THE TERNARY SUBSPACE AND SYMMETRIC PART OF AN OPERATOR SPACE

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Introduction. In 2003, V. I. Paulsen and I defined the ternary subspace of an operator space as the intersection of the space and the adjoint of its quasi-multiplier space. Recently, M. Neal and B. Russo defined the completely symmetric part of an operator space by considering the symmetric part of the matrix of infinite size with entries in the operator space, and posed the question: Under what conditions does it consist of the adjoint of quasi-multipliers? I give a partial answer to this question revealing the relationship between the ternary subspace and the completely symmetric part.

Materials and methods. The key idea is to embed an operator space $\mathcal{X}$ into its linking multiplier algebra $\mathcal{M}_{\text{link}}(\mathcal{X})$ which I introduced. Since the linking multiplier algebra is a unital operator algebra, its symmetric part coincides with the C^*-algebra $\mathcal{M}_{\text{link}}(\mathcal{X}) \cap \mathcal{M}_{\text{link}}(\mathcal{X})^*$ by a result of J. Arazy and B. Sorel, while an off-diagonal corner of this C^*-algebra coincides with the ternary subspace of $\mathcal{X}$. This makes a connection between TER $(\mathcal{X})$ and CS $(\mathcal{X})$.

Results. Definition (M. Kaneda and V. A. Paulsen 2003): The *ternary subspace* of an operator space $\mathcal{X}$ is the ternary ring of operators (TRO) defined by $\text{TER}(\mathcal{X}) := \mathcal{X} \cap \mathcal{QM}(\mathcal{X})^*$, where $\mathcal{QM}(\mathcal{X})$ is the quasi-multiplier space of $\mathcal{X}$.

Definition (M. Neal and B. Russo 2012): Let $\mathcal{X}$ be an operator space.

The *completely symmetric part* of $\mathcal{X}$ is defined to be $\text{CS}(\mathcal{X}) := \mathcal{X} \cap \text{S}(\text{M}_\infty(\mathcal{X}))$, where $\text{S}(\text{M}_\infty(\mathcal{X})$ is the symmetric part of the infinite matrix $\text{M}_\infty(\mathcal{X})$.

The *noncommutative Jordan product* $y \cdot x$ for $x, y \in \mathcal{X}$ corresponding to $v \in \text{CS}(\mathcal{X})$ is defined by

$$\begin{bmatrix} y \cdot x & 0 \\ 0 & 0 \end{bmatrix} = 2\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\}_{M_2(X)},$$

where $\{\cdot,\cdot,\cdot\}_{M_2(X)}$ is the partial Jordan triple product on $M_2(\mathcal{X})$.

Proposition (M. Kaneda 2014): *For any operator space* X, $\text{TER}(\mathcal{X}) \subset \text{CS}(\mathcal{X})$.

Theorem (M. Kaneda 2014): *If $\mathcal{X}$ is an operator space and* $v \in \text{TER}(\mathcal{X})$ *such that* $\|v\| \leq 1$, *then the noncommutative Jordan product is an operator algebra product.*

Discussion. The notion of symmetric part comes from holomorphy (analytical property), while the ternary subspace is defined algebraically. Revealing the relationship between $\text{TER}(\mathcal{X})$ and $\text{CS}(\mathcal{X})$ links these two notions with different origins and may give insights into solving algebraic problems analytically and vice versa. For instance, I am now attempting to solve the algebraic question raised by myself which was presented in one of my posters last year: 'Does the unit ball of an injective operator space always have an extreme point?' analytically using this theorem.