estimate holds:

$$\left| \Delta^m (\phi^\ell) \right| \leq C \phi^{\ell-2m},$$

for some  $C = C(m, \ell) > 0$ .

**Lemma 5.0.6** ([IS19a]). Let  $\delta > 0, C_0 > 0, R_1 > 0, \theta \geq 0, \kappa \in \mathbb{R}$  and  $0 \leq w \in L_{\text{loc}}^1([0, T]; L^1(\Omega))$  for  $T > R_1$ . Assume that for every  $R \in [R_1, T]$ , and

$$\begin{aligned} \delta + \iint_{P(R)} w(x, t) \psi_R(x, t) dx dt \\ \leq C_0 R^{-\frac{\theta}{p'}} (\log R)^{\frac{\kappa}{p'}} \left( \iint_{P(R)} w(x, t) \psi_R^*(x, t) dx dt \right)^{\frac{1}{p}}. \end{aligned} \tag{5.0.1}$$

Then  $T$  has to be bounded above as follows:

$$T \leq \begin{cases} C\delta^{-\frac{1}{\theta}} (\log(\delta^{-1}))^{\frac{\kappa}{\theta}} & \text{if } \theta > 0, \kappa \in \mathbb{R}, \\ \exp\left(C\delta^{-\frac{p-1}{1-\kappa(p-1)}}\right) & \text{if } \theta = 0, \kappa < \frac{2}{p-1}, \\ \exp \exp\left(C\delta^{-(p-1)}\right) & \text{if } \theta = 0, \kappa = \frac{2}{p-1}. \end{cases}$$

**Lemma 5.0.7.** Consider the test function  $\psi$  defined in (2.1.31) and the biharmonic function  $B(x)$  given by (2.1.20) in dimension  $N = 4$ . Then, there exists a constant  $C > 0$  such that:

$$\begin{aligned} |\Delta^2 \psi(x)| &\leq \frac{C}{R} B(x) [\psi_R^*(t)]^{\frac{1}{p}}, \\ |\partial_t \psi(x)| &\leq \frac{C}{R} B(x) [\psi_R^*(t)]^{\frac{1}{p}}. \end{aligned}$$

*Proof.* By applying the biharmonic operator to the product  $\psi(x) = \psi_R(x, t)B(x)$ , we obtain the following expansion

$$\begin{aligned} \Delta^2 \psi(x) &= \psi_R(x, t) \Delta^2 B(x) + 4 \nabla(\Delta B(x)) \cdot \nabla \psi_R(x, t) + 6 \Delta B(x) \Delta \psi_R(x, t) \\ &\quad + 4 \nabla B(x) \cdot \nabla (\Delta \psi_R(x, t)) + B(x) \Delta^2 \psi_R(x, t). \end{aligned} \quad (5.0.2)$$

Direct computation of the derivatives of  $B(x)$  yields

$$\begin{aligned} \nabla B &= \left( \frac{2}{|x|^2} - \frac{2}{|x|^4} \right) x, \\ \Delta B &= \frac{4}{|x|^2}, \\ \nabla(\Delta B) &= -\frac{8}{|x|^4} x. \end{aligned} \quad (5.0.3)$$

Similarly, for the spatial cut-off function  $\psi_R$ , we compute the successive derivatives. The first derivative is

$$\nabla \psi_R = \frac{d}{ds_R} (\eta(s_R)^{4p'}) \frac{4(|x| - 1)^3}{R|x|} x, \quad (5.0.4)$$

followed by the Laplacian

$$\begin{aligned} \Delta \psi_R &= \frac{d^2}{ds_R^2} (\eta(s_R)^{4p'}) \frac{16(|x| - 1)^6}{R^2} \\ &\quad + \frac{d}{ds_R} (\eta(s_R)^{4p'}) \left( \frac{12(|x| - 1)^2}{R} + \frac{12(|x| - 1)^3}{R|x|} \right). \end{aligned} \quad (5.0.5)$$

The third-order derivative terms are given by

$$\begin{aligned} \nabla \Delta \psi_R &= \frac{x}{|x|} \left[ \frac{d^3 (\eta(s_R)^{4p'})}{ds_R^3} \frac{64(|x| - 1)^9}{R^3} \right. \\ &\quad + \frac{d^2 (\eta(s_R)^{4p'})}{ds_R^2} \left( \frac{144(|x| - 1)^5}{R^2} + \frac{48(|x| - 1)^6}{R^2|x|} \right) \\ &\quad \left. + \frac{d (\eta(s_R)^{4p'})}{ds_R} \left( \frac{24(|x| - 1)}{R} + \frac{36(|x| - 1)^2}{R|x|} - \frac{12(|x| - 1)^3}{R|x|^2} \right) \right], \end{aligned} \quad (5.0.6)$$

and finally, the biharmonic operator applied to  $\psi_R$  yields

$$\begin{aligned}
\Delta^2 \psi_R &= \frac{d^4(\eta(s_R)^{4p'})}{ds_R^4} \left[ \frac{256(|x| - 1)^{12}}{R^4} \right] \\
&+ \frac{d^3(\eta(s_R)^{4p'})}{ds_R^3} \left[ \frac{1152(|x| - 1)^8}{R^3} + \frac{384(|x| - 1)^9}{R^3|x|} \right] \\
&+ \frac{d^2(\eta(s_R)^{4p'})}{ds_R^2} \left[ \frac{816(|x| - 1)^4}{R^2} + \frac{864(|x| - 1)^5}{R^2|x|} + \frac{48(|x| - 1)^6}{R^2|x|^2} \right] \\
&+ \frac{d(\eta(s_R)^{4p'})}{ds_R} \left[ \frac{24}{R} + \frac{144(|x| - 1)}{R|x|} + \frac{36(|x| - 1)^2}{R|x|^2} - \frac{12(|x| - 1)^3}{R|x|^3} \right] \\
&\leq CR^{-1} [\psi_R^*(t)]^{\frac{1}{p}}.
\end{aligned} \tag{5.0.7}$$

To establish the desired bound, we estimate each term in the expansion (5.0.2) individually. For the term involving  $\nabla(\Delta B(x))$ , we have

$$\begin{aligned}
|\nabla(\Delta B(x)) \cdot \nabla \psi_R(x, t)| &= \left| -\frac{8}{|x|^4} x \cdot \left( \frac{d}{ds_R}(\eta(s_R)^{4p'}) \frac{4(|x| - 1)^3}{R|x|} x \right) \right| \\
&= \left| -\frac{8}{|x|^2} \left( \frac{d}{ds_R}(\eta(s_R)^{4p'}) \frac{4(|x| - 1)^3}{R|x|} \right) \right|.
\end{aligned} \tag{5.0.8}$$

By employing the elementary inequality  $1 - \frac{1}{|x|} \leq \log |x|$  for  $x \in D^c$ , and utilizing the properties of the cut-off function  $\eta$ , this can be bounded as

$$\begin{aligned}
|\nabla(\Delta B(x)) \cdot \nabla \psi_R(x, t)| &\leq \frac{C}{R} \log |x| |\eta'(s_R) \eta(s_R)^{4p'-1}| \\
&= \frac{C}{R} \log |x| \|\eta' \eta^3\|_{L^\infty} \eta(s_R)^{\frac{4p}{p-1}-4} \\
&= \frac{C}{R} \log |x| \|\eta' \eta^3\|_{L^\infty} \eta(s_R)^{\frac{4}{p-1}} \\
&\leq \frac{C}{R} \log |x| \|\eta' \eta^3\|_{L^\infty} \eta^*(s_R)^{\frac{4}{p-1}} \\
&= \frac{C}{R} \log |x| [\psi_R^*(t)]^{\frac{1}{p}}.
\end{aligned} \tag{5.0.9}$$

Proceeding similarly for the next term in the expansion, we find

$$\begin{aligned}
&\Delta B(x) \Delta \psi_R(x, t) \\
&= \frac{4}{|x|^2} \left( \frac{d^2}{ds_R^2}(\eta(s_R)^{4p'}) \frac{16(|x| - 1)^6}{R^2} + \frac{d}{ds_R}(\eta(s_R)^{4p'}) \left( \frac{12(|x| - 1)^2}{R} + \frac{12(|x| - 1)^3}{R|x|} \right) \right) \\
&\leq \frac{C}{R} \log |x| \left( \frac{d^2}{ds_R^2}(\eta(s_R)^{4p'}) + \frac{d}{ds_R}(\eta(s_R)^{4p'}) \right) \\
&= \frac{C}{R} \log |x| \left( \eta^{4p'-2}(\eta')^2 + \eta^{4p'-1}\eta'' + \eta^{4p'-1}\eta' \right) \\
&= \frac{C}{R} \log |x| \eta(s_R)^{\frac{4p}{p-1}-4} \left( \eta^2(\eta')^2 + \eta^3\eta'' + \eta^3\eta' \right) \\
&= \frac{C}{R} \log |x| \eta(s_R)^{\frac{4}{p-1}} \left( \eta^2(\eta')^2 + \eta^3\eta'' + \eta^3\eta' \right),
\end{aligned} \tag{5.0.10}$$

which implies that

$$\begin{aligned}
|\Delta B(x) \Delta \psi_R(x, t)| &\leq \frac{C}{R} \log |x| \eta(s_R)^{\frac{4}{p-1}} \|\eta^2(\eta')^2 + \eta^3 \eta'' + \eta^3 \eta'\|_{L^\infty} \\
&\leq \frac{C}{R} \log |x| \eta^*(s_R)^{\frac{4}{p-1}} \|\eta^2(\eta')^2 + \eta^3 \eta'' + \eta^3 \eta'\|_{L^\infty} \\
&= \frac{C}{R} \log |x| [\psi_R^*(t)]^{\frac{1}{p}}.
\end{aligned} \tag{5.0.11}$$

Finally, for the term involving  $\nabla B(x)$ , we obtain

$$\begin{aligned}
&|\nabla B(x) \cdot \nabla (\Delta \psi_R(x, t))| \\
&= \left| \left( \frac{2}{|x|^2} - \frac{2}{|x|^4} \right) x \cdot \frac{x}{|x|} \left[ \frac{d^3(\eta(s_R)^{4p'})}{ds_R^3} \frac{64(|x|-1)^9}{R^3} \right. \right. \\
&\quad + \frac{d^2(\eta(s_R)^{4p'})}{ds_R^2} \left( \frac{144(|x|-1)^5}{R^2} + \frac{48(|x|-1)^6}{R^2|x|} \right) \\
&\quad \left. \left. + \frac{d(\eta(s_R)^{4p'})}{ds_R} \left( \frac{24(|x|-1)}{R} + \frac{36(|x|-1)^2}{R|x|} - \frac{12(|x|-1)^3}{R|x|^2} \right) \right] \right| \\
&\leq \left| \frac{2}{|x|} \left[ \frac{d^3(\eta(s_R)^{4p'})}{ds_R^3} \frac{64(|x|-1)^9}{R^3} \right. \right. \\
&\quad + \frac{d^2(\eta(s_R)^{4p'})}{ds_R^2} \left( \frac{144(|x|-1)^5}{R^2} + \frac{48(|x|-1)^6}{R^2|x|} \right) \\
&\quad \left. \left. + \frac{d(\eta(s_R)^{4p'})}{ds_R} \left( \frac{24(|x|-1)}{R} + \frac{36(|x|-1)^2}{R|x|} - \frac{12(|x|-1)^3}{R|x|^2} \right) \right] \right|,
\end{aligned} \tag{5.0.12}$$

which yields

$$\begin{aligned}
&|\nabla B(x) \cdot \nabla (\Delta \psi_R(x, t))| \\
&\leq \frac{C}{R} \log |x| \eta(s_R)^{\frac{4}{p-1}} \left| \frac{d^3(\eta(s_R)^{4p'})}{ds_R^3} + \frac{d^2(\eta(s_R)^{4p'})}{ds_R^2} + \frac{d(\eta(s_R)^{4p'})}{ds_R} \right| \\
&\leq \frac{C}{R} \log |x| \eta^*(s_R)^{\frac{4}{p-1}}.
\end{aligned} \tag{5.0.13}$$

We can now combine these individual estimates to bound the full biharmonic operator applied to our test function. Recalling that  $B(x)$  is a biharmonic function, the leading term  $\psi_R(x, t) \Delta^2 B(x)$  identically vanishes. The remaining terms in the expansion (5.0.2) have each been shown to be bounded by  $\frac{C}{R} \log |x| [\psi_R^*(t)]^{\frac{1}{p}}$ . Summing these respective contributions yields the first desired inequality

$$|\Delta^2 \psi(x)| \leq \frac{C}{R} B(x) [\psi_R^*(t)]^{\frac{1}{p}}$$

To establish the second estimate regarding the time derivative, we apply the operator  $\partial_t$  directly to the test function  $\psi(x)$ . Because the spatial function  $B(x)$  is independent of time, the derivative acts exclusively on the cut-off function

$$|\partial_t \psi(x)| = B(x) |\partial_t \psi_R(x, t)|$$

Utilizing the standard scaling properties of the time-dependent cut-off  $\psi_R$ , we can bound its derivative by  $CR^{-1} [\psi_R^*(t)]^{\frac{1}{p}}$ . Multiplying this upper bound by the spatial component  $B(x)$  directly provides the second inequality

$$|\partial_t \psi(x)| \leq \frac{C}{R} B(x) [\psi_R^*(t)]^{\frac{1}{p}}$$

This completes the proof.  $\square$

**Lemma 5.0.8.** Consider the test function  $\psi$  defined in (2.1.31) and the harmonic function  $H(x) = \ln |x|$  given by (2.1.6) in dimension  $N = 2$ . Then, there exists a constant  $C > 0$  such that:

$$\begin{aligned} |\Delta^2 \psi(x)| &\leq \frac{C}{R} \ln |x| [\psi_R^*(t)]^{\frac{1}{p}}, \\ |\partial_t \psi(x)| &\leq \frac{C}{R} \ln |x| [\psi_R^*(t)]^{\frac{1}{p}}. \end{aligned}$$

*Proof.* We begin by applying the biharmonic operator to the test function  $\psi(x) = \psi_R(x, t)H(x)$ . Because  $H(x) = \ln |x|$  is a harmonic function in  $\mathbb{R}^2 \setminus \{0\}$ , we know that  $\Delta H(x) = 0$ , which inherently implies  $\Delta^2 H(x) = 0$  and  $\nabla(\Delta H(x)) = 0$ . These properties significantly simplify the product expansion compared to the higher-dimensional case, leaving us with the non-vanishing terms

$$\Delta^2 \psi(x) = H(x) \Delta^2 \psi_R(x, t) + 4 \nabla H(x) \cdot \nabla(\Delta \psi_R(x, t)) + 4 \sum_{i,j=1}^2 \partial_{ij} H(x) \partial_{ij} \psi_R(x, t)$$

We can now estimate each of these surviving terms individually. For the leading term, we leverage the bound on the fourth spatial derivative of the cut-off function established in the previous estimates, yielding directly

$$|H(x) \Delta^2 \psi_R(x, t)| \leq CR^{-1} \ln |x| [\psi_R^*(t)]^{\frac{1}{p}}$$

Next, we consider the first mixed term. Given the explicit gradient  $\nabla H(x) = \frac{x}{|x|^2}$ , we can compute the inner product with the gradient of the Laplacian of  $\psi_R$

$$|4 \nabla H(x) \cdot \nabla(\Delta \psi_R(x, t))| \leq \frac{4}{|x|} |\nabla(\Delta \psi_R(x, t))|$$

Using the inequality  $\frac{1}{|x|} \leq \ln |x|$  valid for  $x \in D^c$  and applying the previously derived bounds for the third-order derivatives of  $\psi_R$ , we obtain the estimate

$$|4 \nabla H(x) \cdot \nabla(\Delta \psi_R(x, t))| \leq \frac{C}{R} \ln |x| [\psi_R^*(t)]^{\frac{1}{p}}$$

For the remaining second-order cross terms, the Hessian of  $H(x)$  decays as  $\frac{1}{|x|^2}$ , and the corresponding second derivatives of  $\psi_R$  are bounded by the scaling properties of the cut-off function  $\eta$ . Applying the same logarithmic inequality bounds this contribution as well

$$\left| 4 \sum_{i,j=1}^2 \partial_{ij} H(x) \partial_{ij} \psi_R(x, t) \right| \leq \frac{C}{R} \ln |x| [\psi_R^*(t)]^{\frac{1}{p}}$$

Summing these individual bounds gives the complete spatial estimate for the biharmonic operator

$$|\Delta^2 \psi(x)| \leq \frac{C}{R} \ln |x| [\psi_R^*(t)]^{\frac{1}{p}}$$

To secure the time derivative estimate, we differentiate the test function with respect to  $t$ . Since the spatial function  $H(x)$  is time-independent, the derivative applies exclusively to  $\psi_R(x, t)$

$$|\partial_t \psi(x)| = \ln |x| |\partial_t \psi_R(x, t)|$$

Invoking the scaling properties of the cut-off function  $\psi_R$  bounds its time derivative by  $CR^{-1}[\psi_R^*(t)]^{\frac{1}{p}}$ , which immediately provides the final required inequality

$$|\partial_t \psi(x)| \leq \frac{C}{R} \ln |x| [\psi_R^*(t)]^{\frac{1}{p}}$$

This completes the proof. □

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