

Application of Partial Differential Equation in Human Centered Design Systems

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1 Abstract

Human-Centered Design (HCD) is a concept that prioritizes people-oriented solutions, focusing on sustainability and safety. Partial differential equations are used to model different physical concepts, like fluid dynamics and flow behavior. In this capstone, we considered two cases, urban flooding and pedestrian flow, and connected them, taking into consideration HCD as the main idea. The urban flooding model was based on the Shallow Water Equations and simplified to a 1D steady-state solution, while the crosswalk pedestrian flow was based on the Hughes model and also reduced to 1D steady-state conditions. To mathematically connect these two cases, this research derived the relationship $\frac{Q}{W(h)} = \rho(1 - \rho)$, which links the physical parameters of the flood to the pedestrian crowd dynamics. The increase in floodwater depth h artificially reduces the available safe walking space, forcing the local pedestrian density ρ of the “thinking fluid” to increase. Quantitative results revealed a critical capacity threshold of 50% for crowd density ($\rho = 0.5$); after exceeding it, pedestrians lose their safe free-flow state. Both cases are based on the same class of equations, called conservation laws. The “thinking fluid” in the pedestrian case represents the flow of people, analogous to the water flow in urban flooding. By connecting these two cases, this research provides urban planners with a quantitative tool to assess whether evacuation routes remain safe during urban flood emergencies.

2 Introduction

Flood hazards are one of the most frequent natural disasters globally. With climate change and the rapid urbanization of land use, the influence that comes from flooding is significant and can lead to critical risk for those who are the most vulnerable to facing it (low- and middle-income communities) (Rentschler & Salhab, 2020). Traditional risk management and solutions are based on the assumption that people during critical situations will act rationally, based on logic, and considering all the risks. But in reality, from history, it is defined that human behavior is much more complex, and it can be emotional rather than rational (Brown et al., 2018). Usually, during evacuations, people intuitively tend to follow the crowd around them, listen and copy the actions of their family and friends (Tyler & Sadiq, 2018). This complicates structural planning and creates a “behavioral gap” during emergencies.

In an urban flooding scenario, optimal routes are modeled based on dry-weather conditions, which significantly differ from real flooding. When water enters the streets, it creates physical limitations for people and acts as boundaries, minimizing the accessible safe walking space. This leads to the Flood Hazard Rate (FHR) increasing, resulting in a rise in local pedestrian density, as people have narrowed dry channels to step in (Musolino et al., 2022).

Originally, Human-Centered Design (HCD), was taking roots from the Bauhaus movement, and first introduced in the 20th century, focusing on physical products and mass production (Cardoso, 2008). In the second half of the 20th century, these concepts started to integrate into systemic societal issues (Buchanan, 1992). Nowadays, HCD prioritizes inclusion, active participation, empathy, and creating solutions that are people-oriented (Van Manen et al., 2023). Recently, designers started to implement HCD into disaster

risk management to create socio-economic and physically optimal decisions (Charlesworth & Fien, 2022). As disasters are not just physical issues, but are also shaped by human decisions, HCD creates a space where the complex framework interacts with community behavior (Kelman, 2020). Motivated by this approach, this research shows that a mathematical framework - the conservation law (macroscopic continuity equations) of flood-water (using the Shallow Water Equations) and pedestrian flow (modeled by the Hughes method), was analyzed in 1D steady-state conditions and provides the critical threshold at which evacuation routes will not be safe.

This paper is structured to demonstrate that mathematics is an instrument for human-centered urban design. The framework of the universal conservation law, $\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u) = S(u)$, derived via the divergence theorem, is used in two physically distinct but mathematically unified cases. The shallow water equation is derived from the conservation law, and reduced to a 1D steady state ODE and solved analytically, to obtain the $hu = q_0$ relation. For the pedestrian flow case, the same conservation structure is applied on the Hughes continuity model for pedestrian crowd flow, which is also simplified to achieve the fundamental flux relation $q = \rho(1 - \rho)$. The synthesis of these two solutions results in $\frac{Q}{W(h)} = \rho(1 - \rho)$, which mathematically connects flood water depth to pedestrian crowd density, and the derived result is the critical threshold $\rho_c = 0.5$, the boundary beyond which the evacuation route fails.

The goal of this capstone is to show the connection between mathematics (partial differential equations) and human-oriented solutions. First, the mathematical framework and the background will be given, and in the further sections (Section 4 and 5) the problem will split into two cases. In Section 6, the connection between the two cases will be demonstrated as well as the HCD interpretations. In conclusion, the results and future directions will be provided.

3 Mathematical Framework

3.1 Two Approaches to Modelling Urban Hazards

Mathematical models play a critical role in the prediction of natural hazards and the evaluation of urban design. Without a mathematical framework as a supporting tool, it is impossible to identify where critical bottlenecks will appear or to calculate capacity thresholds during emergencies.

There are two methods that are used to model urban hazards. The first one is microscopic - the main idea of this method is treating and tracking each individual separately in the system (Antonini et al., 2006; Helbing & Molnar, 1995). This model adopts a Lagrangian perspective, and in the context of this capstone, it means treating each water droplet, or each human, as a separate independent entity (particle). This model has its own advantages, such as more accurate results and fewer anomalies, but it strictly depends on the context. In the case of urban floods or pedestrian flow, this model becomes computationally overwhelming and requires unnecessary, similar calculations that do not affect the overall system significantly (Kumar & Pahar, 2024; Zheng et al., 2009). Furthermore, the physical floodwater is modeled by the Shallow Water Equations (see Section 4.2), which are Eulerian. Coupling a discrete Lagrangian pedestrian model with a continuous Eulerian fluid model can cause mathematical inconsistencies.

To resolve this, another method called the macroscopic approach should be adopted, which utilizes the Eulerian perspective. For this model, it is not important how a small droplet of water or a single person moves, as it ‘zooms out’, taking a bigger picture. It takes the group of individuals as a ‘thinking fluid’, treating the group as a continuous whole (Hughes, 2002; Di Francesco et al., 2011). In the case of urban flooding or pedestrians, the

large volume of water or group of people is assumed to behave as a single flowing entity (Maury et al., 2010). To achieve this mathematically, the macroscopic model relies on the continuum assumption and spatial averaging over a Representative Elementary Volume (REV) (Kumar & Pahar, 2024; Slattery, 1967). Due to this averaging over a large number of agents, aggregate quantities like density $u(x, t)$, mean flux $F(u)$, and velocity v are formally defined and assumed to be continuous (or sufficiently smooth) in space and time (Hughes, 2002).

Although the macroscopic approach may ignore individual-level details, it is highly efficient in terms of computation, as it is widely used in broad case scenarios, geographical analyses, and large-scale events, giving an optimal model of what is happening (Helbing, 1998). However, it is critical to consider that the continuum assumption does not work in low-density scenarios where individual decisions dominate, or at extreme micro-bottlenecks (like narrow doorways) where continuous flow breaks into discrete steps. Most importantly, because the crowd is assumed to behave as a continuous, smooth medium within the REV, and number of people remains constant during an evacuation, this fundamental smoothness ensures that the transition to continuous partial differential equations (PDEs) is rigorous. It allows us to mathematically calculate critical anomalies and flow thresholds without calculating each individual separately, by relying on fundamental conservation laws (Di Francesco et al., 2011).

3.2 From Integral Form to PDE - The Divergence Theorem

Building upon the physical intuition of the macroscopic approach, the conservation of the medium within the environment and a defined Representative Elementary Volume (REV) must be formalized mathematically [Kumar & Pahar, 2024]. Let $u(x, t)$ represent the density of the conserved quantity. For example, this could be the water depth during a flood [Kumar & Pahar, 2024] or the concentration of a pedestrian crowd at a certain time and spatial coordinate [Hughes, 2002].

The REV is represented by a fixed control volume V , which is bounded by a closed

surface ∂V equipped with an outward-directed unit normal vector n . It is assumed that the density $u(x, t)$ is locally integrable over the control volume. Moreover, the flux vector will be denoted by $\mathbf{F}(u, x, t)$, representing the flow rate across the boundary. This relationship is crucial because the flow is modulated by the physical environment - such as changing topography in flood models (Kumar & Pahar, 2024), or time-evolving directional variations in pedestrian models (Di Francesco et al., 2011). Finally, $S(u)$ will represent internal sources or sinks, such as pedestrians leaving a building (Di Francesco et al., 2011; Kumar & Pahar, 2024). The fundamental macroscopic conservation law is expressed in its integral form as:

$$\frac{d}{dt} \int_V u dV = - \int_{\partial V} F(u, x, t) \cdot n dA + \int_V S(u) dV \quad (3.1)$$

In this formulation, the left-hand side represents the overall rate of change of the conserved quantity within the volume V . On the right-hand side, the first term calculates the net flux crossing the boundary ∂V ; the negative sign shows that an outward-directed flow corresponds to a decrease in the volume's total mass. The second term represents any localized additions or subtractions generated within the boundaries [Hughes, 2002]. Because the control volume V is fixed in space and time under the Eulerian perspective (Kumar & Pahar, 2024), the Leibniz integral rule allows the time derivative on the left-hand side to be rigorously brought inside the integral as a partial derivative, yielding $\int_V (\partial u / \partial t) dV$.

While this integral form describes the system macroscopically, directly evaluating surface integrals across complex urban systems is mathematically and computationally challenging. Examples of such complex systems might include jagged building walls, parking lots, or twisting streets (Kumar & Pahar, 2024). To make the calculations more accurate and localized to these complex cell interfaces, spatial-averaging techniques and the Gauss-Divergence Theorem are used (Gray & Lee, 1977; Kumar & Pahar, 2024).

$$\int_{\partial V} F(u, x, t) \cdot n dA = \int_V \nabla \cdot F(u, x, t) dV \quad (3.2)$$

3.3 The General Conservation Equation

To calculate the measurement of internal divergence, the Gauss-Divergence Theorem is used, and the main advantage of this theorem is that it translates the complex boundary flow measurement to an internal one (Gray & Lee, 1977). By using this theorem, the surface integral of the macroscopic conservation law is converted into a volume integral. Substituting this relationship back into the original equation from the previous section yields a formula where every term is now evaluated over the exact same control volume V :

$$\int_V \frac{\partial u}{\partial t} dV = - \int_V \nabla \cdot F(u, x, t) dV + \int_V S(u) dV \quad (3.3)$$

Now that all three components - which are the rate of change in density, the divergence of the flux, and the source terms are integrated over the identical volume V , they can be algebraically rearranged and grouped together under a single integral formula:

$$\int_V \left[\frac{\partial u}{\partial t} + \nabla \cdot F(u, x, t) - S(u) \right] dV = 0 \quad (3.4)$$

The main idea of doing this is to generalize the formula so it can be used independently, no matter the size of the city or the chosen urban area, as it holds true for any arbitrarily chosen control volume V . To rigorously pass from this integral form to the pointwise Partial Differential Equation (PDE), we must first assume that both the density $u(x, t)$ and the flux $F(u, x, t)$ are continuously differentiable (C^1 smooth) in space and time. Because this integral equality holds true for any arbitrary volume V within the domain, we apply the standard localization argument (a corollary of the Fundamental Lemma of Calculus of Variations). This mathematical law dictates that if the integral of a continuous function is zero over an arbitrary volume, the continuous integrand itself must identically vanish pointwise everywhere. As a result, the volume integrals can be rigorously eliminated, yielding a localized PDE commonly referred to as the continuity equation:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u, x, t) = S(u) \quad (3.5)$$

This means that the internal sources or sinks should be balanced with the rate of change of the density and divergence, as it relies on the universal physics of continuous media. These differential equations are the basis for modeling the density of pedestrian crowds (Di Francesco et al., 2011; Hughes, 2002) and the depth of urban shallow-water floods (Pathirana et al., 2008; Xing, 2017). The transition from macroscopic boundary evaluation to the continuous PDE can be summarized by the following sequence:

$$\frac{d}{dt} \int_V u dV = - \int_{\partial V} \mathbf{F}(u, x, t) \cdot \mathbf{n} dA + \int_V S(u) dV \quad (3.2)$$

↓ Leibniz rule (V fixed in space)

$$\int_V \frac{\partial u}{\partial t} dV = - \int_{\partial V} \mathbf{F}(u, x, t) \cdot \mathbf{n} dA + \int_V S(u) dV \quad (3.3)$$

↓ Gauss-Divergence Theorem

$$\int_V \frac{\partial u}{\partial t} dV = - \int_V \nabla \cdot \mathbf{F}(u, x, t) dV + \int_V S(u) dV \quad (3.4)$$

↓ Consolidation under a single integral

$$\int_V \left(\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u, x, t) - S(u) \right) dV = 0 \quad (3.5)$$

↓ Localization argument (u and $\mathbf{F}$ are C^1 smooth)

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u, x, t) = S(u) \quad (3.6)$$

3.4 Classification and Properties

The equation derived in the previous section belongs to the class of nonlinear partial differential equations. To show that this conservation law is hyperbolic, the system must be analyzed using a flux Jacobian matrix. For a system written as $\partial_t U + \partial_x F(U) = 0$,

the flux Jacobian is defined as $A(U) = F'(U)$ (Xing, 2016). The “Jacobian matrix” is reduced to the single scalar derivative of the flux function $F'(u)$ in the case of a 1D scalar conservation law (Di Francesco et al., 2011). Because the derivative of a real-valued flux function is a real scalar number, it trivially serves as a real eigenvalues, meaning that any 1D scalar conservation law is inherently hyperbolic.

To apply this classification to the case of urban flooding, the 1D shallow water equation is examined. The conserved variables are represented by the vector $U = (h, hu)^T$, where h is the water depth and u is the velocity, with the corresponding flux vector $F(U) = (hu, hu^2 + \frac{1}{2}gh^2)^T$ (Xing, 2016). Taking the derivative of this flux vector yields the flux Jacobian matrix:

$$A(U) = \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix} \quad (3.6)$$

The eigenvalues of this matrix are calculated as $\lambda_{1,2} = u \pm \sqrt{gh}$ (Xing, 2016). As the water depth h is always positive, both eigenvalues are real, which strictly confirms that the shallow water flood model is hyperbolic.

One of the unique characteristics of a hyperbolic equation is that it can develop “shocks” or sharp discontinuities over time, even if the initial flow started smoothly (Di Francesco et al., 2011; Xing, 2016). These play a crucial role for urban safety, as during a flooding event, water hitting a physical barrier or a dangerous surge is represented by these shocks (Dong et al., 2021).

To demonstrate the connection to the pedestrian problem, the same hyperbolic classification can be applied to the macroscopic Hughes model, which treats crowd movement as a “thinking fluid” using a 1D scalar conservation law (Carlini et al., 2016; Di Francesco et al., 2011). Let ρ represent the pedestrian density with a normalized flux function $F(\rho) = \rho(1 - \rho)$, the scalar Jacobian is simply the derivative $F'(\rho) = 1 - 2\rho$ (Di Francesco et al., 2011). Because the system is hyperbolic, this derivative represents the speed and direction at which density waves (or information) spread through the crowd. The physical

meaning of the sign of $F'(\rho)$ is critical: when the crowd density is low ($\rho < 0.5$), the wave speed is positive, meaning information transfers forward smoothly with the pedestrians. However, when the pedestrian density crosses the critical threshold ($\rho > 0.5$), $F'(\rho)$ becomes negative. This indicates a bottleneck appearance, like if a person stops or slows down, resulting wave of congestion ripples backwards through the people behind them, exactly like brake lights in a traffic jam (Helbing, 1998).

When faster-moving characteristics from a low-density region intersect with slower backwards-moving characteristics from a high-density region, it results in forming a mathematical discontinuity that creates a shock wave (Fischer et al., 2019). In pedestrian dynamics, this mathematical shock demonstrates a sudden crowd crush, a dangerous traffic jam, or a bottleneck at narrow exits (Carlini et al., 2016; Fischer et al., 2019). As pedestrians approach narrower paths or bottlenecks, the density increases rapidly, forcing individuals to line up and leading to a sudden drop in flow capacity directly in front of the exit during emergencies. When the density suddenly jumps during a shock, the standard continuous PDE stops working right at the shockwave. To mathematically compute how dangerous this can be and to find the exact speed of these moving shocks, the Rankine-Hugoniot condition must be used (Di Francesco et al., 2011; Fischer et al., 2019):

$$\llbracket F(u) \rrbracket = \dot{x}_0(t) \llbracket u \rrbracket \quad (3.7)$$

The sudden jump or instantaneous difference in the quantity across the shockwave is represented by the brackets $\llbracket \cdot \rrbracket$, and the speed of the shock is denoted by $\dot{x}_0(t)$. Based on this information, engineers can calculate when a critical bottleneck will appear in the urban environment, helping them to design safer and more effective routes (Fischer et al., 2019).

4 Shallow Water Flow in Urban Environments

4.1 Introduction to Urban Flood Risks

Floods are among the most frequent natural disasters, and the frequency of these events is rising dramatically, causing huge damage to infrastructure and people's lives (Mishra et al., 2022). In recent years, the reasons for the increase in flood events include climate change, rapid urbanization, and land-use transformation (Mishra et al., 2022). Global warming and rising temperatures have caused an increase in atmospheric moisture retention, leading to extreme rainfall and flooding in regions that have historically never experienced it (World Bank Group, 2021). Moreover, over a billion people live on lands that are less than 10 meters above high tide lines, which are under high risk due to sea-level rise (Kulp & Strauss, 2019; Rentschler & Salhab, 2020). As a result, annual finances spent on flood losses are projected to increase from \$6 billion in 2005 to over \$60 billion by 2050 (Alizadeh et al., 2021; Rentschler & Salhab, 2020).

However, not everyone faces these disasters equally. The flood problem is distributed unequally across the world and societies (Rentschler & Salhab, 2020). Approximately 89% of the global population that is at risk of substantial flooding resides in low - and middle-income communities, where risk management practices are often poorly developed or do not even exist (Alizadeh et al., 2021; Rentschler & Salhab, 2020). The burden falls on the most vulnerable groups of the population, such as older people, disabled individuals, and children. They are limited in their physical abilities, making evacuation or other actions during emergency situations such as flooding highly difficult (Musolino et al.,

2022). Moreover, varied risk perceptions and social barriers among marginalized groups complicate the emergency response and evacuation process (Cetara et al., 2025; Tyler & Sadiq, 2018).

The main problem remains in the fact that flooding is usually a sudden event, and it is hard to forecast the direction of the water in urban areas, so disaster management strategies cannot fully cover all situations and be prepared (Alfieri et al., 2012; Mishra et al., 2022). For a long time, national disaster risk reduction (DRR) strategies have relied heavily on top-down, technocratic approaches (Cetara et al., 2025). These traditional frameworks tend to be ineffective, focusing on building large-scale defensive infrastructure - like floodwalls - while fundamentally neglecting the localized, complex vulnerabilities of the people living within the system (Cetara et al., 2025; Mishra et al., 2022). Standard emergency models rely on theoretical static assumptions that do not consider the dynamic, real-time situation of water around buildings, through crosswalks, and into localized community depressions (Kumar & Pahar, 2024; Systematic Review of ABM-GIS, 2026).

4.2 The Shallow Water Equations

To be able to work with the conservation equation that was mentioned earlier in urban flood scenarios, the three-dimensional equations should be adapted to 2D fluid dynamics, which are the Shallow Water Equations (SWE) (Xing, 2016). The full three-dimensional equation is called the Navier-Stokes equation, and it is used to simulate fluid flows directly. Using the Navier-Stokes equation is a more complex model and much more expensive in computation.

To make urban flood modeling more practical, the 3D Navier-Stokes equations are simplified to the 2D SWE. The reason for that is the length in the horizontal direction is significantly larger than the vertical depth, and the vertical velocity and acceleration of the fluid are negligible (Kumar & Pahar, 2024; Xing, 2016). By mathematically averaging the fluid flow properties over the vertical axis, the complex 3-dimensional flow is simplified

into a 2D system of conservation laws.

The 2D Shallow Water Equations consist of a mass conservation equation (the continuity equation) and two momentum conservation equations (for the x and y horizontal directions) (Pathirana et al., 2008). In their standard Eulerian form, the depth-averaged SWE can be written as:

Mass Conservation (Continuity):

$$\frac{\partial h}{\partial t} + \frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} = S_{mass} \quad (4.1)$$

Momentum Conservation (x -direction):

$$\frac{\partial(hu)}{\partial t} + \frac{\partial}{\partial x} \left(hu^2 + \frac{1}{2}gh^2 \right) + \frac{\partial(huv)}{\partial y} = S_{bed,x} + S_{fric,x} \quad (4.2)$$

In the equations above, water depth is denoted by h , u and v represent the velocities in the x and y directions, and g is the gravitational constant.

Source and sink terms are defined on the right-hand side of the equations. S_{mass} represents external mass inputs like rainfall or drainage (Pathirana et al., 2008). The momentum terms are divided into the topographical source term ($S_{bed,x}$) and the friction sink term ($S_{fric,x}$). These are defined as:

$$S_{bed,x} = -gh \frac{\partial b}{\partial x}, \quad S_{fric,x} = -gh S_{fx} \quad (4.3)$$

Here, $b(x, y)$ represents the bottom topography, like the slope of a street, and S_{fx} represents the friction slope or resistance caused by the ground surface. To close the system, this friction slope is empirically modeled using Manning's formula, defined as $S_{fx} = \frac{n^2 u \sqrt{u^2 + v^2}}{h^{4/3}}$, where n is the Manning's roughness coefficient associated with the urban terrain (Pathirana et al., 2008).

In the context of urban flooding, the city grid is filled with some topographic obstacles such as parked cars, walls, and jagged buildings, which disrupt the flow. The standard SWE can be accurately applied to open water. However, in modeling a complex urban

environment, severe computational challenges appear. The standard SWE attempt to calculate based on exact boundaries, while in cities with complex topographical challenges, there are no fixed boundaries (Özgen et al., 2016).

To solve this problem, researchers introduced the Porous Shallow Water Equations, which is an adapted version of the SWE. In a macroscopic approach, it calculates a spatial-average over a Representative Elementary Volume (REV), instead of modeling the exact boundaries of each obstacle in the city (e.g., buildings, cars, etc.) (Kumar & Pahar, 2024). It introduces an “areal and volumetric porosity” parameter (ε) into the SWE, representing the fraction of the volume that is actually available for water to flow through (i.e., the space between buildings) (Kumar & Pahar, 2024; Özgen et al., 2016). Mathematically, this is achieved by scaling the conserved variables and fluxes by the porosity parameter ε , transforming the continuity equation to:

$$\frac{\partial(\varepsilon h)}{\partial t} + \frac{\partial(\varepsilon hu)}{\partial x} + \frac{\partial(\varepsilon hv)}{\partial y} = \varepsilon S_{mass} \quad (4.4)$$

By treating the urban layout as a porous medium, the SWE can utilize macroscopic drag coefficients to accurately simulate how buildings slow down and channel floodwaters, capturing realistic inundation depths on a coarse, highly efficient computational grid (Kumar & Pahar, 2024; Özgen et al., 2016).

4.3 Simplification to a 1D Steady-State

In this capstone, a simplified 1D version of the Shallow Water Equations (SWE) will be considered.

Reducing the 2D Shallow Water equation to a 1D formulation requires a geometric assumption. The 1D approximation is suitable for justified urban settings, for instance, narrow alleyways, streets, and evacuation routes. In these environments, buildings act as boundaries that restrict cross-flow. Consequently, the fluid dynamics are dominated by a single longitudinal direction aligned with the x -axis, meaning the velocity components

along the y -axis can be ignored (Delestre et al., 2016; Pathirana et al., 2008).

The simplified SWE consists of two main equations which represent the conservation of mass and the momentum along the straight street (Xing, 2016):

Mass Conservation:

$$\frac{\partial h}{\partial t} + \frac{\partial(hu)}{\partial x} = 0 \quad (4.5)$$

Momentum Conservation:

$$\frac{\partial(hu)}{\partial t} + \frac{\partial(hu^2 + \frac{1}{2}gh^2)}{\partial x} = -gh\frac{\partial b}{\partial x} - ghS_{fx} \quad (4.6)$$

To understand the physical mechanics of the flood, it is necessary to define the parameters within these equations:

- h : The depth of the floodwater.
- u : The depth-averaged flow velocity along the x -axis (the street).
- t : Time.
- x : The spatial coordinate along the length of the street.
- g : The gravitational constant.
- b : The bottom topography. The derivative term $(\partial b/\partial x)$ represents the physical slope of the street, which provides the gravitational driving force for the water.
- S_{fx} : The friction slope, representing the resistance and drag caused by the ground and urban obstacles (Pathirana et al., 2008; Xing, 2016).

Planners, when designing for human safety, prioritize the consideration of the worst-case scenario, when flood hazards are maximized. To capture this mathematically, the steady-state assumption is applied to remove time dependence from the system. In the context of urban flooding, this does not imply the water is static. Instead, it represents the sustained, peak equilibrium phase of a flood, such as when rainfall intensity is at

its maximum. This will be used as a standard assumption for isolating the maximum sustained depths and velocities (MacDonald, Baines, Nichols, & Samuels, 1997; Xing, 2016). It means that time derivatives for water depth and the flow momentum become zero:

$$\frac{\partial h}{\partial t} = 0, \quad \frac{\partial(hu)}{\partial t} = 0 \quad (4.7)$$

As a result, the flux gradients are perfectly balanced by the topographical and frictional source terms, yielding a constant discharge along the channel. Moreover, because the time-dependent terms are eliminated in steady-state conditions, the PDE simplifies to a 1D ordinary differential equation (ODE) that only depends on the spatial dimension (x), allowing the calculation of an analytical solution.

4.4 Analytical Solution

The PDE of the Shallow Water Equations is simplified to an ODE, as 1D steady-state conditions are assumed, to analytically evaluate the flood parameters (MacDonald et al., 1997).

Let us assume that no additional water sources are added or removed (like rainfall or canalization), then the steady-state mass conservation equation reduces to the form $\partial_x(hu) = 0$. This means that the unit discharge hu should be constant along the entire x -axis. Therefore, we can define:

$$hu = q_0 \quad (4.8)$$

In this analytical solution, h denotes the floodwater depth, u represents the flow velocity, and q_0 corresponds to the steady flow rate per unit width of the street channel.

Because q_0 is constant, any decrease in the available street width forces an increase in either the local flood depth or the flow velocity (Delestre et al., 2016).

To evaluate the exact spatial variation of the flood depth ($h(x)$), the steady-state mo-

momentum equation should be balanced. By substituting the constant discharge relationship ($u = q_0/h$) into the steady-state momentum equation, the convective acceleration term $\partial_x(hu^2)$ can be rewritten as $\partial_x(q_0^2/h)$. Simplification of the momentum conservation law into a single ODE can be done by the chain rule, with respect to the water depth h (Delestre et al., 2016):

$$\partial_x b = \left(\frac{q_0^2}{gh^3} - 1 \right) \partial_x h - S_{fx} \quad (4.9)$$

Here, $b(x)$ is the ground topography (the elevation of the street), g is the gravitational constant, and S_{fx} is the friction slope (Delestre et al., 2016; MacDonald et al., 1997). According to Manning's empirical formula, the friction slope S_{fx} represents the resistance caused by the ground's texture and can be calculated as $S_{fx} = \frac{n^2 u^2}{h^{4/3}}$, where n is the Manning roughness coefficient (Pathirana et al., 2008).

Under strictly ideal conditions - specifically assuming steady flow and zero ground friction ($S_{fx} = 0$) - the momentum ODE can be analytically integrated to yield the classic Bernoulli relationship (Delestre et al., 2016):

$$\frac{q_0^2}{2gh^2} + h(x) + b(x) = \text{constant} \quad (4.10)$$

This equation proves that the flow's kinetic energy, the water depth, and the street elevation are in equilibrium. It is critical to note that this relation relies on these assumptions to conserve energy along the channel. In realistic urban modeling, however, this relation fails because the friction of the urban environment (S_{fx}) causes energy loss, requiring the full ODE to be solved (Delestre et al., 2016). In real cities, buildings create obstacles that disrupt this "perfect" flow. That is why the Porous SWE is used, as it integrates an areal and volumetric parameter (ε), that represents the fraction of space that is available for water to flow (Kumar & Pahar, 2024; Özgen et al., 2016).

Rather than being a purely conceptual parameter, ε explicitly modifies the governing equations by scaling both the conserved variables and the fluxes. Mathematically, it

enters the 1D steady-state continuity equation by transforming it into $\partial_x(\varepsilon hu) = 0$. In the momentum equation, the flux terms are similarly multiplied by ε , while the friction and drag coefficients are scaled to account for the energy lost when water impacts buildings (Kumar & Pahar, 2024; Özgen et al., 2016). By substituting the parameter (ε) into the analytical equation, the model works more accurately by considering obstacles (buildings, etc.), allowing planners and engineers to predict how the spacing between buildings will scale up the maximum depth (h) and velocity (u) during worst-case situations (Kumar & Pahar, 2024).

4.5 Human-Centered Design Interpretation: Bridging the Behavioral Gap

Even though the physical danger of a flood can be calculated mathematically and the optimal evacuation solution can be modeled using the Shallow Water Equations, individuals do not always react to these dangers in a strictly rational manner. Historically, flood risk management relied on the assumption of a purely logical response during critical situations (Mishra et al., 2022). However, in reality, people often make decisions based on intuitive thinking and emotions rather than logical reasoning, creating a substantial “behavioral gap” (Cetara et al., 2025). Moreover, individuals tend to follow the people around them, such as family and friends, prioritizing informal networks. This affects the overall evacuation timing and speed (Tyler & Sadiq, 2018).

To effectively bridge this gap, alongside mathematical calculations, human vulnerabilities should be considered. For example, whether a person will topple in floodwater depends on both their personal characteristics and the physical parameters, specifically the water depth (h) and flow velocity (u). To protect vulnerable populations, engineers utilize a Flood Hazard Rate (FHR) index to identify when these parameters cross a life-threatening threshold (Musolino et al., 2022).

Urban planners can physically “nudge” the floodwater away from critical evacuation routes by retrofitting specific street sections (Musolino et al., 2022). For example, by

slightly raising the street elevation, engineers are actively manipulating the topographical source term ($\partial_x b$) in the steady-state momentum ODE. This intentional grading creates a new equilibrium with the friction slope (S_{fx}), effectively reducing the local flow velocity (u) and depth (h) to keep the route safe for evacuating pedestrians (Musolino et al., 2022).

From a mathematical perspective, this HCD intervention directly increases the local porosity (ε) in the steady-state continuity equation $\partial_x(\varepsilon hu) = 0$. By physically expanding the effective width parameter, the infrastructure can safely accommodate a larger volumetric flow rate, preventing the local water depth (h) from backing up and reaching dangerous levels in narrowed corridors (Kumar & Pahar, 2024). Ultimately, this human-centered mathematical solution demonstrates that integrating behavioral insights directly translates into tangible modifications of the parameters within the physical equations.

5 Pedestrian Flow at Urban Crosswalks

5.1 Motivation and Real-World Context

It is important to consider people’s reactions during emergency situations like flood hazards, and not only prioritize physical boundaries, but also human behavior and its limitations.

During an emergency situation, evacuations, or even in daily life, pedestrian flow creates severe structural bottlenecks. A simple example can be when people from opposite sides, or even more directions, cross the sidewalk at the same time. It creates difficulty in the flow, increases the crossing time, and reduces the overall pedestrian velocity. This is called bidirectional counter flow, which occurs when people from opposite sides interact with each other physically (Zhang et al., 2018).

In section 3, to model fluid dynamics, the macroscopic model was introduced, based on the idea that a whole “group” of water moved, driven by gravity. In the case of pedestrians, the crowd cannot be treated as a mindless flow. People make forward-looking decisions to maximize their utility, optimize the routes they are choosing, try to avoid barriers, and limit friction (Helbing, 1992). To analyze psychological intent in terms of mathematics, researchers treat the group of people as a “thinking fluid” (Di Francesco et al., 2011; Hughes, 2002). This means that movement is limited not only by physical factors, but by complex behavior, such as moving in synchronization with a larger group to feel safer, or walking on the right side of the way (Zhang et al., 2018).

Engineers and architects should pay attention to specific behaviors and design infrastructure considering user experience. Otherwise, high crowd densities may lead to dangerous consequences, such as high concentrations of people, pressure, etc. (Adrian et

al., 2018). Vulnerable groups of people, such as the elderly or disabled, pregnant women, and children, should also be accounted for, as they may require more space and time to cross safely (Musolino et al., 2022).

To optimize planning in urban areas, engineers and designers require a mathematical framework that considers both macroscopic flow and individual, microscopic human decisions, making them human-oriented. By prioritizing Human-Centered Design (HCD) alongside conservation law equations like the pedestrian continuity equation, designers can accurately calculate movements and ensure safe, accessible transit for all people.

5.2 The Pedestrian Continuity Equation

The Shallow Water Equations, introduced in the previous section, demonstrate how the physical flow of a flood is based on the macroscopic conservation of momentum and mass. To model the movement of a dense human crowd, the same mathematical foundation is used. In the computation of flooding, one of the primary parameters is water depth (h). In crowd movement, this can be paralleled to the continuous pedestrian density $\rho(x, t)$. Assuming that no pedestrians are spontaneously created or destroyed within the crosswalk domain, the conservation of pedestrians through space is given by the standard continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0 \quad (5.1)$$

where $V(x, t)$ is the two-dimensional velocity vector field of the crowd.

For floodwater, the velocity vector (V) is determined by computing a complex momentum balance that considers the gravitational conditions of the area and the physical friction of the ground. On the other hand, pedestrians are active, “thinking” matter that are not pushed by gravity. They make conscious, forward-looking decisions to reach a destination point and avoid severe barriers along the way (Helbing, 1992; Henderson, 1971).

To analyze this psychological intent mathematically, physicists use the Hughes model, which replaces the physical momentum equation with a behavioral one (Hughes, 2002). In the Hughes model, the velocity vector is decomposed into a scalar walking speed and a directional unit vector (Di Francesco et al., 2011; Hughes, 2002).

The scalar walking speed is a density-dependent function, typically taking a linear, normalized form such as $f(\rho) = 1 - \rho$. It is critical to clarify that this is not a universal physical law of fluid dynamics, but rather a classical behavioral modeling assumption. It prescribes a “logistic dependency” where the pedestrian walking speed decreases linearly as the local congestion increases. This models pedestrians walking at their maximum free-flow speed when crossing an empty crosswalk ($\rho = 0$). Conversely, as the crowd density reaches the maximum possible limit ($\rho = 1$), the walking speed of people drops to zero, and the crowd stops (Di Francesco et al., 2011).

It is also important to consider the direction of movement, which is controlled by a potential field $\phi(x, t)$. This scalar potential function is introduced to mathematically represent the pedestrians’ “common sense of the task,” calculating the estimated travel cost to reach their destination. Because the goal is to reach the end of the crosswalk, this potential must satisfy the Dirichlet boundary condition $\phi = 0$ at the target boundary (the exit), meaning the remaining travel cost drops to zero upon arrival (Di Francesco et al., 2011; Fischer et al., 2019). At closed boundaries such as walls and kerbs, the Neumann condition $\nabla\phi \cdot \mathbf{n} = 0$ is imposed, ensuring no pedestrian flux crosses these boundaries (Fischer et al., 2019). People intuitively choose the most optimal route to minimize their travel time, meaning they will move in the opposite direction from the gradient of this potential field ($-\nabla\phi$) (Di Francesco et al., 2011; Helbing, 1992).

To compute for the crowd’s intentions of avoiding highly concentrated areas, the continuity equation is coupled with a nonlinear Eikonal equation:

$$|\nabla\phi| = \frac{1}{f(\rho)} = \frac{1}{1 - \rho} \quad (5.2)$$

The core idea of using this Eikonal equation is that it creates the connection between

the crowd density and cost of moving, meaning as the concentration of people (density) increases, the cost of moving will also rise (Fischer et al., 2019; Hughes, 2002). As the density approaches the maximum limit ($\rho \rightarrow 1$), the “thinking fluid” of the crowd will reroute itself. Substituting the achieved velocity back into the mass balance results in the coupled Hughes PDE for pedestrian flow (Di Francesco et al., 2011; Hughes, 2002):

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho f^2(\rho) \nabla \phi) = 0 \quad (5.3)$$

While solving this system in two dimensions is mathematically complex, it can be decoupled in a 1D spatial model by explicitly expanding the gradient terms. In one dimension, the spatial gradient $\nabla \phi$ reduces to the partial derivative $\partial_x \phi$. The elimination of ϕ from the coupled system proceeds as follows:

$$|\partial_x \phi| = \frac{1}{f(\rho)} \quad (5.4)$$

↓ Any real value equals its magnitude times its sign

$$\partial_x \phi = |\partial_x \phi| \cdot \text{sgn}(\partial_x \phi) \quad (5.5)$$

↓ Substitute 1D Eikonal magnitude

$$\partial_x \phi = \frac{\text{sgn}(\partial_x \phi)}{f(\rho)} \quad (5.6)$$

↓ Substitute into 1D momentum flux term

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x} \left(\rho f^2(\rho) \cdot \frac{\text{sgn}(\partial_x \phi)}{f(\rho)} \right) = 0 \quad (5.7)$$

↓ Cancel $f(\rho)$ terms, ϕ eliminated

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x} (\rho f(\rho) \text{sgn}(\partial_x \phi)) = 0 \quad (5.8)$$

This mathematical representation establishes the continuous macroscopic model needed to analyze crosswalk capacity. Just as the Navier-Stokes equations were simplified to the

2D SWE, this equation must also be simplified from 2D to a 1D steady-state to analytically evaluate the safety thresholds of a crosswalk (Di Francesco et al., 2011).

5.3 Simplification to 1D Steady-State and the Fundamental Diagram

Just as with the 2D Shallow Water Equations in section 3, we must also simplify the 2D Hughes model to extract design metrics and work with a pedestrian model that is reduced to a straight, signalized crosswalk (Di Francesco et al., 2011). Assuming a one-dimensional spatial domain, the unidirectional pedestrian platoon is crossing a straight crosswalk, and the movement is restricted to a single axis, the x -axis (Hughes, 2002). As a result, the pedestrian density $\rho(x, y, t)$ is reduced to $\rho(x, t)$, and the potential field ϕ varies only with respect to x (Di Francesco et al., 2011). The targeted destination of pedestrians is determined by the term $\text{sgn}(\partial_x \phi)$ (Hughes, 2002).

We assume the movement of pedestrians is in a 1D crosswalk, from left to right with a fixed exit where the potential is $\phi = 0$. Because pedestrians minimize their travel cost, the potential field ϕ strictly decreases toward this exit and the spatial gradient is negative, fixing the sign as $\text{sgn}(\partial_x \phi) = -1$. This guarantees that the pedestrian velocity vector always points strictly toward the destination (Di Francesco et al., 2011).

In fluid dynamics and traffic flow theory, the macroscopic flux q is defined as the product of the pedestrian density ρ and their mean walking velocity v . As established in the Hughes model, the velocity v is dictated by the behavioral congestion function $f(\rho) = 1 - \rho$.

To calculate the maximum capacity of the crosswalk, the steady-state assumption must be applied. During the peak flow like crossing on a green light, we should assume the overall crowd's density remains constant in time, meaning that the derivative ($\partial_t \rho = 0$) vanishes (Di Francesco et al., 2011). The 2D Hughes model simplified to the 1D steady-state constant flux is derived below:

From 2D Hughes model to 1D conservation law:

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho f^2(\rho) \nabla \phi) = 0 \quad (5.9)$$

↓ Reduce to 1D: $\nabla \rightarrow \partial_x$

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x} (\rho f^2(\rho) \partial_x \phi) = 0 \quad (5.10)$$

↓ Substitute $\partial_x \phi = \frac{\text{sgn}(\partial_x \phi)}{f(\rho)}$ from 1D Eikonal

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x} \left(\rho f^2(\rho) \cdot \frac{\text{sgn}(\partial_x \phi)}{f(\rho)} \right) = 0 \quad (5.11)$$

↓ Cancel $f(\rho)$, unidirectional assumption $\text{sgn}(\partial_x \phi) = -1$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho f(\rho)) = 0 \quad (5.12)$$

From flux definition to fundamental diagram:

$$q = \rho v, \quad v = f(\rho) = 1 - \rho \quad (5.13)$$

↓ Substitute $v = 1 - \rho$

$$q = \rho(1 - \rho) \quad (5.14)$$

Steady-state reduction:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (q) = 0 \quad (5.15)$$

↓ Steady-state assumption: $\frac{\partial \rho}{\partial t} = 0$

$$\frac{d}{dx} (q) = 0 \quad (5.16)$$

↓ Integrate with respect to x

$$q = \rho(1 - \rho) = \text{constant} \quad (5.17)$$

This analytical result mathematically proves that in a steady-state 1D environment, the pedestrian flux is conserved along the entire length of the crosswalk (Di Francesco

et al., 2011). Just as it was shown in Section 4, where the floodwater mass conservation dictated that $h \cdot u = q_0$, the crowd conservation law shows that $\rho \cdot v = q_0$.

This parabolic relationship between crowd density and flux creates the basis of the “Fundamental Diagram” of pedestrian traffic (Henderson, 1971; Yano & Kröger, 2025). This is important as it will be used in the next section to identify the critical capacity threshold of the crosswalk. The assumed linear velocity $f(\rho) = 1 - \rho$ corresponds to the normalized Greenshields fundamental diagram, which is just one of many possible mathematical formulations used to model traffic flow. Alternative models exist, such as the Underwood model, which assumes an exponential drop in velocity, or the Drake model, which utilizes a bell-shaped curve to prevent the velocity from ever reaching zero (Puppo et al., 2016).

While with the linear Greenshields model, the parabolic flux curve for analytical integration is controllable, it is an idealized simplification. The limitation of such a model is that it assumes a perfectly homogeneous crowd following a strict, single-valued density-velocity relationship. In reality, during high density, the data scatters. This happens because real people are heterogeneous, meaning individual pedestrians have different physical sizes, motivations, and maximum walking speeds. While highly complex multi-population kinetic models are required to fully capture this scattered behavioral data, the linear Greenshields simplification remains a widely accepted and robust macroscopic baseline to analytically evaluate the absolute capacity limits of an urban crosswalk (Puppo et al., 2016).

5.4 Analytical Solution and Critical Density Threshold

To calculate the limit of the crosswalk’s capacity, we use the established 1D steady-state equation $q = \rho(1 - \rho)$ (Hughes, 2002). This equation is a downward-opening parabola and defines the Fundamental Diagram of pedestrian traffic, showing the relationship between crowd density and the flow rate (Henderson, 1971; Yano & Kröger, 2025).

By finding the vertex of the parabola for maximizing the macroscopic flux $q(\rho)$, the

maximum capacity of the crosswalk can be determined. We take first derivative of the flux with respect to density and set it equal to zero to find the critical points, and then evaluate the second derivative to confirm a maximum:

First derivative:

$$\frac{dq}{d\rho} = 1 - 2\rho = 0 \quad (5.18)$$

⇓ Solve for critical density

$$\rho_c = 0.5 \quad (5.19)$$

Second derivative test:

$$\frac{d^2q}{d\rho^2} = -2 < 0 \quad (5.20)$$

Since the second derivative is strictly negative ($\frac{d^2q}{d\rho^2} < 0$), the function is concave down. This means that the critical density $\rho_c = 0.5$ yields the absolute maximum possible flux for this system (Di Francesco et al., 2011).

In this mathematical model, the maximum flux occurs when the density of the crowd reaches 50% capacity (Di Francesco et al., 2011). This point is called the “critical density threshold” (Fischer et al., 2019). Meaning, if the crowd’s density is less than 0.5, pedestrians are in a “free flow” state, where there is enough space to walk at their comfortable speed and navigate around others freely. In the case when the density exceeds 0.5, the concentration of people on the crosswalk is much higher, meaning they are packed too tightly to follow their desired speed or walk efficiently. In this scenario, the speed drops dramatically, and the flux of the crosswalk decreases, even though there are more people in the area at the present time (Hughes, 2002).

5.5 Human-Centered Design Interpretation

From a Human-Centered Design (HCD) perspective, the mathematical revelation that the maximum macroscopic flux occurs at the critical density threshold of $\rho_c = 0.5$ dictates how urban planners must design safe intersections. Considering the safety of people, designers

and engineers should pay attention to the mathematical data showing that pedestrian crossings transition into a congested, inefficient state when occupancy exceeds this limit.

Operating a crosswalk beyond this critical borderline might be dangerous, especially for vulnerable groups of people, such as children, elderly people, or individuals with disabilities (Musolino et al., 2022). For these people, going through the crosswalk may require more time and space for comfortable moving. Because these individuals often walk at slower speeds or require mobility aids, a heterogeneous crowd experiences a more severe capacity drop than the idealized homogeneous model predicts (Puppo et al., 2016). Mathematically, planners can prevent the system from crossing this threshold by altering the physical dimensions of the crosswalk to artificially lower the local density ρ . By widening the crosswalk, the available spatial domain increases, which directly decreases ρ for a given number of pedestrians, ensuring the flux $q(\rho)$ remains safely in the free-flow phase.

Designers and urban planners must use the $q = \rho(1 - \rho)$ equation to properly construct the size and time of the signals, ensuring that local density will not go beyond 50%, even during peak hours and with the maximum possible amount of people. In the context of the PDE, traffic signals act as temporal boundary conditions that control the pedestrian injection rate at the entrance of the crosswalk. By optimizing the green-light duration, planners effectively limit the initial influx q_0 , preventing the density ρ from accumulating and forming dangerous mathematical shockwaves (Di Francesco et al., 2011).

Also, specific human behavior must be taken into account. According to cellular automata simulations of bidirectional crosswalks, pedestrians show a preference for walking on the right side of the road, and intuitively follow the person in front of them to avoid head-on collisions (Zhang et al., 2018). This is microscopic human behavior; when people from opposite sides meet in the middle, they start to organize into interlaced, unidirectional lanes.

Designers can “control” how people behave on the crosswalks, relying on behavioral psychology and choice architecture to keep crowd density lower than 50% (Cetara et al., 2025). For instance, this can be achieved by implementing physical channelizations, such

as widening the crosswalk at the center, using directional arrows on the ground, or creating islands in the middle. Mathematically, visual signs like directional arrows manipulate the pedestrians’ “common sense of the task,” directly altering the potential field $\phi(x, t)$ in the Eikonal equation. As the person’s travel cost can be changed, the gradient $\nabla\phi$ is modified to actively steer the direction vector of the crowd into organized lanes (Cetara et al., 2025; Hughes, 2002).

Moreover, the construction of physical pedestrian islands introduces new hard boundaries ($\partial\Omega$) into the middle of the crosswalk. As established in Section 5.2, this enforces a Neumann boundary condition, $\nabla\phi \cdot \mathbf{n} = 0$, which mathematically prohibits pedestrian flux from crossing into opposing lanes (Fischer et al., 2019). By strictly separating the counter-flows, the model avoids the severe velocity reductions caused by head-on avoidance maneuvers. This helps to reduce the risks and the friction between opposing flows, and allows the “thinking fluid” of the crowd to pass the crosswalk safely and efficiently, without risking a capacity decrease (Zhang et al., 2018). Combining Human-Centered Design with the continuous PDE helps to create accessible and convenient urban infrastructure.

6 Synthesis - When Mathematics Becomes a Design Tool

6.1 Linking the Two Models

The physical flow of floodwater and the psychological flow of a pedestrian crowd are based on macroscopic continuity equations independently. However, their connections should be considered, because in real-world emergencies, during evacuations, they share the same urban infrastructure and interact physically. To create sustainable and effective Human-Centered Design (HCD) systems, the mathematical connection between these two models is critical.

During flood hazards where people are involved, it is important to note the connection between these two models, because walking through floodwater is one of the leading causes of risks, damages, and threats to human safety (Musolino et al., 2022). While it is strongly not recommended to enter water during the hazards, emergency evacuations force people to navigate through the floodwater to reach safe zones (Musolino et al., 2022). The evacuation process differs depending on whether it is done in a dry environment or in floodwater, and the reaction of people also differs.

The two models that were discussed in previous chapters interact with each other through physical constraints. During urban flooding, the variables extracted from the Shallow Water Equations, especially water depth $h(x, t)$ and the velocity of flow $u(x, t)$, act like dynamic boundaries that affect the crowd's ability to move. Based on the Mechanics Based Method (MBM), pedestrians navigating in floodwater face the risk of toppling or sliding, which creates a Flood Hazard Rate (FHR), meaning when the rate reaches the

critical point, the water becomes a strict physical barrier for people (Musolino et al., 2022; Xia et al., 2014). As a result, the pedestrian continuity equation is directly disrupted by the floodwater. The water becomes more than just a physical condition, but a limitation to the Eikonal equations that forces the “thinking fluid” of people to change their path, avoid the risk, and reach a safe point.

The results presented in this chapter will explain why this interaction is dangerous. The connection between the flood depth and the physical size (width) of the crosswalk mathematically shows how floodwater forces an evacuating crowd to move towards narrow, dry channels. This limitation artificially increases the pedestrian density, pushing the crowd past the 50% threshold, causing the evacuation routes to fail.

6.2 Effective Walkable Width $W(h)$ as a Function of Flood Depth

Streets in cities are usually crowned, meaning the water starts accumulating on the outer margins like the gutters and curb cuts before reaching the center of the road. It is important to understand how floodwater alters the physical area of the crosswalk to mathematically connect these two systems. People make decisions based on several factors, and one of them is making forward-looking choices to avoid risky areas and maintain their physical stability, so they aim to avoid these submerged margins. As a result, the water serves as a physical limitation and an obstacle for people. According to Fischer et al. (2019), in experiments and simulations, people consider these obstacles located at the margins significant, which significantly affects the corridor’s effective width over a long distance, and as a result, this restricts the movement of the crowd and the queuing dynamics.

Mathematically, this constraint can be expressed by computing the effective walkable width W as a function of the flood depth h , denoted as $W(h)$. Before a flood hazard, the crosswalk is dry ($h = 0$), and the effective width is equal to the total crosswalk width ($W(0) = W_{\max}$). As the flood depth increases, according to the Flood Hazard Rate, the submerged area squeezes at the edges, causing a decrease in the effective width $W(h)$

(Musolino et al., 2022).

Let us consider a standard urban street with a uniform transverse slope (or “crown”) denoted by α . As the depth of the floodwater rises, the width of the unwalkable area on each side of the road can be calculated as h/α . Moreover, as there are two margins on the crosswalk, the total lost space is $2h/\alpha$. As a result, the effective space to walk can be modeled as a linear decrease with respect to flood depth (Hughes, 2002):

$$W(h) = W_{\max} - \frac{2h}{\alpha} \quad (6.1)$$

To link the physical characteristics of the crosswalk with the crowd dynamics, there should be a connection between the absolute total flow of people and the behavioral macroscopic flux (Hughes, 2002). Here, Q denotes the total number of people entering the crosswalk per unit time, and the flux q , which stands for the flow rate per unit width, is defined as $q = \frac{Q}{W(h)}$.

As established in Section 5.3, the macroscopic behavioral flux of the crowd is governed by the equation $q = \rho v$, which simplifies to $q = \rho(1 - \rho)$ based on the linear speed assumption (Puppo et al., 2016). Connecting these two definitions provides a link between the flood environment and the pedestrian response:

From total flow to effective flux:

$$q = \frac{Q}{W(h)} \quad (6.2)$$

⇓ Equate with the behavioral macroscopic flux

$$\frac{Q}{W(h)} = \rho v \quad (6.3)$$

⇓ Substitute linear speed assumption $v = 1 - \rho$

$$\frac{Q}{W(h)} = \rho(1 - \rho) \quad (6.4)$$

This physical condition means that the safe and effective space is narrower at the center. As $W(h)$ geometrically shrinks due to the rising water h , the term $\frac{Q}{W(h)}$ rapidly increases.

To maintain the conservation of mass for a constant total demand Q , the local pedestrian density ρ must increase to balance the equation. Consequently, the entire volume of evacuating pedestrians concentrates in a squeezed dry channel in the center of the road, causing a dangerous capacity drop as the crowd is pushed toward the critical density threshold (Di Francesco et al., 2011; Fischer et al., 2019).

6.3 Crowd Density Response $\rho(h)$ – When Does It Become Dangerous?

As the floodwater reduces the effective walking width of the crosswalk $W(h)$, it is important to evaluate how this impacts the pedestrians' behavior. The local pedestrian flux can be expressed in the form $q = Q/W(h)$, as the total pedestrian flux entering the crosswalk should be equal to the local flux times the available width.

The relationship between flood depth and crowd density is obtained by substituting the local flux into the 1D steady-state equation from Chapter 4; as a result, we have:

$$\frac{Q}{W(h)} = \rho(1 - \rho) \quad (6.5)$$

To understand how the narrowing width forces the density of the crowd to increase toward the critical threshold from the mathematical side, the formula $(\rho^2 - \rho + \frac{Q}{W(h)} = 0)$ should be rearranged into a standard quadratic equation with respect to density. Assuming the crowd starts in a stable, free-flow state with $\rho < 0.5$, the local density $\rho(h)$ is given by the negative root of the quadratic formula:

Quadratic rearrangement:

$$\rho^2 - \rho + \frac{Q}{W(h)} = 0 \quad (6.6)$$

↓ Solve for ρ using the quadratic formula (free-flow root)

$$\rho(h) = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4Q}{W(h)}} \quad (6.7)$$

The solution provided above shows that as the depth of the floodwater increases, the effective width $W(h)$ becomes narrower, meaning as $\frac{4Q}{W(h)}$ grows larger, the square root value approaches zero. Consequently, this forces the overall density $\rho(h)$ upward, converging toward the critical value of 0.5 (Di Francesco et al., 2011).

Furthermore, this equation defines an absolute physical limit for the crosswalk infrastructure. If the floodwaters squeeze the width such that $W(h) < 4Q$, the term inside the square root becomes negative. In macroscopic kinematic modeling, this lack of a real mathematical solution indicates that the available space can no longer process the constant pedestrian demand Q (Puppo et al., 2016).

When the available width becomes more restricted and this mathematical threshold is crossed, it can be dangerous for the people, raising the risk, as the density is forced to increase past the capacity threshold of 50% (Fischer et al., 2019). The path becomes overloaded with people, the speed of the crowd drops dramatically, and the space to safely and comfortably cross the street for the “thinking fluid” is reduced to 0. These conditions create a severe bottleneck, making the evacuation process much harder to proceed, and trapping pedestrians in the hazardous zone (Musolino et al., 2022).

6.4 Combined Safety Threshold and Design Implications

The synthesis of the Shallow Water Equations and the Hughes pedestrian model provides a helpful quantitative tool for urban planners. Using the $\frac{Q}{W(h)} = \rho(1 - \rho)$, planners can determine the maximum total pedestrian flux Q that a passage can handle before the local density ρ reaches the critical limit $\rho_c = 0.5$. Substituting this critical density into the flux equation yields the maximum possible flow rate:

$$q(\rho_c) = 0.5(1 - 0.5) = \frac{1}{4} \tag{6.8}$$

This result allows us to define a strict mathematical safety condition for any evacuation route:

$$\frac{Q}{W(h)} \leq \frac{1}{4} \quad (6.9)$$

When the flux exceeds $\frac{1}{4}$, the model can predict dangerous bottlenecks and show whether the targeted route is effective (Carlini et al., 2016). As long as the inequality is satisfied, it means the crowd remains in a free-flow state, meaning they have enough space to move freely.

Minimum safe crosswalk width: Rearranging the safety condition gives the minimum effective width required to safely accommodate a pedestrian demand Q :

$$W(h) \geq 4Q \quad (6.10)$$

Urban planners should redesign the infrastructure so that the unflooded “dry channel” never drops below four times that demand during the evacuations. If the available width shrinks below this limit, it would create a bottleneck that restricts people’s movement. Consequently, the crowd density might exceed the critical threshold, resulting in dangerous queuing, crowding, and crushing (Fischer et al., 2019).

For a crosswalk of known effective width $W(h)$, the maximum pedestrian flow that can be safely accommodated is:

$$Q \leq \frac{W(h)}{4} \quad (6.11)$$

Critical flood depth h_c : Substituting the linear width model $W(h) = W_{\max} - \frac{2h}{\alpha}$ from Section 6.2 into the safety condition and solving for h results the maximum flood depth at which the crosswalk remains safe:

$$W_{\max} - \frac{2h_c}{\alpha} = 4Q \implies h_c = \frac{\alpha(W_{\max} - 4Q)}{2} \quad (6.12)$$

When the flood depth h is higher than h_c , the effective width $W(h)$ drops below $4Q$, meaning the safety condition is violated, and the crosswalk can no longer effective for safe

pedestrians. One of the core important highlights is the parameter h_c because it unifies the hydrological environment with human behavioral dynamics. It shows the depth of water at the street margins that will squeeze the dry central channel just enough to trigger a pedestrian bottleneck. It translates a hydrological variable (water depth) into a human safety limit. h_c can be used by urban planners as a benchmark to establish early warning systems, to prevent water levels reaching this critical point during an evacuation (Li et al., 2021).

Professionals can test and model different flood scenarios - storms, rapid snowmelt, or heavy rainfall — and predict the maximum water depth h (Mishra et al., 2022) to evaluate whether the effective width $W(h)$ will satisfy the safety condition. If the predicted water depth h exceeds h_c , the crosswalk will fail to safely accommodate evacuating pedestrians. To prevent this, engineers should redesign the street geometry to ensure the condition $\frac{Q}{W(h)} \leq \frac{1}{4}$ is continuously met.

Recent flood studies show that the Flood Hazard Rate at bottlenecks can be reduced by retrofitting targeted routes — for example, by elevating the crosswalk pavement or widening the pathways (Musolino et al., 2022). By physically raising the pavement, the local water depth h decreases, which preserves a larger effective width $W(h)$ and keeps the specific flux safely below the $\frac{1}{4}$ limit. By prioritizing human-oriented approaches alongside mathematical framework fundamentals, urban planners can create evacuation plans to keep people away from areas where $h > h_c$ (Wang & Wang, 2024).

Applying this improvement locally can help ensure that the flow of the crowd during emergency accidents remains below the density threshold $\rho_c = 0.5$, allowing people to evacuate safely.

6.5 Limitations and Future Directions

Although the achieved results are crucial and serve as a foundation for future research, they are limited and need to be evaluated. As the Shallow Water Equations and the Hughes model were reduced from two dimensions to a 1D steady state, assuming operations along

a single axis, they do not consider more complex cases. For example, during a flood event, water comes from different directions, and during emergencies, pedestrians act in multidirectional flows that a 1D steady-state model cannot fully cover.

Furthermore, a 1D analytical model struggles to accurately represent boundary effects caused by complex urban geometries. For example, flow separation around building corners, dead zones, or localized queuing at narrow bottlenecks and exits (Bruwier et al., 2017; Fischer et al., 2019). In those cases, it is harder to analyze because additional friction appears, and it affects the density threshold as well.

Future research should focus on more complicated calculations, moving to 2-dimensional nonlinear partial differential equations for both problems: urban flooding and pedestrian flow. While scaling up the computational model may require much more effort and resources for efficient calculations, it can provide a more accurate model. Because 2D nonlinear PDEs cannot be solved analytically for arbitrary urban geometries, future research must rely on advanced numerical methods. Specifically, robust numerical techniques such as finite volume schemes are highly recommended for solving 2D models, as they naturally conserve mass and momentum across computational grid cells (Kumar & Pahar, 2024; Xing, 2017). With those algorithms, these finite volume schemes can successfully handle complex boundary conditions, capture discontinuous shockwaves, and accurately simulate moving dry-wet fronts, minimizing numerical instability (Begnudelli & Sanders, 2006; Xing, 2017). Such a 2D nonlinear PDE framework can allow engineers to simulate the interaction between the water and the crowd in specific conditions.

For future investigations, this data computation may be available for risk managers during real-time emergencies. For example, crowdsourced street photos and live flood data can be used to model and predict the available width during the ongoing storm and help identify optimal and safe solutions during the emergency case (Alizadeh et al., 2022). The system can work in real-time, analyzing the surroundings, available safe routes, and people’s responses, and based on that data, offer optimal, efficient, sustainable, and safe routes, taking into consideration HCD. By integrating Internet of Things (IoT) sensors,

machine learning, and real-time data synchronization, these computational frameworks can be transformed from fixed plans to dynamic digital solutions, providing continuous, adaptive evacuation routing for vulnerable populations during rapidly evolving disasters (Alizadeh et al., 2022; Mylonas et al., 2021).

7 Conclusion

This capstone introduced a novel mathematical framework demonstrating the connection between the physical parameters of floodwater and the behavioral dynamics of a pedestrian crowd. The mathematical framework showed that both systems are based around a fundamental conservation law, and the optimal model is the macroscopic continuity equation.

The primary goal of this research is the contribution of partial differential equations as a tool for human-oriented design systems in disaster risk reduction. Usually, the planning of evacuation routes is based on dry-weather conditions, which fails during real-life flood hazards, as these plans do not accurately account for the additional physical parameters and human mobility in limited conditions. The integration of the Mechanics Based Method (MBM) and the Flood Hazard Rate (FHR) showed the inconsistencies and challenges that vulnerable groups of pedestrians face during floodwaters.

To overcome these challenges, this research explored the connection between the Shallow Water Equations and the Hughes pedestrian model. This was achieved through the derivation of the effective walkable width function $W(h)$. By proving that rising flood depths systematically squeeze the available dry path, this research mathematically translated a hydrological variable into a direct geometric constraint on human movement. Substituting this constraint into the behavioral flux equation yielded the unifying relationship:

$$\frac{Q}{W(h)} = \rho(1 - \rho) \quad (7.1)$$

Through this original coupling, we proved analytically that reducing the width forces the crowd density toward the critical capacity threshold of 50%, triggering dangerous congestion. By evaluating the flux equation at this critical density, we established the

strict safety inequality:

$$\frac{Q}{W(h)} \leq \frac{1}{4} \quad (7.2)$$

This in turn yielded the central quantitative result of this paper - the critical flood depth threshold:

$$h_c = \frac{\alpha(W_{\max} - 4Q)}{2} \quad (7.3)$$

This equation unifies the hydrological environment and human behavior, connecting the water depth with the human safety limit. Furthermore, it offers mathematical justification for small-scale Human-Centered Design solutions, such as elevating crosswalks or widening pathways, in order to keep the water depth below h_c . Consequently, this influences the FHR by decreasing it, ensuring that evacuation routes are safe and optimal for the “thinking fluid”.

Looking forward, as urbanization continues to develop and extreme weather events become more unpredictable, the risk of urban flooding will increase as well. This will influence vulnerable groups of the population, such as elderly people and children. To generate a solution that is mathematically efficient and prioritizes human safety in complex cases, future research should be expanded toward 2D nonlinear partial differential equations (Pathirana et al., 2008). The 1D steady-state solution is limited, as it only works for cases operating along a single axis. However, when there are multidirectional flows, it is restricted and cannot effectively cover those cases. Further dynamic routing could be powered by additional helping tools like sensors, real-time data synchronizations, and social media data to provide real-time effective evacuation routes (Alizadeh et al., 2022; Mishra et al., 2022). Combining more complex mathematical frameworks with Human-Centered Design, professionals can move beyond theoretical solutions to create sustainable, safe, and flood-resilient cities that prioritize human safety.

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