Extremal Black Hole in a Nonlinear Newtonian Theory of Gravity

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Abstract

This work investigates an upper-limit of charge for a black hole in a nonlinear Newtonian theory of gravity. The charge is accumulated via protons fired isotropically at the black hole. This theoretical study of gravity (known as 'pseudo-Newtonian') is a forced merger of special relativity and Newtonian gravity. Whereas the source of Newton's gravity is purely mass, pseudo-Newtonian gravity includes effects of fields around the mass, giving a more complete picture of how gravity behaves. Interestingly, pseudo-Newtonian gravity predicts such relativistic phenomena as black holes and deviations from Kepler's laws, but of course, provides a less accurate picture than general relativity. Though less accurate, it offers an easier approach to understanding some results of general relativity, and merits interest due to its simplicity. The method of study applied here examines the predictions of pseudo-Newtonian gravity for a particle interacting with a highly charged black hole. A black hole with a suitable charge will reach an upper limit (expressed by pseudo-Newtonian gravity) in charge capacity before Coulomb's law repels like-charge particles away from the hole. In particular, this work attempts to push pseudo-Newtonian gravity to its extreme and discover how its results differ from general relativistic predictions involving the same proton bombardment. It is found that the results for an upper limit of charge in general relativity and this nonlinear theory of Newtonian gravity differ by a factor of four. This may give insight into the importance of space-time curvature effects on the description of particle dynamics around a black hole.

I. INTRODUCTION

The fundamental law of gravitation discovered by Newton has withstood more than three centuries of experimental scrutiny and, at least in the weak gravity region throughout our Solar System, has provided an excellent description of gravitational phenomena. In terms of a field theory, Newtonian gravity is a linear theory completely described by a single scalar field variable, ϕ_N, whose single source is mass.

The comprehensive gravitational theory proposed by Einstein depicts gravitational phenomena as the result of a particular curvature of four-dimensional space. Any such curvature may result from a variety of causes, such as mass, fields, rotations of masses, etc. The complex mathematical nature of Einsteinian gravity as expressed by a system of coupled, nonlinear field equations makes obtaining any complete description of a specific curvature extremely difficult. The rewards of successful efforts, however, have led to some drastic differences from Newtonian gravity in cases of extremely strong gravitational fields. The most notable case, completely absent from Newtonian gravity, is that of gravity in the immediate vicinity of a black hole.

It was shown by Peters[1] that extending the basic notion of the source of Newtonian gravity to include fields as source contributions results in a nonlinear field theory, but one in terms of a single scalar field variable. Solutions in this "pseudo-Newtonian" gravity representing fields in regions exterior to a highly compacted mass, M, are found to possess singularities which have black hole characteristics. It was later shown by Young[2] that if M were imagined to have a net charge, Q, a black hole-like gravitational field would result in which the charge contributed directly and significantly to that gravitational field. Peters' solution bears a strong similarity to the well-known Schwarzschild solution found in Einsteinian gravity, but has the advantage of an enormous mathematical simplicity. Young's solution, on the other hand, contains certain features exhibited by the Reissner-Nordstrom solution of the Einstein-Maxwell equations, but again with the substantial mathematical simplicity offered by the pseudo-Newtonian formulation.

The gravitational field of a charged black hole leads to new issues in the dynamics of charged particles interacting with the field. For example, a particle of mass m and charge q would experience the competing effects of gravitational mass-attraction and Coulomb charge-repulsion. If we imagine starting with an electrically neutral black hole and gathering protons, the resulting charge build-up within the hole would have the effect of eventually repelling additional protons. If this were not so, the hole could ultimately become "over-charged" and consequently destroyed, as discussed in the next sections. The work discussed here will be directed toward determining the maximum charge that proton accretion could add to a black hole. Section II reviews and verifies the nonlinear Newtonian theory, while sections III and IV find low energy and high energy proton bombardment charge limits, respectively. Section V compares the low energy case to the high energy case, while Section VI derives and compares the general relativistic result with the nonlinear Newtonian answer. Motivated by the chance to complement the pedagogical nature of pseudo-Newtonian gravity, we have used S.I. units.

II. PSEUDO-NEWTONIAN GRAVITY

The Newtonian gravitational field due to localized mass distribution ρ_m is completely described by the potential function, ϕ_N, satisfying the linear Poisson equation

$$\nabla^2 \phi_N = 4\pi G \rho_m. \tag{1}$$

Exterior to spherically symmetric ρ_m representing total mass M, this has solution

$$\phi_N(r) = -\frac{GM}{r}. \tag{2}$$

This potential is independent of charge and angular momentum. The sole source of Newtonian gravity is mass.

Peters[1] examined the results of including the effective mass of the field produced by and surrounding the mass, by the following. There is an energy density stored in the gravitational field; this is given by[3]

$$u = -\frac{1}{8\pi G}(\nabla \phi_N)^2. \tag{3}$$

Taken in conjunction with the mass-energy relationship, $E = mc^2$, the energy density can be interpreted as a field-mass density,

$$\rho_f = -\frac{1}{8\pi G c^2}(\nabla \phi_N)^2. \tag{4}$$

If the mass density, the source of Newtonian gravity, were taken to include that associated with the field ($\rho_m \to \rho_m + \rho_f$), (1) and (4) would combine to yield the *nonlinear* field equation

$$\nabla^2 \phi + \frac{1}{2c^2}(\nabla \phi)^2 = 4\pi G \rho_m. \tag{5}$$

This has a spherically symmetric solution, exterior to the spherical region of M,

$$\phi(r; M) = 2c^2 \ln\left(1 - \frac{GM}{2c^2 r}\right) \tag{6}$$

The Peters potential, Eq. (6), has some features worth discussion. First, careful examination shows, as is necessary with all relativistic results, that $\phi(r) \rightarrow \phi_N(\mathrm{r})$ in the limit of infinite c. Further, for large r, there is no distinction between $\phi(r)$ and $\phi_N(\mathrm{r})$. If a body of mass M were imagined to be extremely compact, having radius $R < \frac{GM}{2c^2}$ from its center, it would then be confined to an infinitely deep potential well. The pseudo-Newtonian gravity proposed by Peters thus leads, in a simple mathematical way, to the existence of a black hole which is strikingly analogous to the Schwarzschild black hole of Einsteinian gravity (but completely absent from Newtonian gravity).

The all-inclusive character, in terms of sources, of Einsteinian gravity allows black holes other than the Schwarzschild type. The first such solution to the combined Einstein-Maxwell field equations representing a spherically symmetric mass containing a spherically symmetric charge was found independently by Reissner and Nordstrom. The pseudo-Newtonian counterpart of this solution[2] is realized by including the electric field energy density,

$$u_e = \frac{\varepsilon_0}{2}(\nabla\varphi)^2, \tag{7}$$

where ε_0 is the permittivity of a vacuum and φ is the electrostatic scalar potential produced by the charge, *viz,* $\frac{Q}{4\pi\varepsilon_0 r}$. The field equation in this case where ρ_m has the value of zero, is

$$\nabla^2\phi + \frac{1}{2c^2}(\nabla\phi)^2 - \frac{2\pi G\varepsilon_0}{c^2}(\nabla\varphi)^2 = 0. \tag{8}$$

This can be shown to have a spherically symmetric solution, representing the gravitational field of a mass M carrying uniform charge Q, given by

$$\phi(r; M, Q) = 2c^2 \ln\left[\cosh\left(b\frac{Q}{r}\right) - a\frac{M}{Q}\sinh\left(b\frac{Q}{r}\right)\right], \tag{9}$$

where

$$a \equiv \sqrt{4\pi\varepsilon_0 G}; \qquad b \equiv \frac{1}{2c^2}\sqrt{\frac{G}{4\pi\varepsilon_0}}. \tag{10}$$

The potential given in Eq. (9) is only ostensibly dominated by the charge, and is insensitive to the sign of that charge. It has a limiting form identical to the Peters potential as $Q \rightarrow 0$. Further, it also implies the possible existence of a black hole since ϕ exhibits a singularity as the argument of the logarithm becomes zero. This is the case, for given Q and M, at $r = R_{BH}$, where

$$R_{BH} = 2b\frac{Q}{\ln\left(\frac{1+\frac{Q}{aM}}{1-\frac{Q}{aM}}\right)}. \tag{11}$$

In order that such a radius exist, the argument of the logarithm must be positive. This condition requires that such a charged black hole can exist in pseudo-Newtonian gravity only if the mass is not over-charged. It is convenient to express this condition with the inequality

$$\left(\frac{Q}{M}\right)^2 < 4\pi\varepsilon_0 G \sim 10^{-20}\left[\frac{C}{kg}\right]^2. \tag{12}$$

Rather remarkably, this is precisely the same condition found in the Reissner-Nordstrom solution.[4] Should the condition be violated, no black hole exists. In the case of a proton, the charge to mass ratio has a square value approximately $10^{16}\ [C/kg]^2$. Thus, as has been well documented, a proton is not a black hole!

Graphical comparison of the charged and uncharged potentials is facilitated, as is the ensuing analysis, by first carrying out a rescaling. The unitless parameters λ and σ are defined as

$$\lambda = \frac{r}{R_{bh}}, \qquad \sigma = \frac{1}{\sqrt{4\pi G\varepsilon_0}}\frac{Q}{M} \tag{13}$$

where $R_{bh} = GM/2c^2$, i.e. the uncharged pseudo-Newtonian black hole radius (also the general relativistic spherical black hole radius measured in isotropic coordinates).[1] The inequality given in Eq. (12) restricts σ values to the range $0 \le \sigma < 1$, the upper limit corresponding to a highly-charged mass. In these terms, the potentials of masses with and without charge can be written as, respectively,

$$\frac{1}{2c^2}\phi(r;M,Q) = \ln\left[\cosh\left(\frac{\sigma}{\lambda}\right) - \frac{1}{\sigma}\sinh\left(\frac{\sigma}{\lambda}\right)\right], \qquad \frac{1}{2c^2}\phi(r;M) = \ln\left(1 - \frac{1}{\lambda}\right). \tag{14}$$

III. NON-RELATIVISTIC PARTICLE DYNAMICS

A particle of mass m and charge q interacting with compacted mass M having charge Q will interact gravitationally and electrically with the latter. The particle's mass will interact with $\phi(r;M,Q)$ and its charge with the electrical potential produced by Q. The electrical potential here is not altered from its elementary form by including energy density effects. The inclusion adds only to total mass, altering the gravity because of the mass-energy relationship. Since an equivalent charge-energy relationship does not exist, the equations governing pseudo-Newtonian gravity remain coupled and asymmetric.[2]

Any such particle would have non-relativistic ($v \le c/10$) total energy $E = T + V$, where

$$V(r;M,Q) = m\phi(r;M,Q) + \frac{qQ}{4\pi\varepsilon_0 r}. \tag{15}$$

In the course of its motion, the particle would undergo the competitive effects of gravitational attraction (since $\phi < 0$) and, for $qQ > 0$, Coulomb repulsion. Since the fields in which the particle is considered here are central (i.e., functions of r only), its total energy will be conserved throughout its motion. In the case of the particle accelerated toward M, Q from infinity with kinetic energy E, its kinetic energy at any location will be

$$T = E - V(r; M, Q). \tag{16}$$

In a plot where E is a constant and $V(r; M, Q)$ is superimposed, the difference between these curves at any r would correspond to the particle's kinetic energy at that point. Physically realistic motion can occur only where the kinetic energy is positive, or at least zero; more specifically, it cannot be negative. Equation (16) indicates that the kinetic energy will decrease to zero at the critical distance, r_c. This results in the equality

$$E = V(r_c; M, Q). \tag{17}$$

In the case of a particle approaching M, Q from infinity, where $V = 0$, this equality would correspond to an intersection of the superimposed total and potential energy curves and indicate a turning point, or point of closest approach, in its motion.

The main purpose of the present work is to examine the process of proton accretion by a black hole and, in particular, the charge buildup within the hole due to that process. An extremely large value of Q on M leads to no black hole at all. We imagine starting out with a black hole having little, or even no charge, then firing in protons isotropically to preserve spherical symmetry. The value of Q thus increases as the process continues and so then will the size of the Coulomb repulsion experienced by subsequent protons. Can protons continue to be added indefinitely, thus destroying the black hole, or will the Coulomb effect eventually dominate the gravitational attraction once some value, Q_{max}, is reached? The addition of protons would have the additional effect of increasing the mass of the hole; that addition, however, would be negligible in comparison to the size of any significant M. This investigation has been restricted, for now, to the case of radially-approaching, non-relativistic protons having total energy less than about 4.5 MeV ($v \leq c/10$).

The kinetic energy of an approaching proton would change with distance from the hole. This could increase or decrease depending on the amount of charge, presumably accumulated by earlier protons, already captured by the hole. Gravitational attraction dominates for small Q, allowing all protons with $0 < E \leq 4.5$ MeV to be captured. As Q increases, the Coulomb repulsion of subsequent protons grows in size and reaching R_{BH} becomes more difficult.

The analysis of the motion is most easily carried out by means of the general dynamical considerations discussed above. Writing Eq. (16) explicitly, gives

$$\frac{T}{2mc^2} = \frac{E}{2mc^2} - \ln\left(\cosh\left(\frac{\sigma}{\lambda}\right) - \frac{1}{\sigma}\sinh\left(\frac{\sigma}{\lambda}\right)\right) - \frac{q}{m\sqrt{4\pi\varepsilon_0 G}}\frac{\sigma}{\lambda}. \tag{18}$$

For protons, the last term on the right side of Eq. (18) is approximately $1.1 \times 10^{18}\frac{\sigma}{\lambda}$. The requirement that $T \geq 0$ means that an approaching proton will encounter a turning point in its motion at the value of λ that causes T to become zero.

The kinetic energy, T, is just the difference between a horizontal (constant) $E/2mc^2$ curve and a $V(\lambda; M, Q)/2mc^2$ curve for differing values of charge. The value of E is chosen as the approximate maximum for non-relativistic protons for which $E/2mc^2$ has a value 1/400. No intersection occurs in the case of small Q, meaning that any proton having this energy (or, in fact any $E > 0$) would get into the hole. As charge is added, the scaled-potential energy curves increase in size and an intersection can occur. The critical value of Q is that which maximizes a V curve at value 1/400. Any larger Q will then cause the V curve to intersect with the constant energy curve, indicating a turning point at the location of the intersection. This illustrates that a definite limit on Q exists such that no additional protons at this energy will be able to gain entry into the black hole. Hence, eventually Coulomb will become dominant. It is impossible to destroy a black hole via low energy proton bombardment.

It is now an easy matter to determine the value of Q_{max} that a given M can carry if it is to retain its character as a black hole. The maximum charge corresponds to that critical value, σ_c which makes $V(\lambda; M, Q_{max}) = E$. For the case of $E/2mc^2 = 1/400$ (which corresponds to about 4.5 MeV in the case of a proton), we obtain $\sigma_c \approx 9.6 \times 10^{-19}$. Using this value in Eq. (13) yields the maximum value of charge that non-relativistic protons could add to a black hole of given mass, M:

$$Q_{max} \approx 8.3 \times 10^{-29}\left[\frac{C}{kg}\right] M. \tag{19}$$

IV. RELATIVISTIC PARTICLE DYNAMICS

This section will extrapolate on the low-energy non-relativistic proton treatment. Consider now, depositing charge into a pseudo-Newtonian black hole by radial bombardment of high-energy protons. These particles are known to be abundant in astrophysical studies and to have a maximum observed energy of 10^{20} eV.[5] Bombardment by N protons, each having charge q and mass m, increases the charge, Q, and mass, M, of the hole by $N*q$ and $N*m$, respectively.

The charge addition to the hole significantly alters its character and interaction with additional bombarding protons, whereas the mass addition does not, and so it will be ignored.

A particle having rest mass m_0 and velocity v while experiencing resultant force F_R will have equation of motion $F_R = \frac{d}{dt}(m_0\gamma v)$. The usual Lorentz factor, γ, is defined as $\gamma = \frac{1}{(1-\beta^2)^{1/2}}$ and $\beta = v/c$. A proton radially approaching a charged pseudo-Newtonian black hole would experience competing gravitational and Coulomb forces, each of which is conservative, resulting in an equation of motion:

$$-\frac{d}{dr}\left[m_0\gamma\phi + \frac{qQ}{4\pi\varepsilon_0 r}\right] = \frac{d}{dt}(m_0\gamma v). \tag{20}$$

Using $v = dr/dt$, multiplying through by a differential dr and performing some elementary differentiation will show that:

$$-d\left[m_0\gamma\phi + \frac{qQ}{4\pi\varepsilon_0 r}\right] = \frac{m_0c^2\beta}{(1-\beta^2)^{3/2}}d\beta. \tag{21}$$

This is easily integrated from infinity, the proton's presumed point of origination, to r, giving

$$-m_0\gamma_r\phi_r - \frac{qQ}{4\pi\varepsilon_0 r} = m_0c^2[\gamma_r - \gamma_0]. \tag{22}$$

where γ_0 is the Lorentz factor for the proton at infinity and γ_r is its value at r.

This can be solved for the Lorentz factor, γ_r, for the proton at any r. γ_r will always have a value greater than or equal to 1. For a proton instantaneously at rest, such as at a turning point, R_T, a helpful inequality follows

$$\gamma_0 - 1 \quad \geq \quad \frac{1}{c^2}\phi_r + \frac{1}{m_0c^2}\frac{qQ}{4\pi\varepsilon_0 r}. \tag{23}$$

This leads to a clear picture of the dynamics of a proton approaching the black hole. Suppose the proton has total energy at infinity, $E_0 = \gamma_0 m_0 c^2$. Designate the right hand side of Eq. (23) as $F(r;Q,M)$ with ϕ given in Eqn (9). Defining

$$F(r;Q,M) \equiv 2\ln\left[\cosh\left(b\frac{Q}{r}\right) - \frac{aM}{Q}\sinh\left(b\frac{Q}{r}\right)\right] + \frac{1}{m_0c^2}\frac{qQ}{4\pi\varepsilon_0 r}, \tag{24}$$

it is easy to find the maximum of $F(r;Q,M)$ occurring at $r \equiv r_0$; the result is

$$\tanh\left(b\frac{Q}{r_0}\right) = \frac{\frac{Q}{aM}\Gamma - 1}{\Gamma - \frac{Q}{aM}}, \tag{25}$$

where Γ is a constant, defined as $\Gamma \equiv \frac{q}{am_0}$. For protons $\Gamma \sim 10^{18}$, and in order that the black hole exists, $Q/aM < 1$. Hence, to a highly accurate order of approximation, Eq. (25) can be expressed as

$$\tanh\left(b\frac{Q}{r_0}\right) = \frac{Q}{aM} - \frac{1}{\Gamma}. \tag{26}$$

A rescaling is convenient for graphing purposes. With again, $\lambda \equiv \frac{r}{GM/2c^2}$, and $\sigma \equiv \frac{Q}{aM}$. This gives, from Eq. (26),

$$\tanh\left(\frac{\sigma}{\lambda}\right) = \sigma - \frac{1}{\Gamma}. \tag{27}$$

The condition $\sigma < 1$ in order for a black hole to exist, is demanded by the charge to mass ratio, Eq. (12). Even in the case of a hole possessing a huge amount of charge, it will be the case that $\sigma \ll 1$. This allows for the approximation

$$\lambda_{min} \approx \frac{\sigma}{\sigma - \frac{1}{\Gamma}}. \tag{28}$$

This equation represents a seemingly close approximation of distance of closest approach to the hole. The minimum λ here is independent of the original speed given to the protons, because this equation represents the solution to the maximum of $F(r; Q, M)$ for a certain charge-mass ratio, σ. This maximum value of $F(r; Q, M)$ correlates with the closest approach to the black hole for protons of a certain initial speed.

Expressing $F(r; Q, M)$ in its rescaled from, $F(\lambda, \sigma)$, yields

$$F(\lambda, \sigma) = 2\ln\left[\cosh\left(\frac{\sigma}{\lambda}\right) - \frac{1}{\sigma}\sinh\left(\frac{\sigma}{\lambda}\right)\right] + 2\Gamma\frac{\sigma}{\lambda}. \tag{29}$$

For highly relativistic protons, λ will be approximately equal to 1, for the maximum value of $F(\lambda, \sigma)$. This indicates that indeed the protons get very close to the black hole before turning around. Graphical analysis shows that $\lambda \sim 1^+$, and implies that $F(\lambda, \sigma)$ has a maximum turn around point when $\lambda > 1$. Therefore the protons turn around before falling past the event horizon. To see this algebraically, that indeed $\lambda \sim 1$, from Eq. (9) it can be shown that the black hole has a radius defined when the logarithmic argument goes to zero:

$$\tanh\left(b\frac{Q}{R_{BH}}\right) = \frac{Q}{aM}. \tag{30}$$

Using Eq. (26) combined with Eq. (30) it can be seen that r_0 is approximately equal to R_{BH}, but that r_0 is still slightly larger. These are related as, remembering that for protons, $\Gamma \sim 10^{18}$,

$$\tanh\left(b\frac{Q}{r_0}\right) = \tanh\left(b\frac{Q}{R_{BH}}\right) - \frac{1}{\Gamma}. \tag{31}$$

Now that we know the turning point will occur very close to the hole, for high energy protons, we can find the maximum value of $F(\lambda, \sigma)$. Using $\lambda = 1$ and some trigonometric substitutions, it is a straightforward matter to show from Eq. (29) that

$$F_{max}(\lambda, \sigma) \approx 2\ln\left(\frac{\sigma^2}{3}\right) + 2\Gamma\sigma. \tag{32}$$

To show that an upper limit exists, all that is needed is to prove that $F_{max}(\lambda, \sigma)$ can be numerically greater than $(\gamma_0 - 1)$ where $\gamma_0 = 10^{11}$ and continue to have $\sigma < 1$ as given by the restriction in Eq. (12). This value of γ_0, as mentioned before, belongs to extremely high-energy protons from an 'ultra-high' cosmic ray source.

To ask what the maximum value of σ is when F_{max} is equal to $\gamma_0 - 1$ will be to ask what the charge-to-mass ratio of the hole is when it is maximally charged. The natural log term will be dominated by $2\Gamma\sigma_{max}$ so that

$$F_{max} \approx 2\Gamma\sigma_{max}. \tag{33}$$

From Eq. (23), and safely ignoring the minus one because γ_0 is so large, we have

$$\gamma_0 \approx 2\Gamma\sigma_{max}. \tag{34}$$

Using $\gamma_0 = 10^{11}$ yields $\sigma_{max} \approx 4.5 \times 10^{-8}$ with $\Gamma = q/am_0$ for a proton having value $\approx 1.1 \times 10^{18}$. The value of σ is also found graphically by plotting $F(\lambda, \sigma)$ and γ_0, and finding their intersection. This means that all values of γ_0 for protons less than the energy given to them by 'ultra-high' cosmic rays will be stopped by a black hole with σ_{max} value of 4.5×10^{-8}. Un-scaling, will give the charge-mass relationship

$$Q \approx 3.9 \times 10^{-18} \left[\frac{C}{kg}\right] M. \tag{35}$$

V. COMPARING LOW-ENERGY WITH HIGH-ENERGY

It is helpful to see a how a comparison between low-energy protons and high-energy protons affects the upper-limit of charge. Low-energy protons with velocity $0.1c$ possess energy of about 4.5 MeV, and an upper-limit of charge is given by

$$Q \approx 8.3 \times 10^{-29} \left[\frac{C}{kg}\right] M \tag{36}$$

Compare this to Eq. (35) and note that the order of magnitude of the constant is much less for the non-relativistic protons. It follows that with decreasing energy, the less capable the protons are of overcoming the Coulomb force. The less energetic protons will not be able to saturate the black hole as fully as the highly energetic protons. This explains the reason for a much smaller charge to mass ratio for low-energy protons.

Note that the use of conservation of energy for the calculation of non-relativistic protons yields an equation of exactly the same form as the calculation using force for the relativistic protons. Equation (24) will be familiar in both calculations and set equal to E/mc^2 for the low-energy protons, and equal to $\gamma_0 - 1$ for the high-energy case. Setting $E/mc^2 = \gamma_0 - 1$, it is easy to check that indeed they are the same equation. For a γ_0 value of 1.005 (with speed of $0.1c$) and using $.5mv^2$ for E (with again, the same speed of $0.1c$) yields 0.005 as the answer for both.

It can also be checked with high accuracy that Eq. (28) mirrors the low-energy calculation, which utilized conservation of energy. Using $\lambda_0 \approx \sigma/(\sigma - 1/\Gamma)$ with a very precise σ value of 9.635104×10^{-19} yields 14.81462 for λ_0. Using the low-energy calculation $E/mc^2 = F(r; Q, M)$ with the same σ value, yields 14.81433. The two results require a precise σ, but indicate that both methods of determining λ_0 are accurate. The level of accuracy can be further increased with a more precise σ value.

VI. COMPARING PSEUDO-NEWTONIAN WITH GENERAL RELATIVITY

Understanding how well pseudo-Newtonian gravity conforms to general relativity requires a comparison. Using the radial 'effective potential' equation for a charged particle falling in the Reissner-Nordstrom geometry[6], we obtain the condition for maximal charge on the black hole when the bombardment of the last proton is unable to reach inside the event horizon. A turn-around point, where $dr/d\tau = 0$ at $r = r_+$, yields the condition, in raw form via natural units

$$qQ = E(M + \sqrt{M^2 - Q^2}). \tag{37}$$

Where $E = \gamma_0 m$, the energy of the proton at infinity. Solving for the charge to mass ratio,

$$\frac{Q}{M} = \frac{2E}{q} \frac{1}{1 + E^2/q^2}. \tag{38}$$

Converting units to SI, and making a simplification because $\Gamma = q/am \approx 10^{18} \gg \gamma_0 \approx 10^{11}$, yields

$$\frac{Q}{M\sqrt{4\pi\varepsilon_0 G}} = \frac{2\gamma_0}{\Gamma}. \tag{39}$$

Rescaling gives

$$\sigma_{GR} = \frac{2\gamma_0}{\Gamma}. \tag{40}$$

Comparing this result with the previous result, $\gamma_0 \approx 2\Gamma\sigma_{pn}$, it follows that

$$\sigma_{GR} = 4\sigma_{pn}. \tag{41}$$

It could be assumed that because the foundations of general relativity rely on its interpretation of physical geometry, and because space-time curvature was not taken into account for pseudo-Newtonian gravity, that the two descriptions in this case would differ drastically. However, not only are they within the same magnitude, pseudo-Newtonian gravity is precisely four times too small. The curvature effects present in general relativity may completely explain the factor of four. Note especially that the pseudo-Newtonian radius for an uncharged black hole, derived by Peters[1], $GM/2c^2$, is also 4 times smaller than the Schwarzschild radius for an uncharged black hole, $2GM/c^2$. Because the charges on these black holes are far below the Reissner-Nordstrom condition, yet highly charged so that no known protons can get in, these radii are excellent approximations for the black holes involved.

VII. CONCLUSIONS

A merging of relativistic ideas with a non-relativistic theory yields an enlightening nonlinear theory of Newtonian gravity. An upper limit of charge was found for a black hole and it was made explicitly clear that a black hole cannot be destroyed by forcing protons into it as described by pseudo-Newtonian gravity. General relativistic predictions describe the same upper limit, and remarkably, the two results only differ by a factor of four.

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