On a stochastic interacting particle system with pushing dynamics

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Abstract In this paper we study a stochastic two-particle system on $\mathbb{Z}$ where particles interact each other by *pushing dynamics*. We derive the explicit formulas of the transition probability and of the probability distributions of each particle's position at time t. Finally, we discuss about the generalization of our works to N-particle system.

1. Introduction

In this paper we treat a two-particle system on $\mathbb{Z}$ where particles' moves are governed by the following rules. Each site can be occupied by at most 1 particle and each particle waits a random time exponentially distributed with rate 1 and then jumps to the next right site. If a particle at x tries to jump to $x+1$ which is already occupied by the other particle, then the particle at x jumps to $x+1$ and pushes the particle at $x+1$ to $x+2$. We consider only two-particle system in this paper but the rules of the two-particle system can be naturally extended to N-particle system.

Overall, we follow the ideas in [3-5] to study the model in this paper. The model we are considering is a continuous time Markov process with countable state space. A state of the model at time t is represented by the positions of two particles, denoted by $(x_1, x_2; t)$ where $x_1 < x_2$. The aim of this paper is to find the transition probability, that is, the probability of finding two particles at x_1 and x_2 at time t, provided that two particles were at y_1 and y_2 at time $t = 0$, and the probability distributions of the first particle's position and the second particle's position at time t. We will obtain the probability distributions by using the transition probability. In previous works [3] the probability distribution of the k-th particle's position was obtained for the N-particle system of the asymmetric simple exclusion process and the formula for the probability was expressed as a contour integral with a special form of an integrand. In this project we want to see if there is a similar mathematical structure in our model to that of the asymmetric simple exclusion process.

2. Transition probability

Since our model is a continuous time Markov process with countable state spaces, its transition probability satisfies the master equation as in [5], which describes the time evolution of the probability [2]. Our main goal is to find the transition probability of the two-particle systems but we start with one-particle system because we can learn some ideas to work on multi-particle system. Actually, the one-particle system is a continuous-time totally asymmetric random walk. The one-particle system of the other particle modes reduces to the same one-particle system (See for example, [5]).

2.1. **One-particle system.** Let $P_y(x;t)$ be the probability of finding a particle x at the time t given that the particle was at y when $t = 0$. Then $P_y(x;t)$ satisfies the master equation

$$\frac{d}{dt}P_y(x;t) = P_y(x-1;t) - P_y(x;t). \tag{1}$$

and the initial condition

$$P_y(x;0) = \delta_{xy}$$

for $x \geq y$. We write $P_y(x;t) = P_y(x)T(t)$ for an ansatz of separation of variables and then we obtain $T(t) = e^{\epsilon t}$ for some constant ϵ by an elementary technique of differential equations, where ϵ is to be decided. Then, the spatial part of the master equation becomes

$$\epsilon P(x) = P(x-1) - P(x).$$

It can be verified that a solution of the above equation is ξ^x, where $\xi \in \mathbb{C},\ \xi \neq 0$ and in this case,

$$\epsilon = \frac{1}{\xi} - 1.$$

Since the master equation is linear, $A(\xi)\xi^x e^{\epsilon t}$ is also a solution. Now, we form an integration with respect to ξ over some contour and choose the coefficient $A(\xi)$ so that the solution satisfies the initial condition. Let us choose $A(\xi) = \xi^{-y-1}$ and a contour C to be a circle which includes the origin to form a contour integral

$$\frac{1}{2\pi i}\int_C \xi^{x-y-1} e^{\epsilon t}\, d\xi. \tag{2}$$

It is easy to verify that (2) satisfies (1). Now, we show that (2) satisfis the initial condition. If $x = y$ and $t = 0$, then (2) is 1. If $x > y$ and $t = 0$, then (2) is 0 because the integrand will be analytic. Therefore, we have the transition probability for one-particle system

$$P_y(x;t) = \frac{1}{2\pi i}\int_C \xi^{x-y-1} e^{\epsilon t}\, d\xi. \tag{3}$$

2.2. **Two-particles system.** Let $P_{y_1,y_2}(x_1,x_2;t)$ be the probability of finding two particles at x_1 and x_2 at the time t given that particle x_1 was at y_1 and x_2 was at y_2 at time $t=0$. For a state (x_1,x_2) with $x_1 < x_2 - 1$, the master equation is given by

$$\frac{d}{dt}P_{y_1,y_2}(x_1,x_2;t) = P_{y_1,y_2}(x_1-1,x_2;t) + P_{y_1,y_2}(x_1,x_2-1;t) - 2P_{y_1,y_2}(x_1,x_2;t). \tag{4}$$

For a state $(x, x+1)$ the master equation is

$$\begin{aligned}\frac{d}{dt}P_{y_1,y_2}(x,x+1;t) &= P_{y_1,y_2}(x-1,x+1;t) + P_{y_1,y_2}(x-1,x;t) \\ &\quad -2P_{y_1,y_2}(x,x+1;t).\end{aligned} \tag{5}$$

We unify (4) and (5) to a single differential equation and a boundary condition as follows. First, we assume that a function $u(x_1,x_2;t)$ defined on the set of (x_1,x_2) with $x_1 \leq x_2$ satisfies (4) so that

$$\frac{d}{dt}u(x_1,x_2;t) = u(x_1-1,x_2;t) + u(x_1,x_2-1;t) - 2u(x_1,x_2;t). \tag{6}$$

When $x_1 = x, x_2 = x+1$, we want (6) to be in the form of (5). This can be done if we have a boundary condition

$$u(x,x;t) = u(x-1,x;t) \text{ for all } x \in \mathbb{Z}. \tag{7}$$

Hence, $u(x_1, x_2; t)$ that satisfies (6) and (7) satisfies the master equations (4) and (5). For this $u(x_1, x_2; t)$ to be the transition probability, it needs to satisfy the initial condition

$$u(x_1, x_2; 0) = \delta_{x_1 y_1}\delta_{x_2 y_2}$$

where $x_1 < x_2$, $y_1 < y_2$, $y_1 < x_1$ and $y_2 < x_2$. By the separation of variables, letting $u(x_1, x_2; t) = u(x_1, x_2)T(t)$, we obtain that $T(t) = e^{\epsilon t}$ for some constant $\epsilon \in \mathbb{C}$ to be determined, and we have an equation of spatial part

$$\epsilon u(x_1, x_2) = u(x_1 - 1, x_2) + u(x_1, x_2 - 1) - 2u(x_1, x_2). \tag{8}$$

We can verify that the solution is in the form of $\xi_1^{x_1}\xi_2^{x_2}$, where $\xi_1, \xi_2 \in \mathbb{C}$ and $\xi_1, \xi_2 \neq 0$, and in this case

$$\epsilon = \frac{1}{\xi_1} + \frac{1}{\xi_2} - 2. \tag{9}$$

By the linearity of the differential equation, $A_{12}(\xi_1, \xi_2)\xi_1^{x_1}\xi_2^{x_2}$ is also a solution. But we also see that $\xi_1^{x_2}\xi_2^{x_1}$ is also a solution with the same ϵ so the general solution is

$$u(x_1, x_2) = (A_{12}(\xi_1, \xi_2)\xi_1^{x_1}\xi_2^{x_2} + A_{21}(\xi_1, \xi_2)\xi_1^{x_2}\xi_2^{x_1})e^{\epsilon(\xi_1,\xi_2)t}. \tag{10}$$

This way of constructing of the general solution is known as the Bethe Ansatz [1],[5]. Now, we apply the boundary condition (7) to (10). Then, we have

$$A_{21}(\xi_1, \xi_2) = -\frac{\xi_1\xi_2 - \xi_2}{\xi_1\xi_2 - \xi_1}A_{12}(\xi_1, \xi_2).$$

Let

$$S(\xi_1, \xi_2) = \frac{A_{21}(\xi_1, \xi_2)}{A_{12}(\xi_1, \xi_2)}.$$

We choose $A_{12}(\xi_1, \xi_2) = \xi_1^{-y_1-1}\xi_2^{-y_2-1}$ as in the one-particle system.

Theorem. The transition probability of the two-particle system is given by

$$\left(\frac{1}{2\pi i}\right)^2 \int_C \int_C (\xi_1^{x_1-y_1-1}\xi_2^{x_2-y_2-1} + S\xi_2^{x_1-y_2-1}\xi_1^{x_2-y_1-1})e^{\epsilon t}\, d\xi_1\, d\xi_2 \tag{11}$$

where the contour C is a circle centered at the origin and with radius greater than 1, and ϵ is given by (9).

Proof. It can be easily verified that (11) satisfies (4) and (5). We will show that (11) satisfies the initial condition. It is easy to see that the first term of the integrand contributes to the initial condition. For the second term, take $\xi_2 = \frac{\eta}{\xi_1}$

$$\int_C \int_C \frac{\eta - \frac{\eta}{\xi_1}}{\eta - \xi_1}\eta^{x_1-y_2-1}\xi_1^{x_2-x_1+y_2-y_1-1}\, d\xi_1\, d\eta.$$

When we integrate over ξ_1 there are possibly two singularities at $\xi_1 = \eta$ and $\xi_1 = 0$. But, $|\eta| = |\xi_1\xi_2| > |\xi_1|$ since the radius of contour is greater than 1 so this singularity is outside the contour. Also, note that $x_2 - x_1 + y_2 - y_1 - 1 \geq 1$ since x_1, x_2, y_1, y_2 are integers and $x_2 > x_1$, $y_2 > y_1$. So, actually, there is no singularity at the origin. Therefore, integrating over ξ_1, we obtain zero. and the initial condition is satisfied. □

3. Probability distributions of the first and second particle's position

In section 2, we obtained

$$P_{y_1,y_1}(x_1,x_2;t) = \left(\frac{1}{2\pi i}\right)^2 \int_C \int_C (\xi_1^{x_1-y_1-1}\xi_2^{x_2-y_2-1} + S\xi_2^{x_1-y_2-1}\xi_1^{x_2-y_1-1}) e^{\epsilon t}\, d\xi_1\, d\xi_2.$$

In this section, by using this transition probability we will find the probability that the first particle is at x at time t, denoted by $\mathbb{P}_{y_1,y_2}(x_1(t) = x)$, and the the probability that the second particle is at x at time t, denoted by $\mathbb{P}_{y_1,y_2}(x_2(t) = x)$.

3.1. $\mathbb{P}_{y_1,y_2}(x_2(t) = x)$.

In order to find $\mathbb{P}_{y_1,y_2}(x_2(t) = x)$ we should sum up all the possible probabilities. That is, we want to compute

$$\sum_{i=1}^{\infty} P_{y_1,y_2}(x-i,x;t).$$

We have

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2 \int_C \int_C (\xi_1^{x-i-y_1-1}\xi_2^{x-y_2-1} + S\xi_2^{x-i-y_2-1}\xi_1^{x-y_1-1}) e^{\epsilon t}\, d\xi_1\, d\xi_2 \\
=&\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2 \int_C \int_C (\xi_1^{x-y_1-1}\xi_2^{x-y_2-1})\xi_1^{-i} e^{\epsilon t}\, d\xi_1\, d\xi_2 \\
+&\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2 \int_C \int_C (S\xi_2^{x-y_2-1}\xi_1^{x-y_1-1})\xi_2^{-i} e^{\epsilon t}\, d\xi_1\, d\xi_2.
\end{aligned}$$

The radius of contour is greater than 1 and so the geometric series converges and hence, we obtain

$$\begin{aligned}
&\left(\frac{1}{2\pi i}\right)^2 \int_C \int_C \xi_1^{x-y_1-1}\xi_2^{x-y_2-1}\frac{1}{\xi_1-1} e^{\epsilon t}\, d\xi_1\, d\xi_2 \\
+&\left(\frac{1}{2\pi i}\right)^2 \int_C \int_C S\xi_2^{x-y_2-1}\xi_1^{x-y_1-1}\frac{1}{\xi_2-1} e^{\epsilon t}\, d\xi_1\, d\xi_2 \\
=&\left(\frac{1}{2\pi i}\right)^2 \int_C \int_C \xi_1^{x-y_1-1}\xi_2^{x-y_2-1}\Big(\frac{1}{\xi_1-1} - \frac{\xi_1\xi_2-\xi_2}{(\xi_1\xi_2-\xi_1)(\xi_2-1)}\Big) e^{\epsilon t}\, d\xi_1\, d\xi_2 \\
=&\left(\frac{1}{2\pi i}\right)^2 \int_C \int_C \xi_1^{x-y_1-1}\xi_2^{x-y_2-1}\Big(\frac{(\xi_1-\xi_2)(1-\xi_1\xi_2)}{(\xi_1-1)(\xi_2-1)(\xi_1\xi_2-\xi_1)}\Big) e^{\epsilon t}\, d\xi_1\, d\xi_2.
\end{aligned}$$

As a result,

$$\begin{aligned}
&\mathbb{P}_{y_1,y_2}(x_2(t) = x) = \\
&\qquad\left(\frac{1}{2\pi i}\right)^2 \int_C \int_C \xi_1^{x-y_1-1}\xi_2^{x-y_2-1}\frac{(\xi_1-\xi_2)(1-\xi_1\xi_2)}{(\xi_1-1)(\xi_2-1)(\xi_1\xi_2-\xi_1)} e^{\epsilon t}\, d\xi_1\, d\xi_2.
\end{aligned}$$

As we can see, $\mathbb{P}_{y_1,y_2}(x_2(t) = x)$ depends on not only x and y_2 but also x_1 because the second particle can be pushed by the first particle.

3.2. $\mathbb{P}_{y_1,y_2}(x_1(t)=x)$. In order to find $\mathbb{P}_{y_1,y_2}(x_1(t)=x)$ we compute

$$\begin{aligned}
&\sum_{i=1}^{\infty} P_{y_1,y_2}(x,x+i;t)\\
=&\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2\int_C\int_C(\xi_1^{x-y_1-1}\xi_2^{x+i-y_2-1}+S\xi_2^{x-y_2-1}\xi_1^{x+i-y_1-1})e^{\epsilon t}\,d\xi_1\,d\xi_2\\
=&\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2\int_C\int_C\xi_1^{x-y_1-1}\xi_2^{x-y_2-1}e^{\epsilon t}\xi_2^i\,d\xi_1\,d\xi_2\\
+&\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2\int_C\int_C S\xi_2^{x-y_2-1}\xi_1^{x-y_1-1}e^{\epsilon t}\xi_1^i\,d\xi_1\,d\xi_2.
\end{aligned}$$

Let us consider each term separately. In the first integral

$$\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2\int_C\int_C\xi_1^{x-y_1-1}\xi_2^{x-y_2-1}e^{\epsilon t}\xi_2^i\,d\xi_1\,d\xi_2$$

the radius of the contour is greater than 1 and thus, the geometric series does not converge. Hence, we change the contour for ξ_2 variable to a contour C' with radius less than 1. This is possible because the only singularity is at the origin. Then we have

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2\int_C\int_C\xi_1^{x-y_1-1}\xi_2^{x-y_2-1}e^{\epsilon t}\xi_2^i\,d\xi_1\,d\xi_2\\
=&\left(\frac{1}{2\pi i}\right)^2\int_{C'}\int_C\xi_1^{x-y_1-1}\xi_2^{x-y_2-1}e^{\epsilon t}\frac{\xi_2}{1-\xi_2}\,d\xi_1\,d\xi_2
\end{aligned}$$

Changing back the contour C' to C, we should subtract the residue at $\xi_2=1$. Let us calculate the residue. Since

$$\begin{aligned}
&\left(\frac{1}{2\pi i}\right)^2\int_{C'}\int_C\frac{\xi_1^{x-y_1-1}\xi_2^{x-y_2}e^{\epsilon t}}{1-\xi_2}\,d\xi_1\,d\xi_2\\
=&-\left(\frac{1}{2\pi i}\right)^2\int_{C'}\int_C\frac{\xi_1^{x-y_1-1}\xi_2^{x-y_2}}{\xi_2-1}e^{(\frac{1}{\xi_1}+\frac{1}{\xi_2}-2)t}\,d\xi_1\,d\xi_2\\
=&-\left(\frac{1}{2\pi i}\right)^2\int_{C'}\frac{\int_C\xi_1^{x-y_1-1}\xi_2^{x-y_2}e^{(\frac{1}{\xi_1}+\frac{1}{\xi_2}-2)t}\,d\xi_1}{\xi_2-1}\,d\xi_2,
\end{aligned}$$

the residue at $\xi_2=1$ is

$$-\left(\frac{1}{2\pi i}\right)\int_C\xi_1^{x-y_1-1}e^{(\frac{1}{\xi_1}-1)t}\,d\xi_1$$

Hence, the first integral is

$$\left(\frac{1}{2\pi i}\right)^2\int_C\int_C\frac{\xi_1^{x-y_1-1}\xi_2^{x-y_2}e^{\epsilon t}}{1-\xi_2}\,d\xi_1\,d\xi_2+\left(\frac{1}{2\pi i}\right)\int_C\xi_1^{x-y_1-1}e^{(\frac{1}{\xi_1}-1)t}\,d\xi_1.$$

Now, we compute the second integral

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2\int_C\int_C S\xi_2^{x-y_2-1}\xi_1^{x-y_1-1}e^{\epsilon t}\xi_1^i\,d\xi_1\,d\xi_2\\
=\ & -\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2\int_C\int_C \frac{\xi_1-1}{\xi_2-1}\xi_2^{x-y_2}\xi_1^{x-y_1-2}e^{\epsilon t}\xi_1^i\,d\xi_1\,d\xi_2.
\end{aligned}$$

The only possible singularity for the ξ_1 variable is at the origin. So we change the contour C to the contour C' with radisu less than 1. Then we have

$$\begin{aligned}
&-\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2\int_C\int_C \frac{\xi_1-1}{\xi_2-1}\xi_2^{x-y_2}\xi_1^{x-y_1-2}e^{\epsilon t}\xi_1^i\,d\xi_1\,d\xi_2\\
=\ & -\sum_{i=1}^{\infty}\left(\frac{1}{2\pi i}\right)^2\int_C\int_{C'} \frac{\xi_1-1}{\xi_2-1}\xi_2^{x-y_2}\xi_1^{x-y_1-2}e^{\epsilon t}\xi_1^i\,d\xi_1\,d\xi_2\\
=\ & -\left(\frac{1}{2\pi i}\right)^2\int_C\int_{C'} \frac{\xi_1-1}{\xi_2-1}\frac{\xi_1}{1-\xi_1}\xi_2^{x-y_2}\xi_1^{x-y_1-2}e^{\epsilon t}\,d\xi_1\,d\xi_2\\
=\ & \left(\frac{1}{2\pi i}\right)^2\int_{C_r}\int_{C_r} \frac{\xi_2^{x-y_2}\xi_1^{x-y_1-1}e^{\epsilon t}}{\xi_2-1}\,d\xi_1\,d\xi_2.
\end{aligned}$$

Combining the first and the second integrals, we obtain

$$\begin{aligned}
\mathbb{P}_{y_1,y_2}(x_1(t)=x) &= \left(\frac{1}{2\pi i}\right)^2\int_C\int_C \frac{\xi_1^{x-y_1-1}\xi_2^{x-y_2}e^{\epsilon t}}{1-\xi_2}\,d\xi_1\,d\xi_2\\
&+ \left(\frac{1}{2\pi i}\right)\int_C \xi_1^{x-y_1-1}e^{(\frac{1}{\xi_1}-1)t}\,d\xi_1\\
&+ \left(\frac{1}{2\pi i}\right)^2\int_C\int_C \frac{\xi_2^{x-y_2}\xi_1^{x-y_1-1}e^{\epsilon t}}{\xi_2-1}\,d\xi_1\,d\xi_2\\
&= \left(\frac{1}{2\pi i}\right)\int_C \xi_1^{x-y_1-1}e^{(\frac{1}{\xi_1}-1)t}\,d\xi_1. \qquad (12)
\end{aligned}$$

It should be noted that the formula (12) is equivalent to (3). This makes sense. The first particle is not affected by the second particle because if it wants to jump to the site occupied by the second particle, it can do it by pushing the second particle. So, the first particle behaves like the single particle in the one-particle system.

4. Conclusion

We have calculated transition probability for the one-particle system and the two-particle system. Also, we computed the probabilities for the positions of the first particle and the second particle in the two-particle system. It is expected that it is possible to generalize our work to N-particle system. Also, from our results we may deduce that the formula for the second particle's position in N-particle system should be equivalent to the formula we computed in section 3.1 because the second particle is not affected by the particles ahead of itself.

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