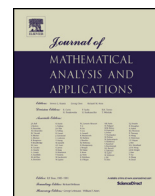




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Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

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ARTICLE INFO

Article history:

Received 18 May 2023

Available online 9 September 2023

Submitted by M.J. Carro

Keywords:

Hardy-Leray inequality

Rellich inequality

Hardy-Sobolev inequality

Gagliardo-Nirenberg inequality

Stein-Weiss inequality

Variable Lebesgue spaces

ABSTRACT

In this paper, we prove the Hardy-Leray inequality and related inequalities in variable Lebesgue spaces. Our proof is based on a version of the Stein-Weiss inequality in variable Lebesgue spaces derived from two weight inequalities due to Melchiori and Pradolini. We also discuss an application of our results to establish an existence result for the degenerate $p(\cdot)$ -Laplace operator.

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1. Introduction and main results

In this paper we consider the problem of extending the classical Hardy-Leray inequality and related inequalities to the scale of variable Lebesgue spaces. The Hardy-Leray inequality, sometimes referred to simply as the Hardy inequality, states that for $1 < p < \infty$, $p \neq n$, and compactly supported smooth functions f ,

$$\left\| \frac{f}{|\cdot|} \right\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)}. \quad (1.1)$$

Originally, this was proved by Hardy [12] in one dimension when $p = 2$, and extended to higher dimensions by Leray [20]. This is a special case of Caffarelli, Kohn, and Nirenberg [2], who proved the following general result. Given $p, q \geq 1$ and exponents a, b such that

[☆] This research was funded by the Science Committee of the Ministry of Education and Science of Kazakhstan [grant number AP09057887]. The authors were partially supported by a Simons Foundation Travel Support for Mathematicians Grant. This research was also funded by Nazarbayev University under Collaborative Research Program Grant 20122022CRP1601. No new data was collected or generated during the course of this research. This project was begun when the authors met at the Department of Mathematics at the University of Alabama in the Fall of 2022. The second author would like to thank the university for their support and hospitality.

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$$0 \leq b - a \leq 1, \quad \frac{-n}{p} < b, \quad \frac{-n}{q} < a, \quad \text{and} \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{n} + \frac{a-b}{n},$$

then

$$\| |\cdot|^a f \|_{L^q(\mathbb{R}^n)} \leq C \| |\cdot|^b \nabla f \|_{L^p(\mathbb{R}^n)}.$$

Note that when $a = b = 0$ this reduces to the classical Hardy-Sobolev inequality [11, Theorem 7.10].

Similar results hold for fractional powers of the Laplacian. Let $\mathcal{S}$ be the Schwartz space of rapidly decaying smooth functions in $\mathbb{R}^n$. The fractional Laplacian of a function can be defined by the following nonlocal operator in $\mathbb{R}^n$ for all $f \in \mathcal{S}$,

$$(-\Delta)^s f(x) := \frac{C(n, s)}{2} \int_{\mathbb{R}^n} \frac{2f(x) - f(x+y) - f(x-y)}{|y|^{n+2s}} dy, \quad 0 < s < 1, \quad (1.2)$$

where $C(n, s) > 0$ is a so-called normalization constant. It is known that (see e.g. [6]) if

$$(-\Delta)^s : \mathcal{S} \rightarrow L^2(\mathbb{R}^n)$$

is the fractional Laplacian given by (1.2). Then for any $f \in \mathcal{S}$,

$$(-\Delta)^s f(x) = \mathcal{F}^{-1} (|\xi|^{2s} \mathcal{F}f)(x), \quad x \in \mathbb{R}^n,$$

where $\mathcal{F}$ is the Fourier transform of a function.

Fix $0 < s \leq 1$; then for $1 \leq p < \infty$ we have the fractional Hardy-Rellich inequality

$$\| |\cdot|^{2s} f \|_{L^p(\mathbb{R}^n)} \leq C \| (-\Delta)^s f \|_{L^p(\mathbb{R}^n)}.$$

This is a special case of the fractional Hardy-Sobolev inequality: given $p, q \geq 1$ and exponents a, b such that

$$0 \leq b - a \leq 2s, \quad \frac{-n}{p} < b, \quad \frac{-n}{q} < a, \quad \text{and} \quad \frac{1}{p} - \frac{1}{q} = \frac{2s}{n} + \frac{a-b}{n},$$

then

$$\| |\cdot|^a f \|_{L^q(\mathbb{R}^n)} \leq C \| |\cdot|^b (-\Delta)^s f \|_{L^p(\mathbb{R}^n)}.$$

For these and related results, see [1, 5, 6, 10, 15].

The variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$ are a generalization of the classical Lebesgue spaces gotten by replacing the constant exponent p by an exponent function $p(\cdot) : \Omega \rightarrow [1, \infty)$. Intuitively, they consist of all the measurable functions f that satisfy

$$\int_{\Omega} |f(x)|^{p(x)} dx < \infty.$$

These functions form a Banach function space when equipped with the Luxemburg norm

$$\|f\|_{p(\cdot)} = \|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

These spaces have been studied for many years; one of the original motivations was the study PDEs and the calculus of variations over these function spaces. For more details, see [3,7,26] and the references they contain.

Because of this, there has been considerable work done in proving Hardy-type inequalities in the variable Lebesgue spaces. A one-dimensional version of Hardy's inequality was first obtained by Kokilashvili and Samko in [18]. The one-dimensional case was also considered in the papers [8] and [21]. In higher dimensions, for a bounded domain Ω , in [13] the authors proved that

$$\left\| \frac{f}{\delta(\cdot)} \right\|_{L^{p(\cdot)}(\Omega)} \leq C \|\nabla f\|_{L^{p(\cdot)}(\Omega)},$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. On $\mathbb{R}^n$ and on bounded domains, the Hardy-Sobolev inequality

$$\|f\|_{q(\cdot)} \leq \|\nabla f\|_{p(\cdot)},$$

where $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{n}$ for all x , has been studied by a number of authors: see [3, Section 6.4] for details and references.

In this paper, we prove the Hardy-Leray inequality and related results on the variable Lebesgue spaces over $\mathbb{R}^n$. We will state our main results here, but for brevity we will defer the definition of our notation (which is standard) to the next section. Our first result is a generalization of inequality (1.1).

Theorem 1.1 (*Hardy-Leray inequality*). *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if $p(\cdot) \in LH(\mathbb{R}^n)$ and $1 < p_- \leq p_+ < n$, then for all $f \in C_0^\infty(\mathbb{R}^n)$,*

$$\left\| \frac{f}{|\cdot|} \right\|_{p(\cdot)} \leq C \|\nabla f\|_{p(\cdot)},$$

where C is a positive constant independent of f .

Remark 1.2. By using the theory of variable exponent Sobolev spaces, these results can be extended to functions f whose (weak) gradient $\nabla f \in L^{p(\cdot)}(\mathbb{R}^n)$. Here and below we leave these extensions to the interested reader.

Theorem 1.1 is a special case of the following result.

Theorem 1.3 (*Hardy-Sobolev inequality*). *Fix exponents $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $p(\cdot), q(\cdot) \in LH(\mathbb{R}^n)$, $1 < p_- \leq p_+ < \infty$, $1 < q_- \leq q_+ < \infty$, and $p(x) \leq q(x)$ for all $x \in \mathbb{R}^n$. Given constants a, b that satisfy*

$$-\frac{n}{q_+} < a \leq b < \frac{n}{(p_-)},$$

suppose further that $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{n} + \frac{a-b}{n}$. Then for all $f \in C_0^\infty(\mathbb{R}^n)$,

$$\| |\cdot|^a f \|_{q(\cdot)} \leq C \| |\cdot|^b \nabla f \|_{p(\cdot)}, \quad (1.3)$$

where C is a positive constant independent of f .

Remark 1.4. If we set $p(\cdot) = q(\cdot)$, $b = 0$ and $a = -1$ in Theorem 1.3, we immediately get Theorem 1.1. Note that these inequalities were studied in [25] in generalized grand Lebesgue spaces.

Remark 1.5. When $a = b = 0$, Theorem 1.3 becomes the version of the Hardy-Sobolev inequality in variable Lebesgue spaces proved in [3, Section 6.4]. However, there this theorem included the case $p_- = 1$; the proof depended on weighted norm inequalities in the constant exponent case, the theory of Rubio de Francia extrapolation and a technique due to Maz'ya. It is an open question whether Theorem 1.3 can be extended to include the case $p_- = 1$.

As a consequence of Theorem 1.3, or more precisely, by adapting its proof, we prove a weighted Gagliardo-Nirenberg inequality in the variable Lebesgue spaces. In the constant exponent case, this result was studied in [27,28]. For related results without weights, see [3, Section 6.5.10].

Theorem 1.6. Fix an exponent $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \leq p_+ < n$ and $p(\cdot) \in LH(\mathbb{R}^n)$. Define $p^*(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ by

$$\frac{1}{p(x)} - \frac{1}{p^*(x)} = \frac{1}{n}.$$

Fix a constant a such that

$$-\frac{n}{(p^*)_+} < a < \frac{n}{(p_-)'}$$

Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, fix $\theta \in [0, 1]$, and define $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ by

$$\frac{1}{r(x)} = \frac{\theta}{p^*(x)} + \frac{1-\theta}{q(x)}.$$

Then for all $f \in C_0^\infty(\mathbb{R}^n)$,

$$\| |\cdot|^a f \|_{r(\cdot)} \leq C \| |\cdot|^a \nabla f \|_{p(\cdot)}^\theta \| |\cdot|^a f \|_{q(\cdot)}^{1-\theta}, \quad (1.4)$$

where C is a positive constant independent of f .

We can also prove Poincaré-type inequalities analogous to the inequalities in Theorem 1.3. Given a set Ω such that $0 < |\Omega| < \infty$ and a locally integrable function f , define

$$\langle f \rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega f(x) dx.$$

Theorem 1.7 (Poincaré inequality). Let Ω be a bounded, convex set. Fix exponents $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$ such that $p(\cdot), q(\cdot) \in LH(\Omega)$, $1 < p_- \leq p_+ < \infty$, $1 < q_- \leq q_+ < \infty$, and $p(x) \leq q(x)$ for all $x \in \mathbb{R}^n$. Given constants a, b that satisfy

$$-\frac{n}{q_+} < a \leq b < \frac{n}{(p_-)'},$$

suppose further that $\frac{1}{p(x)} - \frac{1}{q(x)} \leq \frac{1}{n} + \frac{a-b}{n}$. Then for all $f \in C^\infty(\Omega)$,

$$\| |\cdot|^a [f - \langle f \rangle_\Omega] \|_{L^{q(\cdot)}(\Omega)} \leq C \| |\cdot|^b \nabla f \|_{L^{p(\cdot)}(\Omega)} \quad (1.5)$$

where C is a positive constant independent of f .

Remark 1.8. The assumption that $p(\cdot), q(\cdot) \in LH(\Omega)$ can be replaced with the seemingly stronger assumption that $p(\cdot), q(\cdot) \in LH(\mathbb{R}^n)$, since log-Hölder continuous functions on any set Ω can be extended to functions in $LH(\mathbb{R}^n)$: See [3, Lemma 2.4].

Remark 1.9. When $p(\cdot) = q(\cdot)$ and $a = b = 0$, this result was proved in [3, Theorem 6.21], but again, this result included the case $p_- = 1$.

Our next results are the analogous theorems for the fractional Laplacian.

Theorem 1.10 (*Fractional Hardy-Rellich inequality*). Fix $s \in [0, 1]$. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $1 < p_- \leq p_+ < \frac{n}{2s}$ and $p(\cdot) \in LH(\mathbb{R}^n)$. Then for all $f \in C_0^\infty(\mathbb{R}^n)$,

$$\| |\cdot|^{-2s} f \|_{p(\cdot)} \leq C \| (-\Delta)^s f \|_{p(\cdot)}, \quad (1.6)$$

where C is a positive constant independent of f .

Theorem 1.10 is a special case of the following result.

Theorem 1.11 (*Fractional Hardy-Sobolev inequality*). Fix $s \in [0, 1]$. Given $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that $p(\cdot), q(\cdot) \in LH(\mathbb{R}^n)$, $1 < p_- \leq p_+ < \infty$, $1 < q_- \leq q_+ < \infty$, and $p(x) \leq q(x)$ for all $x \in \mathbb{R}^n$. Given constants a, b that satisfy

$$-\frac{n}{q_+} < a \leq b < \frac{n}{(p_-)'},$$

suppose further that $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{2s}{n} + \frac{a-b}{n}$. Then for all $f \in C_0^\infty(\mathbb{R}^n)$,

$$\| |\cdot|^a f \|_{q(\cdot)} \leq C \| |\cdot|^b (-\Delta)^s f \|_{p(\cdot)}, \quad (1.7)$$

where C is a positive constant independent of f .

Finally, by adapting the proof of Theorem 1.3, we prove a weighted fractional Gagliardo-Nirenberg inequality in the variable Lebesgue spaces. Even in the constant exponent case, this result appears to be new.

Theorem 1.12. Fix $s \in [0, 1]$ and fix an exponent $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \leq p_+ < \frac{n}{2s}$ and $p(\cdot) \in LH(\mathbb{R}^n)$. Define $p_s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ by

$$\frac{1}{p(x)} - \frac{1}{p_s(x)} = \frac{2s}{n}.$$

Fix a constant a such that

$$-\frac{n}{(p_s)_+} < a < \frac{n}{(p_-)'}$$

Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, fix $\theta \in [0, 1]$, and define $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ by

$$\frac{1}{r(x)} = \frac{\theta}{p_s(x)} + \frac{1-\theta}{q(x)}.$$

Then for all $f \in C_0^\infty(\mathbb{R}^n)$,

$$\| |\cdot|^a f \|_{r(\cdot)} \leq C \| (-\Delta)^s f \|_{p(\cdot)}^\theta \| f \|_{q(\cdot)}^{1-\theta}, \quad (1.8)$$

where C is a positive constant independent of f .

The remainder of this paper is organized as follows. In Section 2 we gather some basic definitions and results about variable Lebesgue spaces that we will need. The heart of the proofs of our main results is a generalization of the Stein-Weiss theorem for the fractional integral operator to the scale of variable Lebesgue spaces, which is of independent interest. We state and prove this result in Section 3. In Section 4 we give the proofs of Theorems 1.1–1.12. Finally, in Section 5 we give an application of our results: we show that a Neumann-type problem for the degenerate operator

$$Lu = -\operatorname{div}(|\sqrt{Q}\nabla u|^{p(\cdot)-2}Q\nabla u)$$

has a solution. This operator has been studied previously in [4,16,17,19]. It is a generalization of the $p(\cdot)$ -Laplacian

$$\Delta_{p(\cdot)}u = -\operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u),$$

which arises in the calculus of variations as an example of nonstandard growth conditions, and has been studied by a number of authors: see [9,14,23,26] and the references they contain.

Throughout this paper, n will denote the dimension of the underlying space $\mathbb{R}^n$. By a constant C we will mean a value that may depend on the underlying parameters (such as the exponent functions) but not on a particular function f . Its value may change from line to line. Sometimes, we will write $A \lesssim B$ instead of $A \leq cB$. If $A \lesssim B$ and $B \lesssim A$, we will write $A \approx B$.

2. Properties of variable Lebesgue spaces

In this section we give some basic definitions and properties of variable Lebesgue spaces. For proofs and further information, see [3, Chapter 2].

Given a set $\Omega \subset \mathbb{R}^n$, let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions, denoted by $p(\cdot)$, such that $p(\cdot) : \Omega \rightarrow [1, \infty]$. The functions $p(\cdot)$ are referred to as exponent functions. Given $p(\cdot) \in \mathcal{P}(\Omega)$ and a set $E \subset \Omega$, define

$$p_-(E) = \operatorname{ess\,inf}_{x \in E} p(x), \quad p_+(E) = \operatorname{ess\,sup}_{x \in E} p(x).$$

If the domain is clear we will simply write $p_- = p_-(\Omega)$, $p_+ = p_+(\Omega)$.

Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, define the conjugate exponent $p'(\cdot)$ pointwise by $\frac{1}{p'(\cdot)} + \frac{1}{p(\cdot)} = 1$, with the conventions that $1/\infty = 0$ and $1/0 = \infty$. The following identities follow at once from the definition:

$$(p'(\cdot))_+ = (p_-)', \quad (p'(\cdot))_- = (p_+)'. \quad (2.1)$$

Given $p(\cdot) \in \mathcal{P}(\Omega)$, we say that it belongs to the class of locally log-Hölder continuous exponents $LH_0(\Omega)$, denoted by $p(\cdot) \in LH_0(\Omega)$, if there exists a constant C_0 such that

$$|p(x) - p(y)| \leq \frac{C_0}{-\log(|x - y|)}$$

for all $x, y \in \Omega$ with $|x - y| < \frac{1}{2}$. We say that $p(\cdot)$ belongs to the class of exponents that are log-Hölder continuous at infinity, $LH_\infty(\Omega)$, denoted by $p(\cdot) \in LH_\infty(\Omega)$, if there exists a constant C_∞ and p_∞ such that

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}$$

for all $x \in \Omega$. Define $LH(\Omega) = LH_0(\Omega) \cap LH_\infty(\Omega)$.

Given $p(\cdot) \in \mathcal{P}(\Omega)$ and a Lebesgue measurable function f , define the modular functional (or simply the modular) associated with $p(\cdot)$ by

$$\rho_{p(\cdot), \Omega}(f) = \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} dx + \|f\|_{L^\infty(\Omega_\infty)},$$

where $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$. If $|\Omega_\infty| = 0$, in particular when $p_+ < \infty$, $\|f\|_{L^\infty(\Omega_\infty)} = 0$; when $|\Omega \setminus \Omega_\infty| = 0$, then $\rho_{p(\cdot), \Omega}(f) = \|f\|_{L^\infty(\Omega_\infty)}$. In situations where there is no ambiguity we will simply write $\rho_{p(\cdot)}(f)$ or $\rho(f)$.

Define $L^{p(\cdot)}(\Omega)$ to be the set of Lebesgue measurable functions f such that $\rho(f/\lambda) < \infty$ for some $\lambda > 0$. Define $L_{\text{loc}}^{p(\cdot)}(\Omega)$ to be the set of measurable functions f such that $f \in L^{p(\cdot)}(K)$ for every compact set $K \subset \Omega$. Define

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \rho_{p(\cdot), \Omega}(f/\lambda) \leq 1 \}.$$

If the set on the right-hand side is empty define $\|f\|_{L^{p(\cdot)}(\Omega)} = \infty$. If there is no ambiguity over the domain Ω , we will often write $\|f\|_{p(\cdot)}$ instead of $\|f\|_{L^{p(\cdot)}(\Omega)}$. When $p(\cdot)$ equals a constant p , then $L^{p(\cdot)}(\Omega)$ is equal to the classical Lebesgue space $L^p(\Omega)$ and $\|f\|_{L^{p(\cdot)}(\Omega)} = \|f\|_{L^p(\Omega)}$.

The functional $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ is a norm, and equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach function space. Given $\tau \geq 1$, the norm satisfies the rescaling property that for $\tau > 1$,

$$\|f\|_{L^{\tau p(\cdot)}(\Omega)} = \| |f|^\tau \|_{L^{p(\cdot)}(\Omega)}^\tau. \quad (2.1)$$

It also satisfies a generalized Hölder inequality: given $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}(\Omega)$ such that for all $x \in \Omega$,

$$\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)},$$

then there exists a constant C such that for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$,

$$\|fg\|_{L^{r(\cdot)}(\Omega)} \leq C \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{q(\cdot)}(\Omega)}. \quad (2.2)$$

If $p(\cdot) \in LH(\Omega)$, then for every cube $Q \subset \Omega$ we can estimate the norm of the characteristic function of Q in $L^{p(\cdot)}(\Omega)$ by

$$\|\chi_Q\|_{L^{p(\cdot)}(\Omega)} \approx |Q|^{\frac{1}{p_Q}}, \quad (2.3)$$

where p_Q is the harmonic mean of $p(\cdot)$ on Q :

$$\frac{1}{p_Q} = \int_Q \frac{dx}{p(x)}.$$

See [3, Section 4.6.2].

3. The Stein-Weiss inequality in variable Lebesgue spaces

At the heart of the proofs of our main results is an application of the theory of weighted norm inequalities in the variable Lebesgue spaces. We make use of the following relationships between the gradient, the fractional Laplacian and the Riesz potentials. Given $0 < \alpha < n$, define the Riesz potential I_α to be the integral operator

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y) dy.$$

It is well-known (see, for instance, [30]) that $I_\alpha : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$, whenever $1 < p < \frac{n}{\alpha}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. The analogous inequality holds in the variable Lebesgue spaces: see [3, Chapter 5].

The connection between the Riesz potential and the gradient is given by two inequalities. The first is

$$|f(x)| \lesssim I_1(|\nabla f|)(x), \quad (3.1)$$

which holds for all $f \in C_0^\infty(\mathbb{R}^n)$. The second holds given any bounded, convex domain Ω and $x \in \Omega$:

$$|f(x) - \langle f \rangle_\Omega| \leq I_1(|\nabla f| \chi_\Omega)(x). \quad (3.2)$$

(See [3,11].) Similarly, by the definition of the fractional Laplacian and the properties of the Fourier transform, we have that for $s \in [0, 1]$,

$$I_{2s}((-\Delta)^s f)(x) = f(x). \quad (3.3)$$

We exploit these estimates using the following result, which is a generalization of the classical Stein-Weiss inequality [31] to the variable Lebesgue spaces.

Theorem 3.1. Fix $0 < \alpha < n$. Given $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that $p(\cdot), q(\cdot) \in LH(\mathbb{R}^n)$, $1 < p_- \leq p_+ < \infty$, $1 < q_- \leq q_+ < \infty$, and $p(x) \leq q(x)$ for all $x \in \mathbb{R}^n$. Given constants a, b that satisfy

$$-\frac{n}{q_+} < a \leq b < \frac{n}{(p_-)',} \quad (3.4)$$

suppose further that for all $x \in \mathbb{R}^n$,

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n} + \frac{a-b}{n}. \quad (3.5)$$

Then for all f such that $|\cdot|^b f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\| |\cdot|^a I_\alpha f \|_{q(\cdot)} \leq C \| |\cdot|^b f \|_{p(\cdot)}.$$

We will prove Theorem 3.1 as a consequence of a more general, two weight norm inequality in variable Lebesgue spaces due to Melchiori and Pradolini [22, Theorem 1.1]. By a weight w we mean a non-negative, locally integrable function that satisfies $0 < w(x) < \infty$ almost everywhere.

Theorem 3.2. Fix $0 < \alpha < n$. Given $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that $p(\cdot), q(\cdot) \in LH(\mathbb{R}^n)$, $1 < p_- \leq p_+ < \infty$, $1 < q_- \leq q_+ < \infty$, and $p(x) \leq q(x)$ for all $x \in \mathbb{R}^n$. If for some $r, s > 1$, the pair of weights (u, v) satisfies

$$\sup_Q |Q|^{\frac{\alpha}{n}} \frac{\|\chi_Q\|_{q(\cdot)} \|\chi_Q u\|_{rq^+} \|\chi_Q v^{-1}\|_{s(p^-)'}}{\|\chi_Q\|_{p(\cdot)} \|\chi_Q\|_{rq^+} \|\chi_Q\|_{s(p^-)'}} < \infty. \quad (3.6)$$

Then for all f such that $vf \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|uI_\alpha f\|_{q(\cdot)} \lesssim \|vf\|_{p(\cdot)}.$$

To prove Theorem 3.1 we need one lemma.

Lemma 3.3. Fix $-n < \gamma < \infty$. Given any cube $Q = Q(\ell(Q), c_Q)$, if $\text{dist}(Q, 0) \leq \ell(Q)$, then

$$\left(\int_Q |x|^\gamma dx \right)^{\frac{1}{\gamma}} \lesssim \ell(Q).$$

If $\text{dist}(Q, 0) \geq \ell(Q)$, then

$$\left(\int_Q |x|^\gamma dx \right)^{\frac{1}{\gamma}} \lesssim |c_Q|.$$

In both cases, the implicit constants depend only on n and γ .

Proof. First assume that $\text{dist}(Q, 0) \leq \ell(Q)$. In this case there exists a ball $B_Q = B(0, r)$, where $r \leq (\sqrt{n} + 1)\ell(Q)$ and $Q \subset B_Q$. Therefore, if we shift to polar coordinates,

$$\begin{aligned} \int_Q |x|^\gamma dx &\leq c(n) \int_{B_Q} |x|^\gamma dx \\ &\approx \ell(Q)^{-n} \int_0^{(\sqrt{n}+1)\ell(Q)} \int_{S^{n-1}} r^\gamma r^{n-1} d\theta dr \\ &\approx \ell(Q)^{-n} \int_0^{(\sqrt{n}+1)\ell(Q)} r^{\gamma+n-1} dr \\ &\approx \ell(Q)^{-n} \ell(Q)^{\gamma+n} \\ &= \ell(Q)^\gamma. \end{aligned}$$

Now suppose that $\text{dist}(Q, 0) \geq \ell(Q)$. Let $x_0 \in Q$ be such that $|x_0| = \text{dist}(Q, 0)$. Given any point $x \in Q$, we have

$$|x_0| \leq |x| \leq |x_0| + |x - x_0| \leq |x_0| + \sqrt{n}\ell(Q) \leq (\sqrt{n} + 1)|x_0|.$$

A similar computation shows $|x_0| \approx |c_Q|$. Therefore, we have that

$$\int_Q |x|^\gamma dx \lesssim |c_Q|^\gamma. \quad \square$$

Proof of Theorem 3.1. We need to show that if we let $u(x) = |x|^a$ and $v(x) = |x|^b$, then with our hypotheses on a, b and $p(\cdot), q(\cdot)$, the condition (3.6) holds. Our first step is to replace this condition with a simpler one. Since $p(\cdot), q(\cdot) \in LH(\mathbb{R}^n)$, if we apply (2.3) to the norm of each characteristic function, and rewrite the norms of the weights in terms of Lebesgue spaces norms, we get that it is equivalent to assuming that for some $r, s > 1$,

$$\sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{qQ} - \frac{1}{pQ}} \left(\int_Q |x|^{arq_+} dx \right)^{\frac{1}{rq_+}} \left(\int_Q |x|^{-bs(p_-)'} dx \right)^{\frac{1}{s(p_-)'}} < \infty. \quad (3.7)$$

(We note in passing that this condition should be compared to the result of Perez [24] in the constant exponent case.)

Fix a cube Q . We first choose $r, s > 1$ such that

$$\int_Q |x|^{arq_+} dx < \infty, \quad \int_Q |x|^{-bs(p_-)'} dx < \infty.$$

This is possible by assumption (3.4).

Now suppose first that $\text{dist}(Q, 0) \leq \ell(Q)$. Then by Lemma 3.3, the left-hand side of (3.7) is bounded above by

$$|Q|^{\frac{\alpha}{n} + \frac{1}{qQ} - \frac{1}{pQ}} |Q|^{\frac{a}{n}} |Q|^{-\frac{b}{n}} = |Q|^{\frac{\alpha}{n} + \frac{1}{qQ} - \frac{1}{pQ} + \frac{a-b}{n}} \quad (3.8)$$

However, if we take the integral average of (3.5) over the cube Q , we get that

$$\frac{\alpha}{n} + \frac{1}{qQ} - \frac{1}{pQ} + \frac{a-b}{n} = 0.$$

Therefore, for all such cubes, the left-hand side of (3.7) is uniformly bounded.

Now suppose that $\text{dist}(Q, 0) \geq \ell(Q)$. Then the left-hand side of (3.7) is bounded above by

$$|c_Q|^{\alpha + \frac{n}{qQ} - \frac{n}{pQ}} |c_Q|^a |c_Q|^{-b} = |c_Q|^{\alpha + \frac{n}{qQ} - \frac{n}{pQ} + a - b}.$$

But arguing as above we must have that the exponent on the right-hand side is equal to 0, and so again left-hand side of (3.7) is uniformly bounded for all such cubes. Therefore, (3.7) holds, and so by Theorem 3.2 we get the desired inequality for I_α . $\square$

Remark 3.4. In the proof of Theorem 3.1 we do not actually use the hypothesis $a \leq b$. However, if the opposite inequality holds, then (3.5) implies that

$$\frac{1}{p(x)} - \frac{1}{q(x)} > \frac{\alpha}{n}.$$

In the constant exponent case, it is known that in this case the weighted norm inequality only holds in the trivial case when $u(x) = 0$ whenever $v(x) < \infty$ (see the remarks after [29, Theorem 1]).

4. Proofs of the main results

We can now prove our main results. As we noted in the Introduction, Theorem 1.1 is a special case of Theorem 1.3.

Proof of Theorem 1.3. Fix $f \in C_c^\infty(\mathbb{R}^n)$. Then

$$\| |\cdot|^a f \|_{q(\cdot)} \leq C \| |\cdot|^a I_1(|\nabla f|) \|_{q(\cdot)} \leq C \| |\cdot|^b \nabla f \|_{p(\cdot)};$$

the first inequality follows from (3.1) and the second from our hypotheses and Theorem 3.1 with $\alpha = 1$. $\square$

Proof of Theorem 1.6. Fix $f \in C_c^\infty(\mathbb{R}^n)$. Then by Hölder's inequality (2.2) and the rescaling property (2.1),

$$\| |\cdot|^a f \|_{r(\cdot)} \leq C \| |\cdot|^a f \|_{p^*(\cdot)}^\theta \| |\cdot|^a f \|_{q(\cdot)}^{1-\theta}.$$

By inequality (3.1) and by Theorem 3.1 with $\alpha = 1$, $a = b$ and $q(\cdot) = p^*(\cdot)$, which we can apply because of our hypotheses,

$$\| |\cdot|^a f \|_{p^*(\cdot)} \leq C \| |\cdot|^a I_1(|\nabla f|) \|_{p^*(\cdot)} \leq C \| |\cdot|^a \nabla f \|_{p(\cdot)}.$$

If we combine these two estimates we get the desired result. $\square$

Proof of Theorem 1.7. The proof is nearly the same as the proof of Theorem 1.3. As we noted in Remark 1.8, we may assume without loss of generality that $p(\cdot), q(\cdot) \in LH(\mathbb{R}^n)$. Define $r(\cdot) \in LH(\mathbb{R}^n)$ by

$$\frac{1}{p(x)} - \frac{1}{r(x)} = \frac{1}{n} + \frac{a-b}{n}.$$

Then $q(x) \leq r(x)$, so we can define the exponent $s(\cdot)$ by

$$\frac{1}{q(x)} = \frac{1}{s(x)} + \frac{1}{r(x)}.$$

By the generalized Hölder inequality,

$$\begin{aligned} \| |\cdot|^a [f - \langle f \rangle_\Omega] \|_{L^{q(\cdot)}(\Omega)} &\leq \| |\cdot|^a [f - \langle f \rangle_\Omega] \|_{L^{r(\cdot)}(\Omega)} \| \chi_\Omega \|_{L^{s(\cdot)}(\Omega)} \\ &= C(\Omega, s(\cdot)) \| |\cdot|^a [f - \langle f \rangle_\Omega] \|_{L^{r(\cdot)}(\Omega)}. \end{aligned}$$

By inequality (3.2) and Theorem 3.1,

$$\| |\cdot|^a [f - \langle f \rangle_\Omega] \|_{L^{r(\cdot)}(\Omega)} \leq \| |\cdot|^a I_1(|\nabla f| \chi_\Omega) \|_{L^{r(\cdot)}(\Omega)} \leq C \| |\cdot|^b \nabla f \|_{L^{p(\cdot)}(\Omega)}.$$

If we combine these inequalities we get the desired result. $\square$

As we noted in the Introduction, Theorem 1.10 is a special case of Theorem 1.11 with $a = 2s$, $b = 0$, and $p(\cdot) = q(\cdot)$.

Proof of Theorem 1.11. The proof is essentially the same as the proof of Theorem 1.3, but we use (3.3) instead of (3.1), and apply Theorem 3.1 with $\alpha = 2s$. $\square$

Proof of Theorem 1.12. The proof is essentially the same as the proof of Corollary 1.6 but again using (3.3) and Theorem 3.1 with $\alpha = 2s$, $a = b$ and $q(\cdot) = p_s(\cdot)$. $\square$

5. Solutions to a degenerate $p(\cdot)$ -Laplacian

In this section we give an application of Theorem 1.7 to show the existence of solutions to a degenerate Neumann-type problem studied in [4]. Let Ω be a bounded, open domain in $\mathbb{R}^n$, and let Q be an $n \times n$ self-adjoint, positive-semi-definite matrix function defined on an open neighborhood of $\bar{\Omega}$, and let v be a non-negative, locally integrable function also defined on an open neighborhood of $\bar{\Omega}$. We are interested in the existence of weak solutions to the problem

$$\begin{cases} \operatorname{div} \left(\left| \sqrt{Q} \nabla u \right|^{p(\cdot)-2} Q \nabla u \right) &= |f|^{p(\cdot)-2} f v^{p(\cdot)} \quad \text{in } \Omega \\ \mathbf{n}^T \cdot Q \nabla u &= 0 \quad \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

where $\mathbf{n}$ is the outward unit normal vector of $\partial\Omega$. The precise definition of a weak solution is somewhat technical, and we refer the reader to [4] for complete details. (We note in passing that this definition extends to the case when Ω has a rough boundary and the unit normal is not well-defined.)

In [4] the authors showed that if $1 < p_- \leq p_+ < \infty$, $v, |\sqrt{Q(\cdot)}|_{op} \in L^{p(\cdot)}(\Omega)$, where for each $x \in \Omega$, $|\sqrt{Q(x)}|_{op}$ is the operator norm of the matrix $\sqrt{Q(x)}$, then the existence of a solution to (5.1) was equivalent to the existence of a degenerate Poincaré inequality of the form

$$\|v[f - \langle f \rangle_{\Omega, v}]\|_{L^{p(\cdot)}(\Omega)} \leq C \|\sqrt{Q} \nabla f\|_{L^{p(\cdot)}(\Omega)}, \quad (5.2)$$

where

$$\langle f \rangle_{\Omega, v} = \frac{1}{v(\Omega)} \int_{\Omega} f(x) v(x) dx.$$

By using this result and Theorem 1.7 we can show that solutions exist when v is a power weight and the smallest eigenvalue of Q is bounded below by a power weight.

Theorem 5.1. *Fix an exponent $p(\cdot) \in \mathcal{P}(\Omega)$ such that $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in LH(\Omega)$. Let $a, b \in \mathbb{R}$ be such that*

$$-\frac{n}{p_+} < a \leq b < \frac{n}{(p_-)'} \quad \text{and} \quad \frac{1}{n} + \frac{a-b}{n} \geq 0.$$

Let Q be a self-adjoint, positive semi-definite matrix function such that $|\sqrt{Q}|_{op} \in L^{p(\cdot)}(\Omega)$, and for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,

$$c|x|^b|\xi| \leq |\sqrt{Q(x)}\xi|.$$

Let $v(x) = |x|^a$. Then there exists a solution to the Neumann-type problem (5.1).

Proof. By [4, Theorem 1.8] the existence of solutions follows immediately if we have that $v \in L^{p(\cdot)}(\Omega)$ and the Poincaré inequality (5.2) holds. The first is immediate: since Ω is bounded, by [3, Corollary 2.48],

$$\|\cdot\|^a_{L^{p(\cdot)}(\Omega)} \leq (1 + |\Omega|) \|\cdot\|^a_{L^{p_+}(\Omega)} < \infty;$$

the final inequality follows from our assumptions on a . To prove the Poincaré inequality, note first that by Hölder's inequality in the variable Lebesgue spaces,

$$|\langle f \rangle_\Omega - \langle f \rangle_{\Omega, v}| = \left| \frac{1}{v(\Omega)} \int_\Omega [\langle f \rangle_\Omega - f(x)] v(x) dx \right| \leq \frac{\|\chi_\Omega\|_{L^{p'(\cdot)}(\Omega)}}{v(\Omega)} \|f - \langle f \rangle_\Omega\|_{L^{p(\cdot)}(\Omega)}.$$

Therefore, by Theorem 1.7 and our assumptions on Q ,

$$\begin{aligned} ||| \cdot |^a [f - \langle f \rangle_{\Omega, v}] |||_{L^{p(\cdot)}(\Omega)} &\leq C ||| \cdot |^a [f - \langle f \rangle_\Omega] |||_{L^{p(\cdot)}(\Omega)} \\ &\leq C ||| \cdot |^b \nabla f |||_{L^{p(\cdot)}(\Omega)} \\ &\leq C \|\sqrt{Q} \nabla f\|_{L^{p(\cdot)}(\Omega)}. \quad \square \end{aligned}$$

It is straightforward to construct nontrivial examples of exponents $p(\cdot)$ and matrices Q for which solutions to (5.1) exist. For instance, in $\mathbb{R}^3$ let $\Omega = B(0, 1)$, $a = -3/4$, and $b = 1/4$. Let $p(\cdot) \in \mathcal{P}(B(0, 1))$ be any log-Hölder continuous function such that $p_- = 2$ and $p_+ = 3$. If we define

$$Q(x) = \begin{pmatrix} |x|^{-\frac{3}{4}} & 0 & 0 \\ 0 & |x|^{-\frac{1}{4}} & 0 \\ 0 & 0 & |x|^{\frac{1}{4}} \end{pmatrix},$$

then it is immediate that the hypotheses of Theorem 5.1 are satisfied.

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