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Möbius mirrors

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Abstract

An accelerating boundary (mirror) acts as a horizon and black hole analog, radiating energy with some particle spectrum. We demonstrate that a Möbius transformation on the null coordinate advanced time mirror trajectory uniquely keeps invariant not only the energy flux but the particle spectrum. We clarify how the geometric entanglement entropy is also invariant. The transform allows generation of families of dynamically distinct trajectories, including $\mathcal{PT}$-symmetric ones, mapping from the eternally thermal mirror to the de Sitter horizon, and different boundary motions corresponding to Kerr or Schwarzschild black holes.

Keywords: moving mirrors, Hawking radiation, dynamical Casimir effect

(Some figures may appear in colour only in the online journal)

1. Introduction

Black holes represent a fascinating intersection of spacetime and information, probing the nature of each, and their relation. The relation is intimately tied with the horizon, and one can study other systems with horizons to elucidate the connections. Using the moving mirror model [1–3], the accelerating boundary correspondence (ABC) (e.g. Schwarzschild [4], Reissner–Nordström [5], Kerr [6]) maps hot black holes to particular moving mirror trajectories with horizons. While a close connection between the particle and energy creation by moving mirrors [7, 8] and black holes has been known for sometime, complete investigation of these exact analogs are ongoing. Quasi-thermal solutions typify geometric end-states like black hole remnants [9] (asymptotic constant-velocity mirrors [10–15]). Complete black hole evaporation models are characterized by asymptotic zero-velocity mirrors [16–23].

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While entanglement entropy [24], and hence information [25], is tied directly to the mirror trajectory, the distant observer only detects energy flux and particle production. We investigate the connection between these by considering special transformations of the mirror trajectory such that the energy flux remains invariant. These are Möbius transformations arising from the Schwarzian operator in the quantum stress energy tensor, and correspond to the $SL(2,\mathbb{R})$ group symmetry.

Using this, we can explore the relation between mirrors, and hence spacetimes, with identical flux, such as thermal emission from de Sitter space [26, 27] and the Carlitz–Willey (CW) [28] mirror, investigate cases with merely asymptotically identical flux, and probe the zero energy, but nonzero particle, production of uniformly accelerating motion, whose asymptotic dynamics corresponds to extremal black holes [6, 29–32].

In section 2 we describe the Möbius transform and its effects, applying it to eternally thermal flux and Planckian particle production in section 3. We consider the de Sitter spacetime in particular in section 4, Schwarzschild and Kerr black holes in section 5, and the uniform acceleration case in section 6, concluding in section 7.

2. Möbius transformations

The quantum stress tensor for an ABC indicates an energy flux $\mathcal{F}$ produced from the boundary (horizon) to an observer at infinity,

$$-24\pi\mathcal{F}(u) = \{p,u\} \equiv \frac{p'''}{p'} - \frac{3}{2}\left(\frac{p''}{p'}\right)^2, \tag{1}$$

where $p(u)$ is the trajectory, i.e. mirror position v as a function of u, where $u = t - x$ and $v = t + x$ are the null coordinates, also called the retarded and advanced times. It can be derived via point-splitting [2], or via the Schwinger term in the Virasoro algebra [33]. We use natural units, e.g. $\hbar = k_\mathrm{B} = 1$, throughout.

The notation $\{p,u\}$ denotes the Schwarzian derivative, and this signals that an ABC has an underlying symmetry in the form of the Möbius transformations of $SL(2,\mathbb{R})$,

$$p(u) \rightarrow \frac{ap(u)+b}{cp(u)+d}, \quad ad - bc = 1. \tag{2}$$

For simplicity of calculations, we will often re-scale p by $\kappa = 1$ in order to interpret the trajectory p as the dimensionless quantity κp.

Thus, a trajectory

$$P(u) = \frac{ap(u)+b}{cp(u)+d} \equiv Mp(u), \tag{3}$$

with $ad - bc = 1$ has the same energy flux as the original $p(u)$. Here M is the operator of the transformation, such that $P = M(p)$. Later, we will show that the transformation leaves invariant the particle flux seen by an observer as well.

We can divide the transforms into two cases, when $c = 0$ and when $c \neq 0$. When $c = 0$ then $P = (a/d)p + b/d$. The b/d piece is a shift in p, the translation part of the Möbius transform. This does not contribute to P' and hence not to the energy flux, and since P enters the Bogolyubov beta coefficient $\beta_{\omega\omega'}$ as $\mathrm{e}^{-\mathrm{i}\omega' P}$ then a constant addition in P is a pure phase and will cancel in $|\beta_{\omega\omega'}|^2$. Nor does the translation change the rapidity $\eta(u) = (1/2)\ln p'(u)$ and so it is not of much interest. It can, however, alter the position of the mirror at infinity, e.g. whether the mirror starts at null coordinate $v = 0$ or some finite value.

Thus the $c = 0$ case (hence $d = 1/a$) is

$$P = a^2 p \qquad (c = 0). \tag{4}$$

This represent the dilatation part of the Möbius transform. Note that since $\eta(u) = (1/2)\ln p'$ then a constant multiplicative factor for P is just an additive shift of $\eta \to \eta + \ln a$. This does have physical consequences, as an additive factor to the entropy flux, and a multiplicative factor to the mirror acceleration.

When $c \neq 0$ then we can multiply the numerator and denominator in P by c to get

$$P = \frac{acp + ad - ad + bc}{c(cp + d)} = \frac{a}{c} - \frac{1}{c(cp + d)}, \tag{5}$$

where we have used $ad - bc = 1$. Again ignoring the constant term, we can write ($s \equiv cd$)

$$P = \frac{-1}{c^2 p + s} \qquad (c \neq 0). \tag{6}$$

Now $\eta(P)$ is not simply related to $\eta(p)$, and the entropy and acceleration are likewise different for the two trajectories, though the energy flux is the same. When $d = 0$ ($s = 0$) then the Möbius transform is an inversion of p, with a minus sign.

Equations (4) and (6) are the two transforms exhibiting the symmetry of the ABC due to the Schwarzian, hence leaving the energy flux invariant. We are also interested in whether this carries over to the particle creation and its spectrum. Recall that the particle creation per mode per mode is $N_{\omega\omega'} = |\beta_{\omega\omega'}|^2$, where ω is the outgoing and ω' the ingoing frequency, and the particle spectrum seen by a distant observer is $N(\omega) = \int_0^\infty \mathrm{d}\omega'\, |\beta_{\omega\omega'}|^2$. In particular, for the thermal case $N(\omega)$ has a Planckian spectrum of particles. The Bogolyubov beta coefficient follows

$$\beta_{\omega\omega'} = \frac{-1}{2\pi}\sqrt{\frac{\omega}{\omega'}}\int_{u_{\min}}^{u_{\max}} \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}\omega u - i\omega' p(u)}\ . \tag{7}$$

While we cannot evaluate this for $P(u)$ coming from arbitrary $p(u)$, we examine several physically important cases in the following sections.

3. Eternal thermal mirror

We begin with the classic case of constant thermal energy emission, having particles distributed in a Planck distribution as discovered by CW [28] (see [20, 21, 34] for its trajectory in spacetime coordinates and further detail). This corresponds to a particular nonuniformly accelerating mirror, given by

$$p(u) = \frac{-1}{\kappa}\, e^{-\kappa u}\ . \tag{8}$$

Figure 1 shows the trajectory as the solid black line, starting asymptotically inertial (no acceleration) albeit at light speed, and evolving dynamically toward asymptotic infinite acceleration with horizon at $v_{\mathrm{H}} = 0$ in advanced time v. Since $p(u)$ is the function label for v, we see from equation (8) that the mirror is limited to the bottom and left quadrants of the Penrose diamond.

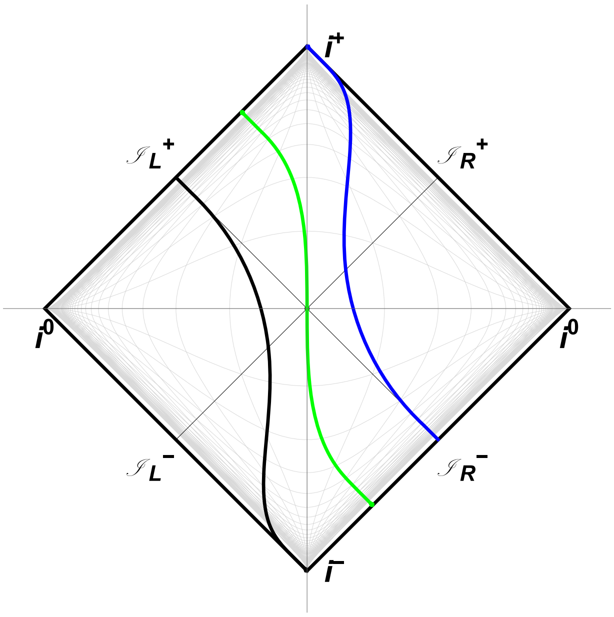

Figure 1. Penrose diagram of constant energy flux mirrors with Planck distributed particles emission, showing CW (black) equation (8), and a transformed case (blue), equation (9) with an equivalent spectrum. The de Sitter mirror (green), equation (18), is also related by Möbius transform. Here $\kappa = 2$ for illustration.

Now let us apply the Möbius transform, specifically the negative inversion of equation (6) with $c = -b = 1$, $a = d = 0$. This gives

$$P(u) = \frac{1}{\kappa} e^{\kappa u}, \tag{9}$$

and is equivalent to the transform $x \to -x$, $t \to -t$, i.e. a $\mathcal{PT}$ symmetry. The trajectory is limited to the right and top quadrants of the Penrose diagram, plotted in figure 1 as the solid blue line. The early time behavior illustrates the mirror climbing out of a horizon and approaching an asymptotic inertial end state at time-like future infinity.

Explicit calculation indeed shows that the energy flux for both trajectories, as observed by a witness at future-null infinity, are identical, here the thermal constant,

$$\mathcal{F} = \frac{\kappa^2}{48\pi}. \tag{10}$$

One also finds that for the transformed case, like the original CW mirror, the particles radiated are in a Planck distributed spectrum,

$$N_{\omega\omega'} = \frac{1}{2\pi\kappa\omega'} \frac{1}{e^{2\pi\omega/\kappa} - 1}, \tag{11}$$

with temperature $T = \kappa/2\pi$.

This result is not void of physical implications. An observer who discovers constant energy flux emission and hot particles (with a temperature) will not be able to distinguish between the accelerated boundary conditions, nor determine the origin of such radiation. Moreover, whether the horizon happened in the past, or in the future, cannot be determined.

To examine the particle creation in detail, we investigate the Bogolyubov beta function under the Möbius transform. For this eternal thermal mirror, equation (8), the transform of $p(u) = -(1/\kappa)e^{-\kappa u}$ in the $c = 0$ case is given by equation (4); evaluating equation (7) yields

$$\beta_{\omega\omega'} \sim (-i\omega' a^2)^{-i\omega/\kappa}\,\Gamma(i\omega/\kappa). \tag{12}$$

We now see how the extra constant a from the (dilatation) transformation does not propagate to $|\beta_{\omega\omega'}|^2$:

$$(ix)^{iy} = x^{iy}\, e^{-(\pi/2)y\,\mathrm{sgn}(x)}, \tag{13}$$

and so when multiplying it by its complex conjugate we simply get $e^{\pi y} = e^{-\pi\omega/\kappa}$, independent of $x = -\omega' a^2$. Alternately, one can view $x^{iy} = \mathrm{e}^{\mathrm{i}y\ln x}$ and so a^2 enters as a pure phase.

Next consider the eternal thermal $c \neq 0$ case. For the $\mathcal{PT}$ transformed mirror, i.e. equation (9), the integral is very similar, giving

$$\beta_{\omega\omega'} \sim (i\omega'/\kappa)^{1-i\omega/\kappa}\,\Gamma(1 - i\omega/\kappa), \tag{14}$$

and this provides an identical $|\beta_{\omega\omega'}|^2$ to that from equation (12) and hence the original CW case: a thermal, Planckian particle spectrum.

The more general case when $d \neq 0$, i.e. equation (6) with $s \neq 0$, is considerably less tractable. The Bogolyubov beta function involves a Whittaker function,

$$\beta_{\omega\omega'} \sim \Gamma\left(\frac{-i\omega}{\kappa}\right)\, W_{i\omega/\kappa,-1/2}(i\omega'/s), \tag{15}$$

but the spectrum per mode squared, $N_{\omega\omega'}$

$$|\beta_{\omega\omega'}|^2 = \frac{1}{4\pi\kappa\omega'\,\sinh(\pi\omega/\kappa)}\left|U\left(\frac{-i\omega}{\kappa}, 0, \frac{i\omega'}{s}\right)\right|^2, \tag{16}$$

is not the orthodox thermal spectrum, equation (11), and thus $|\beta_{\omega\omega'}|^2$ is not invariant for a general trajectory transformation. Here U is the confluent hypergeometric function.

When the argument of W is large, e.g. the high frequency limit, then $W_{a,b}(z) \to e^{-z/2}z^a$. Since $z = i\omega'/s$ is imaginary, the first factor is a pure phase and cancels in the modulus, while z^a gives a factor $e^{-\pi\omega/(2\kappa)}$ using equation (13). Thus the $|\beta_{\omega\omega'}|^2$ is the same as the original, CW mirror,

$$|\beta_{\omega\omega'}|^2 \sim \left|\Gamma\left(\frac{-i\omega}{\kappa}\right)\right|^2 e^{-\pi\omega/\kappa} \sim \frac{1}{e^{2\pi\omega/\kappa} - 1}, \tag{17}$$

a Planckian particle distribution. Outside the high frequency regime the invariance of $|\beta_{\omega\omega'}|^2$ is broken. However, at the level of the actual (observable) particle spectrum, $N(\omega) = \int \mathrm{d}\omega'\,|\beta_{\omega\omega'}|^2$, the thermal spectrum is preserved. See section 7 for the proof.

This indicates that in a Möbius transformation with $cd \neq 0$, the particle production from the new mirror is not identical for the (unobservable) particle per mode per mode distribution. The low-frequency situation is reminiscent of the 'soft' (zero energy) particle per mode per mode distribution of certain mirrors, such as in the uniformly accelerated case; we treat that in more detail in section 6. However, the particles reaching a distant observer would be blueshifted, and therefore in the high frequency limit where the per mode per mode distribution is invariant. And as mentioned in the previous paragraph, the observable particle spectrum $N(\omega)$ is thermal and invariant.

4. De Sitter

A de Sitter space has a horizon and eternally thermal emission; the accelerated boundary correspondence for it is a mirror with trajectory [26]

$$p(u) = \frac{2}{\kappa}\tanh\frac{\kappa u}{2}. \qquad \text{(de Sitter)}. \tag{18}$$

Using the Möbius transformations we can convert this into the CW eternal thermal mirror of equation (8), or the reverse:

$$P_{\text{dS}} = \frac{-1}{c^2 p_{\text{CW}} + s} \quad \text{with } c^2 = \frac{\kappa}{4} = -s, \tag{19}$$

$$P_{\text{CW}} = \frac{-1}{c^2 p_{\text{dS}} + s} \quad \text{with } c^2 = \frac{\kappa}{4},\ \ s = \frac{1}{2}. \tag{20}$$

As before we have dropped constant contributions to P as they only give a constant phase (though they do affect the v location of the horizon). See figure 1 for a plot of the de Sitter mirror (green), equation (18) with $\kappa = 2$.

We can use the same transformations for anti-de Sitter space, where $p = (2/\kappa)\tan(\kappa u/2)$, as long as we take $\kappa_{\text{AdS}} = \pm i\kappa_{\text{dS}}$, since AdS has negative eternal thermal flux $F = -\kappa_{\text{dS}}^2/(48\pi)$.

As for the eternal thermal mirror with $d \neq 0$ (recall $s = cd$), the Bogolyubov beta coefficient resolves to equation (15), which by converting to the confluent hypergeometric function M is equivalent to equation (16) of [26], and has the same 'soft' particles in $|\beta_{\omega\omega'}|^2$, while keeping an invariant thermal particle spectrum $N(\omega)$.

5. Schwarzschild and Kerr

The Schwarzschild ABC has mirror trajectory [4]

$$p(u) = v_{\text{H}} - \frac{1}{\kappa} W\left(e^{-\kappa(u - v_{\text{H}})}\right), \tag{21}$$

where W is the Lambert W function, also known as a product log. Calculating the particle spectrum of this under Möbius transforms is not analytically tractable, so we make some general observations.

Figure 2 plots the Schwarzschild mirror trajectory as the black curve. The other curves in the figure show the application of Möbius transforms, and one can verify that they have identical energy flux to the original Schwarzschild mirror. (In fact, one can even create spacelike trajectories with the same energy flux.)

Note that one of the Möbius transforms, with $a = 0, d = c = 1, b = -1$, of the Schwarzschild mirror (with $\kappa = 1$, $v_{\text{H}} = 0$ for simplicity) gives the purple curve,

$$P(u) = \frac{1}{W\left(e^{-u}\right) - 1}, \tag{22}$$

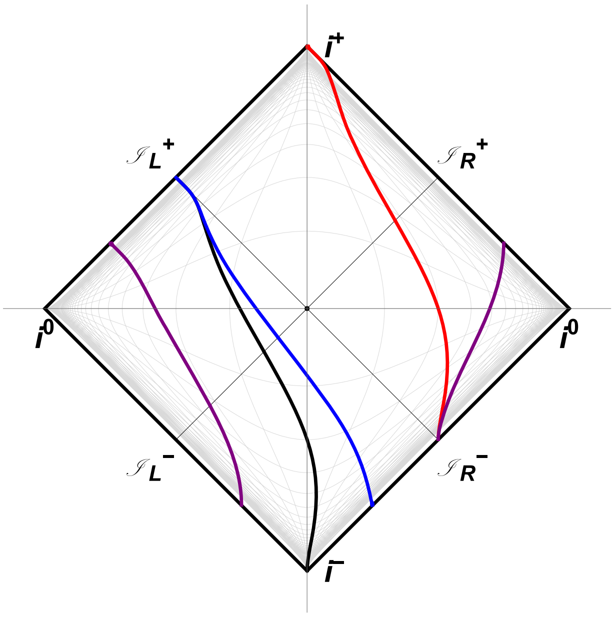

Figure 2. Penrose diagram of some identical flux mirrors. The original Schwarzschild mirror, equation (21), is black. The colors, (red, blue, purple), correspond to $P(u)$ with schematic form: $1/W$, $-W/(W+1)$, $1/(W-1)$, respectively. The purple line (both branches are shown) is a Möbius transform of the Schwarzschild mirror with u-horizon at $u=-1$, where the mirror, equation (22), runs into the observer at right null-infinity. The energy flux detects nothing unusual.

with identical energy flux and a horizon in u, but where the mirror smashes into our intrepid observer at null-infinity. Despite this, the witness will register nothing unusual happening in the energy flux! Interestingly, one can imagine wrapping the space so the points $(u, v) = (-1, +\infty)$ and $(-1, -\infty)$ join, making the purple mirror trajectory continuous (speculatively, a wormhole?) We consider an implication of this sort of idea in section 6.2.

In the hopes of dealing with simpler expressions than product logs, and exploring the Kerr metric for which $p(u)$ is unknown, one might use the label function $f(v)$ describing the null coordinate u. For Kerr (and Schwarzschild as a limiting case),

$$f(v) = v - \frac{1}{2g}\frac{1+\delta}{\delta}\ln|gv| + \frac{1}{2g}\frac{1-\delta}{\delta}\ln|gv-\delta|, \tag{23}$$

where $g = 1/(4M)$ and $\delta = \sqrt{1-a^2/M^2}$, with M and a the mass and spin parameter of the black hole. For the Schwarzschild case $a=0$, $\delta=1$, $g=\kappa$, and the last term in $f(v)$ vanishes. However, the Möbius transformation only applies to $p(u)$. See appendix A for discussion of transformations applied to $f(v)$.

6. Constant acceleration mirrors

Mirrors with constant acceleration have zero energy flux. However, the particle production is a more subtle matter. Let us explore what Möbius transformations of uniform acceleration look

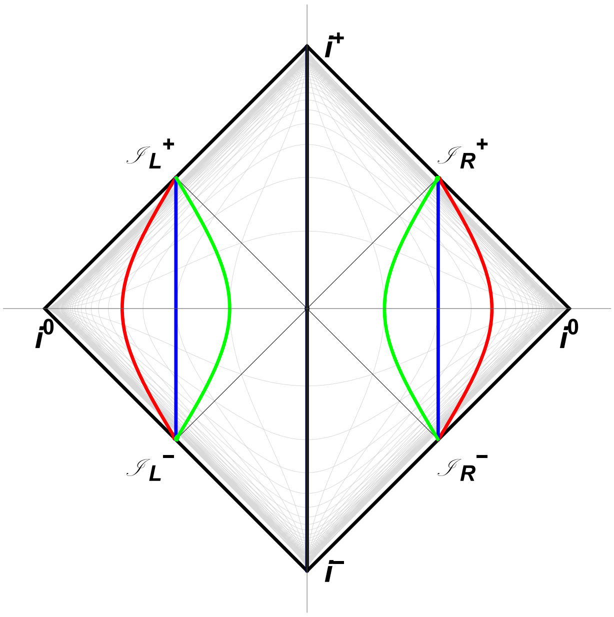

Figure 3. Penrose diagram of the uniformly accelerated mirror, with $\kappa = 1/2, 1, 2$, for red, blue, green, respectively. Note that $\kappa = 1$ (blue) is a straight line on the conformal diagram but is not stationary in space. Horizons exist at $u = 0$ and at $v = 0$. The particle production and energy emission is zero.

like. (Of course they will preserve the zero energy flux.) The classic uniform acceleration mirror trajectory is given by $p(u) = -1/(\kappa^2 u)$, with acceleration $\alpha(u) = p''(u)/[2(p')^{3/2}] = -\kappa$.

6.1. Branches matter

If we Möbius transform with $c = 0$, then $P(u) = a^2 p = -a^2/(\kappa^2 u)$. This dilatation is merely a redefinition of κ, i.e. a different constant acceleration. As mentioned, the energy flux remains zero, but the particle production involves a subtlety: there are two branches, $u < 0$ and $u > 0$ on either side of the horizon at $u = 0$. These can be viewed as two mirrors. A Penrose diagram of the dual-mirror system situation is given in figure 3 depicting the two mirrors (each shown for various κ parameters) that are the two roots of the single spacetime coordinate trajectory function. In spacetime coordinates, x and t, hyperbolic motion is

$$x^2 - t^2 = \alpha^{-2}, \tag{24}$$

so that there are actually two mirrors:

$$x(t) = \pm\sqrt{\alpha^{-2} + t^2}, \tag{25}$$

where $\alpha = -\kappa$ is the constant proper acceleration for each. The mirror in the right quadrant of the diagram traverses $(u, v) = (-\infty, 0)$ to $(0, \infty)$ while the one in the left quadrant traverses $(0, -\infty)$ to $(\infty, 0)$.

6.2. Zero particle creation with two branches

For a $c \neq 0$ Möbius transform,

$$P(u) = \frac{-1}{-c^2/(\kappa^2 u) + s} = -\frac{1}{s} - \frac{c^2}{s(s\kappa^2 u - c^2)}, \tag{26}$$

and we can ignore the first term as contributing a constant phase. Switching to $U = s\kappa^2 u - c^2$, we have $P(U) = -c^2/(sU)$. The integration in $\beta_{\omega\omega'}$, split into $[-\infty, 0]$ and $[0, \infty]$, *but counting both branches* gives

$$\beta_{\omega\omega'} \sim \sqrt{\frac{4q}{\omega}}\, \mathrm{e}^{\mathrm{i}\pi/2}\, \left[K_{-1}\left(-2\sqrt{q\omega}\right) + K_{-1}\left(2\sqrt{q\omega}\right)\right], \tag{27}$$

where $q = \omega'/\kappa^2$ for $p(u)$ and $q = (c^2/s^2)\omega'/\kappa^2$ for the transformed $P(u)$. Since for the modified Bessel function of the second kind $K_{-1}(-z) = -K_{-1}(z)$, we find $\beta_{\omega\omega'} = 0$ regardless of q and hence c, s.

Thus, when taking into account both branches, the transformation leads to both invariant (zero) flux and (zero) particle production for the constant acceleration case.

6.3. Non-zero particle creation with one branch

In the previous subsection we worked with a dual mirror system (two roots) (see also [35]). Now we will look at just the single mirror system (one root) with uniform acceleration, which has long been known to have the surprising result of non-zero particle production with zero energy flux [36]. To compute the soft particle production from a single uniformly accelerating mirror, we can shift it over to the origin to avoid confusion with the previous system, expressing the single uniformly accelerated mirror, again with $\alpha = -\kappa$, as

$$p(u) = \frac{u}{1 + \kappa u}, \tag{28}$$

with early-time horizon positioned at $u_{\mathrm{H}} = -\kappa^{-1}$. It has particle creation that is solved by integrating from the early time horizon onward (this branch only includes one root, and therefore only one mirror) via

$$\beta_{\omega\omega'} = \frac{-1}{2\pi}\sqrt{\frac{\omega}{\omega'}}\int_{-\kappa^{-1}}^{\infty} \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}\omega u - i\omega' u/(1+\kappa u)}. \tag{29}$$

Using a simple substitution $X = u + 1/\kappa$ gives a range of integration from $[0, \infty]$ and

$$\beta_{\omega\omega'} = \frac{-i}{\pi\kappa}\, \mathrm{e}^{\mathrm{i}(\omega-\omega')/\kappa}\, K_1\left(\frac{2}{\kappa}\sqrt{\omega\omega'}\right). \tag{30}$$

The phase pre-factor is unimportant (as well as the sign of the beta) upon complex conjugation, so

$$|\beta_{\omega\omega'}|^2 = \frac{1}{\pi^2\kappa^2}\left|K_1\left(\frac{2}{\kappa}\sqrt{\omega\omega'}\right)\right|^2. \tag{31}$$

Equation (31) is the 'soft'-spectrum per mode per mode of a single uniformly accelerated mirror, distinctly non-thermal (non-Planckian) [2, 3, 36]; in the high frequency limit $K_1 \to 0$ (exponentially), however, and as before the distribution is independent of transformation. Thus, while the union of the two branches preserves the particle flux per mode per mode (zero, for a uniform acceleration) under a Möbius transform, a single branch does not, except in the high frequency limit. However, as we shall see in section 7, the particle spectrum $N(\omega)$ is preserved.

6.4. Thermal uniform acceleration

While a single uniformly accelerated mirror produces soft particles, despite zero energy flux, and we have seen that the spectrum is decidedly non-Planckian [36], the story of the radiation is not told by just the acceleration. Is there a context where a uniformly accelerated moving mirror can have a temperature? Here we find that yes, it almost certainly can, under very specific circumstances. This is similar to the Unruh effect [37] where an eternally uniformly accelerated observer (not mirror) sees a Planck distributed particle radiation with temperature proportional to the acceleration. The situation we will investigate will be in stark contrast to the original situation of the Davies–Fulling effect for a single moving mirror that eternally uniformly accelerates creating a Bessel distributed particle radiation distribution per mode per mode, equation (31), with undefined temperature and zero energy flux (soft particles). If we relax the eternal uniformity, and look at what happens as a mirror *asymptotically* approaches uniform acceleration, we find a particular mirror that can emit constant energy flux indicative of thermal emission.

From the start of this section, recall that $p(u) = -1/(\kappa^2 u)$ has eternal uniform acceleration. This can be written in terms of $f(v) = -1/(\kappa^2 v)$. Now consider the trajectory

$$f(v) = -\frac{1}{\kappa^2 v} - \frac{1}{12\kappa^4 v^3}. \tag{32}$$

(One could obtain $p(u)$ by solving the cubic equation, but dealing with $f(v)$ is simpler.) This asymptotically approaches the uniform acceleration trajectory as $v \to -\infty$. The proper acceleration $\alpha(v) = -f''(v)/[2f'(v)^{3/2}]$ has an early time limit

$$\lim_{v\to-\infty} \alpha(v) = -\kappa \left[1 + \mathcal{O}\left((\kappa v)^{-2}\right)\right], \tag{33}$$

explicitly approaching uniform acceleration (accelerating leftward using the Davies–Fulling convention [2, 3]). The trajectory (green) is shown in figure 4, along with the eternally uniform acceleration case (red).

The energy flux as derived by the quantum stress tensor for the moving mirror model is also easily found in advanced time using the Schwarzian, $24\pi\mathcal{F}(v) = \{f(v)\, v\} f'(v)^{-2}$, which has an early time limit,

$$\lim_{v\to-\infty} \mathcal{F}(v) = \frac{\kappa^2}{48\pi}. \tag{34}$$

This demonstrates conclusively that despite (asymptotically) uniformly accelerating, the energy flux emission is a non-zero constant, unlike the eternally uniformly accelerating mirror [2]. The key here is that $\alpha'(v)$ is not exactly zero, although it is asymptotically zero. The constant energy flux of equation (34) is identical to that of the de Sitter mirror [26] or the eternal thermal mirror of CW [28], which have explicitly been shown to emit particles in a Planck distribution with temperature

$$T = \frac{\kappa}{2\pi}. \tag{35}$$

We study approaches to asymptotes further in appendix A.

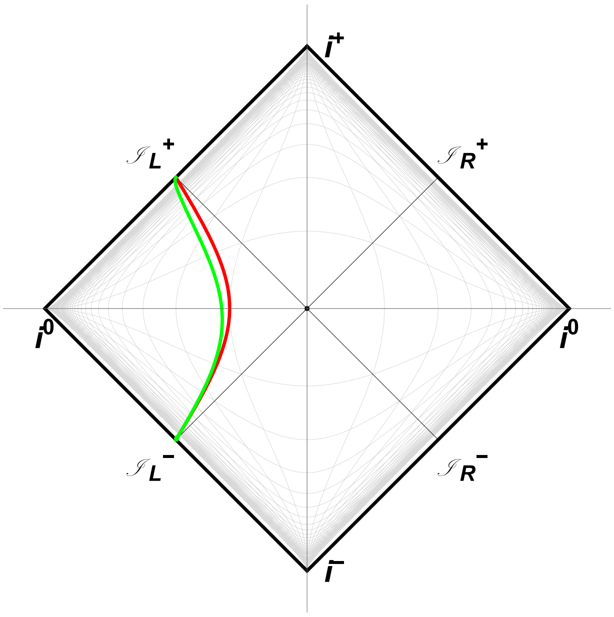

Figure 4. Penrose diagram of a uniformly accelerated mirror (red), $f(v) = -1/(\kappa^2 v)$, (branch $v < 0$ only), and an early-time asymptotically uniformly accelerated mirror (green), equation (32). Here $\kappa = 2$ for illustration.

7. Particle spectrum invariance

In the previous sections we have concentrated on the energy flux and the particle spectrum per mode per mode, i.e. $N_{\omega\omega'} = |\beta_{\omega\omega'}|^2$. Concentrating on the observable particle spectrum itself,

$$N(\omega) = \int_0^\infty \mathrm{d}\omega' |\beta_{\omega\omega'}|^2, \tag{36}$$

its invariance under Möbius transformations can be shown quite generally[4].

To do so, it is most convenient to work with $f(v)$, i.e. the function label for the retarded time u. Here the Bogolyubov beta coefficient takes the form

$$\beta_{\omega\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{v_{\min}}^{v_{\max}} \mathrm{d}v\, \mathrm{e}^{-\mathrm{i}\omega' v - i\omega f(v)}. \tag{37}$$

In computing the particle spectrum, the proof proceeds by carrying out the integration over ω' first, so that

$$\begin{aligned} N(\omega) &= \frac{1}{4\pi^2\omega} \int \mathrm{d}v_1 \int \mathrm{d}v_2\, \mathrm{e}^{-\mathrm{i}\omega[f(v_1) - f(v_2)]} \times \int \mathrm{d}\omega'\, \omega'\, \mathrm{e}^{-\mathrm{i}\omega'(v_1 - v_2)} \\ &= \frac{-1}{4\pi^2\omega} \iint \frac{\mathrm{d}v_1\, \mathrm{d}v_2}{(v_1 - v_2 - i\epsilon)^2}\, \mathrm{e}^{-\mathrm{i}\omega[f(v_1) - f(v_2)]}, \end{aligned} \tag{38}$$

[4] In a late stage of writing we found an argument by [33], of which ours is an inverse (see also [38]); we are indebted to it for improving the derivation to that given here.

where we used a real regulator $\epsilon > 0$ that we then take the limit $\epsilon \to 0$.

Now consider another mirror trajectory that incorporates the Möbius transform of $p(u)$, i.e. the function label for v. We call $V_i = Mv_i$. Its particle spectrum will simply be

$$\tilde{N}(\omega) = \frac{-1}{4\pi^2\omega}\iint \frac{\mathrm{d}V_1\,\mathrm{d}V_2}{(V_1 - V_2 - i\epsilon)^2}\,\mathrm{e}^{-\mathrm{i}\omega[\tilde{f}(V_1)-\tilde{f}(V_2)]}, \tag{39}$$

where we indicate this mirror's quantities with tildes.

However, a property of the Möbius transform is that the quantity

$$\frac{P'(u_1)P'(u_2)}{[P(u_1) - P(u_2)]^2} = \frac{p'(u_1)p'(u_2)}{[p(u_1) - p(u_2)]^2}, \tag{40}$$

is invariant. Since $p(u)$ is the function label for v (and the same relation of $P(u)$ with V), this is precisely the integration 'Jacobian' $\mathrm{d}V_1\,\mathrm{d}V_2/(V_1 - V_2)^2$. Furthermore, since $f(v)$ is the label function for u, and u is kept fixed during the transform $p(u) \to P(u)$, then $\tilde{f}(V = Mv) = f(v)$. This shows that equation (38) is identical to equation (39), and hence that the particle spectrum

$$\tilde{N}(\omega) = N(\omega), \tag{41}$$

is invariant between Möbius transformed mirrors.

In summary, the Möbius transformation preserves the energy flux and the particle spectrum, though the Bogolyubov beta coefficients can differ. The particle distribution per mode per mode can also deviate due to soft particles when $cd \neq 0$, but is preserved in the observable (high frequency) limit.

8. Conclusions

The anomalous quantum effect that breaks conformal symmetry of classical relativity is particularly explicit and manifest in the symmetries of the moving mirror model and its resulting radiation from vacuum. Although gravitation does not extend to the $(1 + 1)$-dimensional spacetime of the model (where $G_{\mu\nu}$ is identically zero), and the full physical significance of the gravitational analog in the ABC is limited to the strength of the analogy between $(1 + 1)$ and $(3 + 1)$-dimensions, the tractability of fully analytical results due to the additional simplicity gained by avoiding spacetime curvature is an asset. As far as quantum fields and relativistic thermodynamics are concerned this approach surrenders some valuable albeit phenomenological insights.

Our work underscores the symmetry governing the conformal anomaly of the moving mirror model. The most fundamental quantities of the model are invariant under Möbius transformations. This includes the energy flux $\mathcal{F}$, the particle spectrum $N(\omega)$, and the von Neumann entanglement (geometric) entropy S_{ren}. In select important cases we have found that this invariance extends to the spectral measure $N_{\omega\omega'} = |\beta_{\omega\omega'}|^2$. We summarize our results:

- Explicit demonstration of the dramatic dynamical change that a Möbius transform inflicts on the mirror trajectory, while not impacting what the observer sees. Humorously phrased, this gives a 'mirror of Dorian Gray', or as seen in figure 1, 'objects in moving mirror are closer (or further) than they appear'; see start of section 3.
- Full particle invariance $|\beta_{\omega\omega'}|^2$ for the eternal thermal mirror under dilatations or inversions ($cd = 0$ case). However, $|\beta_{\omega\omega'}|^2$ can differ for transforms where $cd \neq 0$, though invariance is restored in the high frequency regime, and upon integration over the ingoing modes to form the observable particle spectrum $N(\omega)$; see end of sections 3 and 7.

- Explicit Möbius transforms between the CW mirror and de Sitter mirror. Although both are eternally thermal, the Bogolyubov beta coefficients and $|\beta_{\omega\omega'}|^2$ are not invariant under the transforms, due to changing horizon structure; see section 4.
- A Möbius transform on the Schwarzschild mirror can give a horizon unseen by the stress tensor, equation (22) and figure 2, demonstrating that the energy flux may contain hidden horizons, or a wrapping of the spacetime itself; see section 5.
- Consistency of zero energy and zero particles for dual uniform accelerating mirrors; see section 6.2.
- Shown how asymptotic uniform acceleration can radiate thermal energy flux; see section 6.4.
- Shown that zero total energy time-dependent accelerating mirrors are forbidden; see appendix B.
- Elucidation of invariance from Möbius transformations for entanglement entropy; see appendix C.

The moving mirror model has come a long way over the five decades since DeWitt [1], Davies and Fulling [2, 3] demonstrated accelerated boundaries produced particles from the quantum vacuum. It has flourished, with e.g. Schwinger [39], Unruh *et al* [40], Wilczek [41] helping to establish theoretical and experimental [42, 43] research directions into the nature of particle creation, spacetime, and information. In 2021 alone, the dynamical Casimir effect field has been enriched by diverse, novel ideas ranging over holography, qubits, relativity, quantum power, and harvesting entropy [44–56]. The connection of the Möbius transform to the group $SL(2,\mathbb{R})$, Schwarzian derivatives, and conformal field theory (CFT) offers another avenue to a deeper understanding of the nature of information, horizons, and vacuum particle creation.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Transforming $f(v)$

As seen in the Schwarzschild case, the trajectory $p(u)$ in terms of the null coordinate u is not always tractable to general manipulation. It is worthwhile exploring what the equivalent of the Möbius transformation is for $f(v)$. That is, what mapping $f(v) \to \tilde{f}(V = Mv) \equiv g(v)$ keeps the energy flux invariant. Recall that

$$24\pi\mathcal{F}(v) = \frac{1}{f'(v)^2}\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right]. \tag{A1}$$

We write $g'(v) = h(f)f'(v)$ to obtain a differential equation for $h(f)$. Denoting $\dot{h} = \mathrm{d}h/\mathrm{d}f$, we have

$$\ddot{h} - \frac{3}{2}\frac{\dot{h}^2}{h} + (h - h^3)\,24\pi\mathcal{F}(f) = 0, \tag{A2}$$

where $\mathcal{F}(f)$ is the energy flux we seek to keep invariant.

Consider the case of zero flux, i.e. the constant acceleration mirror of section 6. Here the mapping resolves to

$$\begin{aligned} f(v) &\to \tilde{f}(V = Mv) \equiv g(v) \\ &= k_1 + k_2^{-2} f(v)\,\delta(k_3) - \frac{\not\delta(k_3)}{k_3^2 f + k_2 k_3}, \end{aligned} \tag{A3}$$

where k_1, k_2, k_3 are constants. The δ notation indicates that the second term only exists if $k_3 = 0$ (and $k_2 \neq 0$) and the third term only exists if $k_3 \neq 0$ (so one never has both terms at once). We indicate that when we apply to a Möbius transform to v, written as $V = Mv$, then the function label for u goes from $f(v)$ to $\tilde{f}(V)$. The trajectory $f(v) = -1/(\kappa^2 v)$ gives the constant acceleration, zero energy flux case, so our results indicate that any mirror with trajectory

$$g(v) = k_1 - \frac{\delta(k_3)}{k_2^2\kappa^2 v} - \frac{\not\delta(k_3)}{k_2 k_3 - k_3^2/(\kappa^2 v)}, \tag{A4}$$

will also have zero energy flux. The case with $k_3 = 0$ is a dilatation, a redefinition of κ. When $k_2 = 0$, $k_3 \neq 0$, then $g = \kappa^2 v/k_3$ and this is an inversion, to the trivial constant velocity, zero acceleration case. The case with $k_3 \neq 0$, $k_2 \neq 0$ has $g = -\kappa^2 v/(k_2 k_3 \kappa^2 v - k_3^2)$, and the mirror has $g(v = \pm\infty) = -1/(k_2 k_3) \neq 0$, $g(0^\pm) = 0^\pm$, unlike $f(v = \pm\infty) = 0^\mp$, $f(0^\pm) = \mp\infty$.

When the flux is instead thermal, and hence constant, we can solve equation (A2) to find

$$f(v) \to g(v) = k_1 - \frac{1}{\kappa}\ln\left(\frac{e^{\kappa f} - 1}{e^{\kappa f} + 1}\right). \tag{A5}$$

One can verify that CW $f(v)$ can be transformed into de Sitter, and vice versa, with this formula.

It is also interesting to consider when two ABC have only asymptotically the same energy flux, whether thermal or zero. The transformation teaches us that the approach to thermality is nearly identical for the transformed trajectory relative to the original thermal mirror, even if it is only thermal asymptotically. Consider $g'(v) = [1 + \epsilon(v)]f'(v)$. Then equation (A2) becomes $\ddot{\epsilon} = 48\pi\mathcal{F}\epsilon$ to first order, and

$$\epsilon(f) \sim e^{-\int_{f_0}^{f}\mathrm{d}f_\star\,\sqrt{48\pi\mathcal{F}(f_\star)}}, \tag{A6}$$

as long as the $\dot{h}^2/h \sim \dot{\epsilon}^2$ term is small compared to the leading order. Approaching thermality, $\mathcal{F} \to \mathcal{F}_{\mathrm{th}} = \kappa^2/(48\pi)$, and the distance moved $\Delta f \to \infty$ as seen from equation (23), or equally the eternal thermal or de Sitter cases, so indeed $g'(v) \to f'(v)$ exponentially. Since the energy flux depends only on f' (and its derivatives), then to leading order the transformed and original mirror trajectories approach thermality in the same way.

Similarly, when the flux approaches zero, then on that asymptote we can map the uniform acceleration mirror to, for example, Schwarzschild, which has $\mathcal{F}(v \to -\infty) \sim v^{-3} \to 0$. This can be done through the transformation of equation (A4) with $k_2 = 0$, $k_3 = \kappa$, giving the leading order in $f_{\mathrm{Schw}}(v) = v - \kappa^{-1}\ln(-\kappa v) \approx v$ as $v \to -\infty$. Unfortunately, using the full Schwarzschild flux

$$\mathcal{F}_{\mathrm{Schw}}(v) = \frac{\kappa^2}{48\pi}\frac{1-\kappa v}{(1-\kappa v)^4}, \tag{A7}$$

in equation (A2) is not tractable due to $\mathcal{F}(f)$ leading us back to product logs.

Appendix B. No go zero energy

We have explored transformations that leave invariant the energy flux. If the energy flux is unchanged then of course the total energy emitted is as well. Suppose we now consider mirrors with the same total energy, without requiring the energy flux $\mathcal{F}(u)$ be the same. For mirrors where the total energy is infinite (e.g. the eternal thermal mirrors), this question is not so interesting: there are many ways to add up to infinity. The other total energy of particular interest is zero net energy, such as seen in the constant acceleration case. There, the zero energy arises from uniform zero energy flux. However, we know that negative energy flux can exist as well as positive energy flux, and indeed is required under certain widespread circumstances, such as unitarity [17, 57, 58].

Can we arrange mirror motion such that negative energy flux at some time exactly cancels positive energy flux at some other time, leaving zero total energy? We present here a no-go conjecture against this possibility. What we find is that such a trajectory balancing the negative and positive energy fluxes is always interrupted by a horizon, leaving a net positive or negative total energy. For an interesting example, see the case [59] of an inertial horizon, i.e. dynamics that form a horizon with asymptotic zero acceleration.

It is useful to write the energy flux in terms of the rapidity $\eta(u) = (1/2)\ln p'(u)$. Then the total energy is

$$E = \int_{u_-}^{u_+} \mathrm{d}u\,\left[\eta'' - \left(\eta'\right)^2\right], \tag{B1}$$

where u_-, u_+ are the limits of mirror motion, either finite or infinite horizons.

The first term is a total derivative, so

$$E = \eta'(u_+) - \eta'(u_-) - \int \mathrm{d}u\,\left(\eta'\right)^2. \tag{B2}$$

To achieve zero energy, since the remaining integrand is positive we require $\eta'_+ > \eta'_-$.

Now if $\eta'(u_+) \to \pm\infty$ or $\eta'(u_-) \to \pm\infty$ then the second term, involving η', must win, giving negative total energy. If $\eta' \to \mathrm{const}\ (\neq 0)$, then if u extends to ∞ again the integral will dominate the boundary term, since the sum of a near constant over an infinite interval is infinite. Thus we need either $\eta' \to 0$ (asymptotic inertia) or to cut off u by a horizon at finite coordinate.

A horizon has $\eta = (1/2)\ln p' \to \infty$, so there is a pole in $p'(u)$. Noting that $\eta' = p''/(2p')$, we would normally expect $\eta' \to \infty$, i.e. if the pole is of order m so $p' \sim 1/(u_+ - u)^m$ then $\eta' = m/[2(u_+ - u)] \to \infty$ as $u \to u_+$ (the same holds for obtaining the horizon with $\eta \to \infty$ by $p' \sim (u - u_\star)^m \to 0$, i.e. the previous m is negative). Hence this reduces to the previously considered case, which did not give zero flux. However, there are special cases where $p'' \sim p'$ ($\eta' \to \mathrm{const}$) and $p''/p' \to 0$ ($\eta' \to 0$). The first of these has $p' \sim e^{bu}$, which blows up at $u_+ = \infty$; this is not at finite u and this is the previously considered, failed case of $\eta' \to \mathrm{const}$. The second of these takes $\eta' \to 0$ and is our remaining case to assess.

For this last case, where $\eta' \to 0$, then unless $\eta' = 0$ for all u, the range where it is nonzero will contribute to the integral and again unbalance the terms, preventing zero energy. The only zero total energy case is the zero energy flux case, when $\eta'' = (\eta')^2$, and hence $\eta' = -1/u$. This is precisely the constant acceleration case with $p(u) = -1/(\kappa^2 u)$. This case does have a pole, at $u = 0$, and $\eta' \to \infty$ there, but the terms balance in this one case. For poles of order m, the integral gives $\int \mathrm{d}u\, u^{-2m} = (1-2m)^{-1}u^{-2m+1}$ while the boundary term gives u^{-m}. Only when $-2m+1 = -m$, i.e. the $m = 1$ case, do the terms cancel each other.

Appendix C. Entanglement invariance

Here we derive the $(1+1)$-dimensional entanglement entropy or 'geometric entropy' in CFT and show its fundamental invariance under Möbius transformations of the trajectory. We then break the invariance by referencing a static mirror on one side of the system, which amounts to a choice of a reference frame for the observer, deriving the relationship to the rapidity of the mirror (see e.g. [30, 58, 60, 61]).

We start with the entropy of a system in $(1+1)$-D CFT using ϵ as a UV cut-off [24],

$$S = \frac{1}{6}\ln\frac{L}{\epsilon}, \tag{C1}$$

where L is the size of the arbitrary system in general. For us, the model will be the mirror trajectory which measures the size of the system by keeping track of accessible $(1+1)$ dimensional spacetime in which the quantum field is free to propagate. That is, for a general and arbitrary moving mirror trajectory $p(u)$,

$$L \equiv p(u) - p(u_0), \tag{C2}$$

where u and u_0 are null coordinates that form the region in the system which we are considering, and geometrically ϵ is asymmetrically smeared, i.e. $\epsilon^2 \equiv \epsilon_p \epsilon_{p_0}$. Here $p(u)$ is the trajectory of the mirror in null coordinates (it is the usual advanced time function of retarded time u). The smearing and dynamics of the mirror are related as

$$\epsilon_p = p'(u)\epsilon_u, \qquad \epsilon_{p_0} = p'(u_0)\epsilon_{u_0}. \tag{C3}$$

These ϵ_p smearings are coarse grained parameters at the end of the sub-systems, that parametrize how well the observer distinguishes the subsystem from the rest of the Universe [24].

Substituting equation (C3) into equation (C1) yields the bare entropy of the system,

$$S_{\text{bare}} = \frac{1}{12}\ln\frac{[p(u) - p(u_0)]^2}{p'(u)p'(u_0)\epsilon_u \epsilon_{u_0}}. \tag{C4}$$

The vacuum entropy of the system can be found by considering a static mirror where $L = u - u_0$ and $\epsilon^2 = \epsilon_u \epsilon_{u_0}$. Thus,

$$S_{\text{vac}} = \frac{1}{12}\ln\frac{(u-u_0)^2}{\epsilon_u \epsilon_{u_0}}. \tag{C5}$$

Even though the entropies above are defined in terms of smearing, this dependence can be removed by an intuitive renormalization (see also [58]) via

$$S_{\text{ren}} \equiv S_{\text{bare}} - S_{\text{vac}} = \frac{1}{12} \ln \frac{[p(u) - p(u_0)]^2}{p'(u)p'(u_0)(u - u_0)^2} \ . \tag{C6}$$

The geometric entropy equation (C6) is invariant under Möbius transformations, as seen from equation (40). Thus a Möbius transformation of mirror trajectories preserves energy flux, particle spectrum, and geometric entropy.

Going further, if we define the derivative $p'(u_0)$ as

$$p'(u_0) = \frac{p(u) - p(u_0)}{u - u_0}, \tag{C7}$$

then the dynamic meaning of the entropy becomes clear as we deviate δu away from u_0. We obtain the Möbius invariant,

$$S_{\text{ren}} = \frac{1}{12} \ln \frac{p'(u_0)}{p'(u)} = -\frac{1}{6}(\eta - \eta_0) \ . \tag{C8}$$

Choosing a static reference frame for time u_0 sets $p'(u_0) = 1$ or $\eta_0 = 0$, giving

$$S(u) = -\frac{1}{12} \ln p'(u) = -\frac{\eta}{6} \ , \tag{C9}$$

which is the common definition of entanglement entropy in terms of rapidity, e.g. [58, 62]. Interestingly equation (C9) is not preserved by a Möbius transform. This is clear from $\eta(u) = (1/2)\ln p'(u)$ and $P(u) = (ap + b)/(cp + d)$, hence $P' = p'/(cp + d)^2$. The reason for this is that rapidity $\eta = \tanh^{-1} \dot{z}$ is not by itself invariant, but depends on the coordinate frame in which the mirror velocity $\dot{z}$ is measured. Both the invariant and variant von Neumann entropies, equations (C8) and (C9), measure the degree of quantum entanglement between the two subsystems, past and future. The more fundamental measure of entanglement of the system is equation (C8), which has invariance, like the particles and energy. The operation of choosing a static mirror reference choice breaks the symmetry of the model: rapidity is always measured with respect to some frame.

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