Fractional Fisher-KPP type equations on stratified groups

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Abstract

This work investigates the fractional space-time behavior of the Fisher-KPP equation with initial boundary values. Notably, fractional versions of Fisher-KPP equations describe complex phenomena in cases where the classical local approach is limited. In this work, we combine different techniques from fractional calculus and non-commutative analysis, which provide new results for various fractional models involving the Fisher-KPP equation.

Firstly, we prove that if the initial data lies between 0 and 1, then the global solution also belongs to the interval $[0, 1]$. Secondly, we establish that the solution in the L^2 norm is bounded by the L^2 norm of the initial data. Lastly, we demonstrate that the model exhibits blow-up behavior on a finite time interval under certain conditions.

Importantly, the results obtained in the non-commutative analysis cover many previously known results in the commutative case.

Keywords: fractional Fisher-KPP equation, stratified group, Caputo type derivative, fractional Poincaré inequality, fractional p-Laplacian, fractional p-sub-Laplacian, blow-up solution.

Аңдатпа

Бұл жұмыс бастапқы және шекаралық мәндері бар, кеңістік пен уақыт бойынша бөлшек туынды Фишер-КПП моделін зерттейді. Атап айтқанда, классикалық Фишер-КПП модель сипаттауы дәрменсіз жағдайларда, бөлшек туындылы теңдеу күрделі құбылыстарды сипаттай алады. Стратифицирленген топтар мен бөлшек туындылар қасиеттерін қолдану арқылы, Фишер-КПП теңдеуін қамтитын әртүрлі бөлшек туындылы модельдер үшін жаңа нәтижелер аламыз.

Біріншіден, егер бастапқы деректер 0 мен 1 аралығында жатса, онда ғаламдық шешімі де $[0, 1]$ интервалына жататынын дәлелдейміз. Екіншіден, L^2 нормасындағы шешім бастапқы деректердің L^2 нормасымен шектелетінін анықтаймыз. Соңында, модельдің белгілі бір шарттарда шектеулі уақыт интервалында күйреу әрекетін көрсетеміз.

Маңыздысы, бұл нәтижелер коммутативті жағдайда бұрын белгілі көптеген нәтижелерді қамтып қана қоймай, сонымен қатар оларды коммутативті емес анализге дейін кеңейтеді.

Кілт сөздер: бөлшек туындылы Фишер-КПП теңдеуі, стратифицирленген топ, Капуто типті туынды, бөлшек Пуанкаре теңсіздігі, бөлшек p-Лапласиан, бөлшек p-суб-Лапласиан, шешімнің күйреуі.

Contents

Chapter 1

Introduction

1.1 Problem formulation

The *Fisher-Kolmogorov-Petrovsky-Piskunov (Fisher-KPP) equation* is the nonlinear differential equation

$$v_t - \Delta v = v(1-v), \tag{1.1.1}$$

which has been extensively studied due to its broad applications across various fields [Contri, 2018]. Sometimes, we refer to the equation as a reaction-diffusion equation. Replacing v with $1-u$ in equation (1.1.1), we obtain the equation

$$u_t - \Delta u = u(u-1), \tag{1.1.2}$$

which has particular relevance to the biological sciences [Dipierro and Valdinoci, 2021].

This thesis aims to expand the investigation of the Fisher-KPP equation to its fractional version in the form

$$^{C}\partial_t^{\nu} u + \left(-\Delta_{p,\mathbb{G}}\right)^s u = u(u-1), \quad \Omega \subset \mathbb{G},\ t > 0, \tag{1.1.3}$$

where Ω is an open set on a stratified group $\mathbb{G}$, $p \in (1, \infty)$, $\nu \in (0, 1]$, and $s \in (0, 1]$. In equation (1.1.3), the Caputo fractional derivative $^{C}\partial_t^{\nu}$ replaces the derivative in time, and the Laplace operator is extended by the fractional p-sub-Laplacian $\left(-\Delta_{p,\mathbb{G}}\right)^s$. When $\nu = 1$, the equation reduces to the first-order partial derivative ∂_t, while $s = 1$ with $p = 2$ gives the sub-Laplacian operator $\mathcal{L}$ in the model. The initial and boundary conditions for equation (1.1.3) are given by

$$u(x, 0) = u_0(x), \qquad x \in \Omega, \tag{1.1.4}$$

and

$$u(x, t) = 0, \qquad x \in \partial\Omega, \quad t > 0. \tag{1.1.5}$$

Even though $\mathbb{G}$ is a nilpotent, non-commutative group (in general), it has a closer resemblance to classical analysis than one may expect. As a straightforward example, we can refer to Euclidean space $\mathbb{R}^N$ defined with a linear group law. We cite the following works, as well as the references therein, which contain the key topics below for further discussion.

For the Fisher-KPP (Reaction-diffusion) equation:

- The first appearance of the model with the wave solutions: [Fisher, 1937], [Kolmogorov et al., 1937]
- existence of local and global solutions in one-dimensional case: [Gunzburger et al., 2005]
- Numerical methods: [Ramos, 1983],[Chandraker et al., 2015] [Hasnain et al., 2017]
- Applications in different fields of science: [Volpert and Petrovskii, 2009], [Ablowitz and Zeppetella, 1979, Ai and Huang, 2005, Ducrot et al., 2010], [Nardini and Bortz, 2018],

and especially, for fractional types of the Fisher-KPP model, the existence of:

- unique bounded mild solution: [Cabré and Roquejoffre, 2013]
- unique globally strong solution: [Gal and Warma, 2020]
- local, weak, global, and blow-up solutions: [Ahmad et al., 2015], [Alsaedi et al., 2021], [Borikhanov et al., 2023].

Furthermore, it should be noted that the aforementioned works represent only a bit of the extensive investigation available on this topic. Many additional studies can be found in respectable sources such as Web of Science and Scopus journals.

The wide research mentioned above comes from various domains and a multitude of applications. For instance, the solutions to the classical equation (1.1.1) express the anticipated values for particles that move according to a Brownian process. The motivation for considering the fractional case on $\mathbb{R}^N$, meaning to replace the Laplacian with the fractional operator, is the generator for a stable Lévy process, which is a jump process. The existence of this process in the diffusion phenomena will expedite the invasion of the unstable state ($u = 0$) into the stable state ($u = 1$).

Moreover, when Brownian processes do not properly describe diffusion, fractional diffusion equations are needed in physical phenomena such as plasmas, turbulence, and flames [Cabré and Roquejoffre, 2013].

Another important remark of this research is that it carries a qualitative character. In this work, the existence of a local solution to the problem (1.1.3) is assumed for stratified groups. This assumption makes sense since local existence results are well-established in Euclidean space. However, it is important to note that the findings proven in this thesis are not mere extensions to more general spaces, but also contribute novel results within the Euclidean space. More precisely, we state the following theorems, which can be noted as the main results of this work.

Theorem 1.1.1. *Let $u_0 \in L^\infty(\Omega)$ be such that $0 \leqslant u_0(x) \leqslant 1$. Then the global solution of the problem (1.1.3)-(1.1.5) satisfies*

$$0 \leqslant u(x,t) \leqslant 1, \quad \text{for} \quad (x,t) \in \Omega \times (0,T).$$

Theorem 1.1.2. *Let $u_0 \in L^2(\Omega)$ be such that $0 \leqslant u_0(x) \leqslant 1$. Then the global solution of the problem (1.1.3)-(1.1.5) satisfies the following properties:*

a) When $0 < s < 1$, $\nu = 1$, we have

$$\|u(x,t)\|_{L^2(\Omega)} \leqslant e^{-\lambda_{Q,2,s}t} \|u_0(x)\|_{L^2(\Omega)},$$

b) When $0 < s < 1$, $0 < \nu < 1$, we have

$$\|u(x,t)\|_{L^2(\Omega)} \leq E_\nu\left(-2\lambda_{Q,2,s}t^\nu\right)\|u_0(x)\|_{L^2(\Omega)}.$$

c) When $s = 1$, $\nu = 1$, we obtain

$$\|u(x,t)\|_{L^2(\Omega)} \leqslant e^{-\lambda_{1,2}t} \|u_0\|_{L^2(\Omega)}.$$

d) When $s = 1$, $0 < \nu < 1$, we have

$$\|u(x,t)\|_{L^2(\Omega)} \leq E_\nu\left(-2\lambda_{1,2}t^\nu\right)\|u_0(x)\|_{L^2(\Omega)}.$$

Note that $\lambda_{Q,2,s}(\Omega)$ denotes the constant of the Poincaré inequality (2.1.16), and E_ν is the Mittag-Leffler function. When $s = 1$, $\lambda_{1,2}(\Omega)$ is the first eigenvalue of the sub-Laplacian.

Theorem 1.1.3. *Assume that $u_0 \in L^2(\Omega)$. Let $\lambda_{1,2,s}(\Omega)$ denote the first eigenvalue and u_1 represent the corresponding eigenfunction of the Dirichlet sub-Laplacian problem, such that $\int_\Omega u_1(x) = 1$. Suppose that $1 + \lambda_{1,2,s}(\Omega) < H_0$, where $H_0 := \int_\Omega u_0(x)u_1(x)d\nu$. Then, for $p = 2$, the weak solution u of (1.1.3)-(1.1.5) blows up in a finite time. Specifically, for the case $\nu = 1$ in (1.1.3), the blow-up time is given by*

$$T_0 = \frac{1}{1 + \lambda_{1,2,s}(\Omega)} \log \frac{H_0}{H_0 - (1 + \lambda_{1,2,s}(\Omega))}.$$

This thesis work is organized as follows. In the subsequent sections of this chapter, we include background reviews related to the fractional-type models, fractional Laplacians, and fractional derivatives/integrals. In Chapter 2, we recall some necessary preliminary properties associated with stratified groups, the fractional Sobolev space, and fractional calculus. Notably, in previous works such as [Ahmad et al., 2015, Alsaedi et al., 2021], authors analyzed problem (1.1.3)-(1.1.5) when the regional fractional Laplacian with $p = 2$ was applied. The derived results in these studies are covered in the findings presented in Chapter 3. Specifically, we investigate the case of the Euclidean space by substituting the fractional Laplacian with the p-Laplacian operator. Moving forward, in Chapters 4 and 5, we extend the result in the fractional Sobolev space, and stratified groups. Appendix .1 is dedicated to considering the Fourier definition of the fractional Laplacian.

We note that Theorems 1.1.1-1.1.3 are not proven in a single specific section. Because problem (1.1.3) is divided into three cases based on the domain of the space variables x that appear in each of the chapters (3-5). Furthermore, each chapter is divided into two parts, containing the partial derivative case and the fractional derivative case in time t.

One of the major innovations of this research is that these techniques of solving the problem combine two distinct fields of mathematics: fractional calculus, and non-commutative analysis. In the context of our work, the first one is expressed by derivative in time. While the last one appears on investigating the domain of space variables.

Despite the inclusion of certain aspects of the main results published in our papers [Jabbarkhanov et al., 2022, Jabbarkhanov and Suragan, 2023], this work contains a self-contained, independent study that presents unique and complete findings.

1.2 Background review

1.2.1 Fisher-KPP equations

Fisher-KPP equations (more generally, Reaction-diffusion equations) have their origins in the 30s of the 20th century with the works in population dynamics, combustion theory, and chemical kinetics [Volpert and Petrovskii, 2009]. This term is closely related to the researchers: Fisher and Kolmogorov, Petrovsky, and Piskunov. In this subsection, we provide an overview of their contributions.

Fisher's result: In 1937, Sir Ronald Aylmer Fisher was one of the first to introduce a new approach to understanding spatial heterogeneity in population dynamics. His research work [Fisher, 1937] was focused on studying the spatial propagation of a favorable gene in a population. As a simplification, he considered a one-dimensional space and defined the population proportion by $u(x, t)$, located at point x at time t, that possessed the favorable gene, such that $0 \le u(x, t) \le 1$.

To model a natural selection, he used a frequency equation (see [Bacaër, 2011, equation 14.6]) and the continuous-time variable equation

$$\frac{\partial u}{\partial t} = au(1 - u), \tag{1.2.1}$$

with a positive parameter a.

Equation (1.2.1) is called Verhulst's logistic equation, and it satisfies the equation

$$u(x, \infty) = 1.$$

Moreover, Fisher claimed that the offspring of the favorable gene disperses randomly in the neighborhood of x instead of staying at the same point. In other words, he added a diffusion term to equation (1.2.1), resulting in the following equation:

$$\frac{\partial u}{\partial t} = au(1 - u) + D\frac{\partial^2 u}{\partial x^2}, \tag{1.2.2}$$

where D is a diffusion coefficient.

Equation (1.2.2) with the selection coefficient $a = 0$ was considered by Fourier in his "The Analytical Theory of Heat" [Fourier, 1878].

A solution to equation (1.2.2) exists, and it can be shown by using the form

$$u(x, t) = W(x + vt)$$

that satisfies the following three conditions:

$$\lim_{x\to-\infty} u(x,t) = 0, \quad \lim_{x\to+\infty} u(x,t) = 1, \quad \text{and} \quad 0 \le u(x,t) \le 1. \tag{1.2.3}$$

By making the substitution $y = x + vt$, we can express the solution $u(x,t)$ as a function of a single variable $W(y)$, such that $u(x,t) = W(y)$. Then, it can be easily observed that

$$\frac{\partial u}{\partial t} = vW'(y),$$

$$\frac{\partial u}{\partial x} = W'(y),$$

and

$$\frac{\partial^2 u}{\partial x^2} = W''(y).$$

As a result, assuming u is a solution of equation (1.2.2), the following second-order equation is obtained:

$$vW'(y) = aW(y)(1 - W(y)) + DW''(y). \tag{1.2.4}$$

When $u \to 0$ (or $y \to -\infty$), it is expected that

$$\lim_{y\to-\infty} W(y) = 0 \quad \text{and} \quad \lim_{y\to-\infty} W'(y) = 0,$$

which leads to

$$\lim_{y\to-\infty} \frac{W'(y)}{W(y)} = k.$$

By applying L'Hôpital's rule, it is well known that

$$\lim_{y\to-\infty} \frac{W''(y)}{W'(y)} = k.$$

Therefore, we can write

$$\frac{W''(y)}{W(y)} = \frac{W''(y)}{W'(y)} \cdot \frac{W'(y)}{W(y)} = k^2.$$

We divide equation (1.2.4) by $W(y)$ and with the approach $y \to -\infty$, we obtain the second-order equation

$$Dk^2 - vk + a = 0.$$

It should be noted that k must be a real number, which implies that the discriminant of this equation has to be nonnegative: $v^2 - 4aD \ge 0$, or $v \ge 2\sqrt{aD} = v^*$. Therefore, $v \ge v^*$ represents a necessary condition for the existence of a wave propagating at speed v.

Kolmogorov-Petrovsky-Piskunov's result: In the same year, in 1937, and independently of Fisher's work, Andrey Nikolaevich Kolmogorov, Ivan Georgievich Petrovsky and Nikolay Semenovich Piskunov investigated the same problem of propagation of a dominant gene. In their article [Kolmogorov et al., 1937], they used a mathematical model based on Mendelian genetics, which has a similar form as in (1.2.2), with the exception that the non-linear term $u(u-1)$ is replaced with a function denoted as f, that satisfies the conditions: $f(0) = 0$, $f'(0) > 0$, $f(1) = 0$ for $u \in (0,1)$ and $f'(u) < f'(0)$ for $u \in (0,1]$. Despite the result obtained by the authors being analogous to Fisher's one, they got more rigorous proof. Indeed, if the initial condition satisfies $0 \leq u(x,0) \leq 1, u(x,0) = 0$ for all $x < x_1$ and $u(x,0) = 1$ for all $x > x_2 \geq x_1$, then the gene is propagated at the speed $v^* = 2\sqrt{f'(0)D}$. It is easy to mention that Fisher's result is recovered when $f(u) = au(u-1)$.

The works written by Fisher, Kolmogorov, Petrovsky, and Piskunov became the starting point for constructing many mathematical models with geographic diffusion in ecology, genetics, and epidemiology. These models are well-known as "reaction-diffusion equations (systems)" [Bacaër, 2011].

Nowadays, the theory of reaction-diffusion equation is a well-developed area of research that includes qualitative properties of traveling waves for the scalar reaction-diffusion equation and complex nonlinear dynamics, a system of equations, and numerous applications in medicine, biology, chemistry, and physics [Volpert and Petrovskii, 2009]. We refer to the sources [Ablowitz and Zeppetella, 1979, Ai and Huang, 2005, Ducrot et al., 2010, Nardini and Bortz, 2018] and the references therein for further discussions.

1.2.2 Fractional integrals and derivatives

In basic calculus, differentiation operations can be carried out in the first order, second order, and nth order. Integration can be reformulated as differentiation to (-1)th order using the Fundamental theorem of calculus, and further differentiation to (-2)th order, $(-n)$th order, etc. can be obtained by repeating integration. For instance,

$$\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}f(x) = \int_c^x f(y)\mathrm{d}y, \quad \frac{\mathrm{d}^{-2}}{\mathrm{d}x^{-2}}f(x) = \int_c^x \int_c^y f(z)\mathrm{d}z\,\mathrm{d}y. \tag{1.2.5}$$

Otherwise, it is possible to find the derivatives of a function to any integer order, where positive integers are given by differentiation and negative integers by integration. Note that the 0th derivative of a function is the function itself. From here, it is appropriate to introduce the concepts of fractional derivatives.

The main idea of fractional calculus is to generalize the order of differentiation and integration outside the set of integers. In other words, the study includes non-integer orders of differentiation and integration. For example,

$$\frac{\mathrm{d}^{1/5}}{\mathrm{d}x^{1/5}} f(x). \tag{1.2.6}$$

We refer to the sources [Samko et al., 1993, Miller and Ross, 1993, Podlubny, 1999, Kilbas et al., 2006] for a general introduction to fractional calculus. From a purely mathematical perspective, it is about generalizing fundamental concepts of differentiation and integration. On the other hand, the knowledge is widely studied by physicists, chemists, biologists, signal processing, porous media, economics, and engineers, because of its applications in many parts of science.

Fractional derivatives have been the subject of research by numerous scholars over the years. Various types of fractional derivatives, including but not limited to Riemann–Liouville, Caputo, Grunwald–Letnikov, Erdelyi–Kober, Hadamard, Marchaud, and Riesz, have been studied. Among these, the Caputo derivative and the Riemann–Liouville fractional derivative (integral) are of particular significance in fractional calculus [Herrmann, 2014, Hilfer, 2000], and we provide a detailed discussion on these derivatives in Section 2.2.

1.2.3 Fractional Laplacians

Riesz introduced the concept of fractional integration for functions of multiple variables, which subsequently became known as Riesz potentials (see [Riesz, 1936a], [Riesz, 1936b, Riesz, 1938, Riesz, 1949]). A notable example of these potentials is represented by the negative fractional power $(-\Delta)^{\alpha/2}$ of the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$. It is realized through the following expression:

$$I^\alpha \varphi = c \int_{R^N} \frac{\varphi(y)dy}{|x-y|^{N-\alpha}},$$

where R^N denotes an N-dimensional Euclidean space, $0 < \alpha < N$, and c is a normalizing constant [Samko et al., 1993].

In recent years, researchers have turned to fractional calculus to develop sophisticated mathematical models that can more accurately describe complex anomalous systems [Pozrikidis, 2016, Bucur and Valdinoci, 2016, Meerschaert and Sikorskii, 2019, Vázquez, 2017]. The fractional Laplacian, in particular, has been found to replace the integer-order Laplace operator in many applications, where the classical local approach is limited or inappropriate. For example, it has been used in studying turbulence [Bakunin, 2008], elasticity [Dipierro et al., 2015], anomalous transport and diffusion [Bologna et al., 2000, Meerschaert, 2012], image processing [Gilboa and Osher, 2008], wave propagation in heterogeneous high contrast media [Tieyuan and Harris, 2014], and porous media flow [Vázquez, 2012], [Cabré and Roquejoffre, 2013]. Additionally, the fractional Laplacian serves as the generator of s-stable processes, which find applications in stochastic models such as mathematical finance [Levendorskiĭ, 2004, Pham, 1997].

There are numerous equivalent definitions of the fractional version of the Laplacian, as provided by Kwaśnicki [Kwaśnicki, 2017]. However, these definitions are limited to bounded domains and give rise to different operators depending on the associated boundary conditions. In this subsection, we explore several significant equations that involve the fractional Laplace operator. For a more comprehensive understanding of fractional models, we refer to Lischke's work [Lischke et al., 2020]. Additionally, for a detailed explanation of the definition of fractional Laplacians, please refer to Appendix .1.

Reaction-Diffusion. Yamamoto [Yamamoto, 2012] studied the linear dissipative equation of the form

$$\partial_t u + (-\Delta)^s u + ku = 0,$$

where $N \in \mathbb{N}$, $s \in (1/2, 1]$, $(x,t) \in \mathbb{R}^N \times (0,\infty)$, and the coefficient function $k : (x,t) \to \mathbb{R}$. Case $s = 1$ is understood as the Laplacian operator $(-\Delta)$, and the fractional Laplacian is given by

$$(-\Delta)^s u = \mathcal{F}^{-1}\left[|\xi|^{2s}\mathcal{F}[u]\right],$$

where $\mathcal{F}$ defines the Fourier transform. Note that when $s \in (1/2, 1]$, the fractional Laplacian $(-\Delta)^s$ describes the anomalous diffusion (see [Metzler and Klafter, 2000]). Yamamoto obtained the large-time behavior of decaying solutions by estimating the difference between solutions and their asymptotic expansion. Moreover, the author derived the spatial decay of this difference. Studying a dissipative equation with the fractional Laplacian presents difficulties in deriving the high-order asymptotic expansion of solutions due to the anomalous diffusion process. However, Yamamoto considered the spatial decay of the difference between solutions and their asymptotic expansion, which provides the high-order (arbitrary) asymptotic expansion [Yamamoto, 2012].

Quasi-geostrophic. Constantin and Wu [Constantin and Wu, 1999] investigated solutions to the $2D$ quasi-geostrophic (QGS) equation of the form:

$$\partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^s \theta = f, \tag{1.2.7}$$

where $\kappa > 0$ with $s \in [0, 1]$. Here, $\theta(x, t)$ denotes a scalar function. The non-local fractional operator $(-\Delta)^s$ is defined by the formula of the Fourier transforms multiplied with $|\xi|^{2s}$. QGS equation is a mathematically significant model investigated over the last decades in oceanography, and meteorology [Constantin et al., 1994, Pedlosky, 1987], [Held et al., 1995].

The authors proved the global existence and uniqueness of smooth solutions for $s \in (1/2, 1]$. However, the uniqueness result only holds in the class of strong solutions. They also obtained essential aspects of large-time approximation from equation (1.2.7). In this model, u denotes the fluid velocity and θ represents the potential temperature in the two-dimensional QGS equation. The equation with the particular power case $s = 1/2$ is derived from general quasigeostrophic models (see [Pedlosky, 1987]). Dimensionally, the two-dimensional QGS equation with $s = 1/2$ is analogous to the three-dimensional Navier-Stokes equations [Constantin and Wu, 1999]. It describes the temperature evolution on the two-dimensional boundary of a rapidly rotating half-space.

Cahn-Hilliard. Ainsworth and Mao [Ainsworth and Mao, 2017b] investigated the well-posedness of the Cahn-Hilliard equation with a fractional derivative $(-\Delta)^s$ replacing the gradient term in the free energy, given by the equation

$$\partial_t u + (-\Delta)^s \left(-\varepsilon^2 \Delta u + f(u)\right) = 0, \tag{1.2.8}$$

where $0 < s \leq 1$. Specifically, replacing the Laplace operator by $(-\Delta)^\beta$, they derived an alternative version of equation (1.2.8) in the two-dimensional case. The new fractional operator is expressed by

$$(-\Delta)^\beta u = \sum_{k,l \in \mathbb{Z}} \left(k^2 + l^2\right)^\beta \hat{u}_{kl} e^{ikx_1 + ilx_2},$$

where the Fourier coefficients $\hat{u}_{kl}$ are given by

$$\hat{u}_{kl} = \left\langle u, e^{ikx_1 + ilx_2} \right\rangle = \frac{1}{(2\pi)^2} \int_\Omega u e^{ikx_1 + ilx_2} dx$$

with $x = (x_1, x_2)$.

Ainsworth and Mao demonstrated the existence and uniqueness of a Fourier-Galerkin approximation for the fractional type of the Cahn–Hilliard equation, and used compactness arguments to establish the existence and uniqueness of the solution. They also derived a convergence rate estimate for the Fourier-Galerkin approximation and presented numerical results on the influence of the parameters ε and β on simple solutions of the fractional Cahn–Hilliard equation. Further insights on this topic can be found in [Ainsworth and Mao, 2017b, Akagi et al., 2016, Ainsworth and Mao, 2017a] and other relevant references.

Porous medium. Pablo, Quirós, Rodríguez, and Vázquez [de Pablo et al., 2011] developed a theory of existence and uniqueness of the fractional porous medium model, where the non-linear part is represented by $f = 0$, with an initial value of $u_0 \in L^1(\mathbb{R}^N)$. The authors denoted the nonlocal fractional operator $(-\Delta)^{1/2}$ by an alternative definition by the Fourier transform a function in the Schwartz class of the form:

$$\widehat{(-\Delta)^{1/2}}h(\xi) = |\xi|\widehat{h}(\xi).$$

The motivation behind studying the problem lies in its various applications in materials science. This equation plays a crucial role in mathematical models for phase-transition, damaging, viscoelasticity, complex fluids, and whenever diffuse interfaces are present (see, [Akagi et al., 2016, Cherfils et al., 2011, Novick-Cohen, 1998] and references therein).

Time-space fractional diffusion. In a recent study, Borikhanov, Ruzhansky, and Torebek [Borikhanov et al., 2023] studied the nonlinear diffusion equation (1.1.3), but with the nonlinear part $k|u|^{n-1}u + l|u|^{m-2}u$, where $k, l \in \mathbb{R}, n > 0, m > 1$. They established the existence of a weak solution applying the Galerkin approximation technique. Furthermore, using the comparison principle, they provide the blow-up, the asymptotic properties of global solutions results.

Fisher-KPP equation. In a recent study, Alsaedi, Kirane, and Torebek [Alsaedi et al., 2021] investigated the time-space fractional Fisher-KPP equation (1.1.3) within a bounded domain $\Omega \subset \mathbb{R}^N$ and for $p = 2$. The Laplace operator is replaced with the regional fractional Laplacian.

The authors established the existence of global solutions and finite-time blow-up solutions for certain initial conditions. They also provided a detailed analysis of the asymptotic behavior of bounded solutions. For further insight on this topic, interested readers can refer to [Alsaedi et al., 2021] and related references. Indeed, all the results presented in Chapter 3 are a natural extension of their findings.

1.2.4 Fractional p-sub-Laplacians

The study of sub-Laplacians on stratified groups is of significant importance in theoretical (see, e.g., [Gromov, 1996, Danielli et al., 2007]) and applied mathematics, such as mathematical models of human vision and crystal material (see, for example, [Christodoulou, 1998, Citti et al., 2004]). However, despite its relevance, research in p-sub-Laplacians remains limited. One of the foundational works in this field is Folland's study [Folland, 1975] of the well-behaved fundamental solution for the sub-Laplacian on stratified groups. Here, we mention some progress in this knowledge.

Recent research has focused on eigenvalue problems involving subelliptic operators on stratified Lie groups, as demonstrated by works such as those by [Chen and Chen, 2021, Hansson and Laptev, 2008, Frank and Laptev, 2010]. One area of recent progress is the study of nonlocal Dirichlet eigenvalue problems over fractional Sobolev spaces on stratified groups associated with the fractional p-sub-Laplacian. Ghosh, Kumar, and Ruzhansky [Ghosh et al., 2023] used the strong minimum principle to show that the first eigenfunction of the fractional p-sub-Laplacian is positive and that the first eigenvalue is isolated and simple. More details on this work are provided in Section 2.1.5.

Jinguo and Dengyun [Jinguo and Dengyun, 2021] studied a problem that involves the fractional p-sub-Laplacian. The authors established the existence and multiplicity of non-negative solutions. In particular, they demonstrated that the nonlocal problem on the homogeneous group has at least two nontrivial solutions under certain conditions.

Significant progress has been made in investigating eigenvalue problems associated with the p-sub-Laplacian operator on the Heisenberg group, which is a particular case of stratified groups (see Section 2.1.2). It is crucial to note that several definitions of the fractional sub-Laplacian on the Heisenberg group exist in the literature [Ferrari and Franchi, 2015, Frank et al., 2015, Roncal and Thangavelu, 2016]. For instance, Roncal and Thangavelu [Roncal and Thangavelu, 2016] defined the fractional sub-Laplacian by generalizing the definition obtained by Haagerup and Cowling [Cowling and Haagerup, 1989] regarding the heat semigroup.

Chapter 2

Preliminaries

2.1 Stratified groups and fractional Sobolev spaces

This chapter covers the basic notations, definitions, theorems, propositions, and lemmas that are essential for proving the main results in Chapters 3-5. Many authors have already considered these results, and to avoid repetition, we briefly mention works such as [Ghosh et al., 2023, Ruzhansky and Suragan, 2019, Bonfiglioli et al., 2007], [Baleanu et al., 2021, Fernandez et al., 2022, Fernandez et al., 2023] and provide acknowledgments throughout this thesis work. Additionally, we provide references for properties that require more specific mention.

2.1.1 Stratified groups

Let $\mathbb{R}^N = \{(x_1, \ldots, x_N) : x_1, \ldots, x_N \in \mathbb{R}\}$ for a given $N \in \mathbb{N}$ and let $\Omega \subseteq \mathbb{R}^N$ be a non-empty open set. We indicate a partial derivative operator with respect to the j-th coordinate of $\mathbb{R}^N$ using the notations

$$\partial_j, \qquad \partial_{x_j}, \qquad \partial/\partial x_j, \qquad \frac{\partial}{\partial x_j}.$$

For scalar functions $a_1, \ldots, a_N$ on Ω such that $a_j : \Omega \to \mathbb{R}$ for all $j \in 1, \ldots, N$, we define the linear differential operator X of order one on Ω with components $a_1, \ldots, a_N$ as

$$X = \sum_{j=1}^{N} a_j \partial_j. \tag{2.1.1}$$

This operator is commonly referred to as a *vector field* on Ω.

Assume that $J \subseteq \Omega$ is an open set. For a differentiable function f on J, we define the function Xf on J as

$$Xf(x) = \sum_{j=1}^{N} a_j(x)\partial_j f(x) \quad \text{with} \quad x \in J.$$

If f is a vector-valued function, meaning $f : J \to \mathbb{R}^k$, and

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_k(x) \end{pmatrix},$$

then Xf denotes the component-wise action of X on f, given by

$$Xf(x) = \begin{pmatrix} Xf_1(x) \\ \vdots \\ Xf_k(x) \end{pmatrix}.$$

For the set of infinitely differentiable real-valued functions (smooth) denoted by $C^\infty(J)$, we define X as a *smooth vector field*, if all of its component functions a_j are smooth. In this context, X is considered as an operator that acts on smooth functions of the form

$$X : C^\infty(J) \to C^\infty(J).$$

For a vector field X given by (2.1.1), the column vector of its components is provided by

$$XI := \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix},$$

where I denotes the identity map on $\mathbb{R}^N$. Note that $I = (I_1, \ldots, I_N)$ with $I_j(x) = x_j$, and $X(I_j) = a_j$.

Alternatively, we can define Xf as a row and column product of the form

$$Xf = (\nabla f) \cdot XI,$$

where the gradient operator $\nabla = (\partial_1, \ldots, \partial_N)$ in $\mathbb{R}^N$ is given for a smooth real-valued function f on $\mathbb{R}^N$,

Example 2.1.1. Consider the smooth vector fields X_1 and X_2 on $\mathbb{R}^3$, with coordinates $x = (x_1, x_2, x_3)$ defined as follows:

$$X_1 = 3\partial_{x_1} + x_2\partial_{x_3},$$

and

$$X_2 = 2\partial_{x_1} + 6x_1\partial_{x_2} - x_2\partial_{x_3}.$$

The action of these vector fields on the identity map $I(x)$ is given by

$$X_1 I(x) = \begin{pmatrix} 3 \\ 0 \\ x_2 \end{pmatrix}, \quad \text{and} \quad X_2 I(x) = \begin{pmatrix} 2 \\ 6x_1 \\ -x_2 \end{pmatrix}.$$

For a set of smooth vector fields Y and Z in $\mathbb{R}^N$, we define the operation known as the Lie bracket as follows:

$$[Y, Z] := YZ - ZY.$$

Indeed, if we define vector fields Y, and Z as in (2.1.1) for components a_j, and b_j, respectively, we can easily show that the bracket is the vector field of the form

$$[Y, Z] := \sum_{j=1}^{N} (X\, b_j - Y\, a_j)\partial_j.$$

As an example, consider the vector fields

$$Y = \partial_{x_1} + 2x_2\partial_{x_3}, \quad \text{and} \quad Z = \partial_{x_2} - 2x_1\partial_{x_3},$$

which yield

$$[Y, Z] = -4\partial_{x_3}.$$

Let $T(\mathbb{R}^N)$ be the Lie algebra of vector fields defined on $\mathbb{R}^N$, and let $\mathfrak{a}$ be a sub-algebra of $T(\mathbb{R}^N)$. For any vector fields $Y, Z, W \in T(\mathbb{R}^N)$, a bilinear mapping $(Y, Z) \mapsto [Y, Z]$ satisfies the Jacobi identity

$$[Y, [Z, W]] + [Z, [W, Y]] + [W, [Y, Z]] = 0,$$

for all vector fields $Y, Z, W \in T(\mathbb{R}^N)$. If $\mathfrak{a}$ is closed under the bracket operator

$$[Y, Z] \in \mathfrak{a}$$

for each $Y, Z \in \mathfrak{a}$, then a vector subspace $\mathfrak{a}$ of $T(\mathbb{R}^N)$ is called a Lie algebra of vector fields.

Let $J = (j_1, \ldots, j_k) \in \{1, \ldots, m\}^k$ be a multi-index and let $Z_1, Z_2, \ldots, Z_m \in T(\mathbb{R}^N)$ be a set of vector fields. The commutator of length k of $Z_1, Z_2, \ldots, Z_m$ is given by

$$Z_J := [Z_{j_1}, ...[Z_{j_{k-1}}, Z_{j_k}]...]. \tag{2.1.2}$$

When $J = j_1$, we have a commutator of length 1 in the form $Z_J = Z_j$ of $Z_1, Z_2, ...Z_m$. The commutator given in (2.1.2) is called *nested*.

Definition 2.1.2. [Bonfiglioli et al., 2007]**(The Lie algebra spanned by a set)** Assume that $D \subseteq T(\mathbb{R}^N)$ and the smallest sub-algebra of $T(\mathbb{R}^N)$ that contains D is denoted as $\mathrm{Lie}\{D\}$. Specifically, it is denoted by

$$\mathrm{Lie}\{D\} := \bigcap \mathfrak{h}, \quad D \subseteq \mathfrak{h},$$

where $\mathfrak{h}$ is a sub-algebra of $T(\mathbb{R}^N)$. We denote the dimension of the real vector space generated by the Lie algebra $\mathrm{Lie}\{D\}$ at point x as

$$\mathrm{rank}(\mathrm{Lie}\{D\}(x)) := \dim_{\mathbb{R}}\{YI(x) \mid Y \in \mathrm{Lie}\{D\}\}.$$

Example 2.1.3. Consider the vector fields $X_1 = \partial_{x_1} + 2x_2\partial_{x_3}$ and $X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$. It can be shown that their Lie bracket is given by $[X_1, X_2] = -4\partial_{x_3}$. Any other commutator involves X_1 and X_2 more than twice is equal to zero. Thus, we have

$$\mathrm{Lie}\{X_1, X_2\} = \mathrm{span}\{X_1, X_2, [X_1, X_2]\},$$

and it can be verified that

$$\mathrm{rank}(\mathrm{Lie}\{X_1, X_2\}(x)) = 3$$

for all $x \in \mathbb{R}^3$.

The next proposition claims that every element in $\mathrm{Lie}\{\cdot\}$ can be represented as a linear combination of nested brackets.

Proposition 2.1.4. *Consider a set of smooth vector fields D on $\mathbb{R}^N$. Let us define*

$$D_1 := \mathrm{span}\{D\}, \quad D_n := \mathrm{span}\left\{[u, w] \mid u \in D, w \in D_{n-1}\right\},$$

for $n \geq 2$. Then, it follows that

$$\mathrm{Lie}\{D\} = \mathrm{span}\left\{D_n \mid n \in \mathbb{N}\right\}.$$

Furthermore, for every $u \in D_i$, and $w \in D_i$, we have

$$[u, w] \in D_{i+j}.$$

The definition of D_n in the previous proposition implies that every vector field in D_n can be expressed as a linear combination of the nested brackets

$$[w_1 [w_2 [w_3 [\cdots [w_{n-1}, w_n] \cdots]]]],$$

where $w_1, \ldots, w_n \in D$. We can now state another important proposition in this context.

Proposition 2.1.5. *Let $Z_1, \ldots, Z_m$ be a fixed set of vector fields in $T(\mathbb{R}^n)$. Then, we have*

$$\operatorname{Lie}\{Z_1, \ldots, Z_m\} = \operatorname{span}\left\{Z_J \mid \textit{ with } J = (j_1, \ldots, j_k) \in \{1, \ldots, m\}^k, k \in \mathbb{N}\right\}.$$

In this thesis, we introduce the notation $[V_1, V_2]$ for any subsets V_1, V_2 of $T(\mathbb{R}^n)$, where $v_1 \in V_1$ and $v_2 \in V_2$, to denote the span of the commutator $[v_1, v_2]$.

For any subsets V_1, V_2 of $T(\mathbb{R}^n)$ with $v_1 \in V_1$, and $v_2 \in V_2$, we define

$$[V_1, V_2] := \operatorname{span}\{[v_1, v_2]\}.$$

Definition 2.1.6. (Lie group on $\mathbb{R}^N$). Assume $\circ$ is a group operation on $\mathbb{R}^N$ for $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$, such that there is the smooth mapping

$$(x_1, x_2) \mapsto x_2^{-1} \circ x_1,$$

where $x_2^{-1} \circ x_1 \in \mathbb{R}^N$. Then, the pair $\mathbb{G} := \left(\mathbb{R}^N, \circ\right)$ is called a *Lie group* on $\mathbb{R}^N$.

Let $\alpha \in \mathbb{G}$ be fixed. We define the *left-translation* by α on $\mathbb{G}$ as follows:

$$\tau_\alpha(x) := \alpha \circ x.$$

Let X be a smooth vector field on $\mathbb{R}^N$ and let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a smooth function. Then X is said to be *left-invariant* on a Lie group $\mathbb{G}$, if it satisfies the following condition:

$$X\left(\varphi \circ \tau_\alpha\right) = (X\varphi) \circ \tau_\alpha$$

for each $\alpha \in \mathbb{G}$.

The set of all left-invariant vector fields on $\mathbb{G}$ is denoted by $\mathfrak{g}$. It can be easily shown that $\mathfrak{g}$ is closed under linear combinations and the Lie bracket operation, meaning that for any each vector fields $Y, Z \in \mathfrak{g}$, we get $\lambda Y + \beta Z \in \mathfrak{g}$ and $[Y, Z] \in \mathfrak{g}$, for $\lambda, \beta \in \mathbb{R}$.

Example 2.1.7. Consider the vector fields $Y = \partial_{y_1} + 2y_2\partial_{y_3}$ and $Z = \partial_{y_2} - 2y_1\partial_{y_3}$ on $\mathbb{R}^3$. Let $\circ$ be the operation on $\mathbb{R}^3$ defined by

$$y \circ z = \left(y_1 + z_1, y_2 + z_2, y_3 + z_3 + 2\left(y_2 z_1 - y_1 z_2\right)\right),$$

for $y = (y_1, y_2, y_3)$, and $z = (z_1, z_2, z_3)$.

It can be verified that Y and Z are left-invariant vector fields on $\mathbb{R}^3$ with respect to the operation $\circ$. Therefore, $(\mathbb{R}^3, \circ)$ is a Lie group.

Definition 2.1.8. (Homogeneous Lie group on $\mathbb{R}^N$). A Lie group $\mathbb{G} = (\mathbb{R}^N, \circ)$ is said to be homogeneous if there exists a dilation mapping

$$\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_\lambda(x_1, ..., x_N) := (\lambda^{\sigma_1} x_1, ..., \lambda^{\sigma_N} x_N)$$

that is an automorphism of $\mathbb{G}$ for every positive real number λ. Here, the dilation mapping is given by $\sigma = (\sigma_1, ..., \sigma_N)$, where $1 \leq \sigma_1 \leq ... \leq \sigma_N$. The datum of a homogeneous group is denoted by $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$, where $\circ$ is the composition law and $\{\delta_\lambda\}_{\lambda>0}$ is the dilation group.

Note that the dilations $\{\delta_\lambda\}_{\lambda>0}$ establish a single parameter group of automorphisms of a Lie group $\mathbb{G}$ with the identity mapping

$$\delta_1 = I.$$

Moreover, this family forms a one-parameter group of automorphisms of $\mathbb{G}$, where we have

$$\delta_{rs}(x) = \delta_r\left(\delta_s(x)\right) \quad \text{for} \quad r, s > 0, \forall x \in \mathbb{G}.$$

and

$$(\delta_\lambda)^{-1} = \delta_{\lambda^{-1}}.$$

Additionally, for any $x, z \in \mathbb{G}$ with $\lambda > 0$, we have that

$$\delta_\lambda(x \circ z) = (\delta_\lambda x) \circ (\delta_\lambda z).$$

If we denote the identity of $\mathbb{G}$ by e, then we have

$$\delta_\lambda(e) = e.$$

As an example, the Heisenberg–Weyl group $\mathbb{H}^1 = (\mathbb{R}^3, \circ)$ is a homogeneous Lie group, where the dilation is given by

$$\delta_\lambda\left(x_1, x_2, x_3\right) = \left(\lambda x_1, \lambda x_2, \lambda^2 x_3\right).$$

Definition 2.1.9. (Stratified group). A homogeneous Lie group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is said to be a stratified group if it satisfies the following two properties:

- $\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$ is split of $\mathbb{R}^N$ for some natural number $N = N_1 + N_2 + \ldots + N_r$, and the dilation $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ is an automorphism of the group $\mathbb{G}$ given by

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, x^{(2)}, \ldots, x^{(r)}) := (\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)})$$

  for every $\lambda > 0$, where $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 1, 2, \ldots, r$.

It follows from Definition 2.1.8 that $(\mathbb{R}^N, \circ, \delta_\lambda)$ is a homogeneous Lie group on $\mathbb{R}^N$.

- Assume the left-invariant vector fields $Z_1, \ldots, Z_{N_1}$ on $\mathbb{G}$ with property $Z_k(0) = \frac{\partial}{\partial x_k}\big|_0$, where $k = 1, \ldots, N_1$. Then, for every $x \in \mathbb{R}^N$, we have

$$\operatorname{rank}\big(\operatorname{Lie}\{Z_1, \ldots, Z_{N_1}\}\big) = N,$$

meaning that the iterated commutators of $Z_1, \ldots, Z_{N_1}$ spans the whole Lie algebra of $\mathbb{G}$.

A homogeneous Lie group $\mathbb{G}$ contains N_1 number of generators and r steps. The sum of the products of these two notations denotes the homogeneous dimension of $\mathbb{G}$ of the form

$$Q = \sum_{k=1}^{r} k N_k.$$

The vector fields $Z_1, \ldots, Z_{N_1}$ are known as *Jacobian generators*, while a basis of $\operatorname{span}\{Z_1, \ldots, Z_{N_1}\}$ is called a *system* of generators of $\mathbb{G}$. Also, the horizontal (or canonical) $\mathbb{G}$-gradient is denoted by

$$\nabla_{\mathbb{G}} = (Z_1, \ldots, Z_{N_1}).$$

We also define the horizontal divergence as

$$\operatorname{div}_{\mathbb{G}} u := \nabla_{\mathbb{G}} \cdot u.$$

Definition 2.1.10. (Sub-Laplacian on a stratified group). Assume that $Z_1, \ldots, Z_{N_1}$ is the set of Jacobian generators for a stratified group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$. Then the *canonical sub-Laplacian on* $\mathbb{G}$ is denoted by the differential operator

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{N_1} Z_j^2.$$

By replacing the vector fields $Z_1, \ldots, Z_{N_1}$ with a basis $Y_1, \ldots, Y_{N_1}$ of span $\{Z_1, \ldots, Z_{N_1}\}$(a system of generators), the resulting operator is called a *sub-Laplacian* of the form

$$\mathcal{L} = \sum_{j=1}^{N_1} Y_j^2.$$

The (horizontal) $\mathcal{L}$-gradient is defined as $\nabla_{\mathcal{L}} = (Y_1, \ldots, Y_{N_1})$, and the *p-sub-Laplacian* (or *horizontal p-Laplacian*) as

$$\mathcal{L}_p f := \operatorname{div}_{\mathbb{G}} \left(|\nabla_{\mathbb{G}} f|^{p-2} \nabla_{\mathbb{G}} f \right), \quad p > 1.$$

Each left-invariant vector field Z_k on a stratified group $\mathbb{G}$ has an explicit form given by

$$Z_k = \frac{\partial}{\partial z'_k} + \sum_{i=2}^{r} \sum_{m=1}^{N_i} a_{k,m}^{(i)} \left(x', \ldots, z^{(i-1)}\right) \frac{\partial}{\partial z_m^{(i)}}. \tag{2.1.3}$$

Example 2.1.11. Consider the Heisenberg–Weyl group $\mathbb{H}^1$ equipped with the vector fields $X_1 = \partial_{x_1} + 2x_2\partial_{x_3}$ and $X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$. The canonical sub-Laplacian is defined as

$$\begin{aligned}
\Delta_{\mathbb{H}^1} &= \sum_{j=1}^{N_1} X_j^2 \\
&= (\partial_{x_1} + 2x_2\partial_{x_3}) \cdot (\partial_{x_1} + 2x_2\partial_{x_3}) + (\partial_{x_2} - 2x_1\partial_{x_3}) \cdot (\partial_{x_2} - 2x_1\partial_{x_3}) \\
&= (\partial_{x_1})^2 + (\partial_{x_2})^2 + 4x_2\partial_{x_1,x_3} - 4x_1\partial_{x_2,x_3} + 4\left(x_1^2 + x_2^2\right)(\partial_{x_3})^2 .
\end{aligned}$$

As an example of a non-canonical sub-Laplacian, we can consider

$$\begin{aligned}
\mathcal{L} &= \sum_{j=1}^{N_1} Y_j^2 \\
&= [(\partial_{x_1} + 2x_2\partial_{x_3}) - (\partial_{x_2} - 2x_1\partial_{x_3})] \cdot [(\partial_{x_1} + 2x_2\partial_{x_3}) - (\partial_{x_2} - 2x_1\partial_{x_3})] \\
&\quad + (\partial_{x_2} - 2x_1\partial_{x_3}) \cdot (\partial_{x_2} - 2x_1\partial_{x_3}) \\
&= (\partial_{x_1})^2 + 2(\partial_{x_2})^2 - 2\partial_{x_1,x_2} + 4(x_1 + x_2)\partial_{x_1,x_3} \\
&\quad + 4\left(x_1^2 + (x_1 + x_2)^2\right)(\partial_{x_3})^2 - 4(x_1 + (x_1 + x_2))\partial_{x_2,x_3}.
\end{aligned}$$

2.1.2 Examples of stratified groups

Example 2.1.12. (Euclidean Group). Consider a homogeneous additive group $\mathbb{E} = (\mathbb{R}^N, +, \delta_\lambda)$, also called the Euclidean group, where the dilations are defined as the usual multiplication

$$\delta_\lambda(x) = \lambda x, \quad \text{for} \quad \lambda > 0.$$

Its Jacobian generators are denoted by $\partial_{x_1}, ..., \partial_{x_N}$ with step $r = 1$. In this group, the well-known classical Laplace operator

$$\Delta = \sum_{j=1}^{N} \partial_{x_j}^2$$

can be defined as the canonical sub-Laplacian. Furthermore, any sub-Laplacian on $\mathbb{E}$ can be expressed as

$$\mathcal{L} = \sum_{j=1}^{N} Y_j^2 = \sum_{j=1}^{N} \left(\sum_{i=1}^{N} w_{j,i}\partial_{x_i} \right)^2 ,$$

where $W = (w_{i,j})_{i,j \leq N}$ denotes a non-singular constant matrix. Thus, we may express the operator $\mathcal{L}$ by

$$\mathcal{L} = \sum_{i,k=1}^{N} \left(\sum_{j=1}^{N} w_{j,i} w_{j,k} \right) \partial_{x_i,x_k} = \sum_{i,k=1}^{N} \left(W^T \cdot W \right)_{i,k} \partial_{x_i,x_k}.$$

Conversely, any positive-definite symmetric matrix $U = (u_{i,k})\, i,k \leq N$ can be written as a square of a non-singular symmetric matrix: $U = W^2$. This leads to $Y_j := \sum_{i=1}^{N} W_{j,i} \partial_{x_i}$, and $L = \sum_{j=1}^{N} Y_j^2$, which is a sub-Laplacian on this additive group.

We emphasize that the only stratified group of N generators and step $r = 1$ is $\mathbb{E}$, and only the Euclidean Laplacian is an elliptic operator and all the other sub-Laplacians on stratified groups are not elliptic.

Example 2.1.13. (Heisenberg group). The Heisenberg group $\mathbb{H}^N$ is a manifold given by $\mathbb{R}^{2N+1} = \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, with the group law

$$y \circ z = (y + y', z + z', t + t' + 2\left(y'z - yz'\right)),$$

where $N \in \mathbb{N}$, $y = (y, z, t) \in \mathbb{H}^N$, $z = (y', z', t') \in \mathbb{H}^N$, and $y, z \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$. Here, $y'z$, and yz' represent scalar products on $\mathbb{R}^N$. The dilation for $\lambda > 0$ is denoted by

$$\delta_\lambda\left(y\right) = (\lambda y, \lambda z, \lambda^2 t).$$

Additionally, the Heisenberg group $\mathbb{H}^N$ has a homogeneous dimension of $Q = 2n + 2$. In fact, a basis of the Lie algebra $\mathfrak{g}$ is provided by the left-invariant vector fields $\{X_j, Y_j\}_{j=1}^{N}$, where

$$X_j = \partial_{x_j} + 2x_j\partial_t, \quad Y_j = \partial_{x_j} - 2x_j\partial_t \quad \text{and} \quad [X_j, Y_j] = \partial_t.$$

Note that $[X_j, Y_j] = \partial_t$ is the only nonzero commutator relation of the basis elements. Therefore, the sub-Laplacian on $\mathbb{H}^N$ is denoted by

$$\mathcal{L}_{\mathbb{H}^N} = \sum_{j=1}^{N} (X_j^2 + Y_j^2),$$

and the corresponding horizontal gradient is given by

$$\nabla_{\mathbb{H}^N} = \left(X_1, \ldots, X_N, Y_1 \ldots Y_N\right).$$

Example 2.1.14. **(K-type Groups.)** Consider a special block-form matrix of the following form:

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & B_r & 0 \end{pmatrix}, \tag{2.1.4}$$

where B_j is a $q_j \times q_{j-1}$ block and has rank q_j, for $j = 1, 2, \ldots, r$.

We select the 0 blocks in (2.1.4) such that the resulting matrix B has dimension $N \times N$. Additionally, we assume that $q_0 \geq q_1 \geq \cdots \geq q_r$ and $q_0 + q_1 + \cdots + q_r = N$. We will now establish that the group $\mathbb{B}$ associated with this matrix is a stratified group, commonly known as a group of K-type (Kolmogorov).

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We will now establish that the group $\mathbb{B}$ associated with this matrix is a stratified group, commonly known as a group of K-type (Kolmogorov).

We observe that $\mathbb{R}^N$ can be split as

$$\mathbb{R}^N = \mathbb{R}^{q_0} \times \mathbb{R}^{q_1} \times \cdots \times \mathbb{R}^{q_r}.$$

For each $\lambda > 0$, we define

$$\mathcal{D}_\lambda x = \mathcal{D}_\lambda \left(x^{(0)}, x^{(1)}, \ldots, x^{(r)}\right) = \left(\lambda x^{(0)}, \lambda^2 x^{(1)}, \ldots, \lambda^{r+1} x^{(r)}\right),$$

where $x^{(i)} \in \mathbb{R}^{q_i}$, $0 \leq i \leq r$. We also define the dilation $\delta_\lambda(t, x) = (\lambda t, \mathcal{D}_\lambda x)$, which is an automorphism of $\mathbb{B}$. For further details, we refer to [Bonfiglioli et al., 2007].

2.1.3 Green's identities for sub-Laplacians

Consider the volume element $d\nu(x)$(or simply dx) on the first stratum of $\mathbb{G}$, which can be defined as

$$d\nu := d\nu(x) = \bigwedge_{j=1}^{N_1} dx_j.$$

Then, the expression of pairs between the left-invariant operators Z_k with respect to Euclidean derivatives, and differential volume element $d\nu(x)$ is given by $\langle Z_k, d\nu \rangle$.

Based on the explicit form of vector fields given in (2.1.3) and Theorem 2.1.15, we can also express the pairing $\langle Z_k, d\nu\rangle$ with respect to differential forms that corresponds to the Euclidean coordinates, such that

$$\langle Z_k, d\nu(x)\rangle = \bigwedge_{j=1, j\neq k}^{N_1} dx_j^{(1)} \bigwedge_{l=2}^{r} \bigwedge_{m=1}^{N_l} h_{l,m},$$

with

$$h_{l,m} = -\sum_{k=1}^{N_1} a_{k,m}^{(l)} \left(x^{(1)}, \ldots, x^{(l-1)}\right) dx_k^{(1)} + dx_m^{(l)}$$

where $m = 1, \ldots, N_l$, and $l = 2, \ldots, r$. Here, $a_{k,m}^{(l)}$ is a similar homogeneous polynomial of degree $l-1$ as that considered in equation (2.1.3).

If a boundary $\partial\Omega$ of a bounded open set $\Omega \subset \mathbb{G}$ is simple and piecewise smooth, then Ω is called an admissible domain.

For an admissible domain $\Omega \subset \mathbb{G}$, we provide the following theorem.

Theorem 2.1.15. ***(Divergence formula).*** *Assume a function $f_k \in C^1(\Omega) \cap C(\bar{\Omega})$ for $k = 1, \ldots, N_1$. Then, for each left-invariant operator Z_k, we have*

$$\int_\Omega Z_k f_k d\nu = \int_{\partial\Omega} f_k \langle Z_k, d\nu\rangle .$$

where $k = 1, \ldots, N_1$, Furthermore, we have

$$\int_\Omega \sum_{k=1}^{N_1} Z_k f_k d\nu = \int_{\partial\Omega} \sum_{k=1}^{N_1} f_k \langle Z_k, d\nu\rangle$$

Theorem 2.1.16. *[Ruzhansky and Suragan, 2019]**(Green's identities for p-sub-Laplacian).*** *Consider an admissible domain $\Omega \subset \mathbb{G}$ on a stratified group $\mathbb{G}$, and let $1 < p < \infty$. Suppose that $v \in C^1(\Omega) \bigcap C(\bar{\Omega})$, and $u, w \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$. Then we have the following identities (also known as Green's identities)*

1.

$$\int_\Omega \left(\left(|\nabla_{\mathbb{G}} u|^{p-2} \widetilde{\nabla} v\right) u + v \mathcal{L}_p u\right) d\nu = \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^{p-2} v \langle \widetilde{\nabla} u, d\nu\rangle$$

with

$$\widetilde{\nabla} u = \sum_{k=1}^{N_1} (Z_k u) Z_k,$$

where Z_k denotes the kth component of the left-invariant vector field.

2.

$$\int_\Omega \left(u\mathcal{L}_p w - w\mathcal{L}_p u + \left(|\nabla_{\mathbb{G}} w|^{p-2} - |\nabla_{\mathbb{G}} u|^{p-2}\right)(\widetilde{\nabla} w)u\right) d\nu$$
$$= \int_{\partial\Omega} \left(|\nabla_{\mathbb{G}} w|^{p-2}\, u\langle \widetilde{\nabla} w, d\nu\rangle - |\nabla_{\mathbb{G}} u|^{p-2}\, w\langle \widetilde{\nabla} u, d\nu\rangle\right).$$

It is important to note that $(\widetilde{\nabla} v)u$ is a scalar, as $\widetilde{\nabla} u$ is a vector field. This can be expressed as:

$$(\widetilde{\nabla} v)u = \tilde{\nabla} vu = \sum_{k=1}^{N_1} (Z_k v)(Z_k u) = \sum_{k=1}^{N_1} Z_k v Z_k u.$$

It should be noted that $(\widetilde{\nabla} v)u$ can be considered both an operator and a vector field. Also, in this work, the boundary condition (1.1.5) leads to the right-hand side of Green's second identity becoming zero.

2.1.4 Sharp Poincaré inequality on stratified groups

In this section, we consider the sharp Poincaré inequality on a bounded open connected set $\Omega \subset \mathbb{G}$ that supports the divergence formula on $\mathbb{G}$. This inequality is a crucial tool in the study of spectral properties of the Dirichlet sub-Laplacian on stratified groups. We start by stating the following theorem:

Theorem 2.1.17. *[Ozawa and Suragan, 2020] Let $\Omega \subset \mathbb{G}$ be an open set that supports the divergence formula (Theorem 2.1.15). For any $\phi \in C^2(\Omega)$ and all $u \in C_0^1(\Omega)$, we have*

$$0 \leqslant \int_\Omega \left|\nabla_{\mathbb{G}} u - \frac{\nabla_{\mathbb{G}}\phi}{\phi} u\right|^2 d\nu = \int_\Omega \left(|\nabla_{\mathbb{G}} u|^2 + \frac{\mathcal{L}\phi}{\phi}|u|^2\right) d\nu, \tag{2.1.5}$$

where the equality case holds if and only if u is proportional to ϕ.

Consider the spectral problem of the sub-Laplacian on stratified groups, also known as the Dirichlet sub-Laplacian:

$$\begin{cases} -\mathcal{L}\phi(x) = \lambda(\Omega)\phi(x), & x \in \Omega \subset \mathbb{G}, \\ \phi(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.1.6}$$

The Dirichlet sub-Laplacian is a positive self-adjoint operator that exhibits a spectral gap for a bounded, open, connected set Ω. Specifically, $\lambda_1(\Omega) > 0$ denotes the smallest eigenvalue, which is also a simple eigenvalue. Furthermore, in [Carfagnini and Gordina, 2023, Theorem 3.11], the authors establish the existence of a corresponding eigenfunction ϕ such that $\phi(x) > 0$ for all $x \in \Omega$. Therefore, since $\phi > 0$ and $\lambda_1(\Omega) > 0$ satisfy the system (2.1.6), we have $\frac{\mathcal{L}\phi}{\phi} = -\lambda_1(\Omega)$.

Consequently, by considering the right-hand side of the inequality in Theorem 2.1.17, we obtain an estimate known as the *sharp Poincaré inequality* on $\Omega \subset \mathbb{G}$:

$$\int_{\Omega} |u|^2 d\nu \leqslant \frac{1}{\lambda_1(\Omega)} \int_{\Omega} |\nabla_{\mathbb{G}} u|^2 d\nu, \quad \text{for any} \quad u \in C_0^1(\Omega). \tag{2.1.7}$$

In this direction, researchers [Karazym and Suragan, 2023] have investigated the use of the p-sub-Laplacian operator $\mathcal{L}_p$ in studying spectral problems on stratified groups. Specifically, the authors considered the Dirichlet p-sub-Laplacian spectral problem

$$\begin{cases} -\mathcal{L}_p u(x) = \lambda(\Omega)|u(x)|^{p-2}u(x), & x \in \Omega \subset \mathbb{G}, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.1.8}$$

and proved that there is the Poincaré inequality

$$\lambda_{1,p}(\Omega) \int_{\Omega} |u|^p dx \leq \int_{\Omega} |\nabla_{\mathbb{G}} u|^p dx \tag{2.1.9}$$

for the first eigenvalue $\lambda_{1,p}(\Omega) > 0$ with $1 < p < \infty$.

2.1.5 Fractional Sobolev spaces

If f is measurable on a stratified group $\mathbb{G}$, then its distribution function $\lambda_f : [0, \infty] \to [0, \infty]$ is given by

$$\lambda_f(\beta) := |\{y : |f(y)| > \beta\}|.$$

Definition 2.1.18. (Gagliardo semi-norm). We define a Gagliardo semi-norm as:

$$[u]_{s,p,\Omega} := \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|y^{-1} \circ x|^{Q+ps}} dx dy \right)^{\frac{1}{p}} < \infty,$$

where $0 < s < 1$, and $1 < p < \infty$.

Definition 2.1.19. (Fractional Sobolev spaces). The fractional Sobolev space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : [u]_{s,p,\Omega} < \infty\}, \tag{2.1.10}$$

equipped with

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + [u]_{s,p,\Omega},$$

where $0 < s < 1$, and $1 < p < \infty$.

Additionally, the space $W_0^{s,p}(\Omega)$ is employed to denote the completion of the space of differentiable functions $C_0^\infty(\Omega)$ equipped with the norm $\|u\|_{W^{s,p}(\Omega)}$. It should be noted that $W_0^{s,p}(\Omega)$ is equivalent to $W^{s,p}(\Omega)$.

Furthermore, we use the notation $H_0^{s,p}(\Omega)$ to represent the closure of $C_0^\infty(\Omega)$ under the norm $\|u\|_{L^p(\Omega)} + [u]_{s,p,\mathbb{G}}$, where Ω denotes an open bounded subset of $\mathbb{G}$. It can be observed that $H_0^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ are not similar even in Euclidean space (see [Di Nezza et al., 2012]).

The quasi-ball of radius r centered at $x \in \mathbb{G}$ is denoted by

$$D(x,r) = \left\{y \in \mathbb{G} : \left|y^{-1} \circ x\right| < r\right\}$$

and

$$D_r \subset \mathbb{G} \backslash \Omega.$$

Lemma 2.1.20. *[Ghosh et al., 2023] If $p > 1$, then $H_0^{s,p}(\Omega)$ is a reflexive Banach space.*

We can write

$$|u(x)|^p = \left|y^{-1} \circ x\right|^{Q+ps} \frac{|u(x) - u(y)|^p}{\left|y^{-1} \circ x\right|^{Q+ps}}$$

for every $u \in C_0^\infty(\Omega)$, $x \in \Omega$, and $y \in D_r$, which, after integrating both sides with respect to y and x, yields

$$\int_\Omega |u(x)|^p dx \leq \frac{\operatorname{diam}\left(D_r \cup \Omega\right)^{Q+ps}}{|D_r|} \int_\Omega \int_{D_r} \frac{|u(x) - u(y)|^p}{\left|y^{-1} \circ x\right|^{Q+ps}} dxdy.$$

We can then define

$$C = C(Q,s,p,\Omega) = \inf \left\{ \frac{\operatorname{diam}(D \cup \Omega)^{Q+ps}}{|D|} : \text{ with a ball } \quad D \subset \mathbb{G} \backslash \Omega \right\}$$

and obtain the *Poincaré type inequality* of the form

$$\int_\Omega |u(x)|^p dx \leq C \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u(x) - u(y)|^p}{\left|y^{-1} \circ x\right|^{Q+ps}} dxdy.$$

Alternatively, we have

$$\|u\|_{L^p(\Omega)}^p \leq C[u]_{s,p,\mathbb{G}}^p.$$

We can express the space $H_0^{s,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm $[u]_{s,p,\mathbb{G}}$. Hence, we have the equivalence

$$\|u\|_{H_0^{s,p}(\Omega)} \cong [u]_{s,p,\mathbb{G}} \text{ for all } u \in H_0^{s,p}(\Omega).$$

The space $H_0^{s,p}(\Omega)$ is defined by imposing the boundary condition $u = 0$ in $\mathbb{G}\backslash\Omega$ for functions u in $H_0^{s,p}(\Omega)$. Note that the inclusion map $j : H_0^{s,p}(\Omega) \to W^{s,p}(\mathbb{G})$ is well-defined and continuous. In general, we have

$$H_0^{s,p}(\Omega) \subset \{u \in W^{s,p}(\mathbb{G}) : u = 0 \text{ in } \mathbb{G}\backslash\Omega\}.$$

Furthermore, if Ω is a bounded domain with at least continuous boundary $\partial\Omega$, then we can represent $H_0^{s,p}(\Omega)$ as

$$H_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{G}) : u = 0 \text{ in } \mathbb{G}\backslash\Omega\}.$$

This means that the space $H_0^{s,p}(\Omega)$ is defined as the subspace of functions in $W^{s,p}(\mathbb{G})$ vanishes outside of Ω.

Theorem 2.1.21. *[Ghosh et al., 2023] Assume that $p > 1$, $s \in (0,1)$, and $\mathbb{G}$ is a stratified Lie group with homogeneous dimension Q. Suppose that $p^*_{s,Q} := \frac{Qp}{Q-sp}$ with $Q > sp$. Then the following properties hold:*

- *If $\Omega \subset \mathbb{G}$ is an open subset, the space $H_0^{s,p}(\Omega)$ continuously embedded in $L^r(\Omega)$ for $p \le r \le p^*_{s,Q}$. In other words, for all $u \in H_0^{s,p}(\Omega)$, there exists a positive constant $C = C(Q,s,p,\Omega)$ such that*

$$\|u\|_{L^r(\Omega)} \le C\|u\|_{H_0^{s,p}(\Omega)}.$$

- *If Ω is bounded, then the embedding $H_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for all $r \in [1, p^*_{s,Q})$, and continuous for $r \in [1, p^*_{s,Q}]$.*

Definition 2.1.22. Assume that $p > 1$, and $s \in (0,1)$. The fractional p-sub-Laplacian is defined as

$$(-\Delta_{p,\mathbb{G}})^s u(x) := C_{Q,s,p} P.V. \int_{\mathbb{G}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|y^{-1} \circ x|^{Q+ps}} dy, \quad x \in \mathbb{G}, \quad (2.1.11)$$

where $P.V.$ denotes the Cauchy principal value, and $C_{Q,s,p}$ is a positive constant that depends on Q, s, and p.

Note that the $s = 1$ case of this definition corresponds to the sub-Laplacian $\mathcal{L}_p$ on $\mathbb{G}$.

For any $\varphi \in H_0^{s,p}(\Omega)$, we define the inner product as follows:

$$\langle (-\Delta_{p,\mathbb{G}})^s u, \varphi \rangle = \iint_{\mathbb{G}\times\mathbb{G}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|y^{-1} \circ x|^{Q+ps}} dxdy. \quad (2.1.12)$$

2.1.6 Fractional p-sub-Laplacian eigenvalue problem

Consider the following equation on a stratified Lie group $\mathbb{G}$:

$$\begin{cases} (-\Delta_{p,\mathbb{G}})^s u = \lambda |u|^{p-2}u, \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{G}\backslash\Omega, \end{cases} \tag{2.1.13}$$

Here, Ω represents bounded domain in $\mathbb{G}$, and $\lambda \in \mathbb{R}$. We refer to (2.1.13) as the *fractional p-sub-Laplacian eigenvalue problem.*

Definition 2.1.23. For every $\varphi \in C_0^\infty(\Omega)$, weak solution $u \in H_0^{s,p}(\Omega)$ to the system (2.1.13) is given by

$$\langle (-\Delta_{p,\mathbb{G}})^s u, \varphi \rangle = \lambda \int_\Omega |u|^{p-2} u \, \varphi \, dx, \tag{2.1.14}$$

where a nontrivial solution to (2.1.14) is referred to as a *p-sub-Laplacian eigenfunction* corresponding to the eigenvalue λ.

We introduce the eigenfunction $u_1 \in \widetilde{H}_0^{s,p}(\Omega)\backslash\{0\}$, which corresponds to the first eigenvalue $\lambda_{1,p,s}(\Omega)$. We define the weak solution u_1 of (2.1.14), which satisfies the following equation for all $\varphi \in \widetilde{H}_0^{s,p}(\Omega)$:

$$\begin{aligned} \iint_{\mathbb{G}\times\mathbb{G}} & \frac{|u_1(x) - u_1(y)|^{p-2}(u_1(x) - u_1(y))}{|y^{-1} \circ x|^{Q+ps}} (\varphi(x) - \varphi(y)) dx dy \\ & = \lambda_{1,p,s}(\Omega) \int_\Omega |u_1|^{p-2} u_1 \varphi dx, \end{aligned} \tag{2.1.15}$$

The eigenfunctions mentioned here are closely linked to the minimization problem of the Rayleigh quotient, denoted as $\mathcal{R}$, which is defined as follows

$$\mathcal{R}(u) := \frac{\iint_{\mathbb{G}\times\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}\circ x|^{Q+ps}} dx dy}{\int_\Omega |u(x)|^p dx}, \quad u \in C_0^\infty(\Omega).$$

Note that the minimizer of the Rayleigh quotient preserves its sign. This can be inferred from the triangle inequality, which implies that for

$$|u(x) - u(y)| > ||u(x)| - |u(y)||,$$

where $u(x)$ and $u(y)$ have a opposite sign and $u(x)u(y) < 0$.

Let us introduce the space $\mathcal{W}$ as the set of functions that belong to $H_0^{s,p}(\Omega)$ and satisfy the condition $\|u\|_p = 1$, that is,

$$\mathcal{W} = \{u \in H_0^{s,p}(\Omega), \quad \text{such that} \quad \|u\|_p = 1\}.$$

Note that the eigenfunctions of the system (2.1.13) are the minimizers of the energy functional on $\mathcal{W}$ of the form:

$$I(u) = \iint_{\mathbb{G}\times\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}\circ x|^{Q+ps}} dxdy.$$

We can define the first eigenvalue λ_1 over Ω as either

$$\lambda_1(\Omega) = \inf\left\{\mathcal{R}(\phi) : \phi \in C_0^\infty(\Omega)\right\},$$

or

$$\lambda_1(\Omega) = \inf\{I(u) : u \in \mathcal{W}\}.$$

Based on Theorem 2.1.21, we can express the Sobolev inequality as

$$\left(\int_\Omega |u(x)|^{p^*_{s,Q}} dx\right)^{\frac{1}{p^*_{s,Q}}} \leq C(Q,p,s)\left(\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}\circ x|^{Q+ps}} dxdy\right)^{\frac{1}{p}},$$

where we select the exponents $\frac{p^*_{s,Q}}{p}$ and $\frac{p^*_{s,Q}}{p^*_{s,Q}-p}$ to use the Hölder inequality.

Moreover, the first eigenvalue $\lambda_1(\Omega)$ of the system (2.1.13) is positive since

$$\lambda_{Q,p,s}(\Omega)\int_\Omega |u(x)|^p dx \leq \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}\circ x|^{Q+ps}} dxdy, \tag{2.1.16}$$

where $\lambda_{Q,p,s}(\Omega) = C(Q,p,s)^{-p}|\Omega|^{-\frac{ps}{Q}} > 0$.

Moreover, it is worth mentioning that the eigenvalues of the system (2.1.13) are strictly positive. Moreover, the weak solution of (2.1.13) associated with $\lambda = \lambda_1$ is commonly known as the *first eigenfunction* of the system. The following theorem presents some noteworthy properties of the principal eigenvalue and eigenfunction.

Theorem 2.1.24. *[Ghosh et al., 2023] Assume that $p > 1$, $s \in (0,1)$, and $\Omega \subset \mathbb{G}$ is a bounded domain of $\mathbb{G}$ with homogeneous dimension Q, such that $Q > sp$. Then the problem (2.1.14) exhibits the following properties:*

- *The first eigenfunction is strictly positive;*
- *The first eigenvalue λ_1 is simple, and isolated.*

2.2 Fractional derivatives

2.2.1 Riemann Liouville integral

Lemma 2.2.1. *Let $f(t)$ be a function, and $n \in \mathbb{N}$. Then, the n-times repeated integral of the closed form, also known as Cauchy's formula for repeated integration, is given by*

$$\int_c^t \int_c^{t_1} \int_c^{t_2} \cdots \int_c^{t_{n-2}} \int_c^{t_{n-1}} f(t_n)\, \mathrm{d}t_n\, \mathrm{d}t_{n-1} \ldots\, \mathrm{d}t_2\, \mathrm{d}t_1$$
$$= \frac{1}{(n-1)!} \int_c^t (t-s)^{n-1} f(s) \mathrm{d}s,$$

where c is a lower limit of each integral.

Definition 2.2.2. (Riemann-Liouville fractional integral). For a function $f(t)$, the Riemann-Liouville fractional integral of order ν with respect to t and with any integration constant c is defined as follows:

$$ {}_c^{RL}I_t^{\nu} f(t) := \frac{1}{\Gamma(\nu)} \int_c^t (t-s)^{\nu-1} f(s) \mathrm{d}s, \quad \mathrm{Re}(\nu) > 0, \tag{2.2.1}$$

for general $\nu \in \mathbb{C}$.

Remark 2.2.3. The fractional integral (2.2.1) is well-defined for the finite constant c if there exists $\int_c^d |f(t)| \mathrm{d}t$ for some interval $[c, d]$ with $t \in [c, d]$.

Remark 2.2.4. The fractional integral (2.2.1) is well-defined for the infinite case $c = -\infty$ if $f \in L^1(-\infty, K]$ and its decay is provided by $f(t) = O(t^{-\nu-\epsilon})$ as $t \to -\infty$ for some $\epsilon > 0$.

2.2.2 Riemann-Liouville derivative

Fractional integrals to order ν are defined for any complex number ν with $\mathrm{Re}(\nu) > 0$. The difference between integrals and derivatives is denoted by the sign of the order ν. Since ν is complex, it is defined by the sign of its real part.

It is well-known that differentiating integrals yields derivatives. Given a function $\int_c^t f(s)ds$, the first differentiation is $f(t)$, and the function is twice differentiable if $f'(t)$ exists. One can extend this idea to define fractional derivatives by differentiating fractional integrals. For example,

$$\frac{\mathrm{d}^{1/5}}{\mathrm{d}t^{1/5}} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}^{-4/5}}{\mathrm{d}t^{-4/5}} f(t) \right),$$
$$\frac{\mathrm{d}^{3.6}}{\mathrm{d}t^{3.6}} f(t) = \frac{\mathrm{d}^4}{\mathrm{d}t^4} \left(\frac{\mathrm{d}^{-0.4}}{\mathrm{d}t^{-0.4}} f(t) \right), \quad \cdots$$

This idea leads to the following definition:

Definition 2.2.5. (Riemann-Liouville fractional derivative). The fractional derivative of a function $f(t)$ with respect to t of order $\nu \in \mathbb{C}$ with non-negative real part is defined as

$$ {}_{c}^{RL}D_t^{\nu} f(t) := \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left({}_{c}^{RL}I_t^{n-\nu} f(t) \right), \quad n := \lfloor \mathrm{Re}(\nu) \rfloor + 1, \quad \mathrm{Re}(\nu) \geq 0, \tag{2.2.2} $$

where $\lfloor v \rfloor$ denotes the largest integer less than or equal to v. The constant of integration c is arbitrary, and we choose $n \in \mathbb{N}$ such that $n - \nu$ has a positive real part. We assume that f is chosen so that the formula is well-defined.

Using the Fundamental Theorem of Calculus, Definitions 2.2.2 and 2.2.5 provide a framework for fractional differentiation or integration to any order in $\mathbb{C}$. It is important to note that for all $\nu \in \mathbb{C}$, the agreement holds:

$$ {}_{c}^{RL}D_t^{-\nu} f(t) = {}_{c}^{RL}I_t^{\nu} f(t), $$

which allows us to use the definitions of ${}_{c}^{RL}I_t^{\nu} f(t)$ for $\mathrm{Re}(\nu) > 0$ and ${}_{c}^{RL}D_t^{\nu} f(t)$ for $\mathrm{Re}(\nu) \geq 0$ to denote both ${}_{c}^{RL}I_t^{\nu} f(t)$ and ${}_{c}^{RL}D_t^{\nu} f(t)$.

2.2.3 Caputo derivative

Let us now consider one of the popular alternative definitions of the Riemann-Liouville model.

Definition 2.2.6. In Caputo fractional calculus, formula (2.2.1) is used to define fractional integrals, similar to Riemann- Liouville fractional calculus. For a function $f(t)$ with respect to t, the fractional derivative to order ν (where $\nu \in \mathbb{C}$) with constant of integration c, is denoted with non-negative real part as follows:

$$ {}_{c}^{C}D_t^{\nu} f(t) := {}_{c}^{RL}I_t^{n-\nu} \left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t) \right), \quad n := \lfloor \mathrm{Re}(\nu) \rfloor + 1, \quad \mathrm{Re}(\nu) \geq 0, \tag{2.2.3} $$

where t, c, and f are such that the formula is well-defined. Note that n is the same as in the Riemann-Liouville derivative (2.2.2). However, in this case, a fractional integral of an ordinary derivative is taken, rather than an ordinary derivative of a fractional integral.

Let us consider important remarks about the Caputo model and the Riemann-Liouville model.

Remark 2.2.7. The Caputo derivative of any constant is zero.

Remark 2.2.8. The Riemann-Liouville derivative is the analytic continuation in ν of the fractional integral (2.2.1), not the Caputo derivative since analytic continuations are unique.

Remark 2.2.9. The principal difference between Caputo and Riemann-Liouville is their different behaviors for initial value problems. This distinction is essential in the study of fractional differential equations.

2.2.4 Caputo derivative of order $0 < \nu < 1$

In this work, we focus on the Caputo derivative of order $0 < \nu < 1$, as defined in Definition 2.2.6. This particular case is important for our analysis with $n = 1$ and $c = 0$ in Definition 2.2.6. The definition of the Caputo-type derivative for $0 < \nu < 1$ is given below:

Definition 2.2.10. For $0 < \nu < 1$ and $t > 0$, the Caputo fractional derivative is defined by

$$ {}^{C}\partial_t^{\nu} f(t) := {}^{C}D_t^{\nu} f(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t (t-s)^{-\nu} \frac{\mathrm{d}}{\mathrm{d}s} f(s)\mathrm{d}s. \tag{2.2.4} $$

We will now present two important theorems related to the Caputo fractional derivative of order $0 < \nu < 1$.

Theorem 2.2.11. *[Dorsaf et al., 2014, Page 49, Part 2] Consider the problem*

$$ \begin{cases} {}^{C}\partial_t^{\nu} f(t) = f(t)\big(f(t)+1\big), \\ f(0) > 0 \end{cases} \tag{2.2.5} $$

with $0 < \nu < 1$. Then f blows up in a finite time.

Theorem 2.2.12. *[Vergara and Zacher, 2008, Alikhanov, 2010] For any absolutely continuous function $f(t)$ on $[0, T]$ with $0 < \nu < 1$, the following inequality holds:*

$$ {}^{C}\partial_t^{\nu} f^2(t) \leqslant 2f(t) \cdot {}^{C}\partial_t^{\nu} f(t). $$

The Mittag-Leffler function is an essential function in the analysis of fractional differential equations. The power series $E_\alpha(t)$ of the form

$$ E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \quad \operatorname{Re}\alpha > 0, $$

is called the *Mittag-Leffler* function. In light of the outcomes acquired in Theorem 1.1.2, it is crucial to present the following property given in [Simon, 2014]:

$$E_\alpha(-t) \lesssim \frac{1}{1+t}, \quad t > 0.$$

Finally, we present a theorem related to the solution of fractional differential equations.

Theorem 2.2.13. *[Kilbas et al., 2006, Section 4.1.3] Consider the Cauchy problem for FDEs*

$$\begin{cases} {}^C\partial_t^\nu f(t) - \lambda f(t) = 0 \\ f(0) = f_0 \in \mathbb{R} \end{cases} \tag{2.2.6}$$

with $t > 0$, and $\lambda \in \mathbb{R}$. Then the solution of (2.2.6) can be expressed as the multiplication of the initial value by the Mittag-Leffler function, as follows:

$$f(t) = f_0 E_\nu[\lambda t^\nu],$$

where $0 < \nu < 1$.

Chapter 3

Fractional Fisher-KPP equation on the Euclidean space

3.1 Introduction

As discussed in Section 2.1.2, the Euclidean space $\mathbb{R}^N$ with addition operation is a prime example of a stratified group $\mathbb{G}$. In light of this, we investigate the fractional problem (1.1.3)-(1.1.5) on $\mathbb{R}^N$ and establish the validity of theorems 1.1.1-1.1.3 for this specific case in the following subsections 3.2-3.7. The derived results cover the previous works such as [Ahmad et al., 2015, Alsaedi et al., 2021], where the authors analyzed problem (3.1.1)-(3.1.4) when the regional fractional Laplacian with $p = 2$ was applied. Also, for other similar studies in this direction for time-space-fractional diffusion equations, we refer to sources [Abadias and Álvarez, 2018, Alvarez et al., 2019, Cao et al., 2019]. The study of the fractional version of the Fisher-KPP model in $\mathbb{R}^N$ is important since it provides a valuable framework to deal with real-world problems. For example, solutions for the classical Fisher-KPP equation correspond to expected values for particles moving under a Brownian process. The fractional Laplacian with $s \in (0, 1)$ serves as the operator for the generator for a stable Lévy process-a jump process. It is plausible to anticipate that the existence of such jumps (or flights) in the diffusion process will accelerate the progression of the invasion from the unstable state $(u = 0)$ to the stable state $(u = 1)$[Cabré and Roquejoffre, 2013].

This chapter draws heavily on our recent work [Jabbarkhanov and Suragan, 2023].

In this chapter, we examine two fractional Fisher-KPP equations, represented by equations

$$u_t + (-\Delta_p)^s u = u(u - 1) \quad \text{in} \quad \Omega \subset \mathbb{R}^N, \; t \in (0, \infty), \tag{3.1.1}$$

and

$$ {}^{C}\partial_t^{\nu} u + (-\Delta_p)^s u = u(u-1) \quad \text{in} \quad \Omega \subset \mathbb{R}^N,\ t \in (0,\infty), \tag{3.1.2} $$

where $(-\Delta_p)^s$(or $(-\Delta_{p,\mathbb{R}^N})^s$) is the fractional p-Laplacian with $p \in (1,\infty)$, $s \in (0,1)$, and ${}^{C}\partial_t^{\nu}$ is the fractional Caputo type derivative with $\nu \in (0,1)$. Additionally, we consider the initial condition

$$ u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \tag{3.1.3} $$

and the boundary condition

$$ u(x,t) = 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Omega,\ t \geq 0. \tag{3.1.4} $$

Although the primary concepts and tools are initially presented in Chapter 2, we start with introducing certain concepts within the Euclidean space $\mathbb{R}^N$, such as the fractional Sobolev space, the nonlinear Dirichlet eigenvalue problem, and the Poincaré inequality.

We define the operator

$$ (-\Delta_p)^s u(x) := 2 \lim_{\varepsilon \to 0^+} \int_{\{y \in \mathbb{R}^N : |y-x| \geq \varepsilon\}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy, $$

which is a generalization of the fractional Laplacian for the case $p = 2$. One of the motivations for using the fractional Laplacian instead of the Laplacian operator in equation (3.1.1) is to describe anomalous diffusion processes.

We denote the fractional Sobolev space $\widetilde{H}_0^{s,p}(\Omega)$ as the completion of $C_0^{\infty}(\Omega)$ to the Gagliardo seminorm of the form:

$$ \|u\|_{\widetilde{H}_0^{s,p}(\Omega)} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}, \quad u \in C_0^{\infty}(\Omega). $$

Definition 3.1.1. Let Ω be a bounded open set in $\mathbb{R}^N$ with $N \geq 1$, and let $\widetilde{H}_0^{s,p}(\Omega)$ denote the fractional Sobolev space. For $0 < \nu < 1$ $[\nu = 1]$, we say that a function a function u in (3.1.2)[(3.1.1)] is a weak solution if it satisfies the equation

$$ \begin{aligned} &\int_{\Omega} {}^{C}\partial_t^{\nu} u(x,t)\varphi(x)dx \\ &\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{N+ps}} (\varphi(x)-\varphi(y))dxdy \\ &\quad = \int_{\Omega} u(x,t)(u(x,t)-1)\varphi(x)dx, \end{aligned} \tag{3.1.5} $$

for all $\varphi \in \widetilde{H}_0^{s,p}(\Omega)$ and for pointwise almost everywhere $t \in [0,\infty)$.

Case $\nu = 1$ is understood as the first-order partial derivative ∂_t.

For any $u \in C_0^\infty(\Omega)$, the first (smallest) eigenvalue $\lambda = \lambda_{1,p,s}$, corresponding to the nonlinear Dirichlet eigenvalue problem given by

$$\begin{cases} (-\Delta_p)^s u(x,t) = \lambda |u(x,t)|^{p-2} u(x,t) & \text{in } \Omega, \\ u(x,t) = 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.1.6}$$

coincides with the sharp constant in the Poincaré inequality for the Gagliardo seminorm. This inequality is defined as follows [Brasco et al., 2014, Brasco and Parini, 2016]:

$$\lambda_{1,p,s}(\Omega) \int_\Omega |u(x)|^p dx \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy. \tag{3.1.7}$$

Here, Ω denotes an open bounded subset of $\mathbb{R}^N$, $1 < p < \infty$, and $s \in (0,1)$ is the fractional order of the derivative.

We introduce the eigenfunction $u_1 \in \widetilde{H}_0^{s,p}(\Omega) \backslash \{0\}$, which corresponds to the first eigenvalue $\lambda_{1,p,s}(\Omega)$. We say that u_1 satisfies (3.1.6) weakly if it satisfies the following equation for all $\varphi \in \widetilde{H}_0^{s,p}(\Omega)$:

$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} & \frac{|u_1(x) - u_1(y)|^{p-2}(u_1(x) - u_1(y))}{|x-y|^{N+sp}} (\varphi(x) - \varphi(y)) dxdy \\ &= \lambda_{1,p,s}(\Omega) \int_\Omega |u_1|^{p-2} u_1 \varphi dx, \end{aligned} \tag{3.1.8}$$

In [Alsaedi et al., 2021], the equation (3.1.1) with the initial-boundary conditions (3.1.3)-(3.1.4) was investigated for the regional fractional Laplacian with $p = 2$ and a time-fractional derivative. However, it was shown in [Brasco and Parini, 2016, Remark 2.4] and [Chowdhury et al., 2021, Theorem 1.4] that the localized Gagliardo seminorm does not satisfy a Poincaré inequality when $s \leq 1/p$, which may require additional restrictions on p and s. Notably, some results in this chapter also cover the "regional" versions of the problem.

Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $u \in C_0^\infty(\Omega)$ be arbitrary. Then, there exists a positive constant $C = C(n, s, p, \Omega)$ such that the inequality

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy \leq C \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy,$$

holds for $p \in (1, \infty)$ and $s \in (1/p, 1)$. This inequality follows from [Loss and Sloane, 2010] and can be proven by imitating the proof of [Jin et al., 2021, Lemma 4]. Using this equivalence, we can extend certain results to the regional fractional p-Laplacian case, provided that $1/p < s < 1$.

Finally, we provide a simple and useful observation.

Lemma 3.1.2. *Suppose* $\tilde{v}(\cdot) = \min(v(\cdot), 0)$ *for all* x, y *in the domain of* v*. Then*

$$0 \leq (v(x) - v(y))(\tilde{v}(x) - \tilde{v}(y)).$$

Proof. Notice that $\tilde{v}(x) = \min(v(x), 0) = \frac{1}{2}(v(x) - |v(x)|)$.

Suppose $U := v(x) - v(y) \geq 0$ with $U \geq W$ and $W := |v(x)| - |v(y)|$. Hence, we have $U^2 - UW \geq 0$. Therefore, $(v(x) - v(y))(\tilde{v}(x) - \tilde{v}(y)) = \frac{1}{2}(U^2 - UW) \geq 0$. □

3.2 Positivity and boundedness of the solution

In this section, we establish a positivity and boundedness result for the global solution of the fractional Fisher-KPP equation (3.1.1) with the initial-boundary conditions (3.1.3)-(3.1.4), assuming the existence of a unique global and bounded weak solution to (3.1.1). We remark that a similar result was considered in [Gal and Warma, 2020, Theorem 3.7.1] (cf. [Coulhon and Hauer, 2016, Section 6.2.2]), where u admits a unique strong solution. In contrast, we present a different approach to obtain a global solution of the problem.

Theorem 3.2.1. *Let* $T \in (0, \infty)$ *and* $u_0 \in L^\infty(\Omega)$ *satisfy* $0 \leq u_0(x) \leq 1$*. For all* $(x,t) \in \Omega \times (0,T)$*, the global solution* u *of* (3.1.1) *with the initial-boundary conditions* (3.1.3)-(3.1.4) *satisfies*

$$0 \leq u(x,t) \leq 1.$$

Proof. To show that the solution u is non-negative, we consider the function

$$\tilde{u}(x,t) := \min(u(x,t), 0).$$

We substitute $\tilde{u}(x,t)$ for φ in (3.1.5) with $\nu = 1$ for t, noting that the equality holds pointwise almost everywhere $t \in [0, \infty)$. This yields

$$\begin{aligned} &\int_\Omega u_t(x,t)\tilde{u}(x,t)dx \\ &\qquad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))}{|x-y|^{N+sp}}(\tilde{u}(x,t) - \tilde{u}(y,t))dxdy \\ &\qquad = \int_\Omega u(x,t)(u(x,t) - 1)\tilde{u}(x,t)dx. \end{aligned}$$

Using Lemma 3.1.2, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))}{|x-y|^{N+sp}}(\tilde{u}(x,t) - \tilde{u}(y,t))dxdy \geq 0,$$

which leads to the inequality

$$\int_\Omega u_t(x,t)\tilde{u}(x,t)dx \le \int_\Omega u(x,t)(u(x,t)-1)\tilde{u}(x,t)dx.$$

Furthermore, it is easy to check that

$$\int_\Omega u(u-1)\tilde{u}dx \le -\int_\Omega u\tilde{u}dx. \tag{3.2.1}$$

Thus, we have

$$\int_\Omega u_t(x,t)\tilde{u}(x,t)dx \le -\int_\Omega u(x,t)\tilde{u}(x,t)dx,$$

which is equivalent to the estimate

$$\frac{d}{dt}\int_\Omega |\tilde{u}(x,t)|^2dx + 2\int_\Omega |\tilde{u}(x,t)|^2dx \le 0.$$

Since $\tilde{u}(x,0)=0$, we can deduce that $\int_\Omega |\tilde{u}(x,t)|^2dx = 0$. Consequently, $\tilde{u}(x,t)\equiv 0$, and $u \ge 0$ for $(x,t)\in\Omega\times(0,T)$.

To show that $u \le 1$, we use of the (anti-)invariance property of the fractional operator, which implies that if u satisfies (3.1.5), then $v := 1-u$ also satisfies the same equation. For a pointwise almost everywhere $t\in[0,T]$, we have

$$\begin{aligned}\int_\Omega & v_t(x,t)\varphi(x)dx \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x,t)-v(y,t)|^{p-2}(v(x,t)-v(y,t))}{|x-y|^{N+ps}}(\varphi(x)-\varphi(y))dxdy \\ &\qquad = \int_\Omega v(x,t)(1-v(x,t))\varphi(x)dx.\end{aligned} \tag{3.2.2}$$

Now, let $\hat{v} := \min(v,0)$ for a given t. Then, replacing φ with $\hat{v}$ in equation (3.2.2) yields

$$\begin{aligned}\int_\Omega & v_t(x,t)\hat{v}(x,t)dx \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x,t)-v(y,t)|^{p-2}(v(x,t)-v(y,t))}{|x-y|^{N+ps}}(\hat{v}(x,t)-\hat{v}(y,t))dxdy \\ &\qquad = \int_\Omega v(x,t)(1-v(x,t))\hat{v}(x,t)dx.\end{aligned} \tag{3.2.3}$$

Using Lemma 3.1.2, it follows that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x,t)-v(y,t)|^{p-2}(v(x,t)-v(y,t))}{|x-y|^{N+ps}}(\hat{v}(x,t)-\hat{v}(y,t))dxdy \ge 0.$$

Thus, we obtain

$$\int_{\Omega} \hat{v}_t(x,t)\hat{v}(x,t)dx \leq \int_{\Omega} \hat{v}(x,t)(1-\hat{v}(x,t))\hat{v}(x,t)dx.$$

By the assumption, the problem (3.1.1) has a unique bounded weak solution satisfying the initial-boundary conditions (3.1.3)-(3.1.4), we conclude that there exists a maximum value C of $|1-\hat{v}|$ for $(x,t) \in \Omega \times (0,T)$. As a result, we have

$$\int_{\Omega} \hat{v}_t(x,t)\hat{v}(x,t)dx \leq C\int_{\Omega} \hat{v}^2(x,t)dx, \tag{3.2.4}$$

and

$$\frac{d}{dt}\int_{\Omega} \hat{v}^2(x,t)dx - 2C\int_{\Omega} \hat{v}^2(x,t)dx \leq 0.$$

It implies $\int_{\Omega} \hat{v}^2(x,t)dx = 0$, since $\hat{v}(x,0) = 0$. Hence, $\hat{v}(x,t) \equiv 0$, i.e, $v \leq 0$, and $u \leq 1$ for $(x,t) \in \Omega \times (0,T)$. □

3.3 Asymptotic time-behavior

Theorem 3.3.1. *Under the assumptions $s \in (0,1)$, $p = 2$, and $u_0 \in L^{\infty}(\Omega)$ with $0 \leq u_0(x) \leq 1$, the global solution of (3.1.1) with the initial-boundary conditions (3.1.3)-(3.1.4) satisfies the following inequality:*

$$\|u(x,t)\|_{L^2(\Omega)} \lesssim e^{-\lambda_{1,2,s}(\Omega)t}\|u_0\|_{L^2(\Omega)}, \tag{3.3.1}$$

Here, $\lambda_{1,2,s}(\Omega)$ refers to the first eigenvalue of the fractional Laplacian, which satisfies equation (3.1.8).

Proof. We start by fixing a time t in (3.1.5) and setting $\varphi = u(x,t)$ to obtain

$$\begin{aligned}\int_{\Omega} & u_t(x,t)u(x,t)dx \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x,t)-u(y,t))(u(x,t)-u(y,t))}{|x-y|^{N+2s}}dxdy \\ &\quad = \int_{\Omega} u^2(x,t)(u(x,t)-1)dx.\end{aligned}$$

By applying Theorem 3.2.1, we obtain the estimate $0 \leq u(x,t) \leq 1$, and

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u^2 dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{N+2s}}dxdy \leq 0,\ t > 0.$$

Using the Poincaré inequality (3.1.7), we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u^2 dx + \lambda_{1,2,s}(\Omega)\int_{\Omega} |u(x)|^2 dx \leq 0,\ t > 0.$$

We introduce the substitution $W(t) := \int_\Omega u^2(x,t)dx$ and differentiate it with respect to t to obtain

$$W_t(t) + 2\,\lambda_{1,2,s}(\Omega)W(t) \lesssim 0.$$

Setting $W(0) := \int_\Omega u_0^2(x)dx$, we solve the differential inequality to obtain

$$W(t)e^{2\,\lambda_{1,2,s}(\Omega)t} \lesssim W(0).$$

Therefore, we obtain the estimate

$$\|u(x,t)\|_{L^2(\Omega)} \lesssim e^{-\,\lambda_{1,2,s}(\Omega)t}\|u_0(x)\|_{L^2(\Omega)}.$$

□

3.4 Blow-up solutions

Let us define $H(t) := \int_\Omega u(x,t)u_1(x)dx$, where $u_1(x)$ is the (weak) eigenfunction corresponding to the first eigenvalue of the fractional Laplacian, such that $\int_\Omega u_1(x) = 1$. It is known that $u_1(x)$ can be chosen positive (see [Brasco et al., 2014, Section 3] and [Jin et al., 2021, Lemma 6]). We also use the notation $H_0 := H(0)$.

Theorem 3.4.1. *Assume that $u_0 \in L^2(\Omega)$. Suppose that $1 + \lambda_{1,2,s}(\Omega) < H_0$. Then, the weak solution u of the problem (3.1.1) with the given conditions (3.1.3)-(3.1.4) blows up in finite time for $p = 2$. This can be expressed as $H(t) \to \infty$ as $t \to T^*$, where*

$$T^* = \log\left|\frac{H_0}{H_0 - 1 - \lambda_{1,2,s}(\Omega)}\right|.$$

Proof. By using the eigenfunction u_1 for φ in equation (3.1.5), we obtain

$$\int_\Omega u_t(x,t)u_1(x)dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x,t)-u(y,t))}{|x-y|^{N+2s}}(u_1(x)-u_1(y))dxdy$$
$$= \int_\Omega u(x,t)(u(x,t)-1)u_1(x)dx. \tag{3.4.1}$$

Using the symmetry of the Gagliardo integral in (3.1.8) with $p = 2$, we can simplify the second term on the left-hand side as

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x,t)-u(y,t))}{|x-y|^{N+2s}}(u_1(x)-u_1(y))dxdy = \lambda_{1,2,s}(\Omega)\int_\Omega u(x,t)u_1(x)dx$$

Thus, we obtain

$$\int_\Omega u_t(x,t)u_1(x)dx + (\lambda_{1,2,s}(\Omega)+1)\int_\Omega u(x,t)u_1(x)dx = \int_\Omega u^2(x,t)u_1(x)dx. \tag{3.4.2}$$

On the other hand, by using the assumption $\int_\Omega u_1(x) = 1$, we can apply the Jensen inequality to obtain

$$\int_\Omega u^2(x,t)u_1(x)dx \geq \left(\int_\Omega u(x,t)u_1(x)dx\right)^2 = H^2(t).$$

Therefore, from (3.4.2), we arrive at the inequality

$$H_t(t) + (1+\lambda_{1,2,s}(\Omega))H(t) \geq H^2(t), \; H_0 := \int_\Omega u(x,0)u_1(x)dx. \tag{3.4.3}$$

To establish a lower bound for $H(t)$, we employ arguments from the theory of ordinary differential inequalities. Specifically, we use the basic properties of Bernoulli-type ordinary differential inequalities to obtain

$$H(t) \geq \frac{1+\lambda_{1,2,s}(\Omega)}{1-e^{(1+\lambda_{1,2,s}(\Omega))(t-T^*)}},$$

where $T^* = \log\left|\frac{H_0}{H_0-1-\lambda_{1,2,s}(\Omega)}\right|$. □

3.5 Positivity and boundedness of the solution: Time fractional case

Theorem 3.5.1. *Assume $T \in (0,\infty)$ and $u_0 \in L^\infty(\Omega)$ satisfies $0 \leq u_0(x) \leq 1$. Then, for all $(x,t) \in \Omega \times (0,T)$, the global solution of (3.1.2) with the initial-boundary conditions (3.1.3)-(3.1.4) satisfies*

$$0 \leq u(x,t) \leq 1.$$

Proof. We aim to prove that the solution u is positive and bounded by one.

Assume $\tilde{u}(x,t) := \min(u(x,t),0)$ in order to prove $u \geq 0$. Using equation (3.1.5) with $0<\nu<1$, we replace φ with $\tilde{u}(x,t)$ for a given t. We obtain

$$\begin{aligned}\int_\Omega {}^C\partial_t^\nu u(x,t)\,\tilde{u}(x,t)dx \\ + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{N+sp}}(\tilde{u}(x,t)-\tilde{u}(y,t))dxdy \\ = \int_\Omega u(x,t)(u(x,t)-1)\tilde{u}(x,t)dx.\end{aligned} \tag{3.5.1}$$

By Lemma 3.1.2, there is the non-negative term

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{N+sp}}(\tilde{u}(x,t)-\tilde{u}(y,t))dxdy \geq 0.$$

Moreover, applying Theorem 2.2.12, we have

$$\begin{aligned}\frac{1}{2}\int_{\Omega} {}^{C}\partial_t^{\nu}\,\tilde{u}^2(x,t)dx &\leq \int_{\Omega} {}^{C}\partial_t^{\nu} u(x,t)\,\tilde{u}(x,t)dx \\ &\leq \int_{\Omega} u(x,t)(u(x,t)-1)\tilde{u}(x,t)dx \qquad (3.5.2)\\ &\leq -\int_{\Omega} u(x,t)\tilde{u}(x,t)dx. \qquad (3.5.3)\end{aligned}$$

This is equivalent to the inequality

$${}^{C}\partial_t^{\nu}\int_{\Omega}\tilde{u}^2(x,t)dx + 2\int_{\Omega}\tilde{u}^2(x,t)dx \leq 0.$$

We define $W(t) := \int_{\Omega}\tilde{u}^2(x,t)dx$. Then we have

$${}^{C}\partial_t^{\nu} W(t) + 2W(t) \leq 0.$$

We set $\widetilde{W}(t)$ as the solution of the differential equation

$${}^{C}\partial_t^{\nu}\widetilde{W}(t) + 2\widetilde{W}(t) = 0$$

with the initial condition

$$\widetilde{W}(0) = \widetilde{W}_0 = \int_{\Omega}\tilde{u}_0^2(x)dx = 0.$$

Notice that $0 \leq W(t) \leq \widetilde{W}(t)$. Using Theorem 2.2.13, we can easily show that $\widetilde{W}(t) = 0$. Consequently, we have $\int_{\Omega}\tilde{u}^2(x,t)dx = 0$, which implies that $\tilde{u}(x,t) \equiv 0$ and $u(x,t) \geq 0$ for all $(x,t) \in \Omega \times (0,T)$.

Let us show that $u \leq 1$. For the fractional p-Laplacian, we employ the (anti-) invariance property. Specifically, we observe that if u satisfies (3.1.5), then the function $v := 1-u$ also satisfies the same equation. For t pointwise a.e. in $[0,T]$, we have the following identity

$$\begin{aligned}&\int_{\Omega} {}^{C}\partial_t^{\nu} v(x,t)\varphi(x)dx \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x,t)-v(y,t)|^{p-2}(v(x,t)-v(y,t))}{|x-y|^{N+ps}}(\varphi(x)-\varphi(y))dxdy \\ &\qquad = \int_{\Omega} v(x,t)(1-v(x,t))\varphi(x)dx. \qquad (3.5.4)\end{aligned}$$

To simplify this expression, we replace φ with $\hat{v} := \min(v, 0)$ for a given t. This yields

$$\int_\Omega {}^C\partial_t^\nu v(x,t)\hat{v}(x,t)dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x,t)-v(y,t)|^{p-2}(v(x,t)-v(y,t))}{|x-y|^{N+ps}}(\hat{v}(x,t)-\hat{v}(y,t))dxdy = \int_\Omega u(x,t)(1-u(x,t))\hat{v}(x,t)dx, \tag{3.5.5}$$

Using Lemma 3.1.2, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x,t)-v(y,t)|^{p-2}(v(x,t)-v(y,t))}{|x-y|^{N+ps}}(\hat{v}(x,t)-\hat{v}(y,t))dxdy \geq 0.$$

As a consequence, we claim that

$$\int_\Omega {}^C\partial_t^\nu \hat{v}(x,t)\,\hat{v}(x,t)dx \leq \int_\Omega \hat{v}(x,t)(1-\hat{v}(x,t))\hat{v}(x,t)dx.$$

Since problem (3.1.2) with the given initial-boundary conditions (3.1.3)-(3.1.4) has a unique bounded weak solution, we know that for $(x,t) \in \Omega \times (0,T)$, there exists a constant K such that $|1-\hat{v}| \leq K$.

By applying Theorem 2.2.12, we get

$$\frac{1}{2}\int_\Omega {}^C\partial_t^\nu \tilde{v}^2(x,t)dx \leq \int_\Omega^C \partial_t^\nu \hat{v}(x,t)\hat{v}(x,t)dx \leq K\int_\Omega \hat{v}^2(x,t)dx, \tag{3.5.6}$$

and

$${}^C\partial_t^\nu \int_\Omega \hat{v}^2(x,t)dx - 2K\int_\Omega \hat{v}^2(x,t)dx \leq 0.$$

By a similar argument as above, we deduce that $\int_\Omega \hat{v}^2(x,t)dx = 0$ since $\hat{v}(x,0) = 0$. This implies that $\hat{v}(x,t) \equiv 0$, and therefore $u(x,t) \leq 1$ for $(x,t) \in \Omega \times (0,T)$.

Hence, we have shown that $0 \leq u(x,t) \leq 1$ for all $(x,t) \in \Omega \times (0,T)$. □

3.6 Asymptotic time-behavior: Time fractional case

In this section, we present the large-time behavior of the global solution to (3.1.2) under certain assumptions. The following theorem presents our main result:

Theorem 3.6.1. *Let $s \in (0,1)$ and $p = 2$. Assume that $u_0 \in L^\infty(\Omega)$ with $0 \leq u_0(x) \leq 1$. Then, the global solution of (3.1.2) subject to initial-boundary conditions (3.1.3)-(3.1.4) satisfies the estimate*

$$\|u(x,t)\|_{L^2(\Omega)} \leq E_\nu\left(-2\lambda_{1,2,s}(\Omega)t^\nu\right)\|u_0(x)\|_{L^2(\Omega)}.$$

Here, $\lambda_{1,2,s}(\Omega)$ denotes the first eigenvalue of the fractional Laplacian on the domain Ω, and E_ν is the Mittag-Leffler function.

Proof. We start by fixing t in (3.1.5) and setting $\varphi = u(x,t)$. This gives us

$$\begin{aligned}\int_\Omega {}^C\partial_t^\nu\, u(x,t)\, u(x,t)dx \\ + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x,t)-u(y,t))(u(x,t)-u(y,t))}{|x-y|^{N+2s}}dxdy \\ = \int_\Omega u^2(x,t)(u(x,t)-1)dx.\end{aligned}$$

Since $0 \le u(x,t) \le 1$ (by Theorem 4.2.1), we have

$$\frac{1}{2}\,{}^C\partial_t^\nu \int_\Omega u^2 dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{N+2s}}dxdy \le 0,\ t>0. \tag{3.6.1}$$

Using the Poincaré inequality (3.1.7), we obtain

$$\frac{1}{2}\,{}^C\partial_t^\nu \int_\Omega u^2 dx + \lambda_{1,2,s}(\Omega)\int_\Omega |u|^2 dx \le 0,\ t>0. \tag{3.6.2}$$

We introduce the substitution $U(t) := \int_\Omega u^2(x,t)dx$ to obtain

$${}^C\partial_t^\nu U(t) + 2\,\lambda_{1,2,s}(\Omega)U(t) \lesssim 0,$$

and

$$U(0) := \int_\Omega u_0^2(x)dx.$$

Note that $U(t) \le \widetilde{U}(t)$, where $\widetilde{U}(t)$ is the solution of

$${}^C\partial_t^\nu \widetilde{U}(t) + 2\,\lambda_{1,2,s}(\Omega)\widetilde{U}(t) = 0$$

with

$$\widetilde{U}(0) = \widetilde{U}_0 = \int_\Omega u_0^2(x)dx.$$

We can solve the above differential inequality explicitly to obtain an upper bound for $U(t)$. In fact, using Theorem 2.2.13, we get the solution

$$U(t) \leqslant \widetilde{U}_0 E_\nu\left(-2\lambda_{1,2,s}t^\nu\right),$$

where E_ν is the Mittag-Leffler function. Therefore, we conclude that

$$\|u(x,t)\|_{L^2(\Omega)} \le E_\nu\left(-2\lambda_{1,2,s}t^\nu\right)\|u_0(x)\|_{L^2(\Omega)}. \tag{3.6.3}$$

□

3.7 Blow-up: Time fractional case

As previously defined in Section 3.4, let $W(t) := \int_\Omega u(x,t)u_1(x), dx$, where $W_0 := W(0)$, and $u_1(x)$ denotes the (weak) eigenfunction corresponding to the first eigenvalue of the fractional Laplacian. Note that $\int_\Omega u_1(x) = 1$, and $u_1(x)$ can be chosen to be positive, as shown in [Brasco et al., 2014, Section 3] and [Jin et al., 2021, Lemma 6].

We state the following theorem regarding the blow-up of the solution to problem (3.1.2) for $p = 2$ with the initial-boundary conditions (3.1.3)-(3.1.4).

Theorem 3.7.1. *Assume that $u_0 \in L^2(\Omega)$. If $1 + \lambda_{1,2,s}(\Omega) < W_0$, then the weak solution u to problem (3.1.2) blows up in a finite time.*

Proof. We multiply both sides of equation (3.1.2) by the eigenfunction u_1, integrate over Ω, and obtain

$$\int_\Omega {}^C\partial_t^\nu u(x,t)u_1(x)dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x,t)-u(y,t))}{|x-y|^{N+2s}}(u_1(x)-u_1(y))dxdy$$
$$= \int_\Omega u(x,t)(u(x,t)-1)u_1(x)dx. \tag{3.7.1}$$

Using equation (3.1.8) with $p = 2$, we can take

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x,t)-u(y,t))}{|x-y|^{N+2s}}(u_1(x)-u_1(y))dxdy = \lambda_{1,2,s}(\Omega)\int_\Omega u(x,t)u_1(x)dx$$

Therefore, we have

$$\int_\Omega {}^C\partial_t^\nu u(x,t)u_1(x)dx + (\lambda_{1,2,s}(\Omega)+1)\int_\Omega u(x,t)u_1(x)dx = \int_\Omega u^2(x,t)u_1(x)dx. \tag{3.7.2}$$

Since $\int_\Omega u_1(x) = 1$, the Jensen inequality yields

$$\int_\Omega u^2(x,t)u_1(x)dx \geq \left(\int_\Omega u(x,t)u_1(x)dx\right)^2 = W^2(t),$$

where $W(t) = \int_\Omega u(x,t)u_1(x)dx$. Hence, we have

$${}^C\partial_t^\nu W(t) + (1+\lambda_{1,2,s}(\Omega))W(t) \geq W^2(t),\ W_0 := \int_\Omega u(x,0)u_1(x)dx. \tag{3.7.3}$$

Without loss of generality, we define $\widetilde{W}(t) = W(t) - (1+\lambda_{1,2,s})$. We can then express the differential inequality as

$${}^C\partial_t^\beta \tilde{W}(t) \geqslant \widetilde{W}(t)\left(\tilde{W}(t)+1+\lambda_{1,2,s}\right) \geqslant \widetilde{W}(t)(\widetilde{W}(t)+1). \tag{3.7.4}$$

Since we have assumed $1+\lambda_{1,2,s} < W_0$, it follows that $\widetilde{W}(0) = W_0 - (1+\gamma_1) > 0$. By applying Theorem 2.2.11, we can conclude that the solution of the differential inequality (3.7.4) will blow up in a finite time. □

Remark 3.7.2. Theorems 3.2.1 and 3.5.1 can be extended to the case with a general Fisher-KPP type nonlinearity. Specifically, instead of the special nonlinearity $u(u-1)$ one may consider more general nonlinearity, for instance, some convex function $f(u)$ with $f(0) = f(1) = 0$.

Chapter 4

Fisher-KPP equation in fractional Sobolev spaces

4.1 Introduction

In this chapter, we present results in the study of the fractional Fisher-KPP equation involving fractional p-sub-Laplacian operators on stratified groups, where the solution u of the given problem is defined on fractional Sobolev space. This study is important in theoretical (see, e.g., [Gromov, 1996, Danielli et al., 2007]) and applied mathematics, such as mathematical models of human vision and crystal material (see, for example, [Christodoulou, 1998, Citti et al., 2004]). However, despite its relevance, research in p-sub-Laplacians is limited. Note that the general approach established on stratified groups allows one to get insights also in the commutative groups. Moreover, the results obtained in the setting of stratified groups can be equally used for both elliptic and subelliptic problems [Jinguo and Dengyun, 2021].

The achievement of this chapter was unreachable until its recent publication [Ghosh et al., 2023]. The theorems derived by the authors are employed as tools in the derivation of the results presented herein. Furthermore, this chapter can be considered a natural extension of the previous chapter.

In this chapter, we investigate the fractional type of the Fisher-KPP equation

$$u_t + (-\Delta_{p,\mathbb{G}})^s u = u(u-1) \quad \text{in} \quad \Omega \subset \mathbb{G},\ t > 0, \tag{4.1.1}$$

with the fractional p-sub-Laplacian $(-\Delta_{p,\mathbb{G}})^s$ introduced in Definition 2.1.22, where $s \in (0,1)$, $p \in (1,\infty)$. We also consider the same problem with the Caputo time-fractional derivative ∂_t^ν of order $0 < \nu < 1$ of the form

$${}^C\partial_t^\nu u + (-\Delta_{p,\mathbb{G}})^s u = u(u-1) \quad \text{in} \quad \Omega \subset \mathbb{G},\ t > 0. \tag{4.1.2}$$

The initial condition for the problem is given by

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \tag{4.1.3}$$

while the Dirichlet boundary condition is provided by

$$u(x,t) = 0 \quad \text{in} \quad \mathbb{G}\backslash\Omega,\, t \in [0,\infty). \tag{4.1.4}$$

Definition 4.1.1. Let Ω be a bounded open set of a stratified group $\mathbb{G}$. We denote by $\widetilde{H}_0^{s,p}(\Omega)$ the fractional Sobolev space introduced in Section 2.1.5. Given $0 < \nu < 1$ $[\nu = 1]$, a function u of (4.1.2) [(4.1.1)] is said to be a weak solution if it satisfies the following equation for all $\varphi \in \widetilde{H}_0^{s,p}(\Omega)$ and pointwise $t \in [0,\infty)$:

$$\begin{aligned}&\int_\Omega^C \partial_t^\nu u(x,t)\varphi(x)dx \\ &\quad + \iint_{\mathbb{G}\times\mathbb{G}} \frac{|u(x)-u(y)|^{p-2}(u(x,t)-u(y,t))(\varphi(x)-\varphi(y)}{|y^{-1}\circ x|^{Q+ps}}dxdy \\ &\quad = \int_\Omega u(x,t)(u(x,t)-1)\varphi(x)dx\end{aligned} \tag{4.1.5}$$

where Q is the homogeneous dimension of $\mathbb{G}$ and case $\nu = 1$ stands for the first-order partial derivative ∂_t.

4.2 Positivity and boundedness of the solution

Theorem 4.2.1. *Assume u is a bounded global solution of (4.1.1), and $T \in (0,\infty)$. Let $u_0 \in L^\infty(\Omega)$ such that $0 \le u_0(x) \le 1$. Then, the global solution of the problem (4.1.1) with the given conditions (4.1.3)-(4.1.4) satisfies*

$$0 \le u(x,t) \le 1$$

for all $(x,t) \in \Omega \times (0,T)$.

Proof. To show that $u \ge 0$, let us define $\tilde{u}(x,t) := \min(u(x,t),0)$. By putting $\tilde{u}$ for φ in equation (4.1.5), we obtain

$$\begin{aligned}&\int_\Omega u_t(x,t)\tilde{u}(x,t)dx \\ &\quad + \iint_{\mathbb{G}\times\mathbb{G}} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))(\tilde{u}(x,t)-\tilde{u}(y,t))}{|y^{-1}\circ x|^{Q+ps}}dxdy \\ &\quad = \int_\Omega u(x,t)(u(x,t)-1)\tilde{u}(x,t)dx.\end{aligned} \tag{4.2.1}$$

Using Lemma 3.1.2, we conclude that the second term on the left side is non-negative. Consequently, we get

$$\begin{aligned}\int_\Omega u_t(x,t)\tilde{u}(x,t)dx &\leq \int_\Omega u(x,t)(u(x,t)-1)\tilde{u}(x,t)dx \\ &\leq -\int_\Omega u(x,t)\tilde{u}(x,t)dx. \qquad (4.2.2)\end{aligned}$$

Therefore, we have

$$\int_\Omega u_t(x,t)\tilde{u}(x,t)dx + \int_\Omega u(x,t)\tilde{u}(x,t)dx \leq 0,$$

which is equivalent to

$$\frac{d}{dt}\int_\Omega |\tilde{u}(x,t)|^2 dx + 2\int_\Omega |\tilde{u}(x,t)|^2 dx \leq 0.$$

Using the same method as in Chapter 3, we introduce the substitution

$$K(t) := \int_\Omega |\tilde{u}(x,t)|^2 dx,$$

with

$$K(0) = \int_\Omega \tilde{u}_0^2 dx = 0.$$

It follows that $\tilde{u}(x,t) \equiv 0$ and $u \geq 0$ for $(x,t) \in \Omega \times (0,T)$.

To show that $u \leq 1$, we commence by observing that if u satisfies (4.1.5), then the function $v := 1-u$ satisfies the same equation due to the (anti-) invariance property. Then, we have

$$\begin{aligned}&\int_\Omega v_t(x,t)\varphi(x)dx \\ &\quad + \iint_{\mathbb{G}\times\mathbb{G}} \frac{|v(x,t)-v(y,t)|^{p-2}(v(x,t)-v(y,t))(\varphi(x)-\varphi(y))}{|y^{-1}\circ x|^{Q+ps}}dxdy \\ &\qquad = \int_\Omega v(x,t)(1-v(x,t))\varphi(x)dx. \qquad (4.2.3)\end{aligned}$$

We then substitute $\hat{v} := \min(v, 0)$ for φ and obtain

$$\begin{aligned}&\int_\Omega v_t(x,t)\hat{v}(x,t)dx \\ &\quad + \iint_{\mathbb{G}\times\mathbb{G}} \frac{|v(x,t)-v(y,t)|^{p-2}(v(x,t)-v(t,y))(\hat{v}(x,t)-\hat{v}(t,y))}{|y^{-1}\circ x|^{Q+ps}}dxdy \\ &\qquad = \int_\Omega v(x,t)(1-v(x,t))\hat{v}(x,t)dx. \qquad (4.2.4)\end{aligned}$$

By utilizing Lemma 3.1.2, it becomes evident that the second term on the left-hand side of equation (4.2.4) is positive. Therefore,

$$\int_\Omega v_t(x,t)\hat{v}(x,t)dx \leq \int_\Omega v(x,t)(1-v(x,t))\hat{v}(x,t)dx,$$

and $1 - v(x,t) \leq L$, we have

$$\int_\Omega \hat{v}_t(x,t)\hat{v}(x,t)dx \leq L \int_\Omega \hat{v}^2(x,t)dx. \tag{4.2.5}$$

Moreover, this is equivalent to

$$\frac{d}{dt}\int_\Omega \hat{v}^2(x,t)dx - 2L\int_\Omega \hat{v}^2(x,t)dx \leq 0.$$

Using the initial condition $\hat{v}(x,0) = 0$, we get $\int_\Omega \hat{v}^2(x,t)dx = 0$. Consequently, $\hat{v}(x,t) \equiv 0$ and $u \leq 1$ for $(x,t) \in \Omega \times (0,T)$, which yields $u \leq 1$ for $(x,t) \in \Omega \times (0,T)$. Overall, we have $0 \leq u \leq 1$. This completes the proof. □

4.3 Asymptotic time-behavior

Theorem 4.3.1. *Suppose u is a bounded global solution of problem (4.1.1). Let $u_0 \in L^\infty(\Omega)$ and $0 \leq u_0(x) \leq 1$. For $s \in (0,1)$ and $p = 2$, the global solution of (4.1.1) subject to conditions (4.1.3) and (4.1.4) satisfies the estimate*

$$\|u(x,t)\|_{L^2(\Omega)} \lesssim e^{-\lambda_{Q,2,s}(\Omega)t}\|u_0(x)\|_{L^2(\Omega)}. \tag{4.3.1}$$

Here, $\lambda_{Q,2,s}(\Omega)$ denotes the eigenvalue of the Poincaré inequality (2.1.16).

Proof. By fixing t in (4.1.5) and setting $\varphi = u(x,t)$, we have

$$\int_\Omega u_t(x,t)u(x,t)dx + \iint_{\mathbb{G}\times\mathbb{G}} \frac{(u(x,t)-u(y,t))^2}{|y^{-1}\circ x|^{Q+2s}}dxdy$$
$$= \int_\Omega u(x,t)(u(x,t)-1)u(x,t)dx. \tag{4.3.2}$$

Therefore, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \iint_{\mathbb{G}\times\mathbb{G}} \frac{|u(x,t)-u(y,t)|^2}{|y^{-1}\circ x|^{Q+2s}}dxdy \leq 0,\ t > 0, \tag{4.3.3}$$

since $0 \leq u(x,t) \leq 1$ (by Theorem 4.2.1).

Using the Poincaré inequality provided in Section 2.1.6, for the value $\lambda_{Q,2,s}(\Omega) > 0$ in (2.1.16), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \lambda_{Q,2,s}(\Omega)\int_\Omega |u(x)|^2 dx \leq 0,\ t > 0. \tag{4.3.4}$$

To solve the inequality, we make the substitution $W(t) := \int_\Omega u^2(x,t)dx$. This yields

$$W_t(t) + 2\,\lambda_{Q,2,s}(\Omega)W(t) \lesssim 0.$$

Therefore, we have

$$W(t)e^{2\,\lambda_{Q,2,s}(\Omega)t} \lesssim W(0).$$

Finally, we obtain

$$\|u(x,t)\|_{L^2(\Omega)} \lesssim e^{-\,\lambda_{Q,2,s}(\Omega)t}\|u_0(x)\|_{L^2(\Omega)}. \tag{4.3.5}$$

□

4.4 Blow-up solution

We consider the function $H(t) := \int_\Omega u(x,t)u_1(x)dx$, where $u_1(x)$ is the weak eigenfunction corresponding to the first eigenvalue of the fractional p-sub-Laplacian problem in (2.1.15), with the assumption $\int_\Omega u_1(x) = 1$, and $H_0 := H(0)$.

Theorem 4.4.1. *Assume that* $u_0 \in L^2(\Omega)$. *If* $1 + \lambda_{1,2,s}(\Omega) < H_0$, *then the weak solution* u *of the problem* (4.1.1) *with conditions* (4.1.3)-(4.1.4) *for* $p = 2$ *blows up in the finite time*

$$T = \frac{1}{1 + \lambda_{1,2,s}(\Omega)} \log\left|\frac{H_0}{H_0 - (1 + \lambda_{1,2,s}(\Omega))}\right|,$$

in the sense that $H(t) \to \infty$.

Proof. We use the eigenfunction u_1 instead of φ with the special case $p = 2$ in equation (4.1.5) to get

$$\int_\Omega u_t(x,t)u_1(x)dx + \iint_{\mathbb{G}\times\mathbb{G}} \frac{(u(x,t)-u(y,t))(u_1(x)-u_1(y))}{|y^{-1}\circ x|^{Q+2s}}dxdy = \int_\Omega u(x,t)(u(x,t)-1)u_1(x)dx. \tag{4.4.1}$$

Setting $\varphi = u_1$, $p = 2$, and using the first eigenvalue $\lambda = \lambda_1$ in equation (2.1.14), we have

$$\iint_{\mathbb{G}\times\mathbb{G}} \frac{(u(x,t)-u(y,t))(u_1(x)-u_1(y))}{|y^{-1}\circ x|^{Q+2s}}dxdy = \lambda_{1,2,s}(\Omega)\int_\Omega u(x,t)u_1(x)dx, \tag{4.4.2}$$

which yields

$$\int_\Omega u_t(x,t)u_1(x)dx + (\lambda_{1,2,s}(\Omega)+1)\int_\Omega u(x,t)u_1(x)dx = \int_\Omega u^2(x,t)u_1(x)dx. \tag{4.4.3}$$

Furthermore, from the Jensen inequality and the fact that $\int_\Omega u_1(x) = 1$, it follows that

$$\int_\Omega u^2(x,t)u_1(x)dx \geq \left(\int_\Omega u(x,t)u_1(x)dx\right)^2 = H^2(t).$$

Consequently, we can express equation (4.4.3) as

$$H_t(t) + (1+\lambda_{1,2,s}(\Omega))H(t) \geq H^2(t), \ H_0 := \int_\Omega u(x,0)u_1(x)dx. \tag{4.4.4}$$

The last part of the proof involves using the basic properties of Bernoulli-type ordinary differential inequalities, which allows us to obtain an estimate for $H(t)$. Specifically, we have

$$H(t) \geq \frac{1+\lambda_{1,2,s}(\Omega)}{1-e^{(1+\lambda_{1,2,s}(\Omega))(t-T^*)}},$$

where T^* is defined as

$$T^* = \log\left|\frac{H_0}{H_0 - 1 - \lambda_{1,2,s}(\Omega)}\right|.$$

□

4.5 Positivity and boundedness of the solution: Time fractional case

In this section, we prove the positivity and boundedness of the global solution u to problem (4.1.2), with initial-boundary conditions (4.1.3)-(4.1.4).

Theorem 4.5.1. *Suppose u is a bounded global solution of (4.1.2), and $T \in (0,\infty)$. Let $u_0 \in L^\infty(\Omega)$ satisfy $0 \leq u_0(x) \leq 1$. Then, the global solution of problem (4.1.2) satisfies*

$$0 \leq u(x,t) \leq 1$$

for all $(x,t) \in \Omega \times [0,T]$.

Proof. First, we define $\tilde{u}(x,t) := \min(u(x,t),0)$ to show that $u \geq 0$. Plugging the function $\tilde{u}$ instead of φ in equation (4.1.5) for a given t. Taking into account that the equality holds for $t \in [0,\infty)$ pointwise almost everywhere, we obtain

$$\begin{aligned}&\int_\Omega {}^C\partial_t^\nu u(x,t)\tilde{u}(x,t)dx \\ &\quad + \iint_{\mathbb{G}\times\mathbb{G}} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))(\tilde{u}(x,t)-\tilde{u}(y,t))}{|y^{-1}\circ x|^{Q+ps}}dxdy \\ &= \int_\Omega u(x,t)(u(x,t)-1)\tilde{u}(x,t)dx.\end{aligned} \tag{4.5.1}$$

From Lemma 3.1.2, we conclude that

$$\int_\Omega {}^C\partial_t^\nu u(x,t)\,\tilde{u}(x,t)dx \leq \int_\Omega u(x,t)(u(x,t)-1)\tilde{u}(x,t)dx. \tag{4.5.2}$$

Furthermore, it is easy to check

$$\int_\Omega u(u-1)\tilde{u}dx \leq \int_\Omega |\tilde{u}u||u-1|dx \leq -\int_\Omega |\tilde{u}u|dx. \tag{4.5.3}$$

Using the fractional derivative property given in Theorem 2.2.12, we can derive the following equivalent inequality

$${}^C\partial_t^\nu \int_\Omega \tilde{u}^2(x,t)dx + 2\int_\Omega \tilde{u}^2(x,t)dx < 0.$$

Let us define

$$K(t) := \int_\Omega \tilde{u}^2(x,t)dx,$$

and substitute this expression into the above inequality to obtain

$${}^C\partial_t^\nu K(t) + 2K(t) \leq 0.$$

Using Theorem 2.2.13, we can easily show that the solution of the differential inequality is $K(t) = 0$. This leads to the conclusion that $\tilde{u}(x,t) = 0$, and $u(x,t) \geq 0$.

Similarly, we can deduce that $u(x,t) \leq 1$.

Assume u satisfies the equation (4.1.5). Using the (anti-)invariance property, we find that the function $v := 1-u$ satisfies the following equality

$$\begin{aligned}\int_\Omega {}^C\partial_t^\nu v(x,t)\varphi(x)dx \\ + \iint_{\mathbb{G}\times\mathbb{G}} \frac{|v(x,t)-v(y,t)|^{p-2}(v(x,t)-v(y,t))(\varphi(x)-\varphi(y))}{|y^{-1}\circ x|^{Q+ps}}dxdy \\ = \int_\Omega v(x,t)(1-v(x,t))\varphi(x)dx,\end{aligned} \tag{4.5.4}$$

We can write this equality in a more convenient form by setting $\hat{v} := \min(v,0)$ for φ, which gives

$$\begin{aligned}\int_\Omega {}^C\partial_t^\nu v(x,t)\hat{v}(x,t)dx \\ + \iint_{\mathbb{G}\times\mathbb{G}} \frac{|v(x,t)-v(y,t)|^{p-2}(v(x,t)-v(y,t))(\hat{v}(x,t)-\hat{v}(y,t))}{|y^{-1}\circ x|^{Q+ps}}dxdy \\ = \int_\Omega v(x,t)(1-v(x,t))\hat{v}(x,t)dx,\end{aligned} \tag{4.5.5}$$

According to Lemma 3.1.2, we can deduce that the integral in the second term is non-negative. Otherwise, we have

$$\iint_{\mathbb{G}\times\mathbb{G}} \frac{|v(x,t)-v(y,t)|^{p-2}(v(x,t)-v(y,t))(\hat{v}(x,t)-\hat{v}(y,t))}{|y^{-1}\circ x|^{Q+ps}}dxdy \geq 0.$$

which yields

$$\int_\Omega v_t(x,t)\hat{v}(x,t)dx \leq \int_\Omega v(x,t)(1-v(x,t))\hat{v}(x,t)dx.$$

Given that u is a globally bounded solution to (4.1.2), we have $|1-v(x,t)| \leq L$. This leads to the inequality

$$\int_\Omega \hat{v}_t(x,t)\hat{v}(x,t)dx \leq L\int_\Omega \hat{v}^2(x,t)dx. \tag{4.5.6}$$

and therefore,

$$\frac{d}{dt}\int_\Omega \hat{v}^2(x,t)dx - 2L\int_\Omega \hat{v}^2(x,t)dx \leq 0.$$

This implies that $\int_\Omega \hat{v}^2(x,t)dx = 0$ since $\hat{v}(x,0)=0$. Hence, $\hat{v}(x,t)\equiv 0$, which means $v\leq 0$ and $u\leq 1$ for $(x,t)\in\Omega\times(0,T)$.

Thus, we have shown that $0\leq u\leq 1$, which completes the proof. □

4.6 Asymptotic time-behavior: Time fractional case

In this section, we prove a theorem about the asymptotic time behavior of global solutions to (4.1.2) with initial-boundary conditions (4.1.3)-(4.1.4), where $s\in(0,1)$ and $p=2$.

Theorem 4.6.1. *Let u be a bounded global solution, and let $u_0\in L^\infty(\Omega)$ with $0\leq u_0(x)\leq 1$. Then, the solution satisfies*

$$\|u(x,t)\|_{L^2(\Omega)} \leq E_\nu\left(-2\lambda_{Q,2,s}t^\nu\right)\|u_0(x)\|_{L^2(\Omega)}. \tag{4.6.1}$$

Here, $\lambda_{Q,2,s}(\Omega)$ denotes the first eigenvalue of the fractional sub-Laplacian on Ω, and E_ν is the Mittag-Leffler function.

Proof. First, we rewrite the weak formulation of the problem (4.1.2) with $\varphi=u(x,t)$, and obtain

$$\int_\Omega {}^C\partial_t^\nu u(x,t)\,u(x,t)dx + \iint_{\mathbb{G}\times\mathbb{G}} \frac{(u(x,t)-u(y,t))(u(x,t)-u(y,t))}{|x-y|^{Q+2s}}dxdy = \int_\Omega u^2(x,t)(u(x,t)-1)dx.$$

Using $0 \leq u(x,t) \leq 1$ (by theorem 4.2.1), the above equation implies

$$\frac{1}{2} {}^{C}\partial_t^{\nu} \int_{\Omega} u^2 dx + \iint_{\mathbb{G}\times\mathbb{G}} \frac{|u(x,t)-u(t,y)|^2}{|x-y|^{Q+2s}} dxdy \leq 0,\, t > 0. \tag{4.6.2}$$

Next, we apply the Poincaré inequality (Section 2.1.6) for $\lambda_{Q,2,s}(\Omega) > 0$, and $t > 0$:

$$\frac{1}{2} {}^{C}\partial_t^{\nu} \int_{\Omega} u^2 dx + \lambda_{Q,2,s}(\Omega) \int_{\Omega} |u(x)|^2 dx \leq 0. \tag{4.6.3}$$

We define the substitution $W(t) := \int_{\Omega} u^2(x,t)dx$ and use it to obtain the inequality

$$ {}^{C}\partial_t^{\nu} W(t) + 2\, \lambda_{Q,2,s}(\Omega) W(t) \lesssim 0,$$

and

$$W(0) := \int_{\Omega} u_0^2(x)dx.$$

Note that $W(t) \leq \widetilde{W}(t)$, where $\widetilde{W}(t)$ is the solution of the differential equation

$$ {}^{C}\partial_t^{\nu} \widetilde{W}(t) + 2\, \lambda_{Q,2,s}(\Omega) \widetilde{W}(t) = 0$$

with the condition

$$\widetilde{W}(0) = \widetilde{W}_0 = \int_{\Omega} u_0^2(x)dx.$$

Using Theorem 2.2.13, we obtain

$$U(t) \leqslant \widetilde{W}_0 E_{\nu}\left(-2\lambda_{Q,2,s} t^{\nu}\right),$$

where E_{ν} is the Mittag-Leffler function. Thus, we have

$$\|u(x,t)\|_{L^2(\Omega)} \leq E_{\nu}\left(-2\lambda_{Q,2,s} t^{\nu}\right) \|u_0(x)\|_{L^2(\Omega)},\, t > 0. \tag{4.6.4}$$

□

4.7 Blow-up: Time fractional case

Let $u_1(x)$ be the (weak) eigenfunction corresponding to the first eigenvalue $\lambda_{1,2,s}(\Omega)$ of the fractional Laplacian, and assume that $\int_{\Omega} u_1(x)dx = 1$.

Theorem 4.7.1. *Assume that $u_0 \in L^2(\Omega)$. If $1 + \lambda_{1,2,s}(\Omega) < \int_{\Omega} u_0(x)u_1(x)dx =: H_0$, then the weak solution u of the problem (4.1.2) with initial and boundary conditions (4.1.3)-(4.1.4) and $p = 2$ blows up in finite time.*

Proof. Substituting u_1 for φ in equation (4.1.5) with $p = 2$, we have

$$\int_\Omega {}^C\partial_t^\nu u(x,t)u_1(x)dx + \iint_{\mathbb{G}\times\mathbb{G}} \frac{(u(x,t)-u(y,t))(u_1(x)-u_1(y))}{|y^{-1}\circ x|^{Q+2s}}dxdy \\ = \int_\Omega u(x,t)(u(x,t)-1)u_1(x)dx. \quad (4.7.1)$$

Using $\psi = u_1$, $p = 2$, and the first eigenvalue $\lambda_{1,2,s}(\Omega) = \lambda_1$ in equation (2.1.14), we obtain

$$\iint_{\mathbb{G}\times\mathbb{G}} \frac{(u(x,t)-u(y,t))(u_1(x)-u_1(y))}{|y^{-1}\circ x|^{Q+2s}}dxdy = \lambda_{1,2,s}(\Omega)\int_\Omega u(x,t)u_1(x)dx, \quad (4.7.2)$$

which yields

$$\int_\Omega {}^C\partial_t^\nu u(x,t)u_1(x)dx + (\lambda_{1,2,s}(\Omega)+1)\int_\Omega u(x,t)u_1(x)dx = \int_\Omega u^2(x,t)u_1(x)dx. \quad (4.7.3)$$

Applying the Jensen inequality and using the assumption that $\int_\Omega u_1(x)dx = 1$, we have

$$\int_\Omega u^2(x,t)u_1(x)dx \geq \left(\int_\Omega u(x,t)u_1(x)dx\right)^2 = H^2(t).$$

In order to analyze the behavior of the solution to equation (4.7.3), we first rewrite it as follows:

$${}^C\partial_t^\nu H(t) + (1+\lambda_{1,2,s}(\Omega))H(t) \geq H^2(t), \;\; H_0 := \int_\Omega u(x,0)u_1(x)dx. \quad (4.7.4)$$

Next, we make the substitution $H := \widetilde{H} + (1+\lambda_{1,2,s}(\Omega))$, where $\widetilde{H}(0) > 0$. This yields the differential inequality

$${}^C\partial_t^\nu \widetilde{H}(t) \geq \widetilde{H}(t)(\widetilde{H}(t)+1+\lambda_{1,2,s}(\Omega)) \geq \widetilde{H}(t)(\widetilde{H}(t)+1), \quad (4.7.5)$$

and using the assumption $1+\lambda_{1,2,s} < H_0$ we have $\widetilde{H}(0) = H_0 - (1+\lambda_{1,2,s}) > 0$.

We can now apply Theorem 2.2.11 to conclude that the solution to the differential inequality (5.7.3) blows up in a finite time. □

Remark 4.7.2. Using the same approach as in Theorems 4.2.1 and 4.5.1, the non-linear term $u(u-1)$ in the Fisher-KPP models can be extended to encompass a certain class of convex functions $f(u)$ that satisfies the conditions $f(0) = f(1) = 0$.

Chapter 5

Fractional Fisher-KKP equation on stratified groups

5.1 Introduction

In this chapter, the main purpose is to extend the study to the limiting case as $s \to 1$ of the fractional p-Laplacian operator $(-\Delta_{p,\mathbb{G}})^s$ on stratified groups. This fractional operator corresponds to the sub-Laplacian operator $\mathcal{L}_2$ (or $\mathcal{L}$) when $s = 1$ and $p = 2$. In order to establish a self-contained study, we replace the fractional Laplacian with the generalized p-sub-Laplacian operator denoted as $\mathcal{L}_p$, where $1 < p < \infty$. The validity of the theorems 1.1.1-1.1.3 on stratified groups is established in the subsequent subsections 5.2 through 5.7.

Observe that this chapter not only serves as a natural extension of Chapter 4, but also covers the results presented in the recent publication [Kashkynbayev et al., 2023]. This publication investigated the Fisher-KPP equation on the Heisenberg group and achieved analogous theorems. Notably, the Heisenberg group is an example of stratified groups, as demonstrated in Example 2.1.13.

Some specific properties that appear in this chapter can be found in Section 2.1.4, including spectral properties of the Dirichlet sub-Laplacian problem on stratified groups.

The significance of investigating partial differential equations on stratified Lie groups comes from the lifting theorem presented by [Rothschild and Stein, 1976].

This chapter is an extension of our publication [Jabbarkhanov et al., 2022], where we consider a similar problem with a particular case $p = 2$.

In this chapter, we focus on the generalized p-sub-Laplacian operator $\mathcal{L}_p$ with $1 < p < \infty$. Specifically, we consider the following two equations:

$$u_t - \mathcal{L}_p u = u(u-1), \qquad x \in \Omega, \quad t > 0, \tag{5.1.1}$$

and

$$ {}^{C}\partial_t^{\nu} u - \mathcal{L}_p u = u(u-1), \qquad x \in \Omega, \quad t > 0, \tag{5.1.2} $$

where Ω is a bounded smooth domain on a stratified group $\mathbb{G}$, and $\partial\Omega$ is its boundary. We assume the initial condition

$$ u(x,0) = u_0(x) \quad \text{with} \quad x \in \Omega, \tag{5.1.3} $$

and the Dirichlet boundary condition

$$ u(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0. \tag{5.1.4} $$

We note that the term $u(u-1)$ is locally Lipschitz continuous, and hence, for $p = 2$, the semigroup approach (see, e.g., [Pazy, 1983, Chapter 6]) guarantees the local ($T \leq \infty$) existence of a unique classical solution of (5.1.1). Therefore, we can assume a bounded global solution u for the problem (5.1.1).

5.2 The positivity and boundedness of the solution

Theorem 5.2.1. *Let u be a bounded global solution of (5.1.1) with initial value $u_0 \in L^{\infty}(\Omega)$ such that $0 \leqslant u_0(x) \leqslant 1$. Then the global solution satisfies*

$$ 0 \leqslant u(x,t) \leqslant 1, \quad \textit{for} \quad (x,t) \in \Omega \times (0,T) $$

with the initial-boundary conditions (5.1.3)-(5.1.4).

Proof. Let $\tilde{u}(x,t) := \min(u(x,t), 0)$, which satisfies problem (5.1.1). Multiplying both sides of (5.1.1) by $\tilde{u}(x,t)$ and integrating over Ω, we obtain

$$ \int_{\Omega} \tilde{u}_t(x,t)\, \tilde{u}(x,t)\, d\nu - \int_{\Omega} \tilde{u}(x,t)\, \mathcal{L}_p \tilde{u}(x,t)\, d\nu = \int_{\Omega} \tilde{u}^2(x,t)\, (\tilde{u}(x,t) - 1)\, d\nu. $$

We apply Theorem 2.1.16 to the second term on the left-hand side and obtain

$$ \int_{\Omega} \left(|\nabla_{\mathbb{G}} \tilde{u}(x,t)|^{p-2}\, \widetilde{\nabla} \tilde{u}(x,t) \right) \tilde{u}(x,t) d\nu = - \int_{\Omega} \tilde{u}(x,t) \mathcal{L}_p \tilde{u}(x,t) $$

Since

$$ \begin{aligned} \left(|\nabla_{\mathbb{G}} \tilde{u}(x,t)|^{p-2}\, \widetilde{\nabla} \tilde{u}(x,t) \right) \tilde{u}(x,t) &= |\nabla_{\mathbb{G}} \tilde{u}(x,t)|^{p-2} \sum_{k=1}^{N_1} (X_k \tilde{u}(x,t)) X_k \tilde{u}(x,t) \\ &= |\nabla_{\mathbb{G}} \tilde{u}(x,t)|^{p} \geqslant 0, \end{aligned} $$

we have

$$\begin{aligned}\int_\Omega \tilde{u}_t(x,t)\,\tilde{u}(x,t)\,d\nu &\leq \int_\Omega \tilde{u}_t(x,t)\,\tilde{u}(x,t)\,d\nu + |\nabla_{\mathbb{G}}\tilde{u}(x,t)|^p \\ &= \int_\Omega \tilde{u}^2(x,t)\,(\tilde{u}(x,t)-1)\,d\nu \\ &\leq -\int_\Omega \tilde{u}^2(x,t)d\nu.\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \tilde{u}^2(x,t)d\nu + 2\int_\Omega \tilde{u}^2(x,t)d\nu \leq 0,$$

which is equivalent to the inequality

$$Y_t(t) + 2Y(t) \leq 0, \quad Y(t) := \int_\Omega \tilde{u}^2(x,t)d\nu.$$

We can conclude that $Y(t) \leq 0$ for all $t \geq 0$. This implies that $\int_\Omega \tilde{u}^2(x,t)d\nu = 0$ for all $t \geq 0$, which in turn implies that $\tilde{u}(x,t) = 0$ for all $x \in \Omega$ and $t \geq 0$. Since $\tilde{u}(x,t) = \min(u(x,t), 0)$, we have $u(x,t) \geq 0$ for all $x \in \Omega$ and $t \geq 0$.

To complete the proof, it remains to establish that $u(x,t) \leqslant 1$. We begin by assuming $v := 1 - u$, and defining $\tilde{v} = \min(v(x,t), 0)$. As before, we obtain the following inequalities:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \tilde{v}^2 d\nu + \int_\Omega |\nabla_{\mathbb{G}}\tilde{v}|^p d\nu = \int_\Omega \tilde{v}(\tilde{v}(1-\tilde{v}))d\nu$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \tilde{v}^2 d\nu \leq \int_\Omega \tilde{v}(\tilde{v}(1-\tilde{v}))d\nu.$$

Since u is a bounded global solution, there exists a constant C such that $|1-\tilde{v}| \leq C$. Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \tilde{v}^2(x,t)d\nu \leq C\int_\Omega \tilde{v}^2(x,t)d\nu.$$

Using a similar argument as before, we conclude that $\int_\Omega \tilde{v}^2\big(x,t\big)d\nu = 0$, and $\tilde{v}\big(x,t\big) = 0$. This implies that $1-u(x,t) \geq 0$, and therefore, $u(x,t) \leq 1$. Combining the results from the previous steps, we obtain $0 \leq u(x,t) \leq 1$, as desired. $\square$

5.3 Asymptotic time-behavior

Theorem 5.3.1. *Assume that $u \in L^\infty(\Omega)$ is a bounded global solution, $p = 2$ and $0 \leqslant u_0 \leqslant 1$. Then the solution of the problem (5.1.1) with the initial-boundary conditions (5.1.3)-(5.1.4) satisfies the inequality*

$$\|u(x,t)\|_{L^2(\Omega)} \leqslant e^{-\lambda_{1,2}t}\|u_0(x)\|_{L^2(\Omega)}, \quad t > 0, \tag{5.3.1}$$

Here, $\lambda_{1,2}$ denotes the first eigenvalue of the Dirichlet sub-Laplacian problem.

Proof. By applying Theorem 5.2.1, we have $0 \leqslant u \leqslant 1$ and $u(u-1) \leqslant 0$. Hence, we get the following estimate:

$$u_t(x,t) - \mathcal{L}u(x,t) \leqslant 0, \quad x \in \Omega, \quad t > 0. \tag{5.3.2}$$

Multiplying both sides of (5.3.2) by $u(x,t)$ and integrating over Ω, Theorem 2.1.16 provides

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u^2\, d\nu + \int_\Omega |\nabla_{\mathbb{G}} u(x,t)|^2\, d\nu \leqslant 0.$$

Applying the Poincaré inequality, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u^2(x,t)d\nu + 2\lambda_{1,2}(\Omega)\int_\Omega u^2(x,t)d\nu \leq 0,$$

We define $Y(t) := \int\limits_\Omega u^2(x,t)d\nu$, and $Y_0 = \int_\Omega u_0^2(x)d\nu$ and then obtain

$$Y'(t) + 2\lambda_{1,2}(\Omega)Y(t) \leqslant 0, \quad \text{for} \quad t > 0.$$

Consequently, we have

$$Y(t) \leqslant Y_0 e^{-2\lambda_{1,2}(\Omega)\cdot t},$$

which gives us the time behavior in L^2-norm:

$$\|u(x,t)\|_{L^2(\Omega)} \leqslant e^{-2\lambda_{1,p}(\Omega)\cdot t}\|u_0(x)\|_{L^2(\Omega)}, \quad t > 0.$$

□

5.4 Blow-up in finite time

Let $\lambda_{1,2}(\Omega)$ the first eigenvalue with corresponding eigenfunction u_1, which we introduced in Section 2.1.4, and assume that $\int_\Omega u_1(x)dx = 1$. We prove the following theorem:

Theorem 5.4.1. *Assume that $u_0 \in L^2(\Omega)$. Suppose that $1 + \lambda_{1,2}(\Omega) < H_0$, where $H_0 := \int_\Omega u_0(x)u_1(x), d\nu$. Then, for $p = 2$, the weak solution u of the problem (5.1.1) with the conditions (5.1.3)-(5.1.4) blows up in a finite time*

$$T_0 = \frac{1}{1+\lambda_{1,2}(\Omega)}\log\frac{H_0}{H_0 - (1+\lambda_{1,2}(\Omega))}.$$

Proof. We multiply both sides of the equation (5.1.1) by u_1, integrate it over Ω, and obtain

$$\int_\Omega u_t(x,t)u_1(x)d\nu - \int_\Omega \mathcal{L}u(x,t)u_1(x)d\nu = \int_\Omega u(x,t)\big(u(x,t)-1\big)u_1(x)d\nu. \qquad (5.4.1)$$

Applying Theorem 2.1.16, we get

$$-\int_\Omega \mathcal{L}u(x,t)u_1(x)d\nu = -\int_\Omega u(x,t)\mathcal{L}u_1(x)d\nu = \lambda_{1,2}(\Omega)\int_\Omega u(x,t)u_1(x)d\nu.$$

Using Jensen's inequality, we have

$$\int_\Omega u^2(x,t)u_1(x)dx \geq \left(\int_\Omega u(x,t)u_1(x)dx\right)^2 = H^2(t),$$

with $H(t) = \int_\Omega u(x,t)u_1(x)d\nu$. Therefore, equation (5.4.1) becomes

$$H_t(t) + (1+\lambda_{1,2}(\Omega))H(t) = \int_\Omega u^2(x,t)u_1(x)\,d\nu \geqslant H^2(t). \qquad (5.4.2)$$

with $H_0 := \int_\Omega u(x,0)u_1(x)dx$. Finally, using the basic properties of Bernoulli-type ordinary differential inequalities, we obtain

$$H(t) \geq \frac{1+\lambda_{1,2}(\Omega)}{1-e^{(1+\lambda_{1,2}(\Omega))(t-T^*)}},$$

where $T^* = \log\left|\frac{H_0}{H_0-1-\lambda_{1,2}(\Omega)}\right|$. It completes the proof. □

5.5 Positivity and boundedness of the solution: Time fractional case

Theorem 5.5.1. *Let u be a bounded global solution. Assume that $u_0 \in L^\infty(\Omega)$ and $0 \leqslant u_0(x) \leqslant 1$ for all $x \in \Omega$. Then, the solution of problem (5.1.2) with initial-boundary conditions (5.1.3)-(5.1.4) satisfies*

$$0 \leqslant u(x,t) \leqslant 1, \quad \textit{for} \quad (x,t) \in \Omega \times (0,T).$$

Proof. We define $\tilde{u}(x,t) = \min(u(x,t),0)$, which satisfies problem (5.1.2). By multiplying both sides of (5.1.2) by $\tilde{u}(x,t)$ and integrating over Ω, we obtain

$$\int_\Omega {}^C\partial_t^\nu \tilde{u}(x,t)\,\tilde{u}(x,t)\,d\nu - \int_\Omega \tilde{u}(x,t)\,\mathcal{L}_p\tilde{u}(x,t)\,d\nu = \int_\Omega \tilde{u}^2(x,t)\,(\tilde{u}(x,t)-1)\,d\nu.$$

Using Theorem 2.1.16 for the second term on the left-hand side, we get

$$-\int_\Omega \tilde{u}(x,t)\mathcal{L}_p\tilde{u}(x,t)d\nu = \int_\Omega \left(|\nabla_{\mathbb{G}}\tilde{u}(x,t)|^{p-2}\,\widetilde{\nabla}\tilde{u}(x,t)\right)\tilde{u}(x,t)d\nu.$$

Using the fact that

$$\left(|\nabla_{\mathbb{G}}\tilde{u}(x,t)|^{p-2}\,\widetilde{\nabla}\tilde{u}(x,t)\right)\tilde{u}(x,t) = |\nabla_{\mathbb{G}}\tilde{u}(x,t)|^{p-2}\sum_{k=1}^{N_1}(X_k\tilde{u}(x,t))X_k\tilde{u}(x,t)$$
$$= |\nabla_{\mathbb{G}}\tilde{u}(x,t)|^{p} \geqslant 0,$$

and Theorem 2.2.12, we conclude that

$$\begin{aligned}\frac{1}{2}\,{}^{C}\partial_t^{\nu}\int_{\Omega}\tilde{u}^2(x,t)\,d\nu &\le \int_{\Omega}{}^{C}\partial_t^{\nu}\tilde{u}(x,t)\,\tilde{u}(x,t)\,d\nu\\ &\le \int_{\Omega}{}^{C}\partial_t^{\nu}\tilde{u}(x,t)\,\tilde{u}(x,t)\,d\nu + |\nabla_{\mathbb{G}}\tilde{u}(x,t)|^{p}\\ &= \int_{\Omega}\tilde{u}^2(x,t)\,(\tilde{u}(x,t)-1)\,d\nu\\ &\le -\int_{\Omega}\tilde{u}^2(x,t)d\nu.\end{aligned}$$

Then

$${}^{C}\partial_t^{\nu}\int_{\Omega}\tilde{u}^2(x,t)d\nu + 2\int_{\Omega}\tilde{u}^2(x,t)d\nu \le 0,$$

which is equivalent to the inequality

$${}^{C}\partial_t^{\nu}Y(t) + 2Y(t) \le 0, \quad \text{with} \quad Y(t) = \int_{\Omega}\tilde{u}^2(x,t)d\nu.$$

Setting $Y(t) \le \widetilde{Y}(t)$ and applying Theorem 2.2.13, we obtain

$${}^{C}\partial_t^{\nu}\widetilde{Y}(t) + 2\widetilde{Y}(t) = 0 \quad \text{with} \quad \widetilde{Y}(0) = \widetilde{Y}_0 = \int_{\Omega}\tilde{u}_0^2(x)dx = 0,$$

and

$$Y(t) \leqslant \widetilde{Y}_0 E_{\nu}\left(-2t^{\nu}\right) = 0,$$

where E_{ν} is the Mittag-Leffler function. This implies that $\int_{\Omega}\tilde{u}^2(x,t)d\nu = 0$, and therefore $\tilde{u}(x,t) = 0$. Moreover, $u(x,t) \ge 0$.

To show that $u(x,t) \le 1$, we introduce the function $v(x,t) := 1 - u(x,t)$, and denote by $\tilde{v}(x,t) := \min(v(x,t),0)$ its negative part. Similar to the previous derivation, we obtain the inequality

$$\frac{1}{2}\,{}^{C}\partial_t^{\nu}\int_{\Omega}\tilde{v}^2 d\nu + \int_{\Omega}|\nabla_{\mathbb{G}}\tilde{v}|^p d\nu = \int_{\Omega}\tilde{v}(\tilde{v}(1-\tilde{v}))d\nu.$$

By the assumption, u is a bounded global solution exists. Therefore, there exists a constant L such that $|1-\tilde{v}| \le L$. We can use this estimate to derive

$$\frac{1}{2}\,{}^{C}\partial_t^{\nu}\int_{\Omega}\tilde{v}^2(x,t)d\nu \le L\int_{\Omega}\tilde{v}^2(x,t)d\nu.$$

By applying Theorem 2.2.13 again, we get $\int_{\Omega}\tilde{v}^2(x,t)d\nu = 0$, and hence $\tilde{v}(x,t) = 0$. This implies that $1 - u(x,t) \ge 0$, and therefore $u(x,t) \le 1$.

To summarize, we have shown that $0 \le u(x,t) \le 1$, which completes the proof. □

5.6 Asymptotic time-behavior: Time fractional case

Theorem 5.6.1. *Let $u \in L^\infty(\Omega)$ be a bounded global solution and $0 \leqslant u_0 \leqslant 1$ be the initial condition. Consider the problem (5.1.2) with initial-boundary conditions (5.1.3)-(5.1.4) with $p = 2$. Then the solution satisfies the inequality*

$$\|u(x,t)\|_{L^2(\Omega)} \leq E_\nu\left(-2\lambda_{1,2}t^\nu\right)\|u_0(x)\|_{L^2(\Omega)}, \tag{5.6.1}$$

where E_ν is the Mittag-Leffler function, and $\lambda_{1,2}$ is the first eigenvalue of the Dirichlet sub-Laplacian problem.

Proof. We apply Theorem 5.2.1 with $0 \leqslant u \leqslant 1$, and $u(u-1) \leqslant 0$. Therefore, we obtain

$$^C\partial_t^\nu u(x,t) - \mathcal{L}u(x,t) \leqslant 0, \quad x \in \Omega, \quad t > 0. \tag{5.6.2}$$

Multiplying both sides of (5.6.2) by $u(x,t)$ and integrating over Ω, and using Theorem 2.1.16, we obtain

$$\frac{1}{2}\,{}^C\partial_t^\nu \int_\Omega u^2\, d\nu + \int_\Omega |\nabla_{\mathbb{G}} u(x,t)|^2\, d\nu \leqslant 0.$$

Applying the Poincaré inequality, we get

$$^C\partial_t^\nu \int_\Omega u^2(x,t)d\nu + 2\lambda_{1,2}(\Omega)\int_\Omega u^2(x,t)d\nu \leq 0,$$

Setting $W(t) := \int\limits_\Omega u^2(x,t)d\nu$, and $W_0 := \int_\Omega u_0^2(x)d\nu$, we have

$$^C\partial_t^\nu W(t) + 2\lambda_{1,2}W(t) \leqslant 0, \quad \text{for} \quad t > 0.$$

Let $\widetilde{W}(t)$ be the solution of the equation

$$^C\partial_t^\nu \widetilde{W}(t) + 2\,\lambda_{1,2}(\Omega)\widetilde{W}(t) = 0$$

with

$$\widetilde{W}(0) = \widetilde{W}_0 = \int_\Omega u_0^2(x)dx.$$

such that $W(t) \leq \widetilde{W}(t)$, then by Theorem 2.2.13, we obtain

$$W(t) \leqslant \widetilde{W}_0 E_\nu\left(-2\lambda_{1,2}t^\nu\right),$$

where E_ν is the Mittag-Leffler function. Thus, we obtain

$$\|u(x,t)\|_{L^2(\Omega)} \leq E_\nu\left(-2\lambda_{1,2}t^\nu\right)\|u_0(x)\|_{L^2(\Omega)}. \tag{5.6.3}$$

□

5.7 Blow-up: Time fractional case

In this section, we establish the occurrence of finite-time blow-up phenomena for the solution of the problem (5.1.2) with $p = 2$, and the conditions (5.1.3)-(5.1.4), in the time-fractional setting. Assume that $\lambda_{1,2}(\Omega)$ is the first eigenvalue with corresponding eigenfunction u_1 that satisfy (2.1.7), Here, we use the assumption $\int_\Omega u_1(x)dx = 1$.

Theorem 5.7.1. *Assume that $u_0 \in L^2(\Omega)$. If $1 + \lambda_{1,2}(\Omega) < \int_\Omega u_0(x)u_1(x), d\nu := H_0$, then the weak solution u of (5.1.2) blows up in a finite time.*

Proof. We begin by multiplying both sides of (5.1.1) by u_1 and integrating over Ω, yielding

$$\int_\Omega {}^C\partial_t^\nu u(x,t)u_1(x)d\nu - \int_\Omega \mathcal{L}u(x,t)u_1(x)d\nu = \int_\Omega u(x,t)\big(u(x,t)-1\big)u_1(x)d\nu. \tag{5.7.1}$$

By using Section 2.1.4 and Theorem 2.1.16, we get

$$-\int_\Omega \mathcal{L}u(x,t)u_1(x)d\nu = -\int_\Omega u(x,t)\mathcal{L}u_1(x)d\nu = \lambda_{1,2}(\Omega)\int_\Omega u(x,t)u_1(x)d\nu,$$

where $\lambda_{1,2}(\Omega) > 0$.

Next, we use Jensen's inequality to obtain

$$\int_\Omega u^2(x,t)u_1(x)dx \geq \left(\int_\Omega u(x,t)u_1(x)dx\right)^2 = H^2(t),$$

with $H(t) = \int_\Omega u(x,t)u_1(x)d\nu$. Substituting this into (5.7.1), we have

$${}^C\partial_t^\nu H(t) + (1+\lambda_{1,2}(\Omega))H(t) = \int_\Omega u^2(x,t)u_1(x)\,d\nu \geqslant H^2(t). \tag{5.7.2}$$

with $H_0 := \int_\Omega u(x,0)u_1(x)dx$.

We introduce the substitution $H = \widetilde{H} + (1+\lambda_{1,2}(\Omega))$, with $\widetilde{H}(0) > 0$. Then, (5.7.2) becomes

$${}^C\partial_t^\nu \widetilde{H}(t) \geq \widetilde{H}(t)(\widetilde{H}(t) + 1 + \lambda_{1,2}(\Omega)) \geq \widetilde{H}(t)(\widetilde{H}(t)+1). \tag{5.7.3}$$

We note that the assumption $1+\lambda_{1,2}(\Omega) < H_0$ implies that $\widetilde{H}(0) = H_0 - (1+\lambda_{1,2}(\Omega)) > 0$. By applying Theorem 2.2.11, we can conclude that the solution of the differential inequality (5.7.3) blows up in a finite time. □

Remark 5.7.2. The nonlinear part $u(u-1)$ of the Fisher-KPP models can be extended by employing a similar approach to that used in Theorems 5.2.1 and 5.5.1. This extension involves the utilization of a convex function $f(u)$, satisfying the conditions $f(0) = f(1) = 0$.

Chapter 6

Conclusions

Nonlocal equations with fractional Laplacians have gained significant attention in PDE research groups, and have been used to describe complex phenomena in various fields such as turbulence [Bakunin, 2008], elasticity [Dipierro et al., 2015], anomalous transport and diffusion [Bologna et al., 2000, Meerschaert, 2012], image processing [Gilboa and Osher, 2008], wave propagation in heterogeneous high contrast media [Tieyuan and Harris, 2014], and porous media flow [Vázquez, 2012]. Additionally, the fractional Laplacian is the generator of s-stable processes, which has applications in stochastic models such as mathematical finance [Levendorskiĭ, 2004, Pham, 1997].

This work studies the fractional type of the Fisher-KPP problem given by

$$ {}^{C}\partial_t^{\nu} u + \left(-\Delta_{p,\mathbb{G}}\right)^s u = u(u-1) \quad \text{in} \quad \Omega \subset \mathbb{G}, \tag{6.0.1} $$

for $p \in (1,\infty)$, $\nu \in (0,1]$, $t > 0$ and $s \in (0,1]$, where the derivative in time is given by the Caputo fractional derivative ${}^{C}\partial_t^{\nu}$, and the fractional p-sub-Laplacian $\left(-\Delta_{p,\mathbb{G}}\right)^s$. The initial-boundary conditions are given by

$$ u(x,0) = u_0(x), \qquad x \in \Omega, \tag{6.0.2} $$

and

$$ u(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0. \tag{6.0.3} $$

The examination of the Fisher-KPP equation in the setting of stratified groups is a crucial part of this work. Studying the mathematical models with sub-Laplacians on stratified groups is of significant importance in theoretical (see, e.g., [Gromov, 1996, Danielli et al., 2007]) and applied fields, such as human vision and crystal material (see, for example, [Christodoulou, 1998, Citti et al., 2004]).

We obtained three main results, presented as Theorems 1.1.1 to 1.1.3.

Theorem 1.1.1 is a combination of Theorems 4.2.1, 4.5.1, 5.2.1, and 5.5.1. Theorem 1.1.2 is derived from Theorems 4.3.1, 4.6.1, 5.3.1, and 5.6.1. Finally, Theorem 1.1.3 is obtained by combining Theorems 4.4.1, 4.7.1, 5.4.1, and 5.7.1.

By considering Remarks 3.7.2, 4.7.2, and 5.7.2, we can extend the applicability of the presented approach in Theorem 1.1.1 to analyze the Fisher-KPP equation with a certain class of convex functions $f(u)$ with the conditions $f(0) = f(1) = 0$.

Our results have provided important insights into the behavior of the solutions of the fractional Fisher-KPP equation. In particular, we have demonstrated that the global solution's range remains between 0 and 1 if the initial data falls within this range. Finally, under certain conditions, we have demonstrated that the solution of the fractional Fisher-KPP equation blows up in a finite time. These findings improve our understanding of the behavior of solutions of fractional differential equations and may have important implications for the application of these models in various fields.

.1 Appendix: Fourier definition of the fractional Laplacian

This review section draws on the work of Lischke [Lischke et al., 2020] to introduce the fractional Laplacian, denoted by $(-\Delta)^s$. Here, Δ is the second-order partial differential operator expressed as

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_N^2}.$$

The operator $(-\Delta)^s$ is defined as the fractional power of $(-\Delta)$ to obtain a positive operator. It serves as the negative generator of the standard isotropic s-stable Lévy process and can be reduced to $-\Delta$ when $s = 1$.

Our objective is to construct a fractional power of the Laplacian, denoted as $(-\Delta)^s$, where $0 < s < 1$. To achieve this, we define the positive real powers L^ρ of a positive self-adjoint linear operator $L = -\Delta$, where $\rho \in [-1, 1]$.

The spectral theorem, as introduced by [Reed and Simon, 1980], states that for a self-adjoint, densely defined linear operator L on a Hilbert space $\mathcal{H}$ with a dense subspace $\mathcal{D}(L)$, there exists a projection-valued measure Ψ_λ such that

$$L = \int_{\lambda \in \sigma(L)} \lambda d\Psi_\lambda$$

on $\mathcal{D}(L) \subset \mathcal{H}$.

Here, Ψ_λ is the unique operator-valued spectral measure associated with L, and $\sigma(L) \subset \mathbb{R}$ is the spectrum of L, which is the support of Ψ_λ.

We define the operator L of order $-1 \leq s \leq 1$ as the self-adjoint operator

$$L^s := \int_{\lambda \in \sigma(L)} \lambda^s d\Psi_\lambda. \tag{.1.1}$$

To avoid negative eigenvalues in the resulting fractional operator, we use $-\Delta$ instead of Δ. The operator Δ has negative eigenvalues, which makes it unsuitable for applying the spectral theorem, typically used for positive-definite operators.

The operator L^s can be extended to the Hilbert space $\mathcal{H}$ by continuity. Our focus is on studying the operator $L = -\Delta$ on $\mathbb{R}^N$ in the Sobolev space $\mathcal{H} = H^2\left(\mathbb{R}^N\right)$, equipped with the $\mathcal{H}$-norm, and domain $\mathcal{D}(-\Delta) = C_0^\infty\left(\mathbb{R}^N\right)$, which is a dense subspace of $\mathcal{H}$. This leads to the following representation for $-\Delta$:

$$-\Delta = \int_{\sigma(-\Delta)} \lambda d\Psi_\lambda.$$

In the case of a regular domain Ω, it is known from the work of [Reed and Simon, 1980] that the spectrum $\sigma(-\Delta)$ only contains eigenvalues, and thus $d\Psi_\lambda$ can be regarded as a projection operator onto the eigenspace of λ for each λ. From equation (.1.1), we prove the fractional Laplacian on $\mathbb{R}^N$ as:

$$(-\Delta)^s := \int_{\sigma(-\Delta)} \lambda^s d\Psi_\lambda \tag{.1.2}$$

In order to provide a clear definition of (.1.2), it is important to have a precise understanding of the spectrum $\sigma(-\Delta)$ in $\mathbb{R}^N$, which consists of eigenvalues $|\xi|^2$ for $\xi \in \mathbb{R}^N$ and their corresponding generalized eigenfunctions $e^{-i\xi\cdot x}$. Thus, the projection-valued measure on $\mathcal{D}(-\Delta)$ can be expressed as

$$d\Psi = \frac{1}{(2\pi)^N}\left(\cdot, e^{-i\xi\cdot x}\right) e^{i\xi\cdot x} d\xi$$

where $(v, w) = \int vw dx$ is the L^2 inner product on $\mathbb{R}^N$. The scaling factor $1/(2\pi)^N$ ensures that $\int d\Psi_\lambda = I$, where I is the identity. Hence, the fractional Laplacian in $\mathbb{R}^N$ can be expressed as

$$(-\Delta)^s u(x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} |\xi|^{2s}\left(u, e^{-i\xi\cdot x}\right) e^{i\xi\cdot x} d\xi = \mathcal{F}^{-1}\left\{|\xi|^{2s}\mathcal{F}\{u\}(\xi)\right\}(x),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier and inverse Fourier transforms, respectively. Additionally, we have

$$\mathcal{F}\left\{(-\Delta)^s u\right\}(\xi) = |\xi|^{2s}\mathcal{F}\{u\}(\xi) \tag{.1.3}$$

which implies that $(-\Delta)^s$, defined in (.1.2), is a Fourier multiplier operator with symbol $|\xi|^{2s}$. This generalizes the well-known Fourier multiplier property of $-\Delta$. While some authors define the fractional Laplacian as a pseudodifferential operator using this relation [Stein, 1970], this approach has the disadvantage of rendering the Fourier transform unavailable for bounded domains, although the functional calculus approach (.1.2) using the spectral theorem can still be employed in the case of zero boundary conditions.

The multiplier $|\xi|^{2s}$ decays for $-N < 2s < 0$ (fractional inverse Laplacians) and has a Fourier inverse in the distribution sense, given by $\mathcal{F}^{-1}\left\{|\xi|^{2s}\right\} = C(N,s)|x|^{-d-2s}$, where $C(N,s)$ is a constant given by the formula mentioned in [Stein, 1970] or [Landkof, 1972]:

$$C(N,s) = \frac{2^{2s}\Gamma\left(s+\frac{N}{2}\right)}{\pi^{N/2}\left|\Gamma\left(-s\right)\right|} \tag{.1.4}$$

By using the convolution property of the Fourier transform and equation (.1.3), the fractional inverse Laplacian can be expressed as

$$(-\Delta)^s u(x) = C(N,s)|x|^{-N-2s} * u(x) = C(N,s)\int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N+2s}}dy =: I_{-2s}u(x), \tag{.1.5}$$

for $-1 < s < 0$. This expression is well-defined for a smooth function $u(x)$ with sufficient decay, as explained in [Reed and Simon, 1980] or [Friedlander, 1998]. The operator I_{-2s} is known as the Riesz potential, which is a crucial tool in the analysis of linear PDEs and harmonic analysis [Hörmander, 2003]. The book by Stein [Stein, 1970] discusses the properties of the Riesz potential, such as its L^p boundedness.

However, for $0 < s < 1$, the derivation presented earlier is not applicable because the Fourier transform of the symbol $|\xi|^{2s}$ does not exist even as a distribution. In this case, it is possible to obtain a real-space formula for the fractional Laplacian using an argument based on analytic continuation in s, as presented by Landkof ([Landkof, 1972]). This formula is given by:

$$(-\Delta)^s u = C(N,s)\ \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy \tag{.1.6}$$

Here, $C(N,s)$ is the same constant as in equation (.1.4), and "P.V." denotes the principal value of the integral. The principal value is defined as the limit of the integral over a ball of radius ϵ centered at x as ϵ approaches zero. The regularization provided by the difference $u(x)-u(y)$ in the numerator of (.1.6), which vanishes at the singularity, along with averaging of positive and negative parts, allows the principal value to exist for smooth u with sufficient decay.

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