

Stability Analysis for the Inhomogeneous Fourth-Order Nonlinear Schrödinger Equations with Mixed Dispersions

by

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Abstract

In this work, we study the strong instability of standing waves for a class of fourth-order nonlinear Schrödinger equations with mixed dispersions, focusing on inhomogeneous nonlinearities. For the single equation, we derive ground state solutions by using a profile decomposition. Then, under some conditions, we show that the ground state of the standing wave solution is strongly unstable by showing blow-up.

For the coupled system case, we establish blow-up criteria by using the localized virial identity and prove the existence of non-global solutions. Moreover, the strong instability of standing wave solutions is shown.

In addition, we drop the radially symmetric assumption and extend the results to the general initial condition. Moreover, we consider both positive and negative signs of the spatial weight exponent.

1 Introduction

The study investigates the fourth-order nonlinear Schrödinger equation with mixed dispersions

$$\begin{cases} i\psi_t - \Delta^2\psi + \mu\Delta u + |x|^{-b}|\psi|^p\psi = 0, \\ \psi(0, x) = \psi_0(x) \end{cases} \quad (1.1)$$

where $b \in \mathbb{R}$, $\mu < 0$, $\psi : [0, T^*) \times \mathbb{R}^N \rightarrow \mathbb{C}$ is a complex-valued function, $\psi_0 \in H^2$. $0 < T^* \leq \infty$ and $0 < p < p^*$, where

$$p^* = \begin{cases} +\infty, & \text{if } N \leq 4, \\ \frac{8-2b}{N-4}, & \text{if } N \geq 5. \end{cases}$$

Another problem that we study is the Cauchy problem for the fourth-order nonlinear Schrödinger system with coupled power-type of nonlinearities and mixed dispersions

$$\begin{cases} i\frac{\partial}{\partial t}u_j + \Delta^2u_j - \mu\Delta u_j = \sum_{k=1}^m |x|^{-b}|u_k|^p|u_j|^{p-2}u_j, \\ u_j(0, \cdot) = \psi_j. \end{cases} \quad (1.2)$$

Here $N \geq 2$, $j \in [1, m]$, $u_j : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{C}$, and $b \in \mathbb{R}$, and $p_* < p < p^*$ such that

$$p_* = 1 + \frac{4-b}{N}$$

and

$$p^* = \begin{cases} 1 + \frac{4-b}{N-4}, & \text{if } N > 4, \\ +\infty, & \text{if } 2 \leq N \leq 4. \end{cases}$$

The blow-up phenomenon have been studied in several cases:

The case when $b = 0$ for (1.1) called the classic biharmonic fourth-order NLS. The existence of blow-up solutions with condition $\frac{8}{N} < p < p^*$, $\mu \in \mathbb{R}$ studied in [2]. Moreover, [1] showed the radial ground state standing wave solutions when $\mu \geq 0$ are strongly unstable via blow-up. Similarly, the strong instability for the case $\mu < 0$ was proved in [4].

Blow-up of radial solutions to (1.1) for $b > 0$ and $p = \frac{8-2b}{N-4}$ was studied in [6]. In addition, authors extended their work by showing blow-up for non-radial case. Moreover, [3] has blow-up criteria for non-radial ground state solutions to the L^2 critical inhomogeneous NLS with $0 < b < 2$ and $p = \frac{8-2b}{N}$. In [5] investigates the existence of non-global solutions for the fourth-order nonlinear Schrödinger system with coupled power-type nonlinearities and establish strong instability.

The aim of this work is to show strong instability of (1.1) and (1.2) by blow-up. More precisely, the study proceeds several steps:

- In the first part of work, we focus on (1.1). The study begins from constructing the characterization of ground state solutions for by using a profile decomposition. Then, assume that if $\partial_\lambda^2 S_\omega(u^\lambda)|_{\lambda=1} \leq 0$, then we will show that radial ground state is strongly unstable by blow-up. Furthermore, the results are extended to non-radial setting and we consider to obtain blow-up results the case of positive power of spatial weight.
- In the second part, we study a stability of (1.2). We assume the existence of ground state solutions. Then, we establish blow-up criteria and prove the existence of non-global solutions using the localized virial identity. Finally, we show strong instability. Moreover, we have additional results as in the first part.

We adopt proofs from [4] and [5] to show strong instability for (1.1)-(1.2). The work is organized as follows. Section 2 and Section 3 contain the background material and main results necessary to establish the strong instability of (1.1) and (1.2), respectively.

2 The Fourth-Order Inhomogeneous Nonlinear Schrödinger Equation with Mixed Dispersions

2.1 Background material

Take the energy space H^2 , which is usual inhomogeneous Sobolev space with norm

$$\|\cdot\|_{H^2} := \left(\|\cdot\|_{L^2}^2 + \|\Delta \cdot\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

(1.1) satisfies the following scaling invariance

$$u^\lambda := \lambda^{\frac{4-b}{p}} u(\lambda^4 t, \lambda x) \quad \text{with } \lambda > 0.$$

Define the number

$$s_c = \frac{N}{2} - \frac{4-b}{p}$$

called the *Sobolev index*. We refer the cases $s_c < 0$, $s_c = 0$, and $0 < s_c < 2$ as *mass-subcritical*, *mass-critical*, and *mass-supercritical*, respectively. $s_c = 2$ corresponds the endpoint case $p = \frac{8-2b}{N-4}$ for $N \geq 5$ and called *energy-critical*.

(1.1) has a class of special solutions called standing waves of the form $e^{i\omega t}u$, where $\omega \in \mathbb{R}$ is the frequency and $u \in H^2(\mathbb{R}^N)$ is a nontrivial solution of the related to elliptic equation

$$\Delta^2 u - \mu \Delta u + \omega u - |x|^{-b}|u|^p u = 0. \quad (2.1)$$

Furthermore, (2.1) can be written as $S'_\omega(u) = 0$, where

$$S_\omega(u) := \frac{1}{2} \|\Delta u\|_{L^2}^2 + \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{\omega}{2} \|u\|_{L^2}^2 - \frac{1}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx \quad (2.2)$$

called **action functional**. Moreover, denote following functionals

$$K_\omega(u) := \partial_\lambda S_\omega(\lambda u)|_{\lambda=1} = \|\Delta u\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 + \omega \|u\|_{L^2}^2 - \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx \quad (2.3)$$

and

$$Q(u) := \partial_\lambda S_\omega(u^\lambda)|_{\lambda=1} = 2\|\Delta u\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 - \frac{\frac{Np}{2} + b}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx, \quad (2.4)$$

where $u^\lambda(x) := \lambda^{\frac{N}{2}} u(\lambda x)$. The family of ground states corresponding to (2.1) is given

$$\mathcal{G}_\omega = \{u \in \mathcal{A}_\omega : S_\omega(u) \leq S_\omega(v) \text{ for all } v \in \mathcal{A}_\omega\} \quad (2.5)$$

where

$$\mathcal{A}_\omega = \{v \in H^2 \setminus \{0\} : S'_\omega(v) = 0\}.$$

Define a function

$$f(\lambda) := S_\omega(u^\lambda) = \frac{\lambda^4}{2} \|\Delta u\|_{L^2}^2 + \frac{\mu \lambda^2}{2} \|\nabla u\|_{L^2}^2 + \frac{\omega}{2} \|u\|_{L^2}^2 - \frac{\lambda^\alpha}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx, \quad (2.6)$$

where $\alpha = \frac{Np}{2} + b$, $\lambda \in (0, \infty)$. In addition, at $\lambda = 1$ $f(\lambda)$ reaches its maximum at $(0, \infty)$.

Lemma 2.1. *Let $0 < p < p^*$ and $\psi_0 \in H^2$. Then, $\exists T = T(\|\psi_0\|_{H^2})$ such that (1.1) has a unique solution $\psi \in C([0, T], H^2)$. Assume that T^* is the maximal existence time of solution $\psi(t)$. If $T^* < \infty$, then $\|\Delta \psi(t)\|_{L^2} \rightarrow \infty$ as $t \rightarrow T^*$.*

Moreover, the solution $\psi(t)$ satisfies the mass and energy conservation laws:

$$M(\psi(t)) = \int_{\mathbb{R}^N} |\psi(t, x)|^2 dx = M(\psi_0)$$

and

$$E(\psi(t)) = \frac{1}{2} \|\Delta \psi(t)\|_{L^2}^2 + \frac{\mu}{2} \|\nabla \psi(t)\|_{L^2}^2 - \frac{1}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |\psi(t)|^{p+2} = E(\psi_0)$$

for all $t \in [0, T^*)$.

Introduce a radial function $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $\nabla^j \varphi \in L^\infty(\mathbb{R}^N)$, $1 \leq j \leq 6$,

$$\varphi : r \mapsto \begin{cases} \frac{r^2}{2}, & \text{for } r \leq 1, \\ \text{const}, & \text{for } r \geq 10, \end{cases}$$

and $\varphi''(r) \leq 1$ for $r \geq 0$.

Take $R > 0$ and define the radial function

$$\varphi_R := R^2 \varphi\left(\frac{\cdot}{R}\right).$$

One can check the following properties of φ_R

$$\nabla \varphi_R(r) = R \varphi'\left(\frac{r}{R}\right) \frac{x}{|x|} = \begin{cases} x, & \text{for } r \leq R, \\ 0, & \text{for } r \geq 10R. \end{cases}$$

$$\|\nabla^j \varphi_R\|_{L^\infty} \lesssim R^{2-j} \quad \text{for } 0 \leq j \leq 6.$$

$$\text{supp}(\nabla^j \varphi_R) \subset \begin{cases} \{|x| \leq 10R\}, & \text{for } j = 1, 2, \\ \{R \leq |x| \leq 10R\}, & \text{for } 3 \leq j \leq 6. \end{cases}$$

For $\psi(t) \in H^2$, the localized virial of $\psi(t)$ is defined by

$$M_R(\psi(t)) := 2\Im \int_{\mathbb{R}^N} \overline{\psi}(t, x) \nabla \varphi_R(x) \nabla \psi(t, x) dx. \quad (2.7)$$

Lemma 2.2. (From [9].) Assume that $\{u_n\}_{n=1}^\infty$ is bounded in H^2 . Then, $\exists$ a subsequence of $\{u_n\}_{n=1}^\infty$, a family $\{x_n^j\}_{j=1}^\infty$ of sequences in $\mathbb{R}^N$ and profiles $\{U^j\}_{j=1}^\infty$ in H^2 such that

i) for every $k \neq j$, $|x_n^k - x_n^j| \rightarrow +\infty$, as $n \rightarrow \infty$;

ii) for every $l \geq 1$ and every $x \in \mathbb{R}^N$, we have

$$u_n(x) = \sum_{j=1}^l U^j(x - x_n^j) + r_n^l,$$

with $\limsup_{n \rightarrow \infty} \|r_n^l\|_{L^r} \rightarrow 0$ as $l \rightarrow \infty$ for every $r \in (2, p^*)$. Moreover,

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^l \|U^j\|_{L^2}^2 + \|r_n^l\|_{L^2}^2 + o_n(1), \quad (2.8)$$

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^l \|\nabla U^j\|_{L^2}^2 + \|\nabla r_n^l\|_{L^2}^2 + o_n(1), \quad (2.9)$$

$$\|\Delta u_n\|_{L^2}^2 = \sum_{j=1}^l \|\Delta U^j\|_{L^2}^2 + \|\Delta r_n^l\|_{L^2}^2 + o_n(1), \quad (2.10)$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$.

Basic estimates. We will introduce some estimates that will be useful on this work. Interpolation inequalities with weights (see in [7])

$$\| |x|^\gamma D^j u \|_{L^r} \leq C \| |x|^\alpha D^m u \|_{L^p}^a \| |x|^\beta u \|_{L^q}^{1-a},$$

if and only if

$$\frac{1}{r} + \frac{\gamma - j}{N} = a \left(\frac{1}{p} - \frac{\alpha - m}{N} \right) + (1 - a) \left(\frac{1}{q} + \frac{\beta}{N} \right)$$

and

$$\gamma \leq a\alpha + (1 - a)\beta,$$

where $p, q, \alpha, \beta, \gamma, r$ are real numbers and $p, q \geq 1$, $\frac{1}{p} + \frac{\alpha}{N}, \frac{1}{q} + \frac{\beta}{N}, \frac{1}{r} + \frac{\gamma}{N} > 0$. Moreover, a Strauss inequality [8]

$$\|u\|_{L^\infty(|x|>R)} \leq CR^{-(N-1)/2} \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2}.$$

2.2 Results

Estimation for nonlinear term

Lemma 2.3. Assume $N \geq 1$ and $0 < b < \min(N, 4)$, and $0 < p < p^*$. Then,

$$\int_{\mathbb{R}^N} |x|^{-b} |\psi|^{p+2} dx \leq C \|\Delta \psi\|_{L^2}^{\frac{Np+2b}{4}} \|\psi\|_{L^2}^{p+2 - \frac{Np+2b}{4}}.$$

Proof. See in [7] with $\gamma = -\frac{b}{p+2}$, $j = 0$, $m = p = q = 2$, $\alpha = \beta = 0$, and $r = p + 2$. Moreover, $b < 4$ was obtained from the condition $0 < p < \frac{8-2b}{N-4}$ for $N \leq 5$. $\square$

2.2.1 Localized virial function

Proposition 2.1. Let $N \geq 2$, $R > 0$, and $\psi \in C([0, T^*), \mathcal{H}_{rd})$ be a solution to (1.1). Then, on $[0, T^*)$

$$\begin{aligned} \frac{d}{dt} M_R(\psi(t)) &\leq (2Np + 4b)E(\psi(t)) - (Np + 2b - 8)\|\Delta \psi(t)\|_{L^2}^2 - (Np + 2b - 4)\mu \|\nabla \psi(t)\|_{L^2}^2 \\ &\quad + X_\mu(\psi(t)) + \mathcal{O} \left(\frac{1}{R^4} + \frac{|\mu|}{R^2} + \frac{\|\nabla \psi(t)\|_{L^2}^2}{R^2} + \frac{\|\nabla \psi(t)\|_{L^2}^{\frac{p}{2}}}{R^{\frac{(N-1)p}{2} + b}} \right), \end{aligned} \quad (2.11)$$

where with some constant $A_0 > 0$

$$X_\mu \lesssim A_0 \cdot \begin{cases} 0, & \text{for } \mu \geq 0, \\ |\mu| \|\nabla \psi(t)\|_{L^2}^2, & \text{for } \mu < 0. \end{cases}$$

Proof. First of all,

$$M_R(\psi(t)) = \langle \psi(t), \Gamma_{\varphi_R} \psi(t) \rangle,$$

where

$$\Gamma_{\varphi_R} := -i(\nabla \varphi_R \cdot \nabla + \nabla \cdot \nabla \varphi_R).$$

Then,

$$\frac{d}{dt} M_R(\psi(t)) = \langle \psi(t), [\Delta^2, i\Gamma_{\varphi_R}] \psi(t) \rangle + \langle \psi(t), [-\mu \Delta, i\Gamma_{\varphi_R}] \psi(t) \rangle + \langle \psi(t), [-|x|^{-b} |\psi(t)|^p, i\Gamma_{\varphi_R}] \psi(t) \rangle.$$

From [2] we have estimations for the linear part

$$\begin{aligned} \langle \psi(t), [\Delta^2, i\Gamma_{\varphi_R}] \psi(t) \rangle + \langle \psi(t), [-\mu \Delta, i\Gamma_{\varphi_R}] \psi(t) \rangle &\leq 8\|\Delta \psi(t)\|_{L^2}^2 + 4\mu \|\nabla \psi(t)\|_{L^2}^2 + X_\mu(\psi(t)) \\ &\quad + \mathcal{O} \left(\frac{1}{R^4} + \frac{|\mu|}{R^2} + \frac{\|\nabla \psi(t)\|_{L^2}^2}{R^2} \right). \end{aligned}$$

Now, one needs to calculate the nonlinear part.

$$\begin{aligned}
\mathcal{N}(\psi(t)) &:= \langle \psi(t), [-|x|^{-b}|\psi(t)|^p, i\Gamma_{\varphi_R}] \psi(t) \rangle = 2 \int_{\mathbb{R}^N} |\psi(t)|^2 \nabla \varphi_R \nabla (|x|^{-b}|\psi(t)|^p) dx \\
&= 2 \left(\int_{\mathbb{R}^N} |\psi(t)|^{p+2} \nabla \varphi_R \nabla (|x|^{-b}) dx + \int_{\mathbb{R}^N} |x|^{-b} |\psi(t)|^2 \nabla \varphi_R \nabla (|\psi(t)|^p) dx \right) \\
&= 2 \left(- \int_{\mathbb{R}^N} |x|^{-b} |\psi(t)|^{p+2} \Delta \varphi_R dx - \int_{\mathbb{R}^N} |x|^{-b} \nabla \varphi_R |\psi(t)|^p \nabla (|\psi(t)|^2) dx \right),
\end{aligned}$$

where we apply integration by parts

$$\int_{\mathbb{R}^N} |x|^{-b} |\psi(t)|^2 \nabla \varphi_R \nabla (|\psi(t)|^p) dx = - \int_{\mathbb{R}^N} \nabla (|x|^{-b} |\psi(t)|^2 \nabla \varphi_R) |\psi(t)|^p dx.$$

It can be shown that

$$\nabla (|\psi(t)|^{p+2}) = \frac{p+2}{2} \nabla (|\psi(t)|^2) |\psi(t)|^p.$$

So,

$$\begin{aligned}
\mathcal{N}(\psi(t)) &= -\frac{2}{\frac{p}{2}+1} \int_{\mathbb{R}^N} |x|^{-b} \nabla (|\psi(t)|^{p+2}) \nabla \varphi_R dx - 2 \int_{\mathbb{R}^N} |x|^{-b} |\psi(t)|^{p+2} \Delta \varphi_R dx \\
&= \frac{2}{\frac{p}{2}+1} \int_{\mathbb{R}^N} \nabla (|x|^{-b} \nabla \varphi_R) |\psi(t)|^{p+2} dx - 2 \int_{\mathbb{R}^N} |x|^{-b} |\psi(t)|^{p+2} \Delta \varphi_R dx \\
&= \frac{2}{\frac{p}{2}+1} \int_{\mathbb{R}^N} -b|x|^{-b-2} x \nabla \varphi_R |\psi(t)|^{p+2} dx - \frac{p}{\frac{p}{2}+1} \int_{\mathbb{R}^N} |x|^{-b} |\psi(t)|^{p+2} \Delta \varphi_R dx.
\end{aligned}$$

Now, using the definition of φ_R we get the following: on the region $|x| < R$, $\Delta \varphi = N$ and $\nabla \varphi = x$. Thus, the first term is

$$\frac{Np}{\frac{p}{2}+1} \int_{\mathbb{R}^N} |x|^{-b} |\psi(t)|^{p+2} dx + \frac{p}{\frac{p}{2}+1} \int_{|x| \geq R} |x|^{-b} |\psi(t)|^{p+2} (\Delta \varphi_R - N) dx.$$

Since $\|\Delta \varphi_R - N\|_{L^\infty} \lesssim 1$ and applying a Strauss inequality, we have

$$\begin{aligned}
\int_{|x| \geq R} |x|^{-b} |\psi(t)|^{p+2} dx &\leq R^{-b} \int_{|x| \geq R} |\psi(t)|^{p+2} dx \lesssim R^{-b} \|\psi(t)\|_{L^2}^2 \|\psi(t)\|_{L^\infty(|x| \geq R)}^p \\
&\lesssim R^{-\frac{N(p-1)}{2}-b} \|\psi(t)\|_{L^2}^2 \|\nabla \psi(t)\|_{L^2}^{\frac{p}{2}}.
\end{aligned}$$

The similar analysis we used for the first term.

$$\frac{-2b}{\frac{p}{2}+1} \int_{\mathbb{R}^N} |x|^{-b} |\psi(t)|^{p+2} dx + \frac{2}{\frac{p}{2}+1} \int_{\mathbb{R}^N} -b|x|^{-b-2} x (\nabla \varphi_R - x) |\psi(t)|^{p+2} dx.$$

One can show that $|\nabla \varphi_R| \lesssim R$ and $|x \cdot (\nabla \varphi_R - x)| \leq |x| |\nabla \varphi_R| + |x|^2 \lesssim |x|^2$. Then,

$$|x|^{-b-2} |x \cdot (\nabla \varphi_R - x)| |\psi(t)|^{p+2} \lesssim |x|^{-b} |\psi(t)|^{p+2} \lesssim R^{-b} |\psi(t)|^{p+2}$$

on $|x| \geq R$ and $b > 0$. Now

$$\begin{aligned}
\int_{|x| \geq R} |x|^{-b-2} x (\nabla \varphi_R - x) |\psi(t)|^{p+2} dx &\lesssim R^{-b} \int_{|x| \geq R} |\psi(t)|^{p+2} dx \lesssim R^{-b} \|\psi(t)\|_{L^2}^2 \|\psi(t)\|_{L^\infty(|x| \geq R)}^p \\
&\lesssim R^{-\frac{p(N-1)}{2}-b} \|\psi(t)\|_{L^2}^2 \|\nabla \psi(t)\|_{L^2}^{\frac{p}{2}}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{d}{dt} M_R(\psi(t)) &\leq 8 \|\Delta \psi(t)\|_{L^2}^2 + 4\mu \|\nabla \psi(t)\|_{L^2}^2 + X_\mu(\psi(t)) - \frac{2Np+4b}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |\psi(t)|^{p+2} dx \\
&\quad + \mathcal{O} \left(\frac{1}{R^4} + \frac{|\mu|}{R^2} + \frac{\|\nabla \psi(t)\|_{L^2}^2}{R^2} + \frac{\|\nabla \psi(t)\|_{L^2}^{\frac{p}{2}}}{R^{\frac{(N-1)p}{2}+b}} \right). \tag{2.12}
\end{aligned}$$

□

2.2.2 The ground state characterization

Define

$$L_\omega(u) := \|\Delta u\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 + \omega \|u\|_{L^2}^2.$$

Moreover, define

$$\omega_0 := -\inf \left\{ \|\Delta v\|_{L^2}^2 + \mu \|\nabla v\|_{L^2}^2 \quad \text{s.t.} \quad v \in H^2, \|v\|_{L^2} = 1 \right\}. \quad (2.13)$$

Then, in the view of (2.13) for every $\omega > \omega_0$, we have:

$$L_\omega(v) \sim \|\Delta v\|_{L^2}^2 + \|v\|_{L^2}^2 \sim \|v\|_{H^2}^2. \quad (2.14)$$

Proof. To prove this, we will use the definition of equivalent norms. So, $\exists C_1, C_2$ such that

$$C_1 (\|\Delta v\|_{L^2}^2 + \|v\|_{L^2}^2) \leq L_\omega(v) \leq C_2 (\|\Delta v\|_{L^2}^2 + \|v\|_{L^2}^2).$$

From the fact $\mu < 0$ it is easy to obtain the upper bound

$$\|\Delta v\|_{L^2}^2 + \mu \|\nabla v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 \leq \|\Delta v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 \leq \max\{1, \omega\} (\|\Delta v\|_{L^2}^2 + \|v\|_{L^2}^2).$$

From (2.13),

$$L_\omega(v) \geq (\omega + \omega_0) \|v\|_{L^2}^2. \quad (2.15)$$

Moreover, we have

$$\|\nabla v\|_{L^2}^2 \leq \|\Delta v\|_{L^2} \|v\|_{L^2} \leq \frac{1}{2} \|\Delta v\|_{L^2}^2 + \frac{1}{2} \|v\|_{L^2}^2.$$

Hence, (note that $\mu < 0$)

$$\begin{aligned} L_\omega(v) &\geq \|\Delta v\|_{L^2}^2 + \mu \left(\frac{1}{2} \|\Delta v\|_{L^2}^2 + \frac{1}{2} \|v\|_{L^2}^2 \right) + \omega \|v\|_{L^2}^2 \\ &= \frac{2 - |\mu|}{2} \|\Delta v\|_{L^2}^2 + \left(\omega - \frac{|\mu|}{2} \right) \|v\|_{L^2}^2. \end{aligned} \quad (2.16)$$

Now, choose $\lambda > 0$ such that

$$\omega - \frac{|\mu|}{2} + \lambda(\omega + \omega_0) \geq 0.$$

Using (2.15) and (2.16), we get:

$$(1 + \lambda)L_\omega(v) \geq \frac{2 - |\mu|}{2} \|\Delta v\|_{L^2}^2 + \left(\omega - \frac{|\mu|}{2} + \lambda(\omega + \omega_0) \right) \|v\|_{L^2}^2 \geq \frac{2 - |\mu|}{2} \|\Delta v\|_{L^2}^2.$$

As a result, we obtain

$$L_\omega(v) \geq \frac{2 - |\mu|}{2(1 + \lambda)} \|\Delta v\|_{L^2}^2.$$

With (2.15), it implies

$$2L_\omega(v) \geq \frac{2 - |\mu|}{2(1 + \lambda)} \|\Delta v\|_{L^2}^2 + (\omega + \omega_0) \|v\|_{L^2}^2,$$

which provides the lower bound. $\square$

Based on (2.14) the following minimizing problem can be solved

$$d(\omega) = \inf \{ S_\omega(v) : v \in H^2 \setminus \{0\}, K_\omega(v) = 0 \}. \quad (2.17)$$

Proposition 2.2. *Let $N \geq 2$, $\mu < 0$, $\omega > \omega_0$, and $0 < p < p^*$. Then, $\exists u \in H^2 \setminus \{0\}$ that is, $K_\omega(u) = 0$ and $S_\omega(u) = d(\omega)$.*

Proof. First of all, show $d(\omega) > 0$. From $K_\omega(u) = 0$ we say that

$$\|\Delta u\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 + \omega \|u\|_{L^2}^2 = \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx.$$

Now, using [7] we have

$$L_\omega(v) = \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx \leq C \|v\|_{H^2}^{p+2}.$$

Hence,

$$L_\omega(v) \leq C \|v\|_{H^2}^{p+2} \leq C L_\omega(v)^{\frac{p+2}{2}}.$$

There exists $C_1 > 0$ such that

$$L_\omega(v) \geq C_1 > 0.$$

Thus, one can check that

$$S_\omega(v) = \frac{1}{p+2} K_\omega(v) + \frac{p}{2(p+2)} L_\omega(v) \geq \frac{p}{2(p+2)} C_1 > 0.$$

Taking inf over v , which is $d(\omega)$ by definition

$$d(\omega) := \inf_{v \in H^2 \setminus \{0\}} \{S_\omega(v)\} > 0.$$

Now, assume that $\{v_n\} \subset H^2 \setminus \{0\}$ is minimizing sequence of (2.17), that is, $K_\omega(v_n) = 0$, $S_\omega(v_n) \rightarrow d(\omega)$ as $n \rightarrow \infty$. Hence,

$$L_\omega(v_n) = \frac{2(p+2)}{p} S_\omega(v_n) \rightarrow \frac{2(p+2)}{p} d(\omega),$$

which implies that $\{v_n\}$ is bounded in H^2 .

Using the profile decomposition in H^2 , $\exists$ subsequence $\{v_n\}$, a family $\{x_n^j\}_{j=1}^\infty \in \mathbb{R}^N$, and $\{U^j\}_{j=1}^\infty \subset H^2$ s.t. for every $\ell \geq 1$ and every $x \in \mathbb{R}^N$, we have

$$v_n(x) = \sum_{j=1}^\ell U^j(x - x_n^j) + r_n^\ell, \quad (2.18)$$

where (2.8)-(2.10) hold. From (2.8)-(2.9), we deduce

$$L_\omega(v_n) = \sum_{j=1}^\ell L_\omega(U^j) + L_\omega(r_n^\ell) + o_n(1).$$

This implies

$$\begin{aligned} K_\omega(v_n) &= L_\omega(v_n) - \int_{\mathbb{R}^N} |x|^{-b} |v_n|^{p+2} dx = \sum_{j=1}^\ell L_\omega(U^j) + L_\omega(r_n^\ell) - \int_{\mathbb{R}^N} |x|^{-b} |v_n|^{p+2} dx + o_n(1) \\ &= \sum_{j=1}^\ell K_\omega(U^j) + \sum_{j=1}^\ell \int_{\mathbb{R}^N} |x|^{-b} |U^j|^{p+2} dx + L_\omega(r_n^\ell) - \int_{\mathbb{R}^N} |x|^{-b} |v_n|^{p+2} dx + o_n(1). \end{aligned}$$

Since $K_\omega(v_n) = 0$, and

$$\int_{\mathbb{R}^N} |x|^{-b} |v_n|^{p+2} dx \rightarrow \frac{2(p+2)}{p} d(\omega) \quad \text{as } n \rightarrow \infty$$

and $L_\omega(r_n^\ell) \geq 0$ for all $n \geq 1$, we say that

$$\sum_{j=1}^\ell K_\omega(U^j) + \sum_{j=1}^\ell \int_{\mathbb{R}^N} |x|^{-b} |U^j|^{p+2} dx - \frac{2(p+2)}{p} d(\omega) \leq 0 \quad (2.19)$$

or

$$\sum_{j=1}^{\ell} L_{\omega}(U^j) - \frac{2(p+2)}{p} d(\omega) \leq 0. \quad (2.20)$$

On the other hand,

$$\frac{2(p+2)}{p} d(\omega) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |x|^{-b} |v_n|^{p+2} dx = \sum_{j=1}^{\infty} \int_{\mathbb{R}^N} |x|^{-b} |U^j|^{p+2} dx. \quad (2.21)$$

Combining (2.19)-(2.21) we obtain

$$\sum_{j=1}^{\infty} L_{\omega}(U^j) \leq \frac{2(p+2)}{p} d(\omega), \quad \sum_{j=1}^{\infty} K_{\omega}(U^j) \leq 0. \quad (2.22)$$

Claim: $K_{\omega}(U^j) = 0$ for all $j \geq 1$. Suppose that there exists $j_0 \geq 1$ such that $K_{\omega}(U^{j_0}) < 0$. Note that

$$K_{\omega}(\lambda U^{j_0}) = \lambda^2 L_{\omega}(U^{j_0}) - \lambda^{p+2} \int_{\mathbb{R}^N} |x|^{-b} |U^{j_0}|^{p+2} dx > 0$$

for sufficiently small $\lambda > 0$. Then, $\exists \lambda_0 \in (0, 1)$ such that

$$K_{\omega}(\lambda_0 U^{j_0}) = 0.$$

It follows that,

$$d(\omega) \leq S_{\omega}(\lambda_0 U^{j_0}) = \frac{p}{2(p+2)} L_{\omega}(\lambda_0 U^{j_0}) = \frac{p}{2(p+2)} \lambda_0^2 L_{\omega}(U^{j_0}) < \frac{p}{2(p+2)} L_{\omega}(U^{j_0}).$$

Together with (2.22)

$$d(\omega) < \frac{p}{2(p+2)} L_{\omega}(U^{j_0}) \leq d(\omega),$$

which is a contradiction. Therefore, there exists only one term $U^{j_0} \neq 0$.

By contradiction, assume that $\exists U^{j_1} \neq 0$, $U^{j_2} \neq 0$, then from (2.22)

$$L_{\omega}(U^{j_1}) < \frac{2(p+2)}{p} d(\omega), \quad L_{\omega}(U^{j_2}) < \frac{2(p+2)}{p} d(\omega)$$

and by $K_{\omega}(U^{j_1}) = 0$ we say

$$\frac{2(p+2)}{p} d(\omega) \leq \frac{2(p+2)}{p} S_{\omega}(U^{j_1}) = L_{\omega}(U^{j_1}) < \frac{2(p+2)}{p} d(\omega),$$

which is a contradiction. Thus, only one term $U^{j_0} \neq 0$ in the decomposition, and $K_{\omega}(U^{j_0}) = 0$, which yields the inf of the (2.17) occurs at U^{j_0} . $\square$

Let

$$M_{\omega} := \{v \in H^2 : S_{\omega}(v) = d(\omega), K_{\omega}(v) = 0, v \neq 0\}$$

be a set of all minimizers for (2.17).

Lemma 2.4. *Let $N \geq 2$, $\mu < 0$, $\omega > \omega_0$ and $0 < p < p^*$. Therefore, $M_{\omega} \subset \mathcal{G}_{\omega}$.*

Proof. Let $u \in M_{\omega}$. Then, $\exists \nu \in \mathbb{R}$ is a Lagrange multiplier that is,

$$0 = K_{\omega}(u) = \langle S'_{\omega}(u), u \rangle = \nu \langle K'_{\omega}(u), u \rangle.$$

By $K_{\omega}(u) = 0$ and $u \neq 0$, one gets

$$\langle K'_{\omega}(u), u \rangle = \partial_{\lambda} K_{\omega}(\lambda u)|_{\lambda=1} = 2L_{\omega}(u) - (p+2) \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx = -p \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx < 0.$$

Hence, we have $v = 0$ and $S_\omega(u) = 0$, which provides that $u \in \mathcal{A}_\omega$.

For any $v \in \mathcal{A}_\omega$ we have $K_\omega(v) = \langle S'_\omega(v), v \rangle = 0$ and $v \neq 0$. Therefore, it implies the definition of $d(\omega)$

$$S_\omega(u) = d(\omega) \leq S_\omega(v).$$

As a result, $u \in \mathcal{G}_\omega$ and $M_\omega \subset \mathcal{G}_\omega$. $\square$

Theorem 2.1. Assume $N \geq 2, \mu < 0, \omega > \omega_0$, and $0 < p < p^*$. Then $u \in \mathcal{G}_\omega$ if and only if u solves (2.17).

Proof. Aim is to show that $\mathcal{G}_\omega \subset M_\omega$. Take $u \in \mathcal{G}_\omega$. By proposition 2.2, $\exists v \in M_\omega$ and by lemma 2.4 $v \in \mathcal{A}_\omega$. Moreover,

$$S_\omega(u) \leq S_\omega(v) = d(\omega).$$

On the other hand, u satisfies $K_\omega(u) = 0$ and $u \neq 0$

$$d(\omega) \leq S_\omega(u).$$

As a result, $S_\omega(u) = d(\omega)$ and $u \in M_\omega$. $\square$

2.2.3 Strong instability

Lemma 2.5. Let $N \geq 2, \mu < 0, \omega > \omega_0$, and $\frac{8-2b}{N} < p < p^*$. Suppose that $u \in \mathcal{G}_\omega$ and $\partial_\lambda^2 S_\omega(u^\lambda)|_{\lambda=1} \leq 0$.

Moreover, suppose that $v \in H^2 \setminus \{0\}$ is such that

$$\|v\|_{L^2}^2 \leq \|u\|_{L^2}^2, \quad K_\omega(v) \leq 0, \quad Q(v) \leq 0.$$

Thus,

$$Q(v) \leq 4(S_\omega(v) - S_\omega(u)). \quad (2.23)$$

Remark. From lemma 2.5, for $u \in \mathcal{G}_\omega$ satisfying

$$\partial_\lambda^2 S_\omega(u^\lambda)|_{\lambda=1} \leq 0$$

and

$$\{v \in H^2 \setminus \{0\} : \|v\|_{L^2}^2 \leq \|u\|_{L^2}^2, S_\omega(v) < S_\omega(u), K_\omega(v) < 0, Q(v) = 0\} = \emptyset,$$

then

$$0 = Q(v) \leq 4(S_\omega(v) - S_\omega(u)) < 0$$

which is a contradiction.

Proof. If $K_\omega(u) = 0$, from theorem 2.1 and $Q(v) < 0$, we have

$$S_\omega(u) \leq S_\omega(v) \leq S_\omega(v) - \frac{1}{4}Q(v).$$

Now, consider $K_\omega(v) < 0$ and define

$$f_1(\lambda) := S_\omega(v^\lambda) - \frac{\lambda^4}{4}Q(v) = \frac{(\lambda^4 - 2\lambda^2)|\mu|}{4}\|\nabla v\|_{L^2}^2 + \frac{\alpha\lambda^4 - 4\lambda^\alpha}{4(p+2)} \int_{\mathbb{R}^N} |x|^{-b}|v|^{p+2} dx \quad \text{with} \quad \alpha = \frac{Np}{2} + b.$$

In addition, as shown above

$$K_\omega(v^\lambda) = \lambda^4 \|\Delta v\|_{L^2}^2 - |\mu|\lambda^2 \|\nabla v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - \lambda^\alpha \int_{\mathbb{R}^N} |x|^{-b}|v|^{p+2} dx.$$

Since, $\lim_{\lambda \rightarrow 0} K_\omega(v^\lambda) = \omega \|v\|_{L^2}^2 > 0$ and $K_\omega(v) < 0$, there exists $\lambda_0 \in (0, 1)$ that is, $K_\omega(v^{\lambda_0}) = 0$. If $f_1(\lambda_0) \leq f_1(1)$, then from theorem 2.1 and $Q(v) \leq 0$

$$S_\omega(u) \leq S_\omega(v^{\lambda_0}) \leq S_\omega(v^{\lambda_0}) - \frac{\lambda_0^4}{4}Q(v) \leq S_\omega(v) - \frac{1}{4}Q(v),$$

which is (2.23). Now, it is sufficient to show that $f_1(\lambda_0) \leq f_1(1)$, that is

$$\frac{(\lambda_0^4 - 2\lambda_0^2)|\mu|}{4} \|\nabla v\|_{L^2}^2 + \frac{\alpha\lambda_0^4 - 4\lambda_0^\alpha}{4(p+2)} \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx \leq \frac{-|\mu|}{4} \|\nabla v\|_{L^2}^2 + \frac{\alpha - 4}{4(p+2)} \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx.$$

Thus,

$$\frac{\lambda_0^4 - 2\lambda_0^2 + 1}{4} |\mu| \|\nabla v\|_{L^2}^2 \leq \frac{\alpha - 4 - \alpha\lambda_0^4 + 4\lambda_0^\alpha}{4(p+2)} \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx$$

and

$$|\mu| \|\nabla v\|_{L^2}^2 \leq \frac{\alpha - 4 - \alpha\lambda_0^4 + 4\lambda_0^\alpha}{(p+2)(\lambda_0^4 - 2\lambda_0^2 + 1)} \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx. \quad (2.24)$$

In the view of (2.6), the calculations yield

$$f(\lambda) = S_\omega(u^\lambda) = \frac{\lambda^4}{2} \|\Delta u\|_{L^2}^2 + \frac{\mu\lambda^2}{2} \|\nabla u\|_{L^2}^2 + \frac{\omega}{2} \|u\|_{L^2}^2 - \frac{\lambda^\alpha}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx$$

with $\alpha = \frac{Np}{2} + b$. Moreover,

$$\partial_\lambda^2 S_\omega(u^\lambda)|_{\lambda=1} \leq 0,$$

thus,

$$6\|\Delta u\|_{L^2}^2 - |\mu| \|\nabla u\|_{L^2}^2 - \frac{\alpha(\alpha-1)}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx \leq 0. \quad (2.25)$$

From $K_\omega(u) = Q(u) = 0$ take

$$4K_\omega(u) - 5Q(u) = 0.$$

Then together with (2.25)

$$\begin{aligned} & 4\|\Delta u\|_{L^2}^2 + 4\mu\|\nabla u\|_{L^2}^2 + 4\omega\|u\|_{L^2}^2 - 4 \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx \\ & - 10\|\Delta u\|_{L^2}^2 - 5\mu\|\nabla u\|_{L^2}^2 + \frac{5\alpha}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx = 0, \end{aligned}$$

which gives

$$\begin{aligned} 4\omega\|u\|_{L^2}^2 &= 6\|\Delta u\|_{L^2}^2 - |\mu| \|\nabla u\|_{L^2}^2 + \frac{4p+8-5\alpha}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx \\ &\leq \frac{\alpha^2 + 4p+8-6\alpha}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx. \end{aligned} \quad (2.26)$$

On the other hand, it follows from $K_\omega(v^{\lambda_0}) = 0$ and theorem 2.1:

$$\begin{aligned} \frac{p}{2(p+2)} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx &= S_\omega(u) \leq S_\omega(v^{\lambda_0}) = \frac{p}{2(p+2)} \int_{\mathbb{R}^N} |x|^{-b} |v^{\lambda_0}|^{p+2} dx \\ &= \frac{p}{2(p+2)} \lambda_0^\alpha \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx. \end{aligned}$$

From (2.26) and assumption $\|u\|_{L^2}^2 \leq \|v\|_{L^2}^2$ one gets

$$\omega\|v\|_{L^2}^2 \leq \frac{\alpha^2 + 4p+8-6\alpha}{4(p+2)} \lambda_0^\alpha \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx.$$

Now, from $K_\omega(v^{\lambda_0}) = 0$ and $Q(v) \leq 0$ we have

$$\begin{aligned} |\mu| \|\nabla v\|_{L^2}^2 &\leq \lambda_0^2 \|\Delta v\|_{L^2}^2 + \lambda_0^{-2} \omega \|v\|_{L^2}^2 - \lambda_0^{\alpha-2} \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx \\ &\leq \lambda_0^2 \|\Delta v\|_{L^2}^2 + \frac{\alpha^2 + 4p+8-6\alpha}{4(p+2)} \lambda_0^{\alpha-2} \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx - \lambda_0^{\alpha-2} \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx \\ &\leq |\mu| \frac{\lambda_0^2}{2} \|\nabla v\|_{L^2}^2 + \left(\frac{\lambda_0^2 \alpha}{2(p+2)} + \frac{(\alpha^2 - 6\alpha) \lambda_0^{\alpha-2}}{4(p+2)} \right) \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx. \end{aligned}$$

Hence,

$$|\mu| \left(1 - \frac{\lambda_0^2}{2}\right) \|\nabla v\|_{L^2}^2 \leq \left(\frac{\lambda_0^2 \alpha}{2(p+2)} + \frac{(\alpha^2 - 6\alpha)\lambda_0^{\alpha-2}}{4(p+2)}\right) \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx$$

and

$$|\mu| \|\nabla v\|_{L^2}^2 \leq \frac{2}{2 - \lambda_0^2} \left(\frac{\lambda_0^2 \alpha}{2(p+2)} + \frac{(\alpha^2 - 6\alpha)\lambda_0^{\alpha-2}}{4(p+2)}\right) \int_{\mathbb{R}^N} |x|^{-b} |v|^{p+2} dx. \quad (2.27)$$

From (2.24) and (2.26), we need to show that:

$$\frac{2\lambda_0^{\alpha-2}}{2 - \lambda_0^2} \left(\frac{\alpha}{2} \lambda_0^{4-\alpha} + \frac{\alpha^2 - 6\alpha}{4}\right) \leq \frac{4\lambda_0^\alpha - \alpha\lambda_0^4 + \alpha - 4}{\lambda_0^4 - 2\lambda_0^2 + 1}. \quad (2.28)$$

The rest of direct calculations are the same in [4]. We want to prove:

$$\frac{(2 - \lambda_0^2)(4\lambda_0^\alpha - \alpha\lambda_0^4 + \alpha - 4)}{2\lambda_0^{\alpha-2}(\lambda_0^4 - 2\lambda_0^2 + 1)} \geq \frac{\alpha}{2} \lambda_0^{4-\alpha} + \frac{\alpha^2 - 6\alpha}{4}.$$

Denote

$$g(\lambda) := \frac{g_1(\lambda)}{4\lambda^{\alpha-2}(\lambda^4 - 2\lambda^2 + 1)}$$

where

$$g_1(\lambda) = 2(\alpha - 2)(\alpha - 4)\lambda^\alpha - (\alpha - 2)(\alpha - 4)\lambda^{\alpha+2} - \alpha(\alpha - 6)\lambda^{\alpha-2} - (4\alpha - 8)\lambda^2 + 4\alpha - 16.$$

One can show that $\lim_{\lambda \rightarrow 1} g(\lambda) = 0$ and $4\lambda^{\alpha-2}(\lambda^4 - 2\lambda^2 + 1) \geq 0$. So, it is enough to show that $g_1(\lambda) \geq 1$ for all $\lambda \in (0, 1]$. Differentiate g_1 :

$$\begin{aligned} g_1'(\lambda) &= \lambda(2\alpha(\alpha - 4)(\alpha - 2)\lambda^{\alpha-2} - (\alpha - 4)(\alpha^2 - 4)\lambda^\alpha - \alpha(\alpha - 2)(\alpha - 6)\lambda^{\alpha-4} - 2(4\alpha - 8)) \\ &= \lambda \cdot g_2(\lambda), \end{aligned}$$

$$\begin{aligned} g_2'(\lambda) &= 2\alpha(\alpha - 2)^2(\alpha - 4)\lambda^{\alpha-3} - \alpha(\alpha - 4)(\alpha^2 - 4)\lambda^{\alpha-1} - \alpha(\alpha - 2)(\alpha - 6)(\alpha - 4)\lambda^{\alpha-5} \\ &= \lambda^{\alpha-5} g_3(\lambda), \end{aligned}$$

$$g_3'(\lambda) = 4\alpha(\alpha - 2)^2(\alpha - 4)\lambda - 4\alpha(\alpha - 4)(\alpha^2 - 4)\lambda^3 = \lambda g_4(\lambda),$$

$$g_4(\lambda) = 4\alpha(\alpha - 2)^2(\alpha - 4) - 4\alpha(\alpha - 4)(\lambda^2 - 1)^2 \lambda^2.$$

One can show that

$$g_4(1) = -16\alpha(\alpha - 2)(\alpha - 4) < 0$$

and

$$g_4(0) > 0, \quad g_4'(\lambda) < 0.$$

Thus, $g_4(\lambda) < 0$ and $g_1(\lambda) > 0 \quad \forall \lambda \in (0, 1)$. Now $\exists \lambda_0 \in (0, 1)$ such that

$$g_3'(\lambda) > 0 \quad \forall \lambda \in (0, \lambda_0) \quad \text{and} \quad g_3'(\lambda) \leq 0 \quad \forall \lambda \in (\lambda_0, 1),$$

$$g_2(0) = -2(4\alpha - 8)\lambda < 0 \quad \text{and} \quad g_2(1) = 0,$$

$$g_1'(\lambda) = \lambda g_2(\lambda) < 0 \quad \forall \lambda \in (0, 1).$$

$g_1'(1) \leq 0$, then $g_1(\lambda) > 0 \quad \forall \lambda \in (0, 1)$. □

Define the set

$$\mathcal{B} := \{v \in H^2 \setminus \{0\}; S_\omega(v) \leq S_\omega(u), \|v\|_{L^2} \leq \|u\|_{L^2}, K_\omega(v) < 0, Q(v) < 0\}.$$

Lemma 2.6. *Let $N \geq 2, \mu < 0, \omega > \omega_0, \frac{8-2b}{N} < p < p^*$. Suppose $u \in \mathcal{G}_\omega$ and $\partial_\lambda^2 S_\omega(u^\lambda)|_{\lambda=1} \leq 0$. Then, the set $\mathcal{B}$ is invariant with respect to the flow of (1.1).*

Proof. Let $\psi_0 \in \mathcal{B}$, then by lemma 2.1 $\exists$ a unique solution $\psi \in C([0, T^*), H^2)$, that is, if $\psi_0 \in \mathcal{B}$, then $\psi_0 \in H^2 \setminus \{0\}$ and also $\frac{8}{N} < p < p^*$. Moreover, conservation laws will work, then

$$S_\omega(\psi(t)) = S_\omega(\psi_0) < S_\omega(u) \quad \text{and} \quad \|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2} < \|u\|_{L^2}$$

for any $t \in [0, T^*)$. By the continuity of the function $t \mapsto K_\omega(\psi(t))$ and theorem 2.1, if $\exists t_0 \in [0, T^*)$ such that $K_\omega(\psi(t_0)) = 0$, then

$$S_\omega(u) \leq S_\omega(\psi(t_0))$$

that contradicts to $S_\omega(u) > S_\omega(\psi(t))$ for all $t \in [0, T^*)$. Thus, $\psi(t)$ satisfies $K_\omega(\psi(t)) < 0$ for all $t \in [0, T^*)$.

Now, it suffices to prove that if $Q(\psi_0) < 0$, then

$$Q(\psi(t)) < 0 \quad \forall t \in [0, T^*).$$

By contradiction, assume that there exists $t_0 \in [0, T^*)$ such that $Q(\psi(t_0)) = 0$. By lemma 2.5,

$$S_\omega(u) \leq S_\omega(\psi(t_0)) - \frac{1}{4}Q(\psi(t_0)) = S_\omega(\psi(t_0)).$$

Thus, we have $S_\omega(u) \leq S_\omega(\psi(t_0))$, however, it should be

$$S_\omega(\psi(t)) < S_\omega(u) \quad \text{for all } t \in [0, T^*).$$

Contradiction. □

Lemma 2.7. Let $N \geq 2, \mu < 0, \omega > \omega_0, \frac{8-2b}{N} < p < p^*$. Suppose that $u \in \mathcal{G}_\omega$ and $\partial_\lambda^2 S_\omega(u^\lambda)|_{\lambda=1} \leq 0$. Then, $u^\lambda \in \mathcal{B}$ for all $\lambda > 1$.

Proof. Scaling is $u^\lambda := \lambda^{\frac{N}{2}} u(\lambda x)$ and one can check that

$$\|u^\lambda\|_{L^2} = \|u\|_{L^2} \leq \|u\|_{L^2}.$$

Define $g(\lambda) := K_\omega(u^\lambda) = \lambda^4 \|\Delta u\|_{L^2}^2 + \mu \lambda^2 \|\nabla u\|_{L^2}^2 + \omega \|u\|_{L^2}^2 - \lambda^\alpha \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx$ with $\alpha = \frac{Np}{2} + b$. We assume that

$$\partial_\lambda^2 S_\omega(u^\lambda)|_{\lambda=1} \leq 0.$$

Therefore,

$$\partial_\lambda^2 S_\omega(u^\lambda)|_{\lambda=1} = 6\|\Delta u\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 - \frac{\alpha(\alpha-1)}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx.$$

Thus,

$$g''(\lambda) = 12\lambda^2 \|\Delta u\|_{L^2}^2 + 2\mu \|\nabla u\|_{L^2}^2 - \alpha(\alpha-1)\lambda^{\alpha-2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx < 0$$

for all $\lambda \geq 1$. Pohozaev identity corresponding to (2.1), implies that

$$g'(\lambda) < g'(1) = 4\|\Delta u\|_{L^2}^2 + 2\mu \|\nabla u\|_{L^2}^2 - \alpha \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx < 0.$$

Then $K_\omega(u^\lambda) < 0$ for all $\lambda \geq 1$. Now, from (2.6):

$$f(\lambda) := S_\omega(u^\lambda) = \frac{\lambda^4}{2} \|\Delta u\|_{L^2}^2 + \frac{\mu \lambda^2}{2} \|\nabla u\|_{L^2}^2 + \frac{\omega}{2} \|u\|_{L^2}^2 - \frac{\lambda^\alpha}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx,$$

we say

$$\begin{aligned} f''(\lambda) &= 6\lambda^2 \|\Delta u\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 - \frac{\alpha(\alpha-1)\lambda^{\alpha-2}}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx \\ &\leq f''(1) = \partial_\lambda^2 S_\omega(u^\lambda)|_{\lambda=1} \leq 0. \end{aligned}$$

$$\frac{Q(u^\lambda)}{\lambda} = f'(\lambda) < f'(1) = Q(u) = 0$$

and

$$S_\omega(u^\lambda) < S_\omega(u)$$

for all $\lambda \geq 1$. Hence, we get $u^\lambda \in \mathcal{B}$ for all $\lambda > 1$. □

Theorem 2.2. *Let $N \geq 2, \mu < 0, \omega > \omega_0, \frac{8-2b}{N} < p < p^*$. Suppose that ground state of (2.1) u is the radial and $\partial_\lambda^2 S_\omega(u^\lambda)|_{\lambda=1} \leq 0$. If $|\mu|$ is sufficiently small, then the standing wave $\psi(t, x) = e^{i\omega t}u(x)$ shows strong instability via blow-up. That is, for any $\epsilon > 0$, $\exists \psi_0 \in H^2$ such that $\|\psi_0 - u\|_{H^2} < \epsilon$ and the solution $\psi(t)$ to (1.1) with initial data ψ_0 blows-up in finite time.*

Proof. Take $\epsilon > 0$, $\omega > \omega_0$ and $u \in \mathcal{G}_\omega$. Now, we have $u^\lambda \rightarrow u$ as $\lambda \rightarrow 1$ (by Brezis-Lieb's lemma) and there exists $\lambda_0 > 1$ such that

$$\|u^{\lambda_0} - u\|_{H^2} \leq \epsilon$$

and from lemma 2.7 $u^{\lambda_0} \in \mathcal{B}$. By lemma 2.1 and the assumption that u^{λ_0} is radially symmetric, there exists a unique solution $\psi(t) \in C([0, T^*), H_{rd}^2)$ with initial data $\psi_0 = u^{\lambda_0}$. Here $T^* > 0$ is the maximal time of existence.

Now, we will prove that $\psi(t)$ blows-up.

Step 1. Claim is $\exists C_0 > 0$ such that

$$Q(\psi(t)) \leq -C_0 \quad \text{for any } t \in [0, T^*).$$

Indeed, $\psi(t) \in \mathcal{B}$ for any $t \in [0, T^*)$. From lemma 2.5, we have

$$Q(\psi(t)) \leq 4[S_\omega(\psi(t)) - S_\omega(u)] = 4[S_\omega(u^{\lambda_0}) - S_\omega(u)] = -C_0,$$

where $C_0 = 4(S_\omega(u) - S_\omega(u^{\lambda_0})) > 0$.

Step 2. The next claim is $\exists C_1 > 0$ such that

$$\frac{d}{dt} M_R(\psi(t)) \leq -C_1 \|\Delta\psi(t)\|_{L^2}^2$$

for any $t \in [0, T^*)$, where $M_R(\psi(t))$ is virial function. First of all, we show that $\exists C_2 > 0$ such that

$$\|\Delta\psi(t)\|_{L^2} \geq C_2 \quad \text{for every } t \in [0, T^*).$$

We will prove by contradiction that $\exists \{t_k\} \subseteq [0, T^*)$ such that

$$\|\Delta\psi(t_k)\|_{L^2} \rightarrow 0.$$

However, from mass conservation and the interpolation inequalities with weights ([7]), one can say that

$$\int_{\mathbb{R}^N} |x|^{-b} |\psi(t_k)|^{p+2} dx \rightarrow 0, \quad \|\nabla\psi(t_k)\|_{L^2}^2 \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Therefore, we have

$$Q(\psi(t_k)) := 2\|\Delta\psi(t_k)\|_{L^2}^2 + \mu\|\nabla\psi(t_k)\|_{L^2}^2 - \frac{Np}{2(p+2)} \int_{\mathbb{R}^N} |x|^{-b} |\psi(t_k)|^{p+2} dx \rightarrow 0$$

as $k \rightarrow \infty$, which contradicts $Q(\psi(t)) \leq -C_0$.

Take $t \in [0, T^*)$ and define

$$\eta := \frac{2Np|E(\psi_0)|}{\delta} + \frac{A^2|\mu|^2\|\psi_0\|_{L^2}^2}{4\delta^2},$$

where

$$\delta = \frac{Np + 2b - 8}{3}, \quad A = A_0 + Np + 2b - 4.$$

Case 1:

$$\|\Delta\psi(t)\|_{L^2}^2 \leq \eta.$$

From (2.11) deduce

$$\begin{aligned} \frac{d}{dt} M_R(\psi(t)) &\leq 4Q(\psi(t)) + X_\mu[\psi(t)] + O\left(\frac{1}{R^4} + \frac{\|\Delta\psi(t)\|_{L^2}^2}{R^2} + \frac{\|\Delta\psi(t)\|_{L^{p/2}}^{\frac{p}{4}}}{R^{(N-1)/2p}} + \frac{|\mu|}{R^2}\right) \\ &\leq -4C_0 + A|\mu|\eta + O\left(\frac{1}{R^4} + \frac{\eta}{R^2} + \frac{\eta^{\frac{p}{8}}}{R^{(N-1)/2p}} + \frac{|\mu|}{R^2}\right). \end{aligned}$$

By choosing $|\mu|$ small enough and R large enough, we obtain

$$\frac{d}{dt}M_R(\psi(t)) \leq -2C_0 \leq -\frac{2C_0}{\eta} \|\Delta\psi(t)\|_{L^2}^2.$$

Case 2:

$$\|\Delta\psi(t)\|_{L^2}^2 \geq \eta.$$

It follows from conservation of energy that

$$\begin{aligned} 2NpE(\psi(t)) - (Np-8)\|\Delta\psi(t)\|_{L^2}^2 - (Np-4)\mu\|\nabla\psi(t)\|_{L^2}^2 + X_\mu[\psi(t)] &\leq \\ 2NpE(\psi_0) - 3\delta\|\Delta\psi(t)\|_{L^2}^2 + \frac{A^2|\mu|^2}{4\delta}\|\psi_0\|_{L^2}^2 &\leq -\delta\|\Delta\psi(t)\|_{L^2}^2, \end{aligned}$$

where

$$\|\nabla\psi(t)\|_{L^2}^2 \leq \|\psi_0\|_{L^2}\|\Delta\psi(t)\|_{L^2} \leq \frac{A|\mu|}{4\delta}\|\psi_0\|_{L^2}^2 + \frac{\delta}{A|\mu|}\|\Delta\psi(t)\|_{L^2}^2. \quad (2.29)$$

Therefore,

$$\frac{d}{dt}M_R(\psi(t)) = -\delta\|\Delta\psi(t)\|_{L^2}^2 + \mathcal{O}\left(\frac{1}{R^4} + \frac{\|\nabla\psi(t)\|_{L^2}^2}{R^2} + \frac{\|\nabla\psi(t)\|_{L^{p/2}}^2}{R^{(N-1)/2p}} + \frac{|\mu|}{R^2}\right).$$

Choose $R > 1$ large enough such that

$$\mathcal{O}\left(\frac{1}{R^4} + \frac{\|\nabla\psi(t)\|_{L^2}^2}{R^2} + \frac{\|\nabla\psi(t)\|_{L^{p/2}}^2}{R^{(N-1)/2p}} + \frac{|\mu|}{R^2}\right) \leq \frac{\delta}{2}\|\Delta\psi(t)\|_{L^2}^2.$$

Now, we have

$$\frac{d}{dt}M_R(\psi(t)) \leq -\frac{\delta}{2}\|\Delta\psi(t)\|_{L^2}^2. \quad (2.30)$$

Step 3. Suppose that there exists a global solution. If $T^* = \infty$, from (2.30) follows that $\exists t_1 > 0$ such that $M_R(\psi(t)) < 0$ for all $t > t_1$. Integrating on $[t_1, t]$, we get:

$$M_R(\psi(t)) \leq -\frac{\delta}{2} \int_{t_1}^t \|\Delta\psi(s)\|_{L^2}^2 ds \leq 0.$$

One the other hand,

$$M_R(\psi(t)) \leq C\|\nabla\varphi_R\|_{L^\infty}\|\psi(t)\|_{L^2}\|\nabla\psi(t)\|_{L^2} \leq C(\psi_0)\|\Delta\psi(t)\|_{L^2}^{1/2}.$$

Then,

$$M_R(\psi(t)) \leq -\frac{\delta}{2} \int_{t_1}^t \|\Delta\psi(s)\|_{L^2}^2 ds \leq -C(\delta, R) \int_{t_1}^t M_R(\psi(s))^4 ds.$$

Define $h(t) := \int_{t_1}^t M_R(\psi(s))^4 ds$. Hence, $h(t)$ is strictly increasing and nonnegative.

$$h'(t) = M_R(\psi(t))^4 \geq C(\delta, R)h(t)^4$$

Integrating this inequality on $[t_2, t]$, $t_2 > t_1$, we get:

$$M_R(\psi(t)) \leq -C(\delta, R)h(t) \leq \frac{-C(\delta, R)h(t_2)}{(1 - 3C(\delta, R)^4 h(t_2)^3(t - t_2))^{1/3}}$$

for all $t > t_2$. We now deduce that $M_R(\psi(t)) \rightarrow -\infty$ as $t \rightarrow T^*$.

Thus, $\psi(t)$ of (1.1) with initial data ψ_0 blows-up in finite time T^* .

□

2.2.4 Additional results

We extend the results to the non-radial case. To estimate localized virial function we faced with this error term

$$|E_r| = \int_{|x| \geq R} |x|^{-b} |\psi(t)|^{p+2} dx,$$

which was estimated via Strauss inequality. Now, we show the same blow-up in [2] but with small changes in estimating an error term.

Theorem 2.3. *Let $N \geq 5$, $0 < b < \min\{4, N\}$ and $\psi_0 \in H^2(\mathbb{R}^N)$, and $E(\psi_0) < -g\mu^2 M(\psi_0)$, where $g > 0$ a positive constant. Then,*

1. *If $\frac{8}{N} < p < \frac{8-2b}{N-4}$, then the maximal solution for (1.1) $\psi(t)$ blows-up in finite time or infinite time, that is*

$$\lim_{t \rightarrow T_{\max}} \|\Delta\psi\|_{L^2} = +\infty.$$

2. *If $\frac{8-2b}{N} < p \leq \frac{8}{N}$, then $\psi(t)$ blows-up in finite time that there exists $T^* > 0$ s.t*

$$\lim_{t \rightarrow T^*} \|\Delta\psi\|_{L^2} = +\infty.$$

Proof. Case $\frac{8}{N} < p < \frac{8-2b}{N-4}$.

Suppose that there exists a global solution, that is

$$\|\Delta\psi\|_{L^2} \leq M \quad \forall t.$$

Then, the estimation for error term

$$|E_r| \lesssim R^{-b} \int_{|x| \geq R} |\psi|^{p+2} dx \lesssim R^{-b} \|\Delta\psi\|_{L^2}^{\frac{Np}{4}} \|\psi\|_{L^2}^{p+2-\frac{Np}{4}} \lesssim_{\psi_0, M} R^{-b}.$$

Now, take $R > 0$ large enough and (2.11) with (2.29) become

$$M'_R(\psi(t)) \leq 2NpE(\psi_0) - (Np - 2b - 8)\|\Delta\psi\|_{L^2}^2 + A|\mu|\|\nabla\psi\|_{L^2}^2 \leq Np[E(\psi_0) + g\mu^2 M(\psi_0)],$$

where $A = A_0 + Np + 2b - 4$. This implies that $M'_R(\psi(t)) \leq -C < 0$. Now, using the Fundamental Theorem of Calculus, we have

$$|M_R(\psi(t))| \lesssim \|\nabla\psi\|_{L^2} \|\psi\|_{L^2} \lesssim_{\psi_0} \|\Delta\psi\|_{L^2}^{\frac{1}{2}} \lesssim_{\psi_0} M^{\frac{1}{2}}.$$

Contradiction.

Case $\frac{8-2b}{N} < p \leq \frac{8}{N}$ implies $\frac{Np}{4} \leq 2$.

From the previous case we have

$$|E_r| \lesssim R^{-b} \|\Delta\psi\|_{L^2}^{\frac{Np}{4}} \|\psi\|_{L^2}^{p+2-\frac{Np}{4}} \lesssim_{\psi_0} R^{-b} + R^{-b} \|\Delta\psi\|_{L^2}^2,$$

where $\|\Delta\psi\|_{L^2}^{\frac{Np}{4}} \leq 1 + \|\Delta\psi\|_{L^2}^2$. Again, choose $R > 0$ sufficiently large and

$$M'_R(\psi(t)) \leq 2Np[E(\psi_0) + g\mu^2 M(\psi_0)] - \frac{Np + 2b - 8}{2} \|\Delta\psi\|_{L^2}^2 \leq -C - a \|\Delta\psi\|_{L^2}^2.$$

The rest of results are given in theorem 2.2. □

Moreover, we derive the results in the case of a positive power of $|x|$.

Lemma 2.8. *Let $N \geq 2$, $b > 0$, $p \geq \frac{2b}{N-1}$ and $p \leq \frac{2b+8}{N-4}$ if $N > 4$. Then, $\exists C(N, p, b) > 0$ such that*

$$\int_{\mathbb{R}^N} |x|^b |\psi|^{p+2} dx \leq C(N, p, b) \|\Delta\psi\|_{L^2}^{\frac{Np-2b}{4}} \|\psi\|_{L^2}^{\frac{8+2b-p(N-4)}{4}} \quad \forall \psi \in H_{rd}^2(\mathbb{R}^N).$$

Proof. From the radial Sobolev inequality for $N \geq 2$ and $\psi \in H_{rd}^2(\mathbb{R}^N)$ we have

$$|x|^{\frac{N-1}{2}} |\psi| \leq C(N) \|\nabla \psi\|_{L^2}^{\frac{1}{2}} \|\psi\|_{L^2}^{\frac{1}{2}} \leq C(N) \|\Delta \psi\|_{L^2}^{\frac{1}{4}} \|\psi\|_{L^2}^{\frac{3}{4}}$$

using the Gagliardo-Nirenberg inequality

$$\|\nabla \psi\|_{L^2} \leq \|\Delta \psi\|_{L^2}^{\frac{1}{2}} \|\psi\|_{L^2}^{\frac{1}{2}}.$$

Then,

$$\int_{\mathbb{R}^N} |x|^b |\psi|^{2+\frac{2b}{N-1}} dx = \int_{\mathbb{R}^N} \left(|x|^{\frac{N-1}{2}} |\psi| \right)^{\frac{2b}{N-1}} |\psi|^2 dx \leq C(N, b) \|\Delta \psi\|_{L^2}^{\frac{b}{2(N-1)}} \|\psi\|_{L^2}^{2+\frac{3b}{2(N-1)}}. \quad (2.31)$$

Moreover, for $N > 4$ and $\psi \in H_{rd}^2(\mathbb{R}^N)$ we have

$$|x|^{\frac{N}{2}-2} |\psi| \leq C(N) \|\Delta \psi\|_{L^2}$$

and the Sobolev embedding $\dot{H}^2(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-4}}(\mathbb{R}^N)$ implies

$$\int_{\mathbb{R}^N} |x|^b |\psi|^{\frac{2N+2b}{N-4}} dx = \int_{\mathbb{R}^N} \left(|x|^{\frac{N}{2}-2} |\psi| \right)^{\frac{2b}{N-4}} |\psi|^{\frac{2N}{N-4}} dx \leq C(N, b) \|\Delta \psi\|_{L^2}^{\frac{2b+2N}{N-4}}. \quad (2.32)$$

Now, for $\frac{2b}{N-1} + 2 \leq p+2 \leq \frac{2N+2b}{N-4}$ and by using (2.31)-(2.32) we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^b |\psi|^{p+2} dx &\leq \left(\int_{\mathbb{R}^N} |x|^b |\psi|^{\frac{2b}{N-1}+2} dx \right)^{\theta} \left(\int_{\mathbb{R}^N} |x|^b |\psi|^{\frac{2N+2b}{N-4}} dx \right)^{1-\theta} \\ &\leq C(N, p, b) \|\Delta \psi\|_{L^2}^{\frac{b}{2(N-1)}\theta + \frac{2N+2b}{N-4}(1-\theta)} \|\psi\|_{L^2}^{(2+\frac{3b}{2(N-1)})\theta} \end{aligned}$$

where $\exists \theta \in [0, 1]$ such that

$$p+2 = \left(\frac{2b}{N-1} + 2 \right) \theta + \frac{2N+2b}{N-4} (1-\theta).$$

Calculations yield

$$\frac{b}{2(N-1)}\theta + \frac{2N+2b}{N-4}(1-\theta) = \frac{Np-2b}{4} \quad \text{and} \quad \left(2 + \frac{3b}{2(N-1)} \right) \theta = \frac{2b+8-p(N-4)}{4}.$$

□

3 The Fourth-Order Inhomogeneous Nonlinear Schrödinger System with Mixed Dispersions and Coupled Power-Type Nonlinearity

3.1 Background material

Denote the space

$$\mathcal{H} := H^2 \times \dots \times H^2 = [H^2]^m,$$

where H^2 is Sobolev space

$$\|\cdot\|_{H^2}^2 := \|\cdot\|_{L^2}^2 + \|\Delta \cdot\|_{L^2}^2.$$

The scaling invariance for (1.2) is $\mathbf{u}^\lambda := \lambda^{\frac{4-b}{2p-2}} \mathbf{u}(\lambda^4 \cdot, \lambda \cdot)$.

Moreover, the Sobolev index is

$$s_c := \frac{N}{2} - \frac{4-b}{2p-2}.$$

Let us define respectively, the mass critical and the energy critical exponents

$$p_* = 1 + \frac{4-b}{N},$$

$$p^* = \begin{cases} 1 + \frac{4-b}{N-4} & \text{if } N > 4, \\ \infty & \text{if } 2 \leq N \leq 4. \end{cases}$$

In addition, denote the real numbers

$$B := \frac{N(p-1)+b}{2} \quad \text{and} \quad A := 2p - B.$$

Now, introduce conserved quantities called the masses, the global mass, the Hamiltonian and the energy, respectively

$$M(u_j(t)) := \int_{\mathbb{R}^N} |u_j(t, x)|^2 dx = M(\psi_j),$$

$$M(\mathbf{u}(t)) := \sum_{j=1}^m M(u_j(t)) = M(\Psi),$$

$$\begin{aligned} H(\mathbf{u}(t)) &:= \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} |\Delta u_j(t, x)|^2 dx + \frac{\mu}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} |\nabla u_j(t, x)|^2 dx \\ &\quad - \frac{1}{2p} \sum_{j=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_j|^p |u_k|^p dx = H(\Psi), \end{aligned}$$

$$E(\mathbf{u}(t)) := H(\mathbf{u}(t)) + \frac{1}{2} M(\mathbf{u}(t)).$$

Define a constraint. For $\mathbf{u} := (u_1, \dots, u_m) \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{R}$ we have

$$\begin{aligned} K_{\alpha, \beta}(\mathbf{u}(t)) &:= \frac{1}{2} \sum_{j=1}^m (2\alpha + (N-4)\beta) \|\Delta u_j\|_{L^2}^2 + \frac{\mu}{2} \sum_{j=1}^m (2\alpha + (N-2)\beta) \|\nabla u_j\|_{L^2}^2 \\ &\quad + \frac{1}{2} \sum_{j=1}^m \|u_j\|_{L^2}^2 - \frac{1}{2p} \sum_{j,k=1}^m (2p\alpha + (N-b)\beta) \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p dx. \end{aligned}$$

Moreover, $Q(\mathbf{u}(t)) := K_{1, -\frac{2}{N}}(\mathbf{u}(t))$, thus,

$$Q(\mathbf{u}(t)) := \frac{4}{N} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \frac{2\mu}{N} \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 - \frac{(p-1) + \frac{b}{N}}{p} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p dx.$$

Definition 3.1. A ground state of the problem (1.2) is any solution of the following elliptic problem

$$\begin{aligned} \Delta^2 \phi_j - \mu \Delta \phi_j + \phi_j &= \sum_{k=1}^m |x|^{-b} |\phi_k|^p |\phi_j|^{p-2} \phi_j, \\ 0 \neq \Phi &:= (\phi_1, \dots, \phi_m) \in \mathcal{H}_{rd} \end{aligned} \quad (3.1)$$

and minimizes the problem

$$d_{\alpha,\beta} := \inf_{0 \neq u \in \mathcal{H}} \{E(\mathbf{u}) : K_{\alpha,\beta}(\mathbf{u}) = 0\}. \quad (3.2)$$

Denote some invariant under flow of (1.2) sets:

$$\begin{aligned} A_{\alpha,\beta}^- &:= \{\mathbf{u} \in \mathcal{H}_{rd} : E(\mathbf{u}) < d_{\alpha,\beta} \quad \& \quad K_{\alpha,\beta}(\mathbf{u}) \leq 0\}, \\ A_{\alpha,\beta}^+ &:= \{\mathbf{u} \in \mathcal{H}_{rd} : E(\mathbf{u}) < d_{\alpha,\beta} \quad \& \quad K_{\alpha,\beta}(\mathbf{u}) > 0\}. \end{aligned}$$

In addition, introduce some real numbers

$$\begin{aligned} \underline{\mu} &:= \min\{2\alpha + (N-4)\beta, \quad 2\alpha + (N-2)\beta, \quad 2\alpha + N\beta\}, \\ \bar{\mu} &:= \max\{2\alpha + (N-4)\beta, \quad 2\alpha + (N-2)\beta, \quad 2\alpha + N\beta\}. \end{aligned}$$

Remark. The problem (3.2) satisfies

$$d_{\alpha,\beta} = \inf_{0 \neq \mathbf{u} \in \mathcal{H}_{rd}} \left\{ \left(E - \frac{1}{\bar{\mu}} K_{\alpha,\beta} \right) (\mathbf{u}) : K_{\alpha,\beta}(\mathbf{u}) \leq 0 \right\}. \quad (3.3)$$

For $\mathbf{u} \in \mathcal{H}$ the localized virial function is

$$M_R(\mathbf{u}(t)) := M_{\varphi_R}(\mathbf{u}(t)) := 2 \sum_{j=1}^m \Im \left(\int_{\mathbb{R}^N} \bar{u}_j \nabla \varphi_R \cdot \nabla u_j \, dx \right). \quad (3.4)$$

Lemma 3.1. Let $C > 0$ and $r > 1$ are two real constants. Then, no positive function $f \in C^1(\mathbb{R}_+)$ exists that satisfies $f(0) > 0$ and

$$f' \geq C f^r \quad \text{on } \mathbb{R}_+.$$

3.2 Results

Nonlinear term estimation

Lemma 3.2. Assume that $N \geq 1$ and $0 < b < \min(N, 4)$, and $p_* < p < p^*$. Then,

$$\sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p \, dx \leq C \left(\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \right)^{\frac{b}{2}} \left(\sum_{j=1}^m \|u_j\|_{L^2}^2 \right)^{\frac{A}{2}}.$$

Proof. By [7] one can show

$$\| |x|^{-\frac{b}{2p}} u \|_{L^{2p}} \leq C \|\Delta u\|_{L^2}^{\frac{N(p-1)+b}{4p}} \|u\|_{L^2}^{1 - \frac{N(p-1)+b}{4p}}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$, $0 < b < N$. Hence,

$$\sum_{j=1}^m \| |x|^{-\frac{b}{2p}} u_j \|_{L^{2p}}^{2p} \leq C \sum_{j=1}^m \|\Delta u_j\|_{L^2}^{\frac{N(p-1)+b}{2}} \sum_{j=1}^m \|u_j\|_{L^2}^{2p - \frac{N(p-1)+b}{2}}.$$

Moreover, $b < 4$ obtained from $p_* < p < 1 + \frac{4-b}{N-4}$ for $N > 4$. □

3.2.1 Localized variance identity

The localized variance was given in the form

$$M_R(\mathbf{u}) := 2 \sum_{j=1}^m \Im \left(\int_{\mathbb{R}^N} (\bar{u}_j \nabla \varphi_R \cdot \nabla u_j) dx \right) = \sum_{j=1}^m \langle u_j, \Gamma_{\varphi_R} u_j \rangle := \sum_{j=1}^m M_R(u_j)$$

where

$$\begin{aligned} \Gamma_{\varphi_R} &:= -i(\nabla \varphi_R \cdot \nabla + \nabla \cdot \nabla \varphi_R) \\ \Gamma_{\varphi_R} f &= -i(\nabla \varphi_R \cdot \nabla f + \nabla \cdot (f \nabla \varphi_R)). \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt} M_R(u_j) &= \langle u_j, [\Delta^2, i\Gamma_{\varphi_R}] u_j \rangle + \langle u_j, [-\mu \Delta, i\Gamma_{\varphi_R}] u_j \rangle \\ &+ \langle u_j, [-\sum_{k=1}^m |x|^{-b} |u_k|^p |u_j|^{p-2}, i\Gamma_{\varphi_R}] u_j \rangle := \mathcal{L}(u_j) + \mathcal{M}(u_j) + \mathcal{N}(u_j). \end{aligned}$$

From [2] we have

$$\mathcal{L}(u_j) \leq 8 \int_{\mathbb{R}^N} |\Delta u_j|^2 + \mathcal{O}(R^{-4} + R^{-2} \|\nabla u_j\|^2), \quad (3.5)$$

$$\mathcal{M}(u_j) = 4\mu \int_{\mathbb{R}^N} |\nabla u_j|^2 + X_\mu(u_j) + \mathcal{O}(|\mu| R^{-2}) \quad (3.6)$$

where with some constant $A_0 > 0$

$$X_\mu(u_j) \lesssim A_0 \cdot \begin{cases} 0, & \text{for } \mu \geq 0, \\ |\mu| \|\nabla u_j\|_{L^2}^2, & \text{for } \mu < 0. \end{cases}$$

Now, it is needed to calculate only nonlinear part. Take $[X, Y] = XY - YX$. Then,

$$\begin{aligned} & \left[\sum_{k=1}^m |x|^{-b} |u_k|^p |u_j|^{p-2}, \nabla \varphi_R \cdot \nabla + \nabla \cdot \nabla \varphi_R \right] u_j \\ &= \sum_{k=1}^m |x|^{-b} |u_k|^p |u_j|^{p-2} (\nabla \varphi_R \nabla u_j + \nabla \cdot (u_j \nabla \varphi_R)) \\ &- \left(\nabla \varphi_R \nabla \left(\sum_{k=1}^m |x|^{-b} |u_k|^p |u_j|^{p-2} u_j \right) + \nabla \cdot \left(\sum_{k=1}^m |x|^{-b} |u_k|^p |u_j|^{p-2} u_j \nabla \varphi_R \right) \right) \\ &= 2 \sum_{k=1}^m |x|^{-b} |u_k|^p |u_j|^{p-2} \nabla \varphi_R \nabla u_j + \sum_{k=1}^m |x|^{-b} |u_k|^p |u_j|^{p-2} u_j \Delta \varphi_R \\ &- 2 \sum_{k=1}^m \nabla (|x|^{-b} |u_k|^p |u_j|^{p-2} u_j) \nabla \varphi_R - \sum_{k=1}^m |x|^{-b} |u_k|^p |u_j|^{p-2} u_j \Delta \varphi_R \\ &= 2 \sum_{k=1}^m (|x|^{-b} |u_k|^p |u_j|^{p-2} \nabla u_j - \nabla (|x|^{-b} |u_k|^p |u_j|^{p-2} u_j)) \nabla \varphi_R \\ &= -2 \sum_{k=1}^m \nabla (|x|^{-b} |u_k|^p |u_j|^{p-2}) u_j \nabla \varphi_R. \end{aligned}$$

Thus,

$$\begin{aligned} -\langle u_j, -2 \sum_{k=1}^m \nabla (|x|^{-b} |u_k|^p |u_j|^{p-2}) u_j \nabla \varphi_R \rangle &= 2 \sum_{k=1}^m \int_{\mathbb{R}^N} u_j \overline{\nabla (|x|^{-b} |u_k|^p |u_j|^{p-2}) u_j \nabla \varphi_R} dx \\ &= 2 \sum_{k=1}^m \int_{\mathbb{R}^N} |u_j|^2 \nabla (|x|^{-b} |u_k|^p |u_j|^{p-2}) \nabla \varphi_R dx \\ &= 2 \sum_{k=1}^m \int_{\mathbb{R}^N} (\nabla (|x|^{-b} |u_k|^p) |u_j|^p \nabla \varphi_R + |x|^{-b} |u_k|^p \nabla |u_j|^{p-2} |u_j|^2 \nabla \varphi_R) dx. \end{aligned}$$

The calculation gives the following

$$\begin{aligned}\nabla(|u_j|^{p-2})|u_j|^2 &= \nabla(u_j \bar{u}_j)^{\frac{p-2}{2}}|u_j|^2 = \frac{p-2}{p}|u_j|^{p-4}(\bar{u}_j \nabla u_j + u_j \nabla \bar{u}_j)|u_j|^2 \\ &= \frac{p-2}{p} \cdot \frac{2}{p} \nabla(|u_j|^p) = \frac{p-2}{p} \nabla(|u_j|^p).\end{aligned}$$

Hence,

$$\begin{aligned}2 \sum_{k=1}^m \int_{\mathbb{R}^N} \left(\nabla(|x|^{-b}|u_k|^p)|u_j|^p \nabla \varphi_R + \frac{p-2}{p}|x|^{-b}|u_k|^p \nabla(|u_j|^p) \nabla \varphi_R \right) dx \\ = 2 \sum_{k=1}^m \int_{\mathbb{R}^N} \left(\nabla(|x|^{-b}|u_k|^p)|u_j|^p - \frac{2}{p}|x|^{-b}|u_k|^p \nabla(|u_j|^p) \right) \nabla \varphi_R dx.\end{aligned}$$

Using integration by parts, one gets

$$\int_{\mathbb{R}^N} \nabla(|x|^{-b}|u_k|^p|u_j|^p) \nabla \varphi_R dx = - \int_{\mathbb{R}^N} |x|^{-b}|u_k|^p|u_j|^p \Delta \varphi_R dx.$$

Therefore,

$$-2 \sum_{k=1}^m \int_{\mathbb{R}^N} \left(|x|^{-b}|u_k|^p|u_j|^p \Delta \varphi_R + \frac{2}{p}|x|^{-b}|u_k|^p \nabla(|u_j|^p) \nabla \varphi_R \right) dx.$$

Again, integration by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^{-b}|u_k|^p \nabla(|u_j|^p) \nabla \varphi_R dx &= - \int_{\mathbb{R}^N} \nabla(|x|^{-b}|u_k|^p \nabla \varphi_R) |u_j|^p dx \\ &= - \int_{\mathbb{R}^N} (\nabla(|x|^{-b}|u_k|^p) \nabla \varphi_R |u_j|^p + |x|^{-b}|u_k|^p |u_j|^p \Delta \varphi_R) dx.\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{N}(u_j) &= -2 \sum_{k=1}^m \int_{\mathbb{R}^N} |x|^{-b}|u_k|^p|u_j|^p \Delta \varphi_R dx \\ &\quad + \frac{4}{p} \int_{\mathbb{R}^N} (|x|^{-b}|u_k|^p|u_j|^p \Delta \varphi_R + \nabla(|x|^{-b}|u_k|^p) \nabla \varphi_R |u_j|^p) dx \\ &= -2 \sum_{k=1}^m \left(\left(1 - \frac{2}{p}\right) \int_{\mathbb{R}^N} |x|^{-b}|u_k|^p|u_j|^p \Delta \varphi_R dx - \frac{2}{p} \int_{\mathbb{R}^N} \nabla(|x|^{-b}|u_k|^p) \nabla \varphi_R |u_j|^p dx \right).\end{aligned}$$

Using symmetry, we get

$$\begin{aligned}&\frac{2}{p} \sum_{k,j=1}^m \nabla(|x|^{-b}|u_k|^p|u_j|^p) \\ &= \frac{2}{p} \sum_{k,j=1}^m |x|^{-b} \nabla(|u_k|^p)|u_j|^p + \frac{2}{p} \sum_{k,j=1}^m \nabla(|x|^{-b})|u_k|^p|u_j|^p + \frac{2}{p} \sum_{k,j=1}^m |x|^{-b}|u_k|^p \nabla(|u_j|^p) \\ &= \frac{4}{p} \sum_{k,j=1}^m |x|^{-b} \nabla(|u_k|^p)|u_j|^p + \frac{2}{p} \sum_{k,j=1}^m \nabla(|x|^{-b})|u_k|^p|u_j|^p.\end{aligned}$$

The calculation gives the following

$$\begin{aligned}\nabla(|x|^{-b}|u_k|^p) &= \nabla(|x|^{-b})|u_k|^p + |x|^{-b} \nabla(|u_k|^p) \\ &= -b|x|^{b-2}x|u_k|^p + |x|^{-b} \nabla(|u_k|^p).\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{N}(\mathbf{u}) &= -2 \sum_{k,j=1}^m \left(\left(1 - \frac{2}{p}\right) \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p \Delta \varphi_R dx - \frac{2}{p} \int_{\mathbb{R}^N} \nabla(|x|^{-b} |u_k|^p) \nabla \varphi_R |u_j|^p dx \right) \\
&= 2 \left(\frac{2}{p} - 1 \right) \sum_{k,j=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p \Delta \varphi_R dx + \frac{4}{p} \sum_{k,j=1}^m \int_{\mathbb{R}^N} |x|^{-b} \nabla(|u_k|^p) \nabla \varphi_R |u_j|^p dx \\
&\quad + \frac{4}{p} \sum_{k,j=1}^m \int_{\mathbb{R}^N} -b|x|^{-b-2} x |u_k|^p \nabla \varphi_R |u_j|^p dx \\
&= 2 \left(\frac{2}{p} - 1 \right) \sum_{k,j=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p \Delta \varphi_R dx + \frac{2}{p} \sum_{k,j=1}^m \int_{\mathbb{R}^N} \nabla(|x|^{-b} |u_k|^p |u_j|^p) \nabla \varphi_R dx \\
&\quad + \frac{2}{p} \sum_{k,j=1}^m \int_{\mathbb{R}^N} -b|x|^{-b-2} x |u_k|^p \nabla \varphi_R |u_j|^p dx \\
&= 2 \left(\frac{2}{p} - 1 \right) \sum_{k,j=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p \Delta \varphi_R dx - \frac{2}{p} \sum_{k,j=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p \Delta \varphi_R dx \\
&\quad + \frac{2}{p} \sum_{k,j=1}^m \int_{\mathbb{R}^N} -b|x|^{-b-2} x |u_k|^p \nabla \varphi_R |u_j|^p dx \\
&= -2 \left(\frac{p-1}{p} \right) \sum_{k,j=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p \Delta \varphi_R dx + \frac{2}{p} \sum_{k,j=1}^m \int_{\mathbb{R}^N} -b|x|^{-b-2} x |u_k|^p \nabla \varphi_R |u_j|^p dx.
\end{aligned}$$

Now, use the equality $\Delta \varphi_R - N = 0$ for $|x| \leq R$ we decompose

$$\begin{aligned}
&-2 \left(\frac{p-1}{p} \right) \sum_{k,j=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p \Delta \varphi_R dx \\
&= \frac{-2N(p-1)}{p} \sum_{k,j=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p dx - \frac{2(p-1)}{p} \sum_{k,j=1}^m \int_{|x| \geq R} |x|^{-b} |u_k|^p |u_j|^p (\Delta \varphi_R - N) dx
\end{aligned}$$

Since, $\|\Delta \varphi_R - N\|_{L^\infty} \lesssim 1$ and using a Strauss inequality, for $b > 0$ one gets

$$\begin{aligned}
&\int_{|x| \geq R} |x|^{-b} |u_k|^p |u_j|^p dx \leq R^{-b} \int_{|x| \geq R} |u_k|^p |u_j|^p dx \\
&\lesssim R^{-b} \|u_k\|_{L^\infty(|x| \geq R)}^p \|u_j\|_{L^\infty(|x| \geq R)}^p \|u_j\|_{L^2}^2 \\
&\lesssim R^{-b} R^{-\frac{(N-1)p}{2}} \|u_k\|^{\frac{p}{2}} \|\nabla u_k\|^{\frac{p}{2}} R^{-\frac{(N-1)(p-2)}{2}} \|u_j\|^{\frac{p-2}{2}} \|\nabla u_j\|^{\frac{p-2}{2}} \|u_j\|^2 \\
&\lesssim R^{-(N-1)(p-1)-b} \|\mathbf{u}\|^{p+1} \|\nabla \mathbf{u}\|^{p-1}.
\end{aligned}$$

Conclusion, we obtain

$$-\frac{2N(p-1)}{p} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p dx + \mathcal{O} \left(R^{-(N-1)(p-1)-b} \left(\sum_{j=1}^m \|\nabla u_j\|^2 \right)^{\frac{p-1}{2}} \right).$$

The next part will be estimated in the same way: we have $\nabla \varphi_R - x = 0$ for $|x| \leq R$, then

$$\begin{aligned}
&-\frac{2b}{p} \int_{\mathbb{R}^N} |x|^{-b-2} x |u_k|^p \nabla \varphi_R |u_j|^p dx \\
&= -\frac{2b}{p} \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p dx - \frac{2b}{p} \int_{|x| \geq R} |x|^{-b-2} x |u_k|^p |u_j|^p (\nabla \varphi_R - x) dx.
\end{aligned}$$

Since, $\|\nabla\varphi_R\|_{L^\infty} \lesssim R$ and one can show that

$$|x \cdot (\nabla\varphi_R - x)| \lesssim |x| |\nabla\varphi_R| + |x|^2 \lesssim |x|^2,$$

for $|x| \geq R$. Now,

$$\begin{aligned} \int_{|x| \geq R} |x|^{-b-2} |x \cdot (\nabla\varphi_R - x)| |u_k|^p |u_j|^p dx &\lesssim R^{-b} \int_{|x| \geq R} |u_k|^p |u_j|^p dx \\ &\lesssim R^{-(N-1)(p-1)-b-1} \|\mathbf{u}\|^{p+1} \|\nabla\mathbf{u}\|^{p-1}. \end{aligned}$$

We obtain this estimation

$$-\frac{2b}{p} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p dx + \mathcal{O} \left(R^{-(N-1)(p-1)-b} \left(\sum_{j=1}^m \|\nabla u_j\|^2 \right)^{\frac{p-1}{2}} \right).$$

Finally, we get the following estimation

$$\mathcal{N}(\mathbf{u}) \leq -\frac{2(N(p-1)+b)}{p} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p dx + \mathcal{O} \left(R^{-(N-1)(p-1)-b} \left(\sum_{j=1}^m \|\nabla u_j\|^2 \right)^{\frac{p-1}{2}} \right). \quad (3.7)$$

The final estimate for the localized virial will be

$$\begin{aligned} \frac{d}{dt} M_R(\mathbf{u}) &\leq 8 \sum_{j=1}^m \int_{\mathbb{R}^N} |\Delta u_j|^2 + 4 \sum_{j=1}^m \mu \int_{\mathbb{R}^N} |\nabla u_j|^2 + X_\mu(\mathbf{u}) \\ &\quad - \frac{2(N(p-1)+b)}{p} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p dx \\ &\quad + \mathcal{O} \left(R^{-4} + R^{-2} \|\nabla u_j\|^2 + |\mu| R^{-2} + R^{-(N-1)(p-1)-b} \left(\sum_{j=1}^m \|\nabla u_j\|^2 \right)^{\frac{p-1}{2}} \right). \end{aligned} \quad (3.8)$$

3.2.2 Blow-up criteria

Theorem 3.1. *Let $2 \leq N \leq 7$, $\Psi := (\psi_1, \dots, \psi_m) \in \mathcal{H}_{rd}$, $\mu \in \mathbb{R}$, $\Phi := (\phi_1, \dots, \phi_m)$ be a ground state solution to (3.1) and the maximal solution $\mathbf{u} \in C_{T^*}(\mathcal{H}_{rd})$ of (1.2). Assume that $2 \leq p \leq 5$ and $0 < s_c < 2$.*

1. If $\mu \neq 0$, we assume that

$$H(\Psi) < \begin{cases} 0, & \text{for } \mu > 0, \\ -g\mu^2 M(\Psi), & \text{for } \mu < 0 \end{cases}$$

with some constant $g = g(N, p) > 0$.

2. If $\mu = 0$, we assume either $H(\Psi) < 0$ or $H(\Psi) \geq 0$ with

$$H(\Psi)M(\Psi)^{\frac{2-s_c}{s_c}} < H(\Phi)M(\Phi)^{\frac{2-s_c}{s_c}} \quad (3.9)$$

and

$$\left(\sum_{j=1}^m \|\Delta \psi_j\|_{L^2}^2 \right) \left(\sum_{j=1}^m \|\psi_j\|_{L^2}^2 \right)^{\frac{2-s_c}{s_c}} > \left(\sum_{j=1}^m \|\Delta \phi_j\|_{L^2}^2 \right) \left(\sum_{j=1}^m \|\phi_j\|_{L^2}^2 \right)^{\frac{2-s_c}{s_c}}. \quad (3.10)$$

Then, $\mathbf{u}$ blows-up in finite time, i.e. $0 < T^* < \infty$ and

$$\lim_{t \rightarrow T^*} \sup \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 = +\infty.$$

Proof. **Case 1.** $\mu \geq 0$ and $H(\Psi) < 0$. Then, by (3.8)

$$\begin{aligned}
M'_R(\mathbf{u}(t)) &\leq 4(N(p-1) + b)H(\Psi) - 2(N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \\
&\quad + \mathcal{O} \left(\frac{1}{R^4} + \frac{|\mu|}{R^2} + \frac{\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2}{R^2} + \frac{(\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2)^{\frac{p-1}{2}}}{R^{(N-1)(p-1)+b}} \right) \\
&\leq 2(N(p-1) + b)H(\Psi) - (N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \\
&\quad + \frac{C_0}{R^4} + \frac{C_0|\mu|}{R^2} + 2(N(p-1) + b)H(\Psi) \\
&\quad + \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \left(-(N(p-1) + b - 4) + \frac{C_0}{R^2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^{-1} + \frac{C_0}{R^2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^{\frac{p-1}{2}-2} \right).
\end{aligned}$$

On the other hand, from $H(\Psi) < 0$ one has

$$\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \leq \frac{1}{p} \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p dx \lesssim \left(\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \right)^{\frac{B}{2}} \left(\sum_{j=1}^m \|u_j\|_{L^2}^2 \right)^{\frac{A}{2}}.$$

Since $B > 2$, one can show

$$\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \geq C$$

on $[0, T^*)$. Now, choose $R > 0$ large enough such that

$$M'_R(\mathbf{u}(t)) \leq 2(N(p-1) + b - 4)H(\Psi) - (N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 < 0.$$

By contradiction, suppose that $T^* = \infty$. Then,

$$M_R(\mathbf{u}(t)) \leq M_R(\mathbf{u}(0, \cdot)) + 2(N(p-1) + b - 4)H(\Psi)t - (N(p-1) + b - 4) \int_0^t \sum_{j=1}^m \|\Delta u_j(s)\|_{L^2}^2 ds. \quad (3.11)$$

Thus, $\exists t_1 \geq 0$ such that for $t \geq t_1$ $M_R(\mathbf{u}(t)) \leq 0$. Moreover,

$$|M_R(\mathbf{u}(t))| \leq \|\nabla \varphi_R\|_{L^\infty} \|\mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \leq C(\Psi)R \|\Delta \mathbf{u}\|_{L^2}^{\frac{1}{2}}.$$

For $t > t_1$ (3.11) becomes

$$M_R(\mathbf{u}(t)) \leq M_R(\mathbf{u}(0, \cdot)) + 2(N(p-1) + b - 4)H(\Psi)t - (N(p-1) + b - 4)CR^{-4} \int_0^t |M_R(\mathbf{u}(s))|^4 ds.$$

From $H(\Psi) \leq 0$ we obtain

$$M_R(\mathbf{u}(t)) \leq -\delta \int_0^t |M_R(\mathbf{u}(s))|^4 ds, \quad \forall t > t_1.$$

Define $z(t) := \int_0^t |M_R(\mathbf{u}(s))|^4 ds$ for $t \geq t_1$. Then,

$$M_R^4(\mathbf{u}(t)) = z'(t) \geq \delta^4 z^4(t), \quad \forall t \geq t_1.$$

The rest of proof is closed by using lemma 3.1.

Case 2. $\mu < 0$ and $H(\Psi) < -g\mu^2 M(\Psi)$. Again, (3.8)

$$\begin{aligned} M'_R(\mathbf{u}) &\leq 4(N(p-1) + b)H(\Psi) - 2(N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + A|\mu| \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 \\ &\quad + \mathcal{O}\left(\frac{1}{R^4} + \frac{|\mu|}{R^2} + \frac{\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2}{R^2} + \frac{(\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2)^{\frac{p-1}{2}}}{R^{(N-1)(p-1)+b}}\right) \end{aligned}$$

where $A = A_0 + 2(N(p-1) + b - 2)$. Then, using

$$\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 \leq \frac{A|\mu|}{4(N(p-1) + b - 4)} \sum_{j=1}^m \|u_j\|_{L^2}^2 + \frac{N(p-1) + b - 4}{A|\mu|} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2$$

we obtain

$$\begin{aligned} M'_R(\mathbf{u}) &\leq 4(N(p-1) + b)H(\Psi) - 2(N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \\ &\quad + \frac{A\mu^2}{4(N(p-1) + b)} \sum_{j=1}^m \|u_j\|_{L^2}^2 + (N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \\ &\quad + \mathcal{O}\left(\frac{1}{R^4} + \frac{|\mu|}{R^2} + \frac{\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2}{R^2} + \frac{(\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2)^{\frac{p-1}{2}}}{R^{(N-1)(p-1)+b}}\right). \end{aligned}$$

Using mass conservation, one gets

$$\begin{aligned} M'_R(\mathbf{u}) &\leq 4(N(p-1) + b) [H(\Psi) + g\mu^2 M(\Psi)] - (N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \\ &\quad + \mathcal{O}\left(\frac{1}{R^4} + \frac{|\mu|}{R^2} + \frac{\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2}{R^2} + \frac{(\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2)^{\frac{p-1}{2}}}{R^{(N-1)(p-1)+b}}\right). \end{aligned}$$

Taking large enough $R > 0$ and

$$M'_R(\mathbf{u}) \leq -C - \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2.$$

The same conclusion as in the first case.

Case 3. $\mu = 0$ and $H(\Psi) \geq 0$. One can check that for

$$\begin{aligned} H(\Psi) &= \frac{1}{2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 - \frac{1}{2p} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_k|^p |u_j|^p dx \\ &\geq \frac{1}{2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 - \frac{C_{N,p,m}}{2p} M^{\frac{A}{2}}(\Psi) \left(\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \right)^{\frac{B}{2}} \end{aligned}$$

with

$$C_{N,p,m} = \frac{2p}{A} \left(\frac{A}{B} \right)^{\frac{B}{2}} (M(\Phi))^{1-p}.$$

Denote a real number $q := \frac{C_{N,p,m}}{p} M^{\frac{A}{2}}(\Psi)$ and a function

$$y \mapsto \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \quad \text{and} \quad F : X \mapsto \frac{1}{2} \left(X - qX^{\frac{B}{2}} \right).$$

Then F has a local maximum at

$$X_* = \left(\frac{2}{Bq} \right)^{\frac{2}{B-2}}$$

and

$$F(X_*) = \frac{B-2}{2B} X_*.$$

Using the Pohozaev identity

$$\sum_{j=1}^m \|\Delta \phi_j\|_{L^2}^2 = \frac{B}{A} \sum_{j=1}^m \|\phi_j\|_{L^2}^2 = \frac{B}{2p} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |\phi_k|^p |\phi_j|^p dx$$

we have

$$H(\Phi) = \frac{B-2}{2B} \sum_{j=1}^m \|\Delta \phi_j\|_{L^2}^2 = \frac{B-2}{2A} \sum_{j=1}^m \|\phi_j\|_{L^2}^2.$$

The calculation gives

$$\begin{aligned} X_* &= \left(\frac{2}{Bq} \right)^{\frac{2}{B-2}} = \left(\frac{2p}{B[M(\Psi)]^{\frac{A}{2}}} \cdot \frac{AB^{\frac{B}{2}}[M(\Phi)]^{p-1}}{2p \cdot A^{\frac{B}{2}}} \right)^{\frac{2}{B-2}} \\ &= \left(\frac{B^{\frac{B}{2}-1}}{A^{\frac{B}{2}-1}} \cdot \frac{[M(\Phi)]^{p-1}}{[M(\Psi)]^{\frac{A}{2}}} \right)^{\frac{2}{(p-1)s_c}} = \left(\frac{B^{\frac{(p-1)s_c}{2}}}{A^{\frac{(p-1)s_c}{2}}} \cdot \frac{[M(\Phi)]^{p-1}}{[M(\Psi)]^{\frac{A}{2}}} \right)^{\frac{2}{(p-1)s_c}} \\ &= \frac{B}{A} \cdot \frac{[M(\Phi)]^{\frac{2}{s_c}}}{[M(\Psi)]^{\frac{2-s_c}{s_c}}}. \end{aligned}$$

One can check

$$\begin{aligned} \frac{(p-1)s_c}{2} &= \frac{N(p-1)+b-4}{4} = \frac{N(p-1)+b}{4} - 1 = \frac{B}{2} - 1, \\ \frac{A}{2} \cdot \frac{2}{(p-1)s_c} &= \frac{4p-N(p-1)-b}{2} \cdot \frac{2}{N(p-1)+b-4} = \frac{2-s_c}{s_c}, \end{aligned}$$

moreover,

$$q = \frac{C_{N,p,m}}{p} [M(\Psi)]^{\frac{A}{2}} = \frac{[M(\Psi)]^{\frac{A}{2}}}{p} \cdot \frac{2p}{A[M(\Phi)]^{p-1}} \left(\frac{A}{B} \right)^{\frac{B}{2}}.$$

Now, from Pohozaev identity

$$\frac{2B}{B-2} H(\Phi) = \sum_{j=1}^m \|\Delta \phi_j\|_{L^2}^2,$$

$$\frac{B}{A} M(\Phi) = \frac{2B}{B-2} H(\Phi).$$

Then

$$\begin{aligned} &\frac{B}{A} M(\Phi) (M(\Phi))^{\frac{2-s_c}{s_c}} (M(\Psi))^{\frac{s_c-2}{s_c}} \\ &= \frac{2B}{B-2} H(\Phi) (M(\Phi))^{\frac{2-s_c}{s_c}} (M(\Psi))^{\frac{s_c-2}{s_c}} = X_* \end{aligned}$$

and

$$F(X_*) = \frac{B-2}{2B} X_* = H(\Phi) (M(\Phi))^{\frac{2-s_c}{s_c}} (M(\Psi))^{\frac{s_c-2}{s_c}}.$$

The conditions (3.9) and (3.10) imply that

$$H(\Psi) < F(X_*) \quad \text{and} \quad \sum_{j=1}^m \|\Delta \psi_j\|_{L^2}^2 > X_*.$$

One deduces that

$$\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 > X_* \quad \forall t \in [0, T^*). \quad (3.12)$$

By contradiction, if $\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 < X_*$, then by continuity in time $\exists t_1 \in (0, T^*)$ such that

$$\sum_{j=1}^m \|\Delta u_j(t_1)\|_{L^2}^2 = X_*.$$

However, this holds

$$H(\Psi) \geq F(X_*)$$

which is a contradiction.

Now, choose $0 < \eta < 1$ such that

$$H(\Phi)[M(\Phi)]^{\frac{2-s_c}{s_c}} > \frac{1}{1-\eta} H(\Psi)[M(\Psi)]^{\frac{2-s_c}{s_c}}.$$

$$\begin{aligned} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 > X_* &= \frac{2B}{B-2} H(\Phi) (M(\Phi))^{\frac{2-s_c}{s_c}} (M(\Psi))^{\frac{s_c-2}{s_c}} \\ &> \frac{2B}{B-2} \frac{1}{1-\eta} H(\Psi). \end{aligned}$$

Then

$$\begin{aligned} (B-2)(1-\eta) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 &> 2B \cdot H(\Psi), \\ 2(N(p-1) + b-4)(1-\eta) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 &> 4(N(p-1) + b)H(\Psi). \end{aligned} \quad (3.13)$$

This implies (3.8)

$$\begin{aligned} M'_R(\mathbf{u}) &\leq 2(N(p-1) + b-4)(1-\eta) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 - 2(N(p-1) + b-4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \\ &\quad + \mathcal{O} \left(R^{-4} + R^{-2} \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 + R^{-(N-1)(p-1)-b} \left(\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 \right)^{\frac{p-1}{2}} \right) \\ &\leq -4\eta(B-2) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \frac{C_0}{R^4} + \frac{C_0}{R^2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \frac{C_0}{R^2} \left(\sum_{j=1}^m \|\Delta u_j\|_{L^2} \right)^{\frac{p-1}{2}} \\ &\leq -2\eta(B-2) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \frac{C_0}{R^4} \\ &\quad + \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \left(-2\eta(B-2) + \frac{C_0}{R^2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^{-1} + \frac{C_0}{R^2} \left(\sum_{j=1}^m \|\Delta u_j\|_{L^2} \right)^{\frac{p-1}{2}-2} \right) \\ &\leq -2\eta(B-2) + o_R(1) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + o_R(1), \end{aligned}$$

with $o_R(1) \rightarrow 0$ as $R \rightarrow \infty$ uniformly in t . Taking $R > 0$ large enough and using the fact $\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \geq H(\Psi) > 0$ we say

$$M'_R(\mathbf{u}) \leq -\eta(B-2) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \quad \text{on } [0, T^*).$$

Now, by contradiction, take $T^* = \infty$, then integrating gives the following

$$M_R(\mathbf{u}(t)) \leq M_R(\mathbf{u}(0)) - \eta(B-2) \int_0^t \sum_{j=1}^m \|\Delta u_j(s)\|_{L^2}^2 ds.$$

Then, by (3.12), $\exists t_1 \geq 0$ such that $M_R(\mathbf{u}(t_1)) \leq 0$ for all $t \geq t_1$. By integrating from t_1 to t we have

$$M_R(\mathbf{u}(t)) \leq -\eta(B-2) \int_{t_1}^t \sum_{j=1}^m \|\Delta u_j(s)\|_{L^2}^2 ds \quad \forall t \in [0, T^*].$$

The same conclusion as in the first case. $\square$

3.2.3 Non-global solutions

Lemma 3.3. *The $A_{\alpha,\beta}^-$ and $A_{\alpha,\beta}^+$ are invariant under flow of (1.2).*

Proof. Claim is if $\mathbf{u}(0) \in A_{\alpha,\beta}^-$, then $\mathbf{u}(t) \in A_{\alpha,\beta}^-$.

From the energy conservation

$$E(\mathbf{u}(t)) = E(\mathbf{u}(0)) < d_{\alpha,\beta}.$$

Now, we need to show $Q(\mathbf{u}(t)) \leq 0$. By contradiction, suppose $Q(\mathbf{u}(t)) > 0$ and at $t = 0$ $Q(\mathbf{u}(0)) \leq 0$. By continuity argument, we say that there exists t_1 such that $Q(\mathbf{u}(t_1)) = 0$. However, one can show that

$$E(\mathbf{u}(t_1)) \geq E(\Phi),$$

which is contradiction. Then, $Q(\mathbf{u}(t)) \leq 0$.

For $A_{\alpha,\beta}^+$ we have the same argumentation. $\square$

Lemma 3.4. *Let $\Psi \in A_{\alpha,\beta}^-$ and $\mathbf{u} \in C_{T^*}(\mathcal{H})$ be the maximal solution to (1.2). Then there exists $\delta > 0$ such that*

$$Q(\mathbf{u}(t)) < -\delta \quad \forall t \in [0, T^*].$$

Proof. Suppose that $\exists t_n \in (0, T^*)$ a sequence of positive numbers such that $Q(\mathbf{u}(t_n)) \rightarrow 0$. From (3.3) we get (by definition $\bar{\mu} = \frac{8}{N}$)

$$d \leq \left(E - \frac{N}{8} Q \right) (\mathbf{u}(t_n)) \leq E(\Psi) - \frac{N}{8} Q(\mathbf{u}(t_n)) \xrightarrow{n \rightarrow \infty} E(\Psi).$$

$\Psi \in A_{\alpha,\beta}^-$, then $E(\Psi) < d$. Conclusion is that $Q(\mathbf{u}(t))$ is negative. $\square$

Lemma 3.5. *There exist two positive real numbers $a, t_1 > 0$ such that*

$$M'_R(\mathbf{u}) \leq -a \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \quad \text{on } [0, T^*), \tag{3.14}$$

$$M_R(\mathbf{u}) < 0 \quad \text{on } [t_1, T^*).$$

Proof. 1. $\mu \geq 0$.

Take $t \in [0, T^*)$.

Case 1.

$$\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \leq \frac{4(N(p-1)+b)}{N(p-1)+b-4}.$$

Moreover, choose $R > 0$ large enough such that

$$\mathcal{O} \left(\frac{1}{R^4} + \frac{|\mu|}{R^2} + \frac{\sum_{j=1}^m \|\psi_j\|_{L^2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}}{R^2} + \frac{(\sum_{j=1}^m \|\psi_j\|_{L^2} \sum_{j=1}^m \|\Delta u_j\|_{L^2})^{\frac{p-1}{2}}}{R^{(N-1)(p-1)+b}} \right) \leq \delta.$$

From lemma 3.4 and (3.8) we say

$$\begin{aligned}
M'_R(\mathbf{u}(t)) &\leq 4(N(p-1) + b)H(\Psi) \\
&\quad - 2(N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 - (2N(p-1) + b - 4)\mu \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 \\
&\quad + \mathcal{O}\left(\frac{1}{R^4} + \frac{|\mu|}{R^2} + \frac{\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2}{R^2} + \frac{(\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2)^{\frac{p-1}{2}}}{R^{(N-1)(p-1)+b}}\right) \\
&\leq 2NQ(\mathbf{u}(t)) + \mathcal{O}\left(\frac{1}{R^4} + \frac{|\mu|}{R^2} + \frac{\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2}{R^2} + \frac{(\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2)^{\frac{p-1}{2}}}{R^{(N-1)(p-1)+b}}\right) \\
&\leq -\delta(2N-1) \leq \frac{-\delta(2N-1)(N(p-1) + b - 4)}{4(N(p-1) + b)H(\Psi)} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2.
\end{aligned}$$

Case 2.

$$\begin{aligned}
&\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 > \frac{4(N(p-1) + b)}{N(p-1) + b - 4}. \\
M'_R(\mathbf{u}(t)) &\leq 4(N(p-1) + b)H(\Psi) - 2(N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \\
&\quad + \mathcal{O}\left(\frac{1}{R^4} + \frac{|\mu|}{R^2} + \frac{\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2}{R^2} + \frac{(\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2)^{\frac{p-1}{2}}}{R^{(N-1)(p-1)+b}}\right) \\
&\leq -(N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \\
&\quad + \mathcal{O}\left(\frac{1}{R^4} + \frac{C_0 + \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2}{R^2} + \frac{C_0}{R^2} \left(\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2\right)^{\frac{p-1}{2}} + \frac{|\mu|}{R^2}\right) \\
&\leq -\frac{N(p-1) + b - 4}{2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \frac{C_0}{R^4} + \frac{C_0|\mu|}{R^2} \\
&\quad + \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \left(-\frac{N(p-1) + b - 4}{2} + \frac{C_0}{R^2} \left(1 + \left[\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2\right]^{\frac{p-1}{2}}\right)\right).
\end{aligned}$$

Choose R sufficiently large, one gets

$$M'_R(\mathbf{u}(t)) \leq -\frac{N(p-1) + b - 4}{2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2.$$

Furthermore,

$$\begin{aligned}
M_R(\mathbf{u}(t)) &= M_R(\mathbf{u}(0, \cdot)) + \int_0^t \frac{d}{ds} M_R(\mathbf{u}(s)) ds \\
&\leq M_R(\mathbf{u}(0, \cdot)) - a \int_0^t \sum_{j=1}^m \|\Delta u_j(s)\|_{L^2}^2 ds \\
&\leq M_R(\mathbf{u}(0, \cdot)) - c_0 t.
\end{aligned}$$

Hence, $M_R < 0$ on $\left[\frac{M_R(\mathbf{u}(0, \cdot))}{c_0}; T^*\right)$.

2. Consider that $\mu < 0$.

Take $t \in [0, T^*)$ and

$$\eta := \frac{4(N(p-1) + b)H(\Psi)}{\delta} + \frac{A^2|\mu|^2 \sum_{j=1}^m \|\psi_j\|_{L^2}^2}{4\delta^2},$$

where

$$\delta = \frac{2(N(p-1) + b) - 8}{3}$$

and

$$A = A_0 + 2(N(p-1) + b) - 4.$$

Case 1.

$$\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \leq \eta.$$

From (3.8) and lemma 3.4

$$\begin{aligned} M'_R(\mathbf{u}(t)) &\leq 4(N(p-1) + b)H(\Psi) - 2(N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \\ &\quad - 2\mu(N(p-1) + b - 2) \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 + A_0|\mu| \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 \\ &\quad + \mathcal{O}\left(\frac{1}{R^4} + \frac{\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2}{R^2} + \frac{|\mu|}{R^2} + \frac{(\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2)^{\frac{p-1}{2}}}{R^{(N-1)(p-1)+b}}\right) \\ &\leq 2NQ(\mathbf{u}(t)) + A_0|\mu|\eta + \mathcal{O}\left(\frac{1}{R^4} + \frac{\eta}{R^2} + \frac{\eta^{\frac{p-1}{2}}}{R^2} + \frac{|\mu|}{R^2}\right) \\ &\leq -2N\delta + A_0|\mu|\eta + \mathcal{O}\left(\frac{1}{R^4} + \frac{\eta}{R^2} + \frac{\eta^{\frac{p-1}{2}}}{R^2} + \frac{|\mu|}{R^2}\right). \end{aligned}$$

By choosing $|\mu|$ small enough and $R > 0$ large enough

$$M'_R(\mathbf{u}(t)) \leq -\frac{2N\delta}{\eta} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2.$$

Case 2.

$$\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 > \eta.$$

One can show that

$$\begin{aligned} &4(N(p-1) + b)H(\Psi) - 2(N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \\ &\quad - 2\mu(N(p-1) + b - 2) \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 + A_0|\mu| \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 \\ &= 4(N(p-1) + b)H(\Psi) - 3\delta \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + A|\mu| \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2. \end{aligned}$$

Using Young inequality

$$\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 \leq \frac{A|\mu|}{4\delta} \sum_{j=1}^m \|u_j\|_{L^2}^2 + \frac{\delta}{A|\mu|} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2$$

we obtain

$$\begin{aligned} &4(N(p-1) + b)H(\Psi) - 3\delta \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \frac{A^2|\mu|^2}{4\delta} \sum_{j=1}^m \|u_j\|_{L^2}^2 + \delta \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 \\ &\leq -\delta \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2. \end{aligned}$$

Then,

$$M'_R(\mathbf{u}) \leq -\delta \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \mathcal{O} \left(\frac{1}{R^4} + \frac{\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2}{R^2} + \frac{|\mu|}{R^2} + \frac{(\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2)^{\frac{p-1}{2}}}{R^{(N-1)(p-1)+b}} \right).$$

Now, choosing $R > 0$ sufficiently large that

$$\mathcal{O} \left(\frac{1}{R^4} + \frac{\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2}{R^2} + \frac{|\mu|}{R^2} + \frac{(\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2)^{\frac{p-1}{2}}}{R^{(N-1)(p-1)+b}} \right) \leq \frac{\delta}{2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2.$$

Thus,

$$M'_R(\mathbf{u}) \leq -\frac{\delta}{2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2.$$

The rest of proof is the same as in the case $\mu \geq 0$. □

Theorem 3.2. *Take $2 \leq N \leq 7$ and $2 \leq p \leq 5$ such that $p_* < p < p^*$. Let $\mathbf{u} \in C_{T^*}(\mathcal{H}_{rd})$ be a maximal solution to (1.2). If $\exists t_0 \in [0, T^*)$ such that $\mathbf{u}(t_0) \in A_{1, -\frac{2}{N}}^-$, then $\mathbf{u}$ is non-global solution.*

Proof. Assume that $\mathbf{u}$ is global solution. Then, integrating (3.14) on $[t_1, t]$ and considering that $M_R(\mathbf{u}(t)) \leq 0$ for $t \geq t_1$

$$M_R(\mathbf{u}(t)) \leq -a \int_{t_1}^t \sum_{j=1}^m \|\Delta u_j(s)\|_{L^2}^2 ds \quad \forall t \geq t_1.$$

In addition,

$$|M_R(\mathbf{u})| \leq \|\nabla \varphi\|_{L^\infty} \|\mathbf{u}(t)\|_{L^2} \|\nabla \mathbf{u}(t)\|_{L^2} \leq C(\mathbf{u}(0, \cdot)) R \|\Delta \mathbf{u}(t)\|_{L^2}^{\frac{1}{2}}.$$

Hence,

$$M_R(\mathbf{u}(t)) \leq -a \int_{t_1}^t \sum_{j=1}^m |M_R(\mathbf{u}(s))|^4 ds.$$

Denote $z(t) := \int_{t_1}^t \sum_{j=1}^m |M_R(\mathbf{u}(s))|^4 ds$ and

$$z'(t) \geq C z^4(t) \quad \forall t \geq t_1.$$

Using lemma 3.1 the proof will be completed. □

3.2.4 Strong instability

The scaling is $\mathbf{u}^\lambda := \lambda^{\frac{N}{2}} \mathbf{u}(\lambda \cdot)$ for $\lambda > 0$ and $\mathbf{u} \in \mathcal{H}_{rd}$.

Lemma 3.6. *Let $\mathbf{u} \in \mathcal{H}_{rd}$ be satisfying $Q(u) \leq 0$. Then, there exists a unique $\lambda_0 \in (0, 1]$ such that:*

1. $Q(\mathbf{u}^{\lambda_0}) = 0$;
2. $\lambda_0 = 1$ if and only if $Q(\mathbf{u}) = 0$;
3. $\frac{\partial}{\partial \lambda} E(\mathbf{u}^\lambda) > 0$ for $\lambda \in (0, \lambda_0)$ and $\frac{\partial}{\partial \lambda} E(\mathbf{u}^\lambda) < 0$ for $\lambda \in (\lambda_0, \infty)$;
4. $\frac{\partial}{\partial \lambda} E(\mathbf{u}^\lambda) = \frac{N}{2\lambda} Q(\mathbf{u}^\lambda)$.

Proof. One can show that

$$E(\mathbf{u}^\lambda) = \frac{\lambda^4}{2} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \frac{\lambda^2 \mu}{2} \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 + \frac{1}{2} \sum_{j=1}^m \|u_j\|_{L^2}^2 - \frac{\lambda^{N(p-1)+b}}{2p} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_j|^p |u_k|^p dx.$$

Then, calculations give

$$\begin{aligned} & \frac{\partial}{\partial \lambda} E(\mathbf{u}^\lambda) \\ &= 2\lambda^3 \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \lambda \mu \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 - \frac{N(p-1)+b}{2p} \lambda^{N(p-1)+b-1} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_j|^p |u_k|^p dx \\ &= \frac{N}{2\lambda} \left(\frac{4}{N} \lambda^4 \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \frac{2\mu}{N} \lambda^2 \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 - \frac{p-1+\frac{b}{N}}{p} \lambda^{N(p-1)+b} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_j|^p |u_k|^p dx \right) \\ &= \frac{N}{2\lambda} Q(\mathbf{u}^\lambda). \end{aligned}$$

So, $\frac{\partial}{\partial \lambda} E(\mathbf{u}^\lambda) = \frac{N}{2\lambda} Q(\mathbf{u}^\lambda)$ is proven.

Now, compute

$$\begin{aligned} Q(\mathbf{u}^\lambda) &= \\ &= \frac{4}{N} \lambda^4 \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \frac{2\mu}{N} \lambda^2 \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 - \frac{p-1+\frac{b}{N}}{p} \lambda^{N(p-1)+b} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_j|^p |u_k|^p dx \\ &= \frac{4}{N} \lambda^4 \left(\sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \frac{\mu}{2\lambda^2} \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 - \frac{N(p-1)+b}{4p} \lambda^{N(p-p^*)} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^{-b} |u_j|^p |u_k|^p dx \right). \end{aligned}$$

Now, we will analyze $\lambda^{-4}Q(\mathbf{u}^\lambda)$. From assumption we have $p > p_*$, $Q(\mathbf{u}) \leq 0$ and it can be shown that

$$\text{at } \lambda = 0^+ \text{ then } \lambda^{-4}Q(\mathbf{u}^\lambda) \approx \frac{4}{N} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + \frac{2\mu}{N} \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 > 0;$$

$$\text{as } \lambda \rightarrow \infty, \text{ then } \lambda^{-4}Q(\mathbf{u}^\lambda) \rightarrow -\infty.$$

So, $\lambda^{-4}Q(\mathbf{u}^\lambda)$ is decreasing from a positive to a negative value. By continuity $\exists \lambda_0$ such that $\lambda_0^{-4}Q(\mathbf{u}_0^\lambda) = 0$.

Moreover, it is obvious that at $\lambda_0 = 1$ $Q(\mathbf{u}) = \lambda_0^{-4}Q(\mathbf{u}_0^\lambda) = 0$. □

Lemma 3.7. Let $\Phi \in \mathcal{H}_{rd}$ be a ground state, $\lambda > 1$ and $\mathbf{u}^\lambda$ be a solution to (1.2) with data Φ_λ . Then

$$E(\mathbf{u}^\lambda) < E(\Phi) \quad \text{and} \quad Q(\mathbf{u}^\lambda) < 0 \quad \text{on} \quad [0, T^*).$$

Proof. We will start from the definition of the ground state. It is obvious that $Q(\Phi) = 0$. Then, by using the previous lemma $\exists \lambda_0 = 1$ and for $\lambda > 1$

$$\frac{\partial}{\partial \lambda} E(\Phi_\lambda) = \frac{N}{2\lambda} Q(\Phi_\lambda) < 0.$$

By using the invariance we get

$$Q(\mathbf{u}^\lambda) = Q(\Phi_\lambda) < 0.$$

Again, by using lemma 3.6

$$\begin{aligned} \frac{\partial}{\partial \lambda} E(\Phi_\lambda) &< 0 \quad \text{for } \lambda > 1, \\ \frac{\partial}{\partial \lambda} E(\Phi_\lambda) &> 0 \quad \text{for } \lambda < 1 \end{aligned}$$

we say that $E(\Phi_\lambda)$ has a maximum at $\lambda_0 = 1$ (So, $E(\Phi)$ is the largest value for energy). One can show that

$$E(\mathbf{u}^\lambda) = E(\Phi_\lambda) < E(\Phi).$$

□

Theorem 3.3. *Take $2 \leq N \leq 7$ and $2 \leq p \leq 5$ such that $p_* < p < p^*$. Then, every ground state to (1.2) is strongly unstable.*

Proof. Take $\mathbf{u}^\lambda \in C_{T^*}(\mathcal{H}_{rd})$ which is the maximal solution to (1.2) with data Φ_λ . Then, the previous lemma says that

$$\mathbf{u}^\lambda \in A_{1, -\frac{2}{N}}^- \text{ on } [0, T^*).$$

□

3.2.5 Additional results.

We extend assumptions for b and $\mathbf{u}(t)$.

Consider the case for a non-radial solution. The main difference in the proof is the estimation of an error term

$$|E_r| = \int_{|x| \geq R} |x|^{-b} |u_k|^p |u_j|^p dx$$

which appear in the estimation for $M'_R(\mathbf{u}(t))$. We cannot use a Strauss inequality as before to estimate an error.

Theorem 3.4. *Suppose that $N \geq 5$, $0 < b < \min(4, N)$ and $\Psi \in \mathcal{H}$. Thus:*

i) *If $1 + \frac{4}{N} < p < p^*$ and*

$$H(\Psi) < \begin{cases} 0, & \text{for } \mu > 0, \\ -g\mu^2 M(\Psi), & \text{for } \mu < 0, \end{cases}$$

then the maximal solution to (1.2) $\mathbf{u}(t)$ blows-up in finite or infinite time, that is,

$$\lim_{t \rightarrow T_{\max}} \sum_{j=1}^m \|\Delta u_j\|_{L^2} = +\infty.$$

ii) *If $p_* < p < 1 + \frac{4}{N}$ and either*

$$H(\Psi) < \begin{cases} 0, & \text{for } \mu > 0, \\ -g\mu^2 M(\Psi), & \text{for } \mu < 0, \end{cases}$$

or

$$H(\Psi)M(\Psi)^{\frac{2-s_c}{s_c}} < H(\Phi)M(\Phi)^{\frac{2-s_c}{s_c}} \quad (3.15)$$

with

$$\left(\sum_{j=1}^m \|\Delta \psi_j\|_{L^2}^2 \right) \left(\sum_{j=1}^m \|\psi_j\|_{L^2}^2 \right)^{\frac{2-s_c}{s_c}} > \left(\sum_{j=1}^m \|\Delta \phi_j\|_{L^2}^2 \right) \left(\sum_{j=1}^m \|\phi_j\|_{L^2}^2 \right)^{\frac{2-s_c}{s_c}}, \quad (3.16)$$

and $\mu = 0$, then $\mathbf{u}(t)$ blows-up in finite time in the sense that $\exists T^ > 0$ such that*

$$\lim_{t \rightarrow T^*} \sum_{j=1}^m \|\Delta u_j\|_{L^2} = +\infty,$$

where Φ is the ground state solution to (1.2).

Proof. Case $1 + \frac{4}{N} < p < p^$.*

Assume that the solution is global in time and satisfies

$$\sum_{j=1}^m \|\Delta u_j\|_{L^2} \leq M \quad \forall t.$$

Thus, using Gagliardo-Nirenberg inequality and the mass conservation we estimate an error related to $M'_R(\mathbf{u}(t))$

$$|E_r| \lesssim R^{-b} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |u_j|^p |u_k|^p dx \lesssim R^{-b} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^{\frac{N(p-1)}{2}} \|u_j\|_{L^2}^{2p - \frac{N(p-1)}{2}} \lesssim_{\mathbf{u}(0,\cdot),M} R^{-b}.$$

Then, choosing $R > 0$ large enough for $\mu > 0$ we obtain the following inequality

$$M'_R(\mathbf{u}(t)) \leq 4(N(p-1) + b)H(\Psi).$$

Since, $H(\Psi) < 0$, then $M'_R(\mathbf{u}(t)) \leq -C < 0$. Using the Fundamental Theorem of Calculus we bound

$$|M_R(\mathbf{u})| \lesssim_R \|\nabla \mathbf{u}\|_{L^2} \|\mathbf{u}\|_{L^2} \lesssim_{R,\mathbf{u}(0,\cdot)} \|\Delta \mathbf{u}\|_{L^2}^{\frac{1}{2}} \lesssim_{R,\mathbf{u}(0,\cdot)} M^{\frac{1}{2}},$$

which is a contradiction. Similarly, for $\mu < 0$ and $H(\Psi) < -g\mu^2 M(\Psi)$ we obtain

$$\begin{aligned} M'_R(\mathbf{u}(t)) &\leq 4(N(p-1) + b)H(\Psi) - 2(N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 + A|\mu| \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 \\ &\leq 4(N(p-1) + b)[H(\Psi) + g\mu^2 M(\Psi)] \leq -C < 0, \end{aligned}$$

where

$$\sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 \leq \sum_{j=1}^m \|\Delta u_j\|_{L^2} \|u_j\|_{L^2} \leq \frac{A|\mu|}{4(N(p-1) + b - 4)} \sum_{j=1}^m \|u_j\|_{L^2}^2 + \frac{N(p-1) + b - 4}{A|\mu|} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2.$$

The same conclusion holds as for the case $\mu > 0$.

Case $p_* < p < 1 + \frac{4}{N}$ implies $\frac{N(p-1)}{2} \leq 2$.

Again, by using Gagliardo-Nirenberg and mass conservation

$$|E_r| \lesssim R^{-b} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^{\frac{N(p-1)}{2}} \|u_j\|_{L^2}^{2p - \frac{N(p-1)}{2}} \lesssim_{\mathbf{u}(0,\cdot)} R^{-b} + R^{-b} \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2, \quad (3.17)$$

since $\sum_{j=1}^m \|\Delta u_j\|_{L^2}^{\frac{N(p-1)}{2}} \leq 1 + \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2$.

Negative energy. Choosing $R > 0$ sufficiently large we get

$$M'_R(\mathbf{u}(t)) \leq 4(N(p-1) + b)H(\Psi) - 2(N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 < 0$$

for $\mu > 0$ and

$$M'_R(\mathbf{u}(t)) \leq 4(N(p-1) + b)[H(\Psi) + g\mu^2 M(\Psi)] - (N(p-1) + b - 4) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2 < 0$$

for $\mu < 0$. The same ODE argument as in the proof of theorem 3.1. Hence, it implies finite time blow-up.

Positive energy. By choosing $R > 0$ large enough and combining with (3.13) we have

$$M'_R(\mathbf{u}(t)) \leq -4\eta(B-2) \sum_{j=1}^m \|\Delta u_j\|_{L^2}^2$$

and the rest of proof exactly the same with theorem 3.1 case 3. □

Moreover, we have a result for a positive power of $|x|$.

Lemma 3.8. *Let $N \geq 2$, $b > 0$ and $p \geq \frac{b}{N-1} + 1$ and $p \leq \frac{N+b}{N-4}$ if $N > 4$ and $\mathbf{u} \in \mathcal{H}_{rd}$. Then, the following inequality*

$$\sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^b |u_k|^p |u_j|^p dx \lesssim \sum_{j=1}^m \|\Delta u_j\|_{L^2}^{\frac{N(p-1)-b}{2}} \|u_j\|_{L^2}^{\frac{4+b-(N-4)(p-1)}{2}}$$

is true.

Proof. Using the Cauchy-Schwarz we obtain

$$\sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^b |u_k|^p |u_j|^p dx \leq \sum_{j=1}^m \left(\int_{\mathbb{R}^N} |x|^b |u_j|^{2p} dx \right)^{\frac{1}{2}} \sum_{k=1}^m \left(\int_{\mathbb{R}^N} |x|^b |u_k|^{2p} dx \right)^{\frac{1}{2}}.$$

Then, lemma 2.8 for $\frac{2b}{N-1} + 2 \leq 2p \leq \frac{2N+2b}{N-4}$

$$\sum_{j=1}^m \int_{\mathbb{R}^N} |x|^b |u_j|^{2p} dx \lesssim \sum_{j=1}^m \|\Delta u_j\|_{L^2}^{\frac{N(p-1)-b}{2}} \|u_j\|_{L^2}^{\frac{4+b-(N-4)(p-1)}{2}}.$$

The result is

$$\begin{aligned} & \sum_{j,k=1}^m \int_{\mathbb{R}^N} |x|^b |u_k|^p |u_j|^p dx \\ & \lesssim \sum_{j=1}^m \|\Delta u_j\|_{L^2}^{\frac{N(p-1)-b}{4}} \|u_j\|_{L^2}^{\frac{4+b-(N-4)(p-1)}{4}} \sum_{k=1}^m \|\Delta u_k\|_{L^2}^{\frac{N(p-1)-b}{4}} \|u_k\|_{L^2}^{\frac{4+b-(N-4)(p-1)}{4}}. \end{aligned}$$

Using the symmetry gives the final inequality. □

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