Analysis of Dead Core Phenomena in Reaction-Diffusion Problems

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Abstract

For some semilinear parabolic problems of reaction-diffusion, a dead core - region of zero reactant concentration - may be formed in finite time. We study the large time behavior of the solution and give an estimate for the asymptotic behavior of the solution of a semilinear heat equation with Robin boundary condition.

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1 Introduction

In a heterogeneous reaction sequence, mass transfer of reactants first takes place from the bulk fluid to the external surface of the pellet. The reactants then diffuse from the external surface into and through the pores within the pellet, with reaction taking place only on the catalytic surface of the pores. At some "critical" Thiele modulus the concentration reaches zero at the center of the pellet. This region devoid of reactant is defined as "Dead zone" and/or "Dead core."

Let $\Omega \in \mathbb{R}$ be a bounded smooth domain. In this work, we consider the boundary value problem in $\mathcal{Q} = \Omega \times (0, +\infty)$

$$
\begin{aligned}
u_t - u_{xx} + \lambda u^p &= 0, \quad \text{in} \quad \Omega, && (1.1a)\\
u_x + Bi_m(u-1) &= 0, \quad \text{on} \quad \partial\Omega \times (0,\infty), && (1.1b)\\
u_x &= 0, \quad \text{on} \quad \{0\} \times (0,\infty), && (1.1c)\\
u &= 1, \quad \text{in} \quad \Omega \times \{0\}. && (1.1d)
\end{aligned}
$$

where $p \in (0,1)$ is the order of the reaction, and $w_+ = \max\{w, 0\}$. $Bi_m > 0$ stands for a Biot number.

We study the problem only for $p \in (0,1)$ because if $p \geq 1$ then the dead core is empty, by the maximum principle, while for $p \leq 1$ the dead core phenomena takes place. In particular, a dead core occurs for some sufficiently large λ. The physical explanation is that if the reaction rate remains high as the concentration decreases, diffusion may not be strong enough to draw the reactant from the boundary into the central part of Ω. Mathematically, it means that there exists a t^* such that a set $N_t = \{x \in \Omega : u(x,t) = 0\} \neq \emptyset$ for $t > t^*$. For any $t, N_t \subset N$, where N is the null set of the family of elliptic problems (S_λ)

$$
\begin{aligned}
-u_{xx} + \lambda u^p &= 0, \quad \text{in} \quad \Omega, && (1.2a)\\
u_x + Bi_m(u-1) &= 0, \quad \text{on} \quad \partial\Omega \times (0,\infty) && (1.2b)
\end{aligned}
$$

where $\lambda > 0$. Notice that if $N = \emptyset$, then there is no dead core and $u(x,t) > 0$ for any t. The dimensionless steady-state mass balance for a single n-th order chemical reaction and diffusion in the catalytic pellets of various geometries is given by

$$\frac{1}{x^s}\frac{d}{dx}(x^s\frac{du}{dx}) - \lambda u_+^p = 0,$$
$$\frac{du}{dx}(0) = 0, \quad (1.3)$$
$$\frac{du}{dx}(1) + Bi_m(u(1) - 1) = 0,$$

$s = 0, 1, 2$ is the shape factor corresponding to the planar, infinite length cylinder and spherical geometries, respectively.

This paper aims to to study the asymptotic behavior of the solution of problem (1.1). In this direction, many papers have been written [1-5] and one of them is by Ricci [6], who investigated the asymptotic behavior of the solution of the dead core problem and proved that the convergence is exponentially fast.

We show that the solution converges to a stationary solution as t tends to infinity and evaluate its rate of convergence.

Theorem 1.1 *Let $0 < p < 1$ and let $\Omega \subset \mathbb{R}$ be a bounded $C^{2+\alpha}$ domain. Then there exist two positive constants C_1 and C_2 such that*

$$\| u(x,t) - u_\infty(x) \|_{L^\infty(\Omega)} \leq C_1 e^{-C_2 t}. \quad (1.4)$$

In Section 2 we review essential elements that are relevant to our study and give a brief overview of function spaces used throughout the paper along with their basic properties. In Section 3 we show a result on existence, uniqueness and in Section 4 prove a monotonicity and boundedness of solutions. Section 5 is devoted to prove the Lipschitz continuity of $u(\cdot;\lambda)$ when λ varies in $\mathbb{R}^+$. In Section 6 we construct upper and lower solution for problem (1.1) and we prove Theorem 1.1.

2 Function Spaces

In order to formulate precise theorems of existence, uniqueness, continuous dependence we need to specify the spaces in which solutions lie and to give a precise meaning to convergence in those spaces. Detailed explanation are to be found in [7-11]

If α is a multi-index $(\alpha_1, ..., \alpha_n)$, then $|\alpha| = \sum_n \alpha_n, x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$. Similarly, if $D_i = \partial/\partial x_i$, then $D^\alpha = D_1^{\alpha_1}...D_n^{\alpha_n}$ denotes a differential operator of order $|\alpha|$.

Definition 2.1 A *norm* on a vector space X is a real-valued function f on X satisfying the following conditions for any $x, y \in X$ and $c \in \mathbb{R}$ (or $\mathbb{C}$):

1. $f(x) \geq 0$ and $f = 0$ if and only if $x = 0$,
2. $f(cx) = |c| f(x)$,
3. $f(x + y) \leq f(x) + f(y)$.

A vector space equipped with a norm is known as a *normed space*. The norm on X will be written as $\| \cdot \|_X$. The norm induces a metric on X given by

$$d(x, y) = \| x - y \| .$$

A sequence $\{x_n\}$ in a normed space X is *convergent* to the limit x_0 if and only if $\lim_{n\to\infty} \| x_n - x_0 \| = 0$ in $\mathbb{R}$.

A subset S of a normed space X is said to be *dense* in X if each $x \in X$ is the limit of a sequence of elements of S. The normed space X is called *separable* if it has a countable dense subset.

A sequence $\{x_n\}$ in a normed space X is called a Cauchy sequence if and only if for every $\varepsilon > 0$, there is an integer N such that $\| x_m - x_n \|_X < \varepsilon$ holds whenever $n, m > N$. X is *complete* and a *Banach space* if every Cauchy sequence in X converges to a limit in X. An important class of Banach spaces in functional analysis is L_p spaces, also called Lebesgue spaces.

We say the normed space X is *embedded* in the normed space Y, and we write $X \to Y$ to designate this embedding, provided that

1. X is a vector subspace of Y,
2. The identity operator I is defined on X into Y by $Ix = x$ for all $x \in X$ is continuous.

Let Ω be a domain in $\mathbb{R}^p$. For any nonnegative integer k let $C^k(\Omega)$ denote the vector space consisting of all functions g which, together with all their partial derivatives $D^\alpha g$ of orders $|\alpha| \leq k$ are continuous on Ω.

Definition 2.2 (The Space $L^p(\Omega)$) Let Ω be a domain in $\mathbb{R}^p$ and let p be a positive real number. We denote by $L^p(\Omega)$ the class of all measurable functions u defined on Ω for which

$$\int_\Omega |u(x)|^p dx < \infty$$

for which the norm provided $1 \leq p < \infty$

$$\| f \|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} < \infty.$$

In other words, the Lebesgue spaces are defined using *Lebesgue integration*, not Riemann integration. Since L^p spaces are outside the scope of this work, for detailed explanation see ().

Vector subspaces of Lebesgue spaces $L^p(\Omega)$ are the Sobolev spaces of integer order. They are vector spaces of functions equipped with a norm that is a combination of L^p-norms with its weak derivatives up to a given order. One motivation of studying these spaces is that solutions of partial differential equations may not have strong solutions with the derivatives understood in the classical sense, while in appropriate Sobolev space there exist its weak solutions. For more information, there are excellent books by Maz'ya (1985) called "Sobolev Spaces" and by Adams and Fournier (2003).

Definition 2.3 (Sobolev Norms) We define a functional $\| \cdot \|_{k,p}$, where k is a positive integer and $1 \leq p \leq \infty$ as follows:

$$\begin{aligned} \| u \|_{k,p} &= \left(\sum_{0 \leq |\alpha| \leq k} \| D^\alpha u \|_p^p \right)^{1/p} \quad \text{for} \quad 1 \leq p < \infty \\ \| u \|_{k,\infty} &= \max_{0 \leq |\alpha| \leq k} \| D^\alpha u \|_\infty \end{aligned} \tag{2.1}$$

for any function u for which the right side makes sense, $\| \cdot \|_p$ being the norm in $L^p(\Omega)$.

Definition 2.4 (Sobolev Spaces). For any positive integer k and $1 \leq p \leq \infty$ we consider vector spaces on which $\| \cdot \|_{k,p}$ is a norm:

a $W^{k,p}(\Omega) \equiv \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \quad \text{for} \quad 0 \leq |\alpha| \leq k\}$, where $D^\alpha u$ is the weak partial derivative;

b $W_0^{k,p}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in the space $W^{k,p}(\Omega)$.

Equipped with the appropriate norm (2.1), these are called *Sobolev spaces* over Ω.

Observe that $W^{0,p}(\Omega) = L^p(\Omega)$, and if $1 \le p \le \infty$, $W_0^{0,p}(\Omega) = L^p(\Omega)$ because $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$. For any k, we have the following chain of embeddings

$$W_0^{k,p}(\Omega) \to W^{k,p}(\Omega) \to L^p(\Omega).$$

Remark It is worth noting here that the claim of Lipschitz continuity will serve to guarantee the fulfillment of the properties of density and embedding in the generalized Sobolev spaces. We will prove it in the next section.

Further, we will need to construct upper and lower solutions, thus let us define the spaces $V = L^p(0,T; W^{1,p}(\Omega)), p > 2$, $V_0 = L^p(0,T; W_0^{1,p}(\Omega))$ and its dual as $V_0' = L^{p'}(0,T; W^{-1,p'}(\Omega))$.

We assume that for any function u defined in Ω the function $\zeta(x,t,u(x,t), u_{xx}(x,t))$ satisfies the Caratheodory conditions. That is, ζ is measurable in $(x,t) \in \mathcal{Q}$ and continuous for almost any fixed (x,t).

Definition 2.5 If $\bar{u}$ is an upper solution then $\bar{u} \ge u$ a.e. in $\Omega \times (0,T)$. Similarly, if $(\underline{u})$ is a lower solution then $\underline{u} \le u$ a.e. in $\Omega \times (0,T)$.

3 Existence & Uniqueness of Weak Solutions

The existence of weak solutions follows from a very general second order nonlinear parabolic boundary value problem proved in [12] under the assumption that a lower $\underline{u}$ and an upper $\bar{u}$ solutions are known.

Definition 3.1 u is a weak solution of the problem (1.1) in (0, T) if $u \in V_0, u_t \in V', u(\cdot,t) \to 0$ in $L^2(\Omega)$ as $t \to 0+$ and

$$\int_0^T dt \int_\Omega u_t v dx + \int_0^T dt \int_\Omega u_{xx} \cdot v_{xx} dx = \lambda \int_0^T dt \int_\Omega u^p v dx.$$

Definition 3.2 A function $\underline{u} \in V$ with $\underline{u}_t \in V'$ is called a lower solution of the boundary value problem if $\underline{u}_x(x,t) \le Bi_m(1-\underline{u})$ on $\Gamma_T, \underline{u}(x,0) \le u_0(x)$ in Ω, and

$$\int_0^T dt \int_\Omega \underline{u}_t v dx + \int_0^T dt \int_\Omega \underline{u}_{xx} \cdot v_{xx} dx \le \lambda \int_0^T dt \int_\Omega \underline{u}^p v dx,$$

for any nonnegative $v \in V_0$. Conversely, $\bar{u}$ is an upper solution, if the inequality signs are reversed.

Uniqueness of a weak solution follows from works in [13-15].

4 Boundedness & Monotonicity

Lemma 4.1 Let $u(x)$ be the weak solution to the boundary value problem (1.3). Then, it holds true $0 \leq u(x) \leq 1$ for all $0 \leq x \leq 1$.

Proof Multiplying the differential equation in (1) by $x^s(u(x)-1)_+$, and integrating by parts over the domain $\Omega = (0,1)$ yields

$$(u(x)-1)_+ x^s \frac{\partial u}{\partial x}(x)\Big|_0^1 - \int_0^1 z^s \frac{\partial (u-1)}{\partial z}(z) \frac{\partial (u-1)_+}{\partial z}(z) dz \\ - \int_0^1 \lambda z^s (u(z)-1)_+ u_+^p(z) dz = 0$$

from which we infer

$$-\int_0^1 z^s \left[\frac{\partial (u-1)_+}{\partial z}(z)\right]^2 dz = Bi_m (u(1)-1)_+^2 + \int_0^1 \lambda z^s (u(z)-1)_+ u_+^p(z) dz.$$

Consequently, $(u(x)-1)_+ = 0$ due to the fact that the right hand side of the last equation is non-negative but the left hand side is non-positive. This means that $u(x) - 1 \leq 0$ for all $0 \leq x \leq 1$, i.e. $u(x) \leq 1$. To find the lower boundary, let $u_- = \min\{u, 0\}$ denote the negative part of u. In order to show that the solution $u(x)$ is non-negative for all $0 \leq x \leq 1$, we multiply the differential equation (1.3) by $x^s u_-(x)$ and integrate by parts. We obtain

$$u_-(x) x^s \frac{\partial u}{\partial x}(x)\Big|_0^1 - \int_0^1 z^s \frac{\partial u}{\partial z}(z) \frac{\partial u_-}{\partial z}(z) dz - \int_0^1 \lambda z^s u_-(z) u_+^p(z) dz = 0$$

from which we deduce

$$-\int_0^1 z^s \left[\frac{\partial u_-}{\partial z}(z)\right]^2 dz + Bi_m u_-(1) = Bi_m u_-(1) u(1) + \int_0^1 \lambda z^s u_-(z) u_+^p(z) dz.$$

Here notice that the left-hand side is non-positive, while the right-hand side is non-negative. It implies that $u_-(x) = const$. For the Robin boundary condition to hold, $\frac{du}{dx}(1) = Bi_m(1-u(1)) = 0$, we infer that $u(1) = 1$. Consequently, $u_-(x) = 0$. Thus, $u(x) \geq 0$ for all $0 \leq x \leq 1$.

Let us show the monotonicity of solutions to problem (1) with respect to the reaction rate constant $k > 0$.

Lemma 4.2 Let $k_1 \geq k_2 \geq 0, n \in (0,1)$, and u_1 and u_2 be solutions to the boundary value problem (1.3) with reaction rate constants k_1 and k_2, respectively. Then, $u_1(x) \leq u_2(x)$ for all $0 \leq x \leq 1$ with $Bi_{m_1} = Bi_{m_2}$.

Proof Since u_1 and u_2 solve the boundary value problem (1.3) with reaction rate constants k_1 and k_2, respectively, we have

$$\frac{1}{x^s}\frac{\partial}{\partial x}\left(x^s\frac{\partial u_1}{\partial x}\right) - \frac{1}{x^s}\frac{\partial}{\partial x}\left(x^s\frac{\partial u_2}{\partial x}\right) = \lambda_1[u_1]_+^p - \lambda_2[u_2]_+^p$$

in Ω. Multiplying the above equation by $x^s(u_1-u_2)_+$ and integrating by parts yields

$$\begin{aligned}\int_0^1 z^s \left(\frac{\partial u_1}{\partial z}(z) - \frac{\partial u_2}{\partial z}(z)\right)\left(\frac{\partial([u_1-u_2]_+)}{\partial z}(z)\right)dz \\ = \left(Bi_{m_1}(1-u_1(1)) - Bi_{m_2}(1-u_2(1))\right)\left(u_1(1)-u_2(1)\right) \\ - \int_0^1 z^s(\lambda_1[u_1]_+^p(z) - \lambda_2[u_2]_+^p(z))(u_1-u_2)_+(z)dz\end{aligned}$$

from which follows

$$\int_0^1 z^s\left[\frac{\partial([u_1-u_2]_+)}{\partial z}(z)\right]^2 dz = (Bi_{m_1}(1-u_1(1)) - Bi_{m_2}(1-u_2(1)))(u_1(1)-u_2(1))$$

$$- \int_0^1 z^s(\lambda_1[u_1]_+^p(z) - \lambda_2[u_2]_+^p(z))(u_1-u_2)_+(z))dz = -Bi_m(u_1(1)-u_2(1))^2$$

$$- \int_0^1 z^s(\lambda_1[u_1]_+^p(z) - \lambda_1[u_2]_+^p(z) + \lambda_1[u_2]_+^p(z) - \lambda_2[u_2]_+^p(z))(u_1-u_2)_+(z))dz.$$

$$\begin{aligned}= -Bi_m(u_1(1)-u_2(1))^2 - \lambda_1\int_0^1 z^s([u_1]_+^p(z) - [u_2]_+^p(z))(u_1-u_2)_+(z)dz \\ -(\lambda_1-\lambda_2)\int_0^1 z^s[u_2]_+^p(z)(u_1-u_2)_+(z)dz.\end{aligned}$$

Then one can observe that

$$\begin{aligned}\int_0^1 z^s\left[\frac{\partial([u_1-u_2]_+)}{\partial z}(z)\right]^2 dz \leq -Bi_m(u_1(1)-u_2(1))^2 - \lambda_1\int_0^1 z^s([u_1]_+^p(z) \\ - [u_2]_+^p(z))(u_1-u_2)_+(z)dz\end{aligned}$$

under the assumption that $k_1 \geq k_2 \geq 0$ and $(u_+^p - v_+^p)(u-v)_+ \geq 0$ for $n \in (0,1)$. Notice that the left hand side is non-negative, while the right hand side is non positive. As a result, $[u_1 - u_2]_+ = const = 0$ in Ω. This means that $u_1 \leq u_2$ in Ω.

Lemma 4.3 Let $k_1 \geq k_2 \geq 0, n \in (0,1)$, and u_1 and u_2 be solutions to the boundary value problem (1) with reaction rate constants k_1 and k_2, respectively. Then, $u_1(x) \leq u_2(x)$ for all $0 \leq x \leq 1$ with $Bi_{m_1} > Bi_{m_2}$.

Proof As in the previous lemma, since u_1 and u_2 solve the boundary value problem (1) with reaction rate constants k_1 and k_2, respectively, we have

$$\frac{1}{x^p}\frac{\partial}{\partial x}\left(x^p\frac{\partial u_1}{\partial x}\right) - \frac{1}{x^p}\frac{\partial}{\partial x}\left(x^p\frac{\partial u_2}{\partial x}\right) = \lambda_1[u_1]_+^p - \lambda_2[u_2]_+^p$$

in Ω. Multiplying the above equation by $x^p(u_1-u_2)_+$ and integrating by parts yields

$$\begin{aligned}
&\int_0^1 z^s\left(\frac{\partial u_1}{\partial z}(z) - \frac{\partial u_2}{\partial z}(z)\right)\left(\frac{\partial([u_1-u_2]_+)}{\partial z}(z)\right)dz \\
&\quad = (Bi_{m_1}(1-u_1(1)) - Bi_{m_2}(1-u_2(1)))(u_1(1)-u_2(1)) \\
&\qquad - \int_0^1 z^s(\lambda_1[u_1]_+^p(z) - \lambda_2[u_2]_+^p(z))(u_1-u_2)_+(z)dz
\end{aligned}$$

from which follows

$$\int_0^1 z^s\left[\frac{\partial([u_1-u_2]_+)}{\partial z}(z)\right]^2 dz = (Bi_{m_1}(1-u_1(1)) - Bi_{m_2}(1-u_2(1)))\,(u_1(1)-u_2(1))$$

$$-\int_0^1 z^s(\lambda_1[u_1]_+^p(z) - \lambda_2[u_2]_+^p(z))(u_1-u_2)_+(z))dz = (Bi_{m_1}(1-u_1(1)) - Bi_{m_1}(1-u_2(1))$$

$$+Bi_{m_1}(1-u_2(1)) - Bi_{m_2}(1-u_2(1)))(u_1(1)-u_2(1))$$

$$-\int_0^1 z^s(\lambda_1[u_1]_+^p(z) - \lambda_1[u_2]_+^p(z) + \lambda_1[u_2]_+^p(z) - \lambda_2[u_2]_+^p(z))(u_1-u_2)_+(z))dz$$

$$= (Bi_{m_1}(u_2(1)-u_1(1)) + (Bi_{m_1} - Bi_{m_2})(1-u_2(1)))(u_1-u_2)_+(z)$$

$$-\lambda_1\int_0^1 z^s([u_1]_+^p(z) - [u_2]_+^p(z))(u_1-u_2)_+(z)dz - (\lambda_1-\lambda_2)\int_0^1 z^s[u_2]_+^p(z)(u_1-u_2)_+(z)dz.$$

Using the Lemma 1, observe that

$$(Bi_{m_1}(u_2(1)-u_1(1))+(Bi_{m_1}-Bi_{m_2})(1-u_2(1)))(u_1-u_2)_+(z)\leq$$

$$(Bi_{m_1}(u_2(1)-u_1(1))-Bi_{m_1}(u_2(1)-u_1(1)))(u_1-u_2)_+(z)=0$$

This implies that

$$0\leq\int_0^1 z^s\left[\frac{\partial([u_1-u_2]_+)}{\partial z}(z)\right]^2 dz\leq -k_1\int_0^1 z^s([u_1]_+^p(z)-[u_2]_+^p(z))(u_1-u_2)_+(z)dz\leq 0$$

under the assumption that $k_1\geq k_2\geq 0$ and $(u_+^p-v_+^p)(u-v)_+\geq 0$ for $n\in(0,1)$. As a result, $[u_1-u_2]_+=const=0$ in Ω due to $[u_1-u_2]_+=0$. This means that $u_1\leq u_2$ in Ω.

In similar fashion, we can work out for every parameter and all results will lead to the monotonicity of the solution.

Another important property is the Lipschitz continuity w.r.t. λ of the solution in the $C^0(\Omega)$ norm.

5 Lipschitz Continuity

Lemma 5.1 (Lipschitz continuity) There exists a constant K, depending only on Ω, such that

$$\| u(\cdot;\lambda_1)-u(\cdot;\lambda_2)\|_{C^0(\overline{\Omega})}\leq K|\lambda_1-\lambda_2|. \tag{5.1}$$

Proof Take $\lambda_1<\lambda_2$, then from the monotonicity lemma, $v(x)=u(x;\lambda_2)-u(x;\lambda_1)\leq 0$. Let $v(x)$ be such that $v(x)\leq z(x)$ where z is the solution of $z_{xx}=\lambda_2-\lambda_1, \partial z\setminus\partial x+Bi_m z=0$.

Observe that $v_{xx}=\lambda_2 u^p(x_2;\lambda_2)-\lambda_1 u^p(x_1;\lambda_1)$ and the boundary condition for $v(x)$ becomes $v_x=-Bi_m v$ on $\partial\Omega$.

Then

$$v_{xx}=\lambda_2 u^p(x_2;\lambda_2)-\lambda_1 u^p(x_2;\lambda_2)+\lambda_1 u^p(x_2;\lambda_2)-\lambda_1 u^p(x_1;\lambda_1)$$

$$=(\lambda_2-\lambda_1)u_2^p+\lambda_1(u_2^p-u_1^p)\leq(\lambda_2-\lambda_1)u_2^p\leq(\lambda_2-\lambda_1)$$

since $u_2^p-u_1^p\leq 0$ and $u(x)\leq 1$ for $p\in(0,1)$.

But if $z(x) = (\lambda_1 - \lambda_2)\tilde{z}(x)$ where $\tilde{z}$ solves $\tilde{z}_{xx} = 1$ with BC $\partial\tilde{z} \setminus \partial x + Bi_m\tilde{z} = 0$, then for equation (4) to hold we can deduce that from the last equation

$$K = \max_{\Omega} \tilde{z}(x) = -\min_{\Omega} \tilde{z}(x) \tag{5.2}$$

The proof is a standard application of the method of upper and lower solutions.

Remark. This paper works with a derivative of u w.r.t. λ in the following way. From Lemma 4 and the fact that Ω is bounded in $(0, 1)$ we have, for any $q > 1$,

$$\| u(\cdot;\lambda_1) - u(\cdot;\lambda_2) \|_{L^q(\Omega)} \leq K|\Omega|^{1/q}|\lambda_1 - \lambda_2|$$

i.e., the map $\Lambda : \lambda \to u(\cdot;\lambda)$ is uniformly Lipschitz continuous from $\mathbb{R}^+$ into any $L^q(\Omega)$. According to Ricci (1988), since $L^q(\Omega)$ is a reflexive Banach space, there exists the derivative of Λ for almost every λ; i.e. for a.e. λ there exists the limit

$$\lim_{\varepsilon\to 0} \frac{u(\cdot;\lambda+\varepsilon) - u(\cdot;\lambda)}{\varepsilon} = \frac{\partial}{\partial\lambda}u(\cdot;\lambda) \in L^q(\Omega) \tag{5.3}$$

in the sense of the L^q convergence. Moreover, since the L^∞ norm is preserved in the L^q limit we can say that

$$\left\| \frac{\partial}{\partial\lambda}u(\cdot;\lambda) \right\|_{L^\infty(\Omega)} \leq K.$$

From the monotonicity property we have $\partial u(\cdot;\lambda)/\partial\lambda \leq 0$ and we want to refine it by giving a sharp lower bound.

The average rate of change for v:

$$v_\varepsilon(x) = \frac{u(\cdot;\lambda+\varepsilon) - u(\cdot;\lambda)}{\varepsilon}$$

Let us take $\varepsilon > 0$, then $v_\varepsilon(x) \leq 0$ in Ω because of the monotonicity and $v_\varepsilon(x) = 0$ if there exists a dead-core, and $\partial v_\varepsilon/\partial x + Bi_m v_\varepsilon = 0$ on $\partial\Omega$. Furthermore, observe that

$$\Delta v_\varepsilon = \frac{1}{\varepsilon}\{(\lambda+\varepsilon)u^p(x;\lambda+\varepsilon) - \lambda u^p(x;\lambda)\}. \tag{5.4}$$

Since $u(x;\lambda+\varepsilon) \leq u(x;\lambda)$, we have

$$u^p(x;\lambda+\varepsilon) - u^p(x;\lambda) = p\xi^{p-1}(x)(u(x;\lambda+\varepsilon) - u(x;\lambda))$$

for some $\xi(x) \in (u(x;\lambda+\varepsilon); u(x;\lambda))$, and since $p<1$

$$u^p(x;\lambda+\varepsilon) - u^p(x;\lambda) < pu^{p-1}(x;\lambda)(u(x;\lambda+\varepsilon) - u(x;\lambda)) \tag{5.5}$$

Substituting the latter into former gives

$$\Delta v_\varepsilon(x) - \lambda p u^{p-1}(x;\lambda) v_\varepsilon(x) < u^p(x;\lambda+\varepsilon) < u^p(x;\lambda), \tag{5.6}$$

in $\Omega_\lambda = \Omega\backslash N_\lambda$. We can now obtain the requested estimate applying the following:

Lemma 5.2 Let $0<p<1$, and let $w \in C^0(\bar{\Omega}_\lambda)$ satisfy

$$\begin{aligned} \Delta w - \lambda p u^{p-1}(x) w = g(x), \quad &\text{in} \quad \Omega_\lambda, \\ w = 0, \quad &\text{on} \quad \partial\Omega, \\ w = 0, \quad &\text{in} \quad N_\lambda, \end{aligned} \tag{5.7}$$

Proof: For $\varepsilon > 0$, we consider $v_\varepsilon(x) = (u(\cdot;\lambda+\varepsilon) - u(\cdot;\lambda))/\varepsilon$. We have $v_\varepsilon(x) \leq 0$ and $v_\varepsilon \equiv 0$ in N_λ if $N_\lambda \neq 0$, and $\partial v_\varepsilon \ \partial v + \beta v_\varepsilon = 0$ on $\partial\Omega$. Inside Ω_λ Eq. 5.6 holds. Based on the work by Ricci [cf], we define

$$z(x) = v_\varepsilon(x) + \frac{1}{\lambda(1-p)}\left(\frac{u^p(x:\lambda)}{p} - u(x;\lambda)\right).$$

Then $z \in C^0(\Omega_\lambda)$ and $z=0$ on $\partial\Omega_\lambda$. Furthermore

$$\Delta z - \lambda p u^{p-1} z = g(x) + \frac{1}{\lambda(1-p)}(p(p-1)u^{p-2}|\nabla u|^2 + pu^{p-1}\Delta u - \lambda p u^{p-1} u)$$
$$-\frac{1}{\lambda(1-p)}(\Delta u - \lambda p u^{p-1} u) = g(x) + \frac{1}{\lambda(1-p)}(p(p-1)u^{p-2}|\nabla u|^2) - u^p < 0$$

where $u(\cdot;\lambda)$ is indicated by u. Since z cannot have interior negative minima, $z(x) \geq 0$ on $\bar{\Omega}_\lambda$. Moreover $z \equiv 0$ in N_λ, and the lemma is proved.

Then $z(x)$ solves

$$\begin{aligned} \Delta z - \lambda p u^{p-1} z < 0, \quad &\text{in} \quad \Omega_\lambda \\ z = 0, \quad &\text{on} \quad \partial N_\lambda \\ \frac{\partial z}{\partial x} + Bi_m z = \gamma(x), \quad &\text{on} \quad \partial\Omega \end{aligned} \tag{5.8}$$

where

$$\gamma(x) = Bi_m \frac{1}{\lambda(1-p)} \left(\frac{1-p}{p} u^p(x;\lambda) + u^{p-1}(x;\lambda) - 1 \right)$$

Since $u(x;\lambda) < 1$ on $\partial\Omega, \gamma(x) > 0$, so z cannot have a negative minimum on $\partial\Omega$, and $z(x) \geq 0$ in Ω, i.e.,

$$v_\varepsilon(x) \geq -\frac{1}{\lambda(1-p)} \left(\frac{u^p(x;\lambda)}{p} - u(x;\lambda) \right). \tag{5.9}$$

Taking the limit $\varepsilon \to 0^+$, the L^∞ estimate is preserved and we get

$$0 \geq \frac{\partial u}{\partial \lambda}(x;\lambda) \geq -\frac{1}{\lambda(1-p)} \left(\frac{u^p(x;\lambda)}{p} - u(x;\lambda) \right). \tag{5.10}$$

To show the asymptotic behavior of the solution we need to construct upper and lower solutions.

6 Construction of Upper and Lower Solutions

The candidate for the lower (or upper) solution is a curve $\underline{u} : t \to u(\cdot;\lambda(t))$ from $\mathbb{R}^+$ into the family of the solutions of elliptic problems.

Let $\mu \in C^1([0,+\infty))$ be a positive function. We define

$$\tilde{u}(x,t) = u(x;\mu(t)),$$

where $u(x;\mu(t))$ is the unique solution of elliptic/Dirichlet problem with $\lambda = \mu(t)$. That is because if Bi_m, p, and Ω are fixed then $u(x,\lambda)$ is a decreasing function of λ, as $\mu \to \infty$, Robin type problem becomes the Dirichlet problem.

Observe that $\tilde{u} \in C^0(\bar{\mathcal{Q}}_T), \tilde{u}(\cdot,t) \in C^{2+\beta}(\bar{\Omega})$ for any t, and $\tilde{u} \in L^\infty(\mathcal{Q}_T)$. In particular this implies $\tilde{u} \in V$ and $\tilde{u}_t \in V'$, for any $T > 0$. Then we can rewrite the problem as

$$\mathcal{L}(\tilde{u}) = \tilde{u}_t - \Delta\tilde{u} + \tilde{u}^p = \frac{\partial u}{\partial \lambda}(x;\mu(t))\mu'(t) - \mu(t)u^p(x;\mu(t)) + u^p(x;\mu(t)).$$

Here note that $\Delta\tilde{u} = \lambda u^p = \mu(t)u^p(x;\mu(t))$ since $\tilde{u}$ is the solution of elliptic problem.

Suppose now that $\mu'(t) > 0$. Then, be simplifying the inequality Eq. 5.10 as

$$\frac{\partial u}{\partial \lambda}(x;\lambda) \geq -\frac{u^p(x;\lambda)}{\lambda(1-p)p},$$

we get

$$\mathcal{L}(\tilde{u}) \geq \left[-\frac{\mu'}{\mu(1-p)p} + 1 - \mu\right] u^p(x;\mu(t)) \tag{6.1}$$

Conversely, if $\mu'(t) < 0$, the sign in Eq. 6.1 is reversed and it becomes:

$$\mathcal{L}(\tilde{u}) \leq \left[-\frac{\mu'}{\mu(1-p)p} + 1 - \mu\right] u^p(x;\mu(t)) \tag{6.2}$$

To find an upper solution, now it is enough to choose $\mu(t)$ such that

$$\mu' = \mu p(1-p)(1-\mu). \tag{6.3}$$

Observe that depending on $(1-\mu)$ term in Eq. 6.3, we consider two cases: If $\mu(0) < 1$, then $\mu'(t) > 0$, from the monotonicity property we know that $\mathcal{L}(\tilde{u}) \geq 0$ and $\tilde{u}(x,t)$ is an upper solution of the Equation 1.1 with initial datum

$$u(x;1) \leq u_0(x) = u(x;\mu(0)) < 1, \text{ in } \Omega. \tag{6.4}$$

If $\mu(0) > 1$, then $\mu'(t) < 0$, $\mathcal{L}(\tilde{u}) \leq 0$ and $\tilde{u}(x,t)$ is a lower solution of the parabolic problem (P) with initial datum

$$u(x;1) \geq u_0(x) = u(x;\mu(0)) \geq 0, \text{ in } \Omega. \tag{6.5}$$

where $u(x;1)$ is the stationary solution of our problem because of $\mu(0) = 1$ in Eq. 6.3.

Let us derive $\mu(t)$ from Eq. 6.3 to summarize the results in the following theorem.

$$\begin{aligned}
\mu' &= \mu p(1-p)(1-\mu) \\
\int_{\mu_0}^{\mu} \frac{d\mu}{\mu(1-\mu)} &= \int_0^T p(1-p)dt \\
\log(\mu) - \log(1-\mu)\Big|_{\mu_0}^{\mu} &= p(1-p)t \\
\frac{\frac{1}{\mu_0} - 1}{\frac{1}{\mu} - 1} &= \exp(p(1-p)t) \\
\mu &= [1 + (1/\mu_0 - 1)\exp(-p(1-p)t)]^{-1}
\end{aligned}$$

Theorem 6.1 Let $\mu(t) = [1+(1/\mu_0-1)\exp(-(1-p)pt)]^{-1}$. Then, if $\mu_0 < 1$ ($\mu_0 > 1$), the function $\tilde{u}(x,t) = u(x;\mu(t))$ is an upper (lower) solution of the Eq. 1.1, corresponding to the datum $u_0(x) = u(x;\mu_0) \geq u(x;1)(\leq)$. Moreover

$$\| u(x;1) - \tilde{u}(x;t) \|_{C^0(\bar{\Omega})} \leq K \frac{|1/\mu_0 - 1| \exp(-p(1-p)t)}{1 + (1/\mu_0 - 1)\exp(-p(1-p)t)}, \qquad (6.6)$$

where K is the constant in Eq.(5.2).

Note that this estimate resulted from the Lipschitz continuity of $u(\cdot;\lambda)$.

Finally, to prove the asymptotic behavior of the solution of the parabolic problem given by Theorem 1.1, by the same technique used in [15] we show the finite time extinction of the solution of Eq. 1.1 at an interior point of the dead core so that it satisfies

$$u(x,t_0) \leq u(x;\mu_0), \text{ in } \Omega. \qquad (6.7)$$

For $\mu_0 \in (0,1)$, define

$$\tilde{u}(x;t) = u(x;\mu_0) + v(t) \quad \text{where} \quad v(t) = [1-(1-p)(1-\mu_0)pt]_+^{1/p(1-p)}, \quad 0 < t < T$$

where $\tilde{u}(x;t)$ is an upper solution of the problem (1.1) and $u(x;\mu_0)$ is the solution of the stationary problem (1.2). For $t_0 = [p(1-p)(1-\mu_0)]^{-1}$ we have $u(x;t_0) \leq \tilde{u}(x;t_0) = u(x;\mu_0)$ and we can apply Theorem 6.1 starting from $t = t_0$. ■

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