

Research Article

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Explicit inverse of symmetric, tridiagonal near Toeplitz matrices with strictly diagonally dominant Toeplitz part

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Abstract: Let $T_n = \text{tridiag}(-1, b, -1)$, an $n \times n$ symmetric, strictly diagonally dominant tridiagonal matrix ($|b| > 2$). This article investigates tridiagonal near-Toeplitz matrices $\tilde{T}_n := [\tilde{t}_{i,j}]$, obtained by perturbing the $(1, 1)$ and (n, n) entry of T_n . Let $\tilde{t}_{1,1} = \tilde{t}_{n,n} = \tilde{b} \neq b$. We derive exact inverses of \tilde{T}_n . Furthermore, we demonstrate that these results hold even when $|\tilde{b}| < 1$. Additionally, we establish upper bounds for the infinite norms of the inverse matrices. The row sums and traces of the inverse provide insight into the matrix's spectral properties and play a key role in understanding the convergence of fixed-point iterations. These metrics allow us to derive tighter bounds on the infinite norms and improve computational efficiency. Numerical results for Fisher's problem demonstrate that the derived bounds closely match the actual infinite norms, particularly for $b > 2$ with $\tilde{b} \leq 1$ and $b < -2$ with $\tilde{b} \geq -1$. For other cases, further refinement of the bounds is possible. Our results contribute to improving the convergence rates of fixed-point iterations and reducing the computation time for matrix inversion.

Keywords: Toeplitz matrices, diagonal dominance, exact inverses, upper bounds

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1 Introduction

Banded near-Toeplitz matrices are a special class of structured matrices that arise in various applications such as interpolation problems, differential equation modeling [10,16,35,41], time series analysis [26,32], and autoregressive models [1,2,37]. For instance, in solving differential equations [30], the finite difference discretization often results in a system of equations involving a banded near-Toeplitz matrix, where boundary conditions slightly perturb the strict Toeplitz structure. In nonlinear differential equations, a fixed-point iteration method is commonly used to solve the system, and the convergence of this method is closely related to the properties of the inverse of the underlying near-Toeplitz matrix.

As an illustration, consider the differential equation

$$\frac{d^2u}{dx^2} = f(u), \quad x \in \Omega = (0, L),$$

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with some boundary conditions at $x \in \{0, L\}$ and some nonlinear function u . Examples are the so-called Fisher equation, with $f(u) = ku(1 - u)$, and the Gelfand-Bratu equation, with $f(u) = ke^u$. Consider an approximation to the second derivative using the central difference scheme

$$\frac{d^2u}{dx^2}(x_i) \approx \frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}), \quad (1.1)$$

where $x_i = ih \in [0, L]$, with h being a constant step size and $i = 1, \dots, n = L/h$, and $u_i \equiv u(x_i)$. This approximation process results in a system of nonlinear equations, which can be written as

$$\widehat{T}_n \mathbf{u} = h^2 f(\mathbf{u}), \quad (1.2)$$

where \widehat{T}_n is an n -by- n tridiagonal near Toeplitz matrix and $\mathbf{u} = (u_1, \dots, u_n)^T \in \mathbb{R}^n$. Starting from an initial guess of the solution \mathbf{u}^0 , an approximate solution of the differential equation can be computed via the fixed-point recurrence

$$\mathbf{u}^{k+1} = h^2 \widehat{T}_n^{-1} f(\mathbf{u}^k), \quad k = 1, 2, \dots, \quad (1.3)$$

provided that the recurrence converges. Assuming that f is a Lipschitz continuous function in the domain Ω , the sequence of solutions generated by the fixed-point iteration satisfies the inequality, for $p \in \{1, 2, \infty\}$,

$$\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_p = \|h^2 \widehat{T}_n^{-1} (f(\mathbf{u}^k) - f(\mathbf{u}^{k-1}))\|_p \leq h^2 \|\widehat{T}_n^{-1}\|_{pL_c} \|\mathbf{u}^k - \mathbf{u}^{k-1}\|_p,$$

where L_c is the Lipschitz constant, which depends on the nature of $f(\mathbf{u})$. To ensure the convergence of the fixed-point iteration method, we require $h^2 L_c \|\widehat{T}_n^{-1}\|_p < 1$. Here, an upper bound of $\|\widehat{T}_n^{-1}\|_p$ is useful in the convergence analysis.

The study of inverses of certain classes of Toeplitz matrices has a rich history. Early work explored explicit inversion formulas, such as those presented in [11,20], which were later extended in [24,33,39,40]. These explicit formulas have proven valuable for investigating properties such as row sums, traces, and bounds, particularly in statistical modeling and econometrics. Historical studies [28] also reviewed inversion techniques for symmetric tridiagonal matrices and discussed efficient computational algorithms for their inversion [15,19,31]. Moreover, the inverses of tridiagonal Jacobi matrices and k -Toeplitz matrices were discussed in [9,38], respectively. Subsequent research built on these foundations, focusing on bounds for the inverse matrix norms and the rate of decay of its elements along rows and columns [3,6,22,23,29].

Determinants and recursive properties of Toeplitz matrices and their inverses have also gathered considerable attention. Recursive formulas and estimates for determinants were discussed in [4,27,36], while further results on Toeplitz determinants and their inverses were provided in [5,21,34]. The periodicity properties of determinants of $(0, 1)$ double banded matrices have been extensively discussed in [12–14]. The determinants of near-Toeplitz matrices have been expressed in terms of Fibonacci numbers in [17], with their inverses being described using Lucas numbers.

Eigenvectors and eigenvalues of near-Toeplitz and Toeplitz matrices have also been extensively studied. Initial work began with [25] and was further developed in [7], which considered matrices arising from differential equations after discretization. Additionally, a comprehensive summary of research related to eigenvalues of Toeplitz matrices has been presented in [8].

In this article, we present properties of the inverse of an $n \times n$ symmetric tridiagonal near-Toeplitz matrix

$$\widehat{T}_n = \begin{bmatrix} \bar{b} & \hat{c} & & & \\ \hat{c} & \hat{b} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \hat{b} & \hat{c} \\ & & & \hat{c} & \bar{b} \end{bmatrix} = -\hat{c} \underbrace{\begin{bmatrix} \bar{b} & -1 & & & \\ -1 & \hat{b} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \hat{b} & -1 \\ & & & -1 & \bar{b} \end{bmatrix}}_{\widetilde{T}_n} = -\hat{c} \widetilde{T}_n, \quad (1.4)$$

with $n \geq 3$, such as the one arising in (1.2). Here, $b = -\hat{b}/\hat{c}$ and $\bar{b} = -\bar{b}/\hat{c}$. Since \widehat{T}_n is a factor of \widetilde{T}_n , we will primarily focus on the latter. The eigenvalues of related near-Toeplitz matrices have been examined in [18].

The near-Toeplitz matrix \tilde{T}_n in (1.4) can be expressed as a rank-2 decomposition

$$\tilde{T}_n = T_n + (\tilde{b} - b)UU^T, \quad (1.5)$$

where $T_n = \text{tridiag}(-1, b, -1)$ is the Toeplitz part of \tilde{T}_n and the matrix U^T is given by

$$U^T = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}. \quad (1.6)$$

The inverse of the Toeplitz part T_n , if it exists, can be characterized and explicitly expressed by the roots of the polynomial $p(r) = -r^2 + br - 1$ [11]. If T_n is diagonally dominant, meaning $|b| \geq 2$, then $p(r)$ has either two distinct real roots or equal roots, with the latter occurring when $b = 2$. Otherwise, $p(r)$ has two complex roots.

Suppose the inverse of \tilde{T}_n exists. Applying the Sherman-Morrison formula, the inverse of \tilde{T}_n is given by

$$\tilde{T}_n^{-1} = T_n^{-1} - (\tilde{b} - b)T_n^{-1}UM^{-1}U^T T_n^{-1}, \quad (1.7)$$

where $M = I_2 + (\tilde{b} - b)U^T T_n^{-1}U \in \mathbb{R}^{2 \times 2}$. Let $\tilde{T}_n^{-1} = [\tilde{t}_{i,j}^{-1}]$, $T_n^{-1} = [t_{i,j}^{-1}]$, and $\beta = \tilde{b} - b$. Then, for $M = [m_{i,j}]_{i,j=1,2}$, we have

$$\begin{aligned} m_{11} &= m_{22} = 1 + (\tilde{b} - b)t_{1,1}^{-1} = 1 + \beta t_{n,n}^{-1}, \\ m_{12} &= m_{21} = (\tilde{b} - b)t_{1,n}^{-1} = \beta t_{n,1}^{-1}, \end{aligned}$$

due to the symmetry and centrosymmetry of T_n . Furthermore,

$$M^{-1} = \frac{1}{\Delta} \begin{bmatrix} m_{11} & -m_{12} \\ -m_{12} & m_{11} \end{bmatrix}, \quad (1.8)$$

where $\Delta = m_{11}^2 - m_{12}^2 = (1 + \beta t_{n,n}^{-1})^2 - \beta^2 (t_{n,1}^{-1})^2$. With this, the (i, j) -entry (with $i \geq j$) of the inverse of \tilde{T}_n is given explicitly by the formula

$$\tilde{t}_{i,j}^{-1} = t_{i,j}^{-1} - \frac{\beta}{\Delta} [t_{i,1}^{-1}(m_{11}t_{1,j}^{-1} - m_{12}t_{n,j}^{-1}) + t_{i,n}^{-1}(-m_{12}t_{1,j}^{-1} + m_{11}t_{n,j}^{-1})]. \quad (1.9)$$

The trace of \tilde{T}_n^{-1} can be calculated using the formula

$$\text{Tr}(\tilde{T}_n^{-1}) = \sum_{i=1}^n \tilde{t}_{i,i}^{-1} = \sum_{i=1}^n t_{i,i}^{-1} - \frac{\beta}{\Delta} \left\{ m_{11} \sum_{i=1}^n ((t_{i,1}^{-1})^2 + (t_{n,i}^{-1})^2) - 2m_{12} \sum_{i=1}^n t_{i,1}^{-1} t_{n,i}^{-1} \right\}, \quad (1.10)$$

due to the symmetry of T_n . The sum of the elements in the i th row of the inverse is given by

$$\begin{aligned} \sum_{j=1}^n \tilde{t}_{i,j}^{-1} &= \sum_{j=1}^n t_{i,j}^{-1} - \frac{\beta}{\Delta} \left[t_{i,1}^{-1} \left(m_{11} \sum_{j=1}^n t_{1,j}^{-1} - m_{12} \sum_{j=1}^n t_{n,j}^{-1} \right) + t_{i,n}^{-1} \left(-m_{12} \sum_{j=1}^n t_{1,j}^{-1} + m_{11} \sum_{j=1}^n t_{n,j}^{-1} \right) \right] \\ &= \sum_{j=1}^n t_{i,j}^{-1} - \frac{\beta}{\Delta} \left[t_{i,1}^{-1} (m_{11} - m_{12}) \sum_{j=1}^n t_{n,j}^{-1} + t_{i,n}^{-1} (-m_{12} + m_{11}) \sum_{j=1}^n t_{n,j}^{-1} \right] \\ &= \sum_{j=1}^n t_{i,j}^{-1} - \frac{\beta}{m_{11} + m_{12}} (t_{i,1}^{-1} + t_{i,n}^{-1}) \sum_{j=1}^n t_{n,j}^{-1}, \end{aligned} \quad (1.11)$$

after making use of the centrosymmetry of T_n and T_n^{-1} , with

$$m_{11} + m_{12} = 1 + \beta (t_{n,1}^{-1} + t_{n,n}^{-1}).$$

We note that a similar class of matrices was explored in [37] and [24]. In [37], the inverse of these matrices was expressed using a recurrence formula involving Chebyshev polynomials of the second kind, while [24] derived more general results for their inverses. This article also investigates properties of the inverse, such as row sums, trace, and bounds, with applications to statistical models like the AR(1) plus noise model. In particular, explicit formulas for the trace and bounds for the row sums of the inverse are derived when $b = 2$. Our work extends this analysis by establishing new bounds for the row sums of the inverse matrices and

further explores bounds for norms. To provide effective bounds, we consider several distinct cases, namely: (i) $|b| > 2$, (ii) $|b| = 2$, and (iii) $|b| < 2$. This article focuses on the first case, $|b| > 2$, where the Toeplitz part is strictly diagonally dominant.

This article is organized as follows. In Section 3, we will derive explicit formulas for the trace and bounds for the rowsum and the norms of the inverse matrix when $b > 2$. Section 4 will delve in the cases where $b < -2$. Section 5 presents numerical experiments demonstrating the effectiveness of the upper bounds and a convergence rate comparison to Fisher's problem.

2 Preliminary results

The inverse of the Toeplitz matrix $T_n = \text{tridiag}(-1, b, -1)$, with $|b| > 2$, in (1.5) can be characterized by the two roots of the polynomial $p(r) = -r^2 + br - 1$: $r_1 = \frac{1}{2}(b + \sqrt{b^2 - 4})$ and $r_2 = \frac{1}{2}(b - \sqrt{b^2 - 4})$ [11]. Let $\gamma_k = r_1^k - r_2^k$ with $\gamma_0 = 0$ and $\gamma_1 = \sqrt{b^2 - 4}$. Its inverse is given by

$$t_{i,j}^{-1} = \begin{cases} \frac{\gamma_j \gamma_{n+1-i}}{\sqrt{b^2 - 4} \gamma_{n+1}}, & i \geq j, \\ t_{j,i}^{-1}, & i < j. \end{cases} \quad (2.1)$$

Now, we list a couple of intermediate results, pertaining to r_1 , γ_k , and b .

Lemma 2.1. *Let $b > 2$. For any integer $k \geq 2$, the following holds true:*

- (i) $\gamma_k + \gamma_{k-2} = b\gamma_{k-1}$;
- (ii) $\frac{b}{2} < r_1 \leq \frac{\gamma_{k+1}}{\gamma_k} \leq b$;
- (iii) $\gamma_{n+1-i}\gamma_{i+1} - \gamma_i\gamma_{n-i} = \gamma_{n+1}\gamma_1$,

where $\gamma_k = r_1^k - r_2^k$.

Proof. Using the definition of γ_k ,

$$\gamma_k + \gamma_{k-2} = r_1^{k-1}(r_1 + r_1^{-1}) - r_2^{k-1}(r_2 + r_2^{-1}),$$

from which Part (i) is derived via a direct calculation. From Part (i), $\gamma_{k+1} = b\gamma_k - \gamma_{k-1} \leq b\gamma_k$. Also,

$$\frac{\gamma_{k+1}}{\gamma_k} = \frac{r_1^{k+1} - r_2^{k+1}}{r_1^k - r_2^k} = r_1 + r_2 - \frac{r_1^k r_2 - r_2^k r_1}{r_1^k - r_2^k} = b - \frac{r_1^{k-1} - r_2^{k-1}}{r_1^k - r_2^k} = b - \frac{r_2 - r_1(r_2/r_1)^k}{1 - (r_2/r_1)^k},$$

after using the fact that $r_1 r_2 = 1$. Since $r_1 > r_2$,

$$\frac{\gamma_{k+1}}{\gamma_k} \geq b - \frac{r_2(1 - (r_2/r_1)^k)}{1 - (r_2/r_1)^k} = b - r_2 = r_1 > \frac{b}{2},$$

completing Part (ii). For Part (iii), using $r_1 r_2 = 1$,

$$\begin{aligned} \gamma_{n+1-i}\gamma_{i+1} - \gamma_i\gamma_{n-i} &= r_1^{n+2} - r_1^{n-2i} - r_2^{n-2i} + r_2^{n+2} - r_1^n + r_1^{n-2i} + r_2^{n-2i} - r_2^n \\ &= r_1^{n+1}r_1 - r_1^{n+1}r_2 - r_2^{n+1}r_1 + r_2^{n+1}r_2 = \gamma_{n+1}\gamma_1. \end{aligned} \quad \square$$

Lemma 2.2. *Let $|b| > 2$. For any integer $p \geq 1$, the following relations hold:*

- (i) $\sum_{k=1}^p \gamma_k = \frac{1}{b-2}(\gamma_{p+1} - \gamma_p - \gamma_1)$;
- (ii) $\sum_{k=1}^p k\gamma_k = \frac{1}{b-2}(p\gamma_{p+1} - (p+1)\gamma_p)$,

where $\gamma_k = r_1^k - r_2^k$.

Proof. For Part (i),

$$\begin{aligned} \sum_{k=1}^p \gamma_k &= \sum_{k=1}^p (r_1^k - r_2^k) = \sum_{k=0}^p r_1^k - 1 - \sum_{k=0}^p r_2^k + 1 = \frac{r_1^{p+1} - 1}{r_1 - 1} - \frac{r_2^{p+1} - 1}{r_2 - 1} \\ &= \frac{r_1^{p+1} - r_2^{p+1} - (r_1^p - r_2^p) - (r_1 - r_2)}{b - 2} = \frac{1}{b - 2}(\gamma_{p+1} - \gamma_p - \gamma_1). \end{aligned}$$

For Part (ii),

$$\begin{aligned} \sum_{k=1}^p k\gamma_k &= \sum_{k=1}^p k(r_1^k - r_2^k) = r_1 \sum_{k=1}^p kr_1^{k-1} - r_2 \sum_{k=1}^p kr_2^{k-1} \\ &= r_1 \left(\sum_{k=0}^p r_1^k \right)' - r_2 \left(\sum_{k=0}^p r_2^k \right)' = r_1 \left(\frac{r_1^{p+1} - 1}{r_1 - 1} \right)' - r_2 \left(\frac{r_2^{p+1} - 1}{r_2 - 1} \right)' \\ &= \frac{pr_1^{p+2} - (p+1)r_1^{p+1} + r_1}{(r_1 - 1)^2} - \frac{pr_2^{p+2} - (p+1)r_2^{p+1} + r_2}{(r_2 - 1)^2} \\ &= \frac{pr_1^{p+1} - (p+1)r_1^p + 1}{b - 2} - \frac{pr_2^{p+1} - (p+1)r_2^p + 1}{b - 2} \\ &= \frac{1}{b - 2}(p\gamma_{p+1} - (p+1)\gamma_p). \quad \square \end{aligned}$$

Lemma 2.3. For $b > 2$ and $n \geq 3$,

$$b - 1 < b - \frac{2}{b} \leq \frac{\gamma_{n+1}}{\gamma_n + \gamma_1} < r_1 < b.$$

The detailed proof of Lemma 2.3 is provided in Appendix A.

3 Trace, rowsum, and norms of the inverse matrix: $b > 2$

In this section, we derive inverse properties of the near Toeplitz matrix \tilde{T}_n , when $b > 2$. Our first result is on non-singularity of \tilde{T}_n .

Lemma 3.1. (Non-singularity of \tilde{T}_n) Let \tilde{T}_n be as defined in (1.4) with $b > 2$. Then \tilde{T}_n is nonsingular if $\tilde{b} \neq b - \frac{\gamma_{n+1}}{\gamma_n + \gamma_1} =: \tilde{b}_1$ or $\tilde{b} \neq b - \frac{\gamma_{n+1}}{\gamma_n - \gamma_1} =: \tilde{b}_2$. Furthermore, it holds that $0 \leq \tilde{b}_2 < \tilde{b}_1 < 1$.

Proof. The first term on the right-hand side of (1.9) exists for $b > 2$, and nonsingularity requires that the second term involving Δ does not vanish. In this case, with $t_{n,1}^{-1} = \gamma_1/\gamma_{n+1}$ and $t_{n,n}^{-1} = \gamma_n/\gamma_{n+1}$, we have

$$\Delta = \frac{\gamma_{n+1}^2 + 2(\tilde{b} - b)\gamma_n\gamma_{n+1} + (\tilde{b} - b)^2(\gamma_n^2 - \gamma_1^2)}{\gamma_{n+1}^2} = 0,$$

for $\tilde{b} \in \{\tilde{b}_1, \tilde{b}_2\}$ defined in the lemma. It is worth noting that $\tilde{b}_1 - \tilde{b}_2 = \frac{2\gamma_1\gamma_{n+1}}{\gamma_n^2 - \gamma_1^2} > 0$, which implies that $\tilde{b}_1 > \tilde{b}_2$.

Furthermore, with $b - 1 < \frac{\gamma_{n+1}}{\gamma_n + \gamma_1}$ (Lemma 2.3), we have $b - \frac{\gamma_{n+1}}{\gamma_n + \gamma_1} = \tilde{b}_1 < 1$. Finally, from the non-negativity of \tilde{b}_2 ,

$$\tilde{b}_2 = b - \frac{\gamma_{n+1}}{\gamma_n - \gamma_1} = \frac{b\gamma_n - b\gamma_1 - \gamma_{n+1}}{\gamma_n - \gamma_1} = \frac{\gamma_{n-1} - b\gamma_1}{\gamma_n - \gamma_1} = \frac{\gamma_{n-1} - \gamma_2}{\gamma_n - \gamma_1} \geq 0,$$

after using Lemma 2.1(i) and the fact that $\gamma_2 = b\gamma_1$, with equality occurring when $n = 3$. □

Next, we introduce the compact form of the element of \tilde{T}_n^{-1} . This form is particularly important when we derive some upper bounds when considering the case $\tilde{b} < 1$. The proof is presented in Appendix B.

Lemma 3.2. Let $b > 2$ and \tilde{b} such that \tilde{T}_n is invertible. For $i \geq j$, we have

$$\tilde{t}_{i,j}^{-1} = C(\gamma_{n+1-i} + \beta\gamma_{n-i})(\gamma_j + \beta\gamma_{j-1}),$$

where $C = \frac{\gamma_{n+1}}{\gamma_1((\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2)}$ and $\beta = \tilde{b} - b$.

3.1 Trace of the inverse

The general formula for the trace of the inverse matrix \tilde{T}_n^{-1} is given by (1.10). With $t_{n,1}^{-1} = \gamma_1/\gamma_{n+1}$, $t_{n,n}^{-1} = \gamma_n/\gamma_{n+1}$, $m_{11} = 1 + (\tilde{b} - b)\gamma_n/\gamma_{n+1}$, and $m_{12} = (\tilde{b} - b)\gamma_1/\gamma_{n+1}$, the diagonal entries of the inverse matrix \tilde{T}_n^{-1} are given by

$$\begin{aligned} \tilde{t}_{i,i}^{-1} &= t_{i,i}^{-1} - \frac{\tilde{b} - b}{\Delta} \left[\frac{\gamma_{n+1} + (\tilde{b} - b)\gamma_n}{\gamma_{n+1}} \left(\left(\frac{\gamma_{n+1-i}}{\gamma_{n+1}} \right)^2 + \left(\frac{\gamma_i}{\gamma_{n+1}} \right)^2 \right) - 2(\tilde{b} - b) \frac{\gamma_1}{\gamma_{n+1}} \frac{\gamma_i \gamma_{n+1-i}}{\gamma_{n+1}^2} \right] \\ &= \left[1 + \frac{2(\tilde{b} - b)^2(b^2 - 4)}{\Delta \gamma_{n+1}^2} \right] t_{i,i}^{-1} - \frac{(\tilde{b} - b)(\gamma_{n+1} + (\tilde{b} - b)\gamma_n)}{\Delta \gamma_{n+1}^3} (\gamma_i^2 + \gamma_{n+1-i}^2). \end{aligned} \quad (3.1)$$

The trace formula (1.10) can then be written as

$$\text{Tr}(\tilde{T}_n^{-1}) = \left[1 + \frac{2(\tilde{b} - b)^2(b^2 - 4)}{\Delta \gamma_{n+1}^2} \right] \sum_{i=1}^n t_{i,i}^{-1} - \frac{(\tilde{b} - b)(\gamma_{n+1} + (\tilde{b} - b)\gamma_n)}{\Delta \gamma_{n+1}^3} \sum_{i=1}^n \{\gamma_i^2 + \gamma_{n+1-i}^2\}. \quad (3.2)$$

The first sum in the right-hand side is the trace of T_n^{-1} , whose formula is given by the following lemma:

Lemma 3.3. For $b > 2$ in T_n ,

$$\text{Tr}(T_n^{-1}) = \frac{n+1}{\sqrt{b^2-4}} \left(1 + \frac{2r_2^{n+1}}{\gamma_{n+1}} \right) - \frac{b}{b^2-4}. \quad (3.3)$$

Proof. Expressing the trace of the inverse matrix T_n^{-1} in terms of its elements given by (2.1), we have

$$\begin{aligned} \text{Tr}(T_n^{-1}) &= \sum_{i=1}^n t_{i,i}^{-1} = \frac{1}{\gamma_1 \gamma_{n+1}} \sum_{i=1}^n (r_1^{n+1} + r_2^{n+1} - r_1^{n+1-2i} - r_2^{n+1-2i}) \\ &= \frac{n+1}{\gamma_1 \gamma_{n+1}} (r_1^{n+1} + r_2^{n+1}) - \frac{1}{\gamma_1 \gamma_{n+1}} \sum_{i=0}^n (r_1^{n+1-2i} + r_2^{n+1-2i}) \\ &= \frac{n+1}{\gamma_1 \gamma_{n+1}} (r_1^{n+1} + r_2^{n+1}) - \frac{1}{\gamma_1 \gamma_{n+1}} \left[\frac{r_1^{n-1} + r_2^{n-1} - r_1^{n+3} - r_2^{n+3}}{(1-r_1^2)(1-r_2^2)} \right] \\ &= \frac{n+1}{\gamma_1 \gamma_{n+1}} (r_1^{n+1} + r_2^{n+1}) + \frac{1}{\gamma_1 \gamma_{n+1}} \frac{\gamma_{n+1} \gamma_2}{4-b^2} \quad [\text{by } \gamma_2 = b\gamma_1] \\ &= \frac{n+1}{\sqrt{b^2-4}} \left(1 + \frac{2r_2^{n+1}}{\gamma_{n+1}} \right) - \frac{b}{b^2-4}. \quad \square \end{aligned}$$

For the second sum, we have the following lemma, which is proved in Appendix C.

Lemma 3.4. For $b > 2$,

$$\sum_{i=1}^n \{\gamma_i^2 + \gamma_{n+1-i}^2\} = \frac{b\gamma_{n+1}}{\sqrt{b^2-4}} \left(\gamma_{\frac{n+1}{2}}^2 + 2 \right) - \gamma_{n+1}^2 - 4(n+1).$$

The following theorem gives an explicit formula for $\text{Tr}(\tilde{T}_n^{-1})$, which is proved using Lemmas 3.3 and 3.4:

Theorem 3.5. (Trace) Let \tilde{b} be such that, for $b > 2$, \tilde{T}_n is nonsingular. Then,

$$\text{Tr}(\tilde{T}_n^{-1}) = K_1 \left(\frac{n+1}{\sqrt{b^2-4}} \frac{\gamma_{n+1}^2 + 2}{\gamma_{n+1}} - \frac{b}{b^2-4} \right) - K_2 \left(\frac{b}{\sqrt{b^2-4}} \left(\gamma_{n+1}^2 + 2 \right) - \gamma_{n+1} - \frac{4(n+1)}{\gamma_{n+1}} \right),$$

where $K_1 = \frac{[\gamma_{n+1} + (\tilde{b}-b)\gamma_n]^2 + (\tilde{b}-b)^2(b^2-4)}{[\gamma_{n+1} + (\tilde{b}-b)\gamma_n]^2 - (\tilde{b}-b)^2(b^2-4)}$ and $K_2 = \frac{(\tilde{b}-b)[\gamma_{n+1} + (\tilde{b}-b)\gamma_n]}{[\gamma_{n+1} + (\tilde{b}-b)\gamma_n]^2 - (\tilde{b}-b)^2(b^2-4)}$.

3.2 Rowsums of the inverse

The general formula for the rowsum of the inverse of \tilde{T}_n is given by (1.11). We need to evaluate each term in the right-hand side of equation (1.11).

For the first term of the right-hand side of (1.11), with $t_{i,j}^{-1}$ given in (2.1),

$$\begin{aligned} \sum_{j=1}^n t_{i,j}^{-1} &= \sum_{j=1}^n t_{i,j}^{-1} + \sum_{j=i+1}^n t_{i,j}^{-1} = \sum_{j=1}^n t_{i,j}^{-1} + \sum_{j=i+1}^n t_{j,i}^{-1} \\ &= \frac{\gamma_{n+1-i}}{\sqrt{b^2-4}} \sum_{j=1}^i \gamma_j + \frac{\gamma_i}{\sqrt{b^2-4} \gamma_{n+1}} \sum_{k=1}^{n-i} \gamma_k \\ &= \frac{\gamma_{n+1-i}}{\sqrt{b^2-4} \gamma_{n+1}} \left(\frac{1}{b-2} (\gamma_{i+1} - \gamma_i - \gamma_1) \right) + \frac{\gamma_i}{\sqrt{b^2-4} \gamma_{n+1}} \left(\frac{1}{b-2} (\gamma_{n+1-i} - \gamma_{n-i} - \gamma_1) \right) \\ &= \frac{1}{(b-2)\sqrt{b^2-4} \gamma_{n+1}} (\gamma_{n+1-i} \gamma_{i+1} - \gamma_1 \gamma_{n+1-i} - \gamma_i \gamma_{n-i} - \gamma_1 \gamma_i) \\ &= \frac{\gamma_{n+1} - \gamma_{n+1-i} - \gamma_i}{(b-2)\gamma_{n+1}}, \end{aligned} \quad (3.4)$$

after the use of Lemmas 2.2 and 2.1(iii).

For the second term of the right-hand side of (1.11), with $t_{n,1}^{-1} = \gamma_1/\gamma_{n+1}$ and $t_{n,n}^{-1} = \gamma_n/\gamma_{n+1}$, we have

$$m_{11} + m_{12} = 1 + (\tilde{b}-b) \frac{\gamma_1 + \gamma_n}{\gamma_{n+1}}. \quad (3.5)$$

Next, with $t_{i,1}^{-1} = \gamma_{n+1-i}/\gamma_{n+1}$ and $t_{i,n}^{-1} = t_{n,i}^{-1} = \gamma_i/\gamma_{n+1}$, we obtain

$$t_{i,1}^{-1} + t_{i,n}^{-1} = \frac{\gamma_{n+1-i} + \gamma_i}{\gamma_{n+1}}. \quad (3.6)$$

Finally,

$$\sum_{j=1}^n t_{n,j}^{-1} = \sum_{j=1}^n \frac{\gamma_j \gamma_1}{\sqrt{b^2-4} \gamma_{n+1}} = \frac{1}{\gamma_{n+1}} \sum_{j=1}^n \gamma_j = \frac{1}{(b-2)\gamma_{n+1}} (\gamma_{n+1} - \gamma_n - \gamma_1), \quad (3.7)$$

due to Lemma 2.2. Substitution of (3.4)–(3.7) to equation (1.11) gives

$$\begin{aligned} R(i) &:= \sum_{j=1}^n \tilde{t}_{i,j}^{-1} = \underbrace{\frac{\gamma_{n+1} - \gamma_{n+1-i} - \gamma_i}{(b-2)\gamma_{n+1}}}_{R_1(i)} + \underbrace{\frac{(b-\tilde{b})(\gamma_{n+1-i} + \gamma_i)(\gamma_{n+1} - \gamma_n - \gamma_1)}{(b-2)\gamma_{n+1}(\gamma_{n+1} + (\tilde{b}-b)(\gamma_1 + \gamma_n))}}_{R_2(i)} \\ &= \frac{1}{b-2} + \frac{b-\tilde{b}-1}{b-2} \frac{\gamma_i + \gamma_{n+1-i}}{\gamma_{n+1} + (\tilde{b}-b)(\gamma_n + \gamma_1)}. \end{aligned} \quad (3.8)$$

We begin with the positivity of the inverse matrix \tilde{T}_n when $\tilde{b} \geq 1$, summarized in the following lemma.

Lemma 3.6. (Positivity) Let $b > 2$, $\tilde{b} \geq 1$, and $n \geq 3$. The inverse matrix \tilde{T}_n^{-1} is positive.

Proof. Note that, by Lemma 3.1, the matrix \tilde{T}_n is nonsingular. Consider the decomposition $\tilde{T}_n = S_n + \tilde{D}_n$, where S_n is defined by \tilde{T}_n in (1.4) with $b = 2$, $\tilde{b} = 1$, and $\tilde{D}_n = \text{diag}(\tilde{b} - 1, b - 2, \dots, b - 2, \tilde{b} - 1)$. For any $x \in \mathbb{R}^n$, we have

$$x^T \tilde{T}_n x = x^T S_n x + x^T \tilde{D}_n x = x^T S_n x + (\tilde{b} - 1)x_1^2 + (b - 2) \sum_{k=2}^{n-1} x_k^2 + (\tilde{b} - 1)x_n^2 > 0,$$

because S_n is symmetric positive semidefinite, $\tilde{b} - 1 > 0$, and $b - 2 > 0$. Therefore, \tilde{T}_n is a Stieltjes matrix, whose inverse is positive. \square

Lemma 3.7. Let $b > 2$ and $\tilde{b} \geq 1$. For the rowsums of \tilde{T}_n^{-1} , the following holds:

- (i) if $\tilde{b} > b - 1$, then the maximum of rowsums is attained at $i = (n + 1)/2$ and the minimum is attained at $i = 1$ and $i = n$;
- (ii) if $\tilde{b} = b - 1$, then the rowsums are constant;
- (iii) if $b - \frac{\gamma_{n+1}}{\gamma_n + \gamma_1} < \tilde{b} < b - 1$, then the maximum of rowsums is attained at $i = 1$ and $i = n$ and the minimum is attained at $i = (n + 1)/2$; and
- (iv) if $\tilde{b} < b - \frac{\gamma_{n+1}}{\gamma_n + \gamma_1}$, then the maximum of rowsums is attained at $i = (n + 1)/2$ and the minimum is attained at $i = 1$ and $i = n$;

Proof. Note that \tilde{T}_n and, hence, \tilde{T}_n^{-1} are centrosymmetric. Thus, the rowsum $R(i)$ is symmetric along $i = (n + 1)/2$. Differentiating $R(i)$ gives $R'(i) = R'_1(i) + R'_2(i)$, where

$$R'_1(i) = \frac{1}{(b - 2)\gamma_{n+1}} ((r_1^{n+1-i} - r_1^i) \ln r_1 - (r_2^{n+1-i} - r_2^i) \ln r_2),$$

$$R'_2(i) = \frac{(\tilde{b} - b)(\gamma_{n+1} - \gamma_n - \gamma_1)}{(b - 2)\gamma_{n+1}(\gamma_{n+1} + (\tilde{b} - b)(\gamma_1 + \gamma_n))} ((r_1^{n+1-i} - r_1^i) \ln r_1 - (r_2^{n+1-i} - r_2^i) \ln r_2).$$

Thus,

$$R'(i) = \frac{1 + \tilde{b} - b}{(b - 2)\gamma_{n+1} + (\tilde{b} - b)(\gamma_1 + \gamma_n)} ((r_1^{n+1-i} - r_1^i) \ln r_1 - (r_2^{n+1-i} - r_2^i) \ln r_2).$$

The aforementioned derivative vanishes if (i) $\tilde{b} = b - 1$, $\forall i$, implying that $R(i)$ is constant, or (ii) $i = (n + 1)/2$, the critical point of $R(i)$. The type of the critical point can be evaluated using the second derivative of R :

$$R''(i) = - \frac{1 + \tilde{b} - b}{(b - 2)\gamma_{n+1} + (\tilde{b} - b)(\gamma_1 + \gamma_n)} \left[\ln^2 r_2 \left((r_1^{n+1-i} + r_1^i) \frac{\ln^2 r_1}{\ln^2 r_2} - (r_2^{n+1-i} + r_2^i) \right) \right].$$

The bracketed term is always positive. So the sign of $R''((n + 1)/2)$ depends on \tilde{b} , which can be determined via standard algebra and leads to the results stated in the lemma. \square

Theorem 3.8. Let \tilde{T}_n be as in (1.4), with $b > 2$ and $\tilde{b} \geq 1$. Then, the rowsums of the inverse matrix \tilde{T}_n^{-1} satisfy the following:

- (i) for $\tilde{b} > b - 1$,

$$\frac{1}{\tilde{b} - 1} < R(i) < \frac{1}{b - 2} \left[1 - \frac{2(1 + \tilde{b} - b)(b - 1)}{\tilde{b}} \frac{\gamma_{\frac{n+1}{2}}}{\gamma_{n+1}} \right];$$

- (ii) for $\tilde{b} = b - 1$,

$$R(i) = \frac{1}{b - 2};$$

(iii) for $\frac{2}{b} < \tilde{b} < b - 1$,

$$\frac{1}{b-2} \left(1 + \frac{2(b-\tilde{b}-1)(b-1)}{\tilde{b}} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1}} \right) < R(i) < \frac{b+1}{\tilde{b}b-2};$$

(iv) for $\tilde{b} < r_2$,

$$\frac{r_1-1}{(b-2)(\tilde{b}-r_2)} < R(i) < \frac{1}{b-2} \left(1 + \frac{2(b-\tilde{b}-1)(b-1)}{\tilde{b}-1} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1}} \right).$$

Proof. From the formula of the rowsum (3.8),

$$R(1) = \frac{\mathcal{V}_{n+1} - \mathcal{V}_n - \mathcal{V}_1}{(b-2)(\mathcal{V}_{n+1} + (\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1))}, \quad (3.9)$$

$$R\left(\frac{n+1}{2}\right) = \frac{1}{b-2} - \frac{2(1+\tilde{b}-b)}{b-2} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1} + (\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)}. \quad (3.10)$$

Setting $\tilde{b} = b - 1$, we have $R(1) = 1/(b-2)$. By Lemma 3.7(ii), $R(i) = R(1)$, proving Part (ii).

For Part (i), with $\tilde{b} > b - 1$,

$$R(1) = \frac{1}{(b-2) \left(1 + \frac{(1+\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)}{\mathcal{V}_{n+1} - \mathcal{V}_n - \mathcal{V}_1} \right)} > \frac{1}{(b-2) \left(1 + \frac{1+\tilde{b}-b}{b-2} \right)} = \frac{1}{\tilde{b}-1},$$

after using Lemma 2.3. For (3.10), note that

$$\begin{aligned} 1 + \frac{(\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)}{\mathcal{V}_{n+1}} &= 1 - \frac{\mathcal{V}_n + \mathcal{V}_1}{\mathcal{V}_{n+1}} + \frac{(1+\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)}{\mathcal{V}_{n+1}} \\ &< 1 - \frac{1}{b} + \frac{(1+\tilde{b}-b)}{b-1} = -\frac{1}{b} + \frac{\tilde{b}}{b-1} < \frac{\tilde{b}}{b-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} R\left(\frac{n+1}{2}\right) &= \frac{1}{b-2} - \frac{2(1+\tilde{b}-b)}{b-2} \frac{1}{1 + \frac{(\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)}{\mathcal{V}_{n+1}}} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1}} \\ &< \frac{1}{b-2} - \frac{2(1+\tilde{b}-b)}{b-2} \frac{1}{\tilde{b}/(b-1)} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1}} = \frac{1}{b-2} \left(1 - \frac{2(1+\tilde{b}-b)(b-1)}{\tilde{b}} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1}} \right). \end{aligned}$$

Combining the two inequalities and Lemma 3.7 leads to Part (i).

For Part (iii), due to Lemma 3.7, with $1 + \tilde{b} - b < 0$, and using inequalities we have, we can see that

$$1 + \frac{(1+\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)}{\mathcal{V}_{n+1} - \mathcal{V}_n - \mathcal{V}_1} > 1 + \frac{1+\tilde{b}-b}{b-1-\frac{2}{b}} = \frac{\tilde{b}-\frac{2}{b}}{b-1-\frac{2}{b}}$$

and since $1 + \frac{(1+\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)}{\mathcal{V}_{n+1} - \mathcal{V}_n - \mathcal{V}_1} > 0$, for $\frac{2}{b} < \tilde{b} < b - 1$, we obtain the following upper bound:

$$R(i) \leq R(1) = \frac{1}{b-2} \frac{1}{1 + \frac{(1+\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)}{\mathcal{V}_{n+1} - \mathcal{V}_n - \mathcal{V}_1}} < \frac{b-1-\frac{2}{b}}{(b-2)(\tilde{b}-\frac{2}{b})} = \frac{b^2-b-2}{(b-2)(\tilde{b}b-2)} = \frac{b+1}{\tilde{b}b-2}.$$

For the lower bound,

$$\begin{aligned}
R(i) &\geq R\left(\frac{n+1}{2}\right) = \frac{1}{b-2} - \frac{2(1+\tilde{b}-b)}{b-2} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1} + (\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)} \\
&= \frac{1}{b-2} + \frac{2(b-\tilde{b}-1)}{b-2} \frac{1}{1 + \frac{(1+\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)}{\mathcal{V}_{n+1} - \mathcal{V}_n - \mathcal{V}_1}} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1} - \mathcal{V}_n - \mathcal{V}_1} \\
&> \frac{1}{b-2} + \frac{2(b-\tilde{b}-1)}{b-2} \frac{1}{1 + (1+\tilde{b}-b)\frac{1}{b-1}} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1} - \mathcal{V}_n - \mathcal{V}_1} \\
&= \frac{1}{b-2} + \frac{2(b-\tilde{b}-1)(b-1)}{(b-2)\tilde{b}} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1}} \frac{\mathcal{V}_n + \mathcal{V}_1}{\mathcal{V}_n + \mathcal{V}_1} \frac{\mathcal{V}_n + \mathcal{V}_1}{\mathcal{V}_{n+1} - \mathcal{V}_n - \mathcal{V}_1} \\
&> \frac{1}{b-2} \left(1 + \frac{2(b-\tilde{b}-1)(b-1)}{\tilde{b}} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1}} \right),
\end{aligned}$$

after applying Lemma 2.3.

For Part (iv), first, note that

$$\frac{\mathcal{V}_{n+1}}{\mathcal{V}_n + \mathcal{V}_1} + \tilde{b} - b < r_1 + \tilde{b} - b = \tilde{b} - r_2.$$

Since $\mathcal{V}_{n+1} + (\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1) < 0$, for all $\tilde{b} < r_2$, we have

$$R(1) = \frac{1}{b-2} \frac{\mathcal{V}_{n+1} - \mathcal{V}_n - \mathcal{V}_1}{\mathcal{V}_n + \mathcal{V}_1} \frac{1}{\frac{\mathcal{V}_{n+1}}{\mathcal{V}_n + \mathcal{V}_1} + \tilde{b} - b} > \frac{1}{(b-2)(\tilde{b} - r_2)} \frac{\mathcal{V}_{n+1} - \mathcal{V}_n - \mathcal{V}_1}{\mathcal{V}_n + \mathcal{V}_1} > \frac{r_1 - 1}{(b-2)(\tilde{b} - r_2)}.$$

Furthermore,

$$1 + \frac{(\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)}{\mathcal{V}_{n+1}} > 1 + \frac{\tilde{b}-b}{b-1} = \frac{\tilde{b}-1}{b-1}.$$

Therefore,

$$\begin{aligned}
R(i) &\leq R\left(\frac{n+1}{2}\right) = \frac{1}{b-2} + \frac{2(b-1-\tilde{b})}{b-2} \cdot \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1}} \frac{1}{1 + \frac{(\tilde{b}-b)(\mathcal{V}_n + \mathcal{V}_1)}{\mathcal{V}_{n+1}}} \\
&< \frac{1}{b-2} + \frac{2(b-1-\tilde{b})(b-1)}{(b-2)(\tilde{b}-1)} \frac{\mathcal{V}_{\frac{n+1}{2}}}{\mathcal{V}_{n+1}}.
\end{aligned}$$

This completes the proof. \square

For $b - \frac{\mathcal{V}_{n+1}}{\mathcal{V}_n + \mathcal{V}_1} < \tilde{b} \leq \frac{2}{b}$, we, however, cannot obtain a meaningful upper bound; the derived upper bound grows faster toward infinity and does not capture the growth of the exact trace of the inverse toward infinity as \tilde{b} approaches either end of this interval. Similarly, we at the moment are not able to derive a meaningful lower bound for the trace of the inverse when $r_2 \leq \tilde{b} < b - \frac{\mathcal{V}_{n+1}}{\mathcal{V}_n + \mathcal{V}_1}$.

3.3 Norms of the inverse matrix

Since $\tilde{T}_n^{-1} \geq 0$ for $\tilde{b} \geq 1$, we have $|\tilde{t}_{i,j}^{-1}| = \tilde{t}_{i,j}^{-1}$. Hence, $\sum_{j=1}^n |\tilde{t}_{i,j}^{-1}| = \sum_{j=1}^n \tilde{t}_{i,j}^{-1}$, and the ∞ -norm of the inverse matrix can be bounded from above using the same upper bound as for the rowsum. Thus,

Theorem 3.9. Let $b > 2$ and $\tilde{b} \geq 1$ such that \tilde{T}_n is nonsingular. Then,

$$\|\tilde{T}_n^{-1}\|_\infty \leq \begin{cases} \frac{1}{b-2} \left(1 - \frac{2(1+\tilde{b}-b)(b-1)}{\tilde{b}} \frac{\gamma_{n+1}^{\frac{n+1}{2}}}{\gamma_{n+1}} \right), & \text{for } \tilde{b} > b-1, \\ \frac{1}{b-2}, & \text{for } \tilde{b} = b-1, \\ \frac{b+1}{\tilde{b}b-2}, & \text{for } 1 \leq \tilde{b} < b-1. \end{cases}$$

For $\tilde{b} < 1$, an upper bound for the ∞ -norm of \tilde{T}_n^{-1} is given in the following theorem:

Theorem 3.10. Let $\tilde{b} < 1$ be such that, for $b > 2$, the matrix \tilde{T}_n is nonsingular. Then, the following inequality holds:

$$\|\tilde{T}_n^{-1}\|_\infty \leq \frac{\gamma_{n+1} - 2\gamma_{\frac{n+1}{2}}}{(b-2)\gamma_{n+1}} + \frac{|K|}{b-2}(\gamma_n + \gamma_1)(\gamma_{n+1} - \gamma_n - \gamma_1) + \frac{|K\beta|}{b-2}\gamma_{n-1}\gamma_{n+1}, \quad (3.11)$$

where $K = \frac{\beta}{(\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2}$ and $\beta = \tilde{b} - b$.

Proof. We can rewrite (A2) in Appendix B as follows for $i \geq j$:

$$\tilde{t}_{i,j}^{-1} = \frac{\gamma_j \gamma_{n+1-i}}{\gamma_1 \gamma_{n+1}} - K(\gamma_{n+1-i} \gamma_{n+1-j} + \gamma_i \gamma_j) - K\beta(\gamma_{n+1-i} \gamma_{n-j} + \gamma_i \gamma_{j-1}),$$

where $K = \frac{\beta}{(\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2}$. Therefore, we can bound its absolute value as

$$|\tilde{t}_{i,j}^{-1}| \leq \frac{\gamma_j \gamma_{n+1-i}}{\gamma_1 \gamma_{n+1}} + |K|(\gamma_{n+1-i} \gamma_{n+1-j} + \gamma_i \gamma_j) + |K\beta|(\gamma_{n+1-i} \gamma_{n-j} + \gamma_i \gamma_{j-1}).$$

For $j \geq i + 1$, we have

$$|\tilde{t}_{i,j}^{-1}| = |\tilde{t}_{j,i}^{-1}| \leq \frac{\gamma_i \gamma_{n+1-j}}{\gamma_1 \gamma_{n+1}} + |K|(\gamma_{n+1-i} \gamma_{n+1-j} + \gamma_i \gamma_j) + |K\beta|(\gamma_{n+1-j} \gamma_{n-i} + \gamma_j \gamma_{i-1}).$$

First, note that

$$\begin{aligned} \sum_{j=1}^n \{\gamma_{n+1-i} \gamma_{n+1-j} + \gamma_i \gamma_j\} &= \gamma_{n+1-i} \sum_{j=1}^n \gamma_{n+1-j} + \gamma_i \sum_{j=1}^n \gamma_j = (\gamma_{n+1-i} + \gamma_i) \sum_{j=1}^n \gamma_j \\ &= \frac{1}{b-2} (\gamma_{n+1-i} + \gamma_i) (\gamma_{n+1} - \gamma_n - \gamma_1) \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^n \{\gamma_{n+1-j} \gamma_{n-i} + \gamma_j \gamma_{i-1}\} &= \sum_{j=1}^i \{\gamma_{n+1-i} \gamma_{n-j} + \gamma_j \gamma_{j-1}\} + \sum_{j=i+1}^n \{\gamma_{n+1-j} \gamma_{n-i} + \gamma_j \gamma_{i-1}\} \\ &= \gamma_{n+1-i} \sum_{j=1}^i \gamma_{n-j} + \gamma_{i-1} \sum_{j=1}^i \gamma_{j-1} + \gamma_{n-i} \sum_{j=i+1}^n \gamma_{n+1-j} + \gamma_{i-1} \sum_{j=i+1}^n \gamma_j \\ &= \frac{\gamma_{n+1-i}}{b-2} (\gamma_n - \gamma_{n-1} - \gamma_{n-i} + \gamma_{n-i-1}) + \frac{\gamma_{i-1}}{b-2} (\gamma_i - \gamma_{i-1} - \gamma_1) \\ &\quad + \frac{\gamma_{n-i}}{b-2} (\gamma_{n+1-i} - \gamma_{n-i} - \gamma_1) + \frac{\gamma_{i-1}}{b-2} (\gamma_{n+1} - \gamma_n - \gamma_{i+1} + \gamma_i) \\ &= \frac{1}{b-2} (\gamma_{n+1} \gamma_{n-i} - \gamma_{n-1} \gamma_{n+1-i} + \gamma_{n+1} \gamma_{i-1}). \end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^n |\tilde{t}_{i,j}^{-1}| &\leq \sum_{j=1}^n t_{i,j}^{-1} + |K| \sum_{j=1}^n \{\gamma_{n+1-i} \gamma_{n+1-j} + \gamma_i \gamma_j\} + |K\beta| \sum_{j=1}^n \{\gamma_{n+1-i} \gamma_{n-j} + \gamma_i \gamma_{j-1}\}, \\
&\leq \frac{\gamma_{n+1} - \gamma_{n+1-i} - \gamma_i}{(b-2)\gamma_{n+1}} + \frac{|K|}{b-2} (\gamma_{n+1-i} + \gamma_i)(\gamma_{n+1} - \gamma_n - \gamma_1) \\
&\quad + \frac{|K\beta|}{b-2} (\gamma_{n+1} \gamma_{n-i} - \gamma_{n-1} \gamma_{n+1-i} + \gamma_{n+1} \gamma_{i-1}).
\end{aligned} \tag{3.12}$$

Now, note that for the first term on the right-hand side, with $\ln r_2 = -\ln r_1$,

$$\begin{aligned}
(\gamma_{n+1} - \gamma_{n+1-i} - \gamma_i)' &= (r_1^{n+1-i} - r_1^i) \ln r_1 + (r_2^i - r_2^{n+1-i}) \ln r_2 \\
&= (r_1^{n+1-i} + r_2^{n+1-i} - r_1^i - r_2^i) \ln r_1 \\
&= \gamma_{\frac{n+1-2i}{2}} \gamma_{\frac{n+1}{2}} \ln r_1.
\end{aligned}$$

The critical point corresponds to $\gamma_{\frac{n+1-2i}{2}} = 0$, which is attained at $i = \frac{n+1}{2}$ (i.e., $\gamma_0 = 0$). Thus,

$$\max_i \{\gamma_{n+1} - \gamma_{n+1-i} - \gamma_i\} = \gamma_{n+1} - 2\gamma_{\frac{n+1}{2}}.$$

Applying a similar analysis to $\gamma_{n+1-i} + \gamma_i$ in the second term of the right-hand side, we obtain

$$\max_i \{\gamma_{n+1-i} + \gamma_i\} \leq \gamma_n + \gamma_1.$$

For the last term, note that $\gamma_{n+1} \gamma_{n-i} - \gamma_{n-1} \gamma_{n+1-i} + \gamma_{n+1} \gamma_{i-1} < \gamma_{n+1} (\gamma_{n-i} + \gamma_{i-1})$, where

$$\max_i \{\gamma_{n-i} + \gamma_{i-1}\} \leq \gamma_{n-1} + \gamma_0 = \gamma_{n-1}.$$

Substituting the aforementioned bounds for the maximum to inequality (3.12) leads to the bound stated in the theorem. \square

To derive a sharper upper bound, consider the case where $i \geq j$. Using (A3), we express $|\tilde{t}_{i,j}^{-1}|$ as follows:

$$|\tilde{t}_{i,j}^{-1}| = \frac{|\gamma_j + \beta \gamma_{j-1}|}{\gamma_1} |\tilde{t}_{i,1}^{-1}|.$$

For the case when $j \geq i + 1$, the expression becomes

$$|\tilde{t}_{i,j}^{-1}| = \frac{|\gamma_{n+1-j} + \beta \gamma_{n-j}|}{\gamma_1} |\tilde{t}_{n,i}^{-1}|.$$

Combining these cases, the overall summation is represented by equation (3.13)

$$\sum_{j=1}^n |\tilde{t}_{i,j}^{-1}| = \sum_{j=1}^i |\tilde{t}_{i,j}^{-1}| + \sum_{j=i+1}^n |\tilde{t}_{i,j}^{-1}| = \frac{|\tilde{t}_{i,1}^{-1}|}{\gamma_1} \sum_{j=1}^i |\gamma_j + \beta \gamma_{j-1}| + \frac{|\tilde{t}_{n,i}^{-1}|}{\gamma_1} \sum_{j=i+1}^n |\gamma_{n+1-j} + \beta \gamma_{n-j}|. \tag{3.13}$$

This representation allows us to study the sign change of $\gamma_j + \beta \gamma_{j-1}$ and $\gamma_{n+1-j} + \beta \gamma_{n-j}$, providing a more accurate upper bound.

4 Trace, rowsum, and norms of the inverse matrix: $b < 2$

In this case, let $b_+ = -b$ and $\tilde{b}_+ = -\tilde{b}$, where $b < -2$, so $b_+ > 2$. We make the following observations:

$$\begin{aligned}
r_1 &= \frac{1}{2}(b + \sqrt{b^2 - 4}) = -\frac{1}{2}(b_+ - \sqrt{b_+^2 - 4}) =: -r_{2,+}, \\
r_2 &= \frac{1}{2}(b - \sqrt{b^2 - 4}) = -\frac{1}{2}(b_+ + \sqrt{b_+^2 - 4}) =: -r_{1,+}.
\end{aligned}$$

Furthermore,

$$\begin{aligned} \gamma_k &= r_1^k - r_2^k = (-1)^{k+1}(r_{1,+}^k - r_{2,+}^k) = (-1)^{k+1}\gamma_{k,+}, \\ \beta &= \tilde{b} - b = -\tilde{b}_+ + b_+ = -\beta_+. \end{aligned}$$

Using the aforementioned relations, entries of T_n^{-1} with $b < -2$ can be related to the $b > 2$ case in the following way: for $i \geq j$,

$$t_{i,j}^{-1} = \frac{\gamma_j \gamma_{n+1-i}}{\gamma_1 \gamma_{n+1}} = \frac{(-1)^{j+1} \gamma_{j,+} (-1)^{n+2-i} \gamma_{n+1-i,+}}{(-1)^2 \gamma_{1,+} (-1)^{n+2} \gamma_{n+1,+}} = (-1)^{j+1-i} \frac{\gamma_{j,+} \gamma_{n+1-i,+}}{\gamma_{1,+} \gamma_{n+1,+}} = (-1)^{i-j-1} t_{i,j,+}^{-1},$$

where $t_{i,j,+}^{-1}$ is as in (2.1).

Now, denote the index of diagonals of T_n by $s_d = i - j$, where $s_d = 0$ and 1 , e.g., indicate, respectively, the main diagonal and one diagonal below the main diagonal, and so on. If s_d is odd, then $t_{i,j}^{-1} = t_{i,j,+}^{-1}$, and if s_d is even, then $t_{i,j}^{-1} = -t_{i,j,+}^{-1}$. Utilizing (A3) in Appendix B, for $i \geq j$,

$$\begin{aligned} \tilde{t}_{i,j}^{-1} &= C(\gamma_{n+1-i} + \beta \gamma_{n-i})(\gamma_j + \beta \gamma_{j-1}) \\ &= C[(-1)^{n+2-i} \gamma_{n+1-i,+} - (-1)^{n+1-i} \beta_+ \gamma_{n-i,+}] (-1)^{j+1} \gamma_{j,+} - (-1)^j \beta_+ \gamma_{j-1,+}] \\ &= C(-1)^{n+1-i+j} (\gamma_{n+1-i,+} + \beta_+ \gamma_{n-i,+})(\gamma_{j,+} + \beta_+ \gamma_{j-1,+}), \end{aligned}$$

where $C = \frac{\gamma_{n+1}}{\gamma_1(\gamma_{n+1} + \beta \gamma_n)^2 - (\beta \gamma_1)^2} = (-1)^n \frac{\gamma_{n+1,+}}{\gamma_{1,+}(\gamma_{n+1,+} + \beta_+ \gamma_{n,+})^2 - (\beta_+ \gamma_{1,+})^2}$. Using this, we have

$$\tilde{t}_{i,j}^{-1} = (-1)^{i-j-1} \tilde{t}_{i,j,+}^{-1}, \quad i \geq j,$$

where $\tilde{t}_{i,j,+}^{-1}$ is the (i, j) th element of the $\tilde{T}_{n,+}^{-1}$ matrix in (1.4) with $b = b_+$ and $\tilde{b} = -\tilde{b}_+$.

Lemma 4.1. (Nonsingularity) *Let \tilde{T}_n be as defined in (1.4), with $b < -2$. Then, \tilde{T}_n is nonsingular if $\tilde{b} = b - \frac{\gamma_{n+1}}{\gamma_n + \gamma_1} = \tilde{b}_1$ or $\tilde{b} = b - \frac{\gamma_{n+1}}{\gamma_n - \gamma_1} = \tilde{b}_2$.*

Proof. $t_{i,j}^{-1}$ in (2.1) is always defined for $b < -2$. So, the existence is $\tilde{t}_{i,j}^{-1}$ given by (1.9), which requires that $\Delta \neq 0$.

$$\Delta = (1 + \beta t_{n,n}^{-1})^2 - (\beta t_{n,1}^{-1})^2 = \left(1 + \beta \left(\frac{\gamma_n + \gamma_1}{\gamma_{n+1}}\right)\right) \left(1 + \beta \left(\frac{\gamma_n - \gamma_1}{\gamma_{n+1}}\right)\right) \neq 0.$$

From here, the result follows. \square

Theorem 4.2. (Trace) *Let \tilde{b} be such that, for $b < -2$, \tilde{T}_n is nonsingular. Then, the trace of the inverse of \tilde{T}_n is given by*

$$\text{Tr}(\tilde{T}_n^{-1}) = K_1 \left(\frac{n+1}{\sqrt{b^2-4}} \frac{\gamma_{\frac{n+1}{2}}^2 + 2}{\gamma_{n+1}} - \frac{b}{b^2-4} \right) - K_2 \left(\frac{b}{\sqrt{b^2-4}} \left(\gamma_{\frac{n+1}{2}}^2 + 2 \right) - \gamma_{n+1} - \frac{4(n+1)}{\gamma_{n+1}} \right),$$

where $K_1 = \frac{[\gamma_{n+1} + \beta \gamma_n]^2 + \beta^2(b^2-4)}{[\gamma_{n+1} + \beta \gamma_n]^2 - \beta^2(b^2-4)}$ and $K_2 = \frac{\beta[\gamma_{n+1} + \beta \gamma_n]}{[\gamma_{n+1} + \beta \gamma_n]^2 - \beta^2(b^2-4)}$ with $\beta = \tilde{b} - b$.

Proof. As we know for $i \geq j$, we have $\tilde{t}_{i,j}^{-1} = (-1)^{i-j-1} \tilde{t}_{i,j,+}^{-1}$. This implies

$$\text{Tr}(\tilde{T}_n^{-1}) = -\text{Tr}(\tilde{T}_{n,+}^{-1}),$$

where $\tilde{T}_{n,+}^{-1}$ with $b = b_+ > 2$ and $\tilde{b} = -\tilde{b}_+$. Using the trace formula from the case $b > 2$, we have

$$\text{Tr}(\tilde{T}_{n,+}^{-1}) = K_{1,+} \left(\frac{n+1}{\sqrt{b_+^2-4}} \frac{\gamma_{\frac{n+1}{2},+}^2 + 2}{\gamma_{n+1,+}} - \frac{b_+}{b_+^2-4} \right) - K_{2,+} \left(\frac{b_+(\gamma_{\frac{n+1}{2},+}^2 + 2)}{\sqrt{b_+^2-4}} - \gamma_{n+1,+} - \frac{4(n+1)}{\gamma_{n+1,+}} \right),$$

where $K_{1,+} = \frac{[\gamma_{n+1,+} + \beta_+ \gamma_{n,+}]^2 + \beta_+^2(b_+^2 - 4)}{[\gamma_{n+1,+} + \beta_+ \gamma_{n,+}]^2 - \beta_+^2(b_+^2 - 4)}$ and $K_{2,+} = \frac{\beta_+[\gamma_{n+1,+} + \beta_+ \gamma_{n,+}]}{[\gamma_{n+1,+} + \beta_+ \gamma_{n,+}]^2 - \beta_+^2(b_+^2 - 4)}$.

Now, we want to express this formula in terms of b , β , and γ 's. Note

$$\frac{\gamma_{\frac{n+1}{2},+}^2 + 2}{\gamma_{n+1,+}} = -\frac{r_{1,+}^{n+1} + r_{2,+}^{n+1}}{r_{1,+}^{n+1} - r_{2,+}^{n+1}} = -\frac{r_1^{n+1} + r_2^{n+1}}{r_1^{n+1} - r_2^{n+1}} = -\frac{\gamma_{\frac{n+1}{2}}^2 + 2}{\gamma_{n+1}}$$

and

$$\gamma_{\frac{n+1}{2},+}^2 + 2 = r_{1,+}^{n+1} + r_{2,+}^{n+1} = (-1)^{n+1}(r_1^{n+1} + r_2^{n+1}) = (-1)^{n+1} \left(\gamma_{\frac{n+1}{2}}^2 + 2 \right).$$

Therefore,

$$\begin{aligned} \frac{n+1}{\sqrt{b_+^2 - 4}} \frac{\gamma_{\frac{n+1}{2},+}^2 + 2}{\gamma_{n+1,+}} - \frac{b_+}{b_+^2 - 4} &= - \left(\frac{n+1}{\sqrt{b^2 - 4}} \frac{\gamma_{\frac{n+1}{2}}^2 + 2}{\gamma_{n+1}} - \frac{b}{b^2 - 4} \right), \\ \frac{b_+(\gamma_{\frac{n+1}{2},+}^2 + 2)}{\sqrt{b_+^2 - 4}} - \gamma_{n+1,+} - \frac{4(n+1)}{\gamma_{n+1,+}} &= (-1)^{n+2} \left(\frac{b(\gamma_{\frac{n+1}{2}}^2 + 2)}{\sqrt{b^2 - 4}} - \gamma_{n+1} - \frac{4(n+1)}{\gamma_{n+1}} \right). \end{aligned}$$

In the same way,

$$\begin{aligned} K_{1,+} &= \frac{[\gamma_{n+1} + \beta \gamma_n]^2 + \beta^2(b^2 - 4)}{[\gamma_{n+1} + \beta \gamma_n]^2 - \beta^2(b^2 - 4)} = K_1, \\ K_{2,+} &= \frac{(-1)^{n+3} \beta [\gamma_{n+1} + \beta \gamma_n]}{[\gamma_{n+1} + \beta \gamma_n]^2 - \beta^2(b^2 - 4)} = (-1)^{n+3} K_2. \end{aligned}$$

Combining all of this leads to the lemma. □

Since $r_1 r_2 = 1$ and $r_1 + r_2 = b$, we conclude that the rowsum of \tilde{T}_n^{-1} would be the same as in equation (3.8), so

Theorem 4.3. (Row sums) *Let $b < -2$ and \tilde{b} such that the matrix \tilde{T}_n is nonsingular. Then, the i th row sum $R(i)$ of the inverse matrix \tilde{T}_n^{-1} is given by*

$$R(i) = \frac{1}{b-2} - \frac{\beta+1}{b-2} \cdot \frac{\gamma_{n+1-i} + \gamma_i}{\gamma_{n+1} + \beta(\gamma_1 + \gamma_n)}, \quad (4.1)$$

where $\beta = \tilde{b} - b$.

Theorem 4.4. (Upper Bound 1) *Let $\tilde{b} \leq -1$ be such that, for $b < -2$, the matrix \tilde{T}_n is nonsingular. Then, the following inequality holds:*

$$\|\tilde{T}_n^{-1}\|_\infty \leq \begin{cases} \frac{1}{b+2} \left(\frac{2(1-\tilde{b}+b)(b+1)}{\tilde{b}} \frac{\gamma_{\frac{n+1}{2}}}{\gamma_{n+1}} - 1 \right), & \text{if } \tilde{b} < b+1, \\ \frac{1}{b+2}, & \text{if } \tilde{b} = b+1, \\ \frac{1-b}{\tilde{b}b-2}, & \text{if } b+1 < \tilde{b} \leq -1. \end{cases}$$

Proof. We observe that

$$\|\tilde{T}_n^{-1}\|_\infty = \max_i \sum_{j=1}^n |\tilde{t}_{i,j}^{-1}| = \max_i \sum_{j=1}^n |\tilde{t}_{i,j,+}^{-1}| = \|\tilde{T}_{n,+}^{-1}\|_\infty.$$

Using the relation

$$\begin{aligned} \frac{\gamma_{\frac{n+1}{2},+}^{n+1}}{\gamma_{n+1,+}} &= \frac{r_{1,+}^{\frac{n+1}{2}} - r_{2,+}^{\frac{n+1}{2}}}{(-1)^{n+2}\gamma_{n+1}} \\ &= \begin{cases} \frac{r_{2,+}^{\frac{n+1}{2}} - r_{1,+}^{\frac{n+1}{2}}}{(-1)^{n+2}\gamma_{n+1}}, & \text{if } n \text{ is odd,} \\ \frac{r_{1,+}^{\frac{n+1}{2}} - r_{2,+}^{\frac{n+1}{2}}}{(-1)^{n+2}\gamma_{n+1}}, & \text{if } n \text{ is even,} \end{cases} \\ &= (-1)^n \frac{r_{1,+}^{\frac{n+1}{2}} - r_{2,+}^{\frac{n+1}{2}}}{(-1)^{n+2}\gamma_{n+1}} \\ &= \frac{\gamma_{\frac{n+1}{2},+}^{n+1}}{\gamma_{n+1}}, \end{aligned}$$

we conclude the proof of the theorem. \square

Theorem 4.5. (Upper Bound 2) *Let $\tilde{b} > -1$ be such that, for $b < -2$, the matrix \tilde{T}_n is nonsingular. Then, the following inequality holds:*

$$\|\tilde{T}_n^{-1}\|_{\infty} \leq -\frac{\gamma_{n+1} - 2\gamma_{\frac{n+1}{2}}}{(b+2)\gamma_{n+1}} - \frac{|K|}{b+2} [(-1)^{n+1}\gamma_n + \gamma_1][(-1)^n(\gamma_{n+1} + \gamma_n) - \gamma_1] - \frac{|K\beta|}{b+2}\gamma_{n-1}\gamma_{n+1},$$

where $K = -\frac{\beta}{(\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2}$.

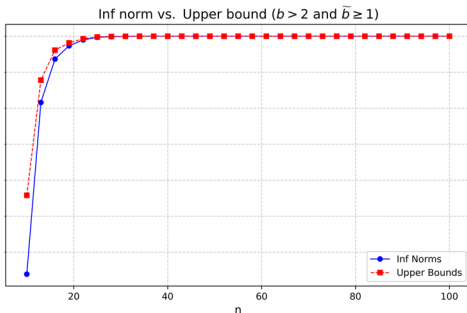
Proof. We begin by noting that $\|\tilde{T}_n^{-1}\|_{\infty} = \|\tilde{T}_{n,+}^{-1}\|_{\infty}$. Then, applying the inequality

$$\|\tilde{T}_{n,+}^{-1}\|_{\infty} \leq \frac{\gamma_{n+1,+} - 2\gamma_{\frac{n+1}{2},+}}{(b_+ - 2)\gamma_{n+1,+}} + \frac{|K_+|}{b_+ - 2}(\gamma_{n,+} + \gamma_{1,+})(\gamma_{n+1,+} - \gamma_{n,+} - \gamma_{1,+}) + \frac{|K_+\beta_+|}{b_+ - 2}\gamma_{n-1,+}\gamma_{n+1,+},$$

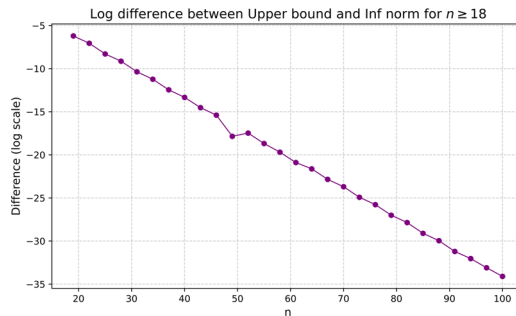
where $K_+ = \frac{\beta_+}{(\gamma_{n+1,+} + \beta_+\gamma_{n,+})^2 - (\beta_+\gamma_{1,+})^2}$, we make the substitutions

$$\begin{aligned} \beta_+ &= -\beta, \\ \gamma_{k,+} &= (-1)^{k+1}\gamma_k, \quad \text{for any } k \in \mathbb{Z}^+, \\ b_+ &= -b, \end{aligned}$$

to obtain the desired result. \square



(a)

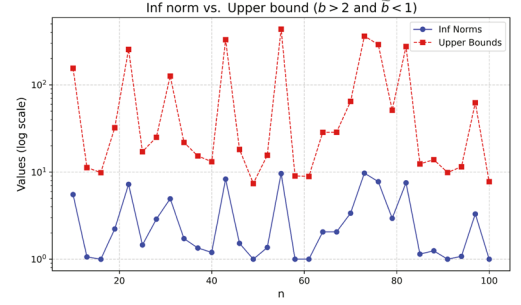


(b)

Figure 1: Comparison of $\|\tilde{T}_n^{-1}\|_{\infty}$ vs upper bounds when $b > 2$ and $\tilde{b} \geq 1$: (a) $\|\tilde{T}_n^{-1}\|_{\infty}$ and upper bounds and (b) log difference comparison.

n	\tilde{b}	$\ \tilde{T}_n^{-1}\ _\infty$	upper_bound
10	0.09	5.532	154.994
13	-1.14	1.062	11.274
16	-1.32	0.999	9.859
19	-0.35	2.222	32.339
22	0.16	7.238	254.557
25	-0.73	1.454	17.118
28	0.94	2.888	25.091

(a)

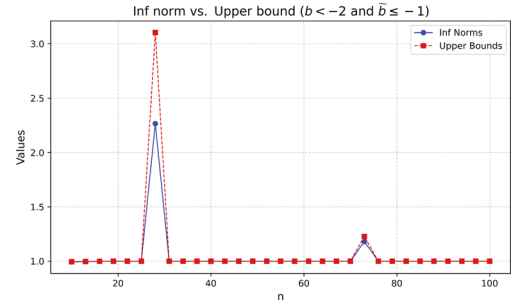


(b)

Figure 2: Comparison of $\|\tilde{T}_n^{-1}\|_\infty$ vs upper bounds when $b > 2$ and $\tilde{b} < 1$: (a) comparison of $\|\tilde{T}_n^{-1}\|_\infty$ and upper bounds and (b) $\|\tilde{T}_n^{-1}\|_\infty$ and upper bounds.

n	\tilde{b}	$\ \tilde{T}_n^{-1}\ _\infty$	upper_bound
10	-2.52	0.993	0.996
13	-4.57	0.996	0.997
16	-4.87	0.999	0.999
19	-3.24	0.999	0.999
22	-2.40	1.000	1.000
25	-3.88	1.000	1.000
28	-1.10	2.266	3.104

(a)

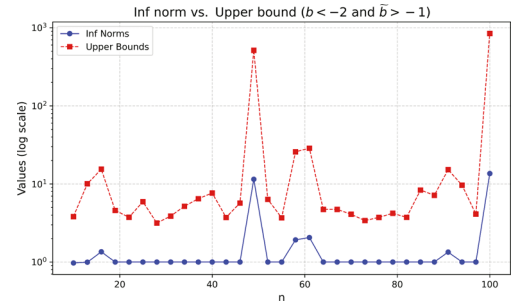


(b)

Figure 3: Comparison of $\|\tilde{T}_n^{-1}\|_\infty$ vs upper bounds when $b < -2$ and $\tilde{b} \leq -1$: (a) comparison of $\|\tilde{T}_n^{-1}\|_\infty$ and upper bounds and (b) $\|\tilde{T}_n^{-1}\|_\infty$ and upper bounds.

n	\tilde{b}	$\ \tilde{T}_n^{-1}\ _\infty$	upper_bound
10	4.57	0.973	3.826
13	1.29	0.995	10.071
16	0.81	1.352	15.470
19	3.41	1.000	4.578
22	4.76	1.000	3.754
25	2.38	1.000	5.926
28	6.85	1.000	3.161

(a)



(b)

Figure 4: Comparison of $\|\tilde{T}_n^{-1}\|_\infty$ vs upper bounds when $b < -2$ and $\tilde{b} > -1$: (a) comparison of $\|\tilde{T}_n^{-1}\|_\infty$ and upper bounds and (b) log comparison of values.

5 Numerical experiments

In this section, we present an analysis of upper bounds and compare the maximum observed convergence rates for the fixed-point iteration method applied to Fisher's problem. Specifically, we note consistently tight upper bounds in cases where $b > 2$ with $\tilde{b} \geq 1$ and when $b < -2$ with $\tilde{b} \leq -1$. In other cases, there exists potential for improvement in alternate upper bounds (Figures 1–4 and Tables 1, 2).

Table 1: Observed maximum convergence rate for Fisher's problem with $n = 20$, $b = \bar{b} = 4.0$, and $L = 2.0$

k	Iterations	Numerical rate	Expected rate
1/2	4	0.0024	0.0025
1	4	0.0048	0.005
2	4	0.0097	0.01
4	5	0.0196	0.02
8	6	0.0391	0.04
16	7	0.0789	0.08
32	9	0.1564	0.16

The expected rates are determined using upper bounds from equation (3.9) and Theorem 3.10 applied to the norms of the inverse matrix.

Table 2: Observed maximum convergence rate for Fisher's problem with $n = 50$, $b = \bar{b} = -4.0$, and $L = 1.0$

k	Iterations	Numerical rate	Expected rate
1	3	0.0001	0.0002
3	3	0.0003	0.0006
9	3	0.001	0.0018
27	4	0.0046	0.0054
81	5	0.0153	0.0162
273	6	0.0461	0.0486
729	8	0.137	0.1458

The expected rates are determined using upper bounds from Theorems 4.4 and 4.5 applied to the norms of the inverse matrix.

6 Conclusion

The objective of this work is to present the explicit or the exact formula of bounds for the norm of inverse of some tridiagonal near-Toeplitz matrices. In addition, we provide the value of the traces and row sums of the inverse matrices. We show bounds for $|b| > 2$. Theoretical results were validated by the numerical experiments. The findings suggest that these bounds could be useful for the fixed-point iterations in the convergence analysis. Future research outcomes should explore scenarios where the Toeplitz part of tridiagonal near-Toeplitz matrices is weakly dominant, specifically when $|b| = 2$.

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Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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Appendix

A Proof of Lemma 2.3

Using Lemma 2.1,

$$\frac{y_{n+1} - y_n - y_1}{y_n + y_1} = \frac{y_{n+1}}{y_n + y_1} - 1 = b - 1 - \frac{y_{n-1} + by_1}{y_n + y_1}.$$

Since $y_n < by_{n-1}$ (Lemma 2.1),

$$\frac{y_{n+1} - y_n - y_1}{y_n + y_1} < b - 1 - \frac{y_{n-1} + by_1}{by_{n-1} + y_1} = b - 1 - \frac{1}{b} \frac{y_{n-1} + by_1}{y_{n-1} + \frac{1}{b}y_1} < b - 1 - \frac{1}{b} < b - 1,$$

for $b > 2$. Now, let us have a look at

$$\begin{aligned} \frac{y_{n-1} + by_1}{y_n + y_1} &= \frac{y_{n-1} + y_2}{y_n + y_1} = \frac{r_1^{n-1} - r_2^{n-1} + r_1^2 - r_2^2}{r_1^n - r_2^n + r_1 - r_2} \\ &= \frac{\left(r_1^{\frac{n+1}{2}} - r_2^{\frac{n+1}{2}} \right) \left(r_1^{\frac{n-3}{2}} + r_2^{\frac{n-3}{2}} \right)}{\left(r_1^{\frac{n+1}{2}} - r_2^{\frac{n+1}{2}} \right) \left(r_1^{\frac{n-1}{2}} + r_2^{\frac{n-1}{2}} \right)} \\ &= \frac{r_1^{\frac{n-3}{2}} + r_2^{\frac{n-3}{2}}}{r_1^{\frac{n-1}{2}} + r_2^{\frac{n-1}{2}}}. \end{aligned}$$

Let us look at the difference

$$\begin{aligned} \frac{y_{n-1} + by_1}{y_n + y_1} - \frac{y_n + by_1}{y_{n+1} + y_1} &= \frac{r_1^{\frac{n-3}{2}} + r_2^{\frac{n-3}{2}}}{r_1^{\frac{n-1}{2}} + r_2^{\frac{n-1}{2}}} - \frac{r_1^{\frac{n-2}{2}} + r_2^{\frac{n-2}{2}}}{r_1^{\frac{n}{2}} + r_2^{\frac{n}{2}}} \\ &= \frac{r_1^{\frac{3}{2}} + r_2^{\frac{3}{2}} - r_1^{\frac{1}{2}} - r_2^{\frac{1}{2}}}{\left(r_1^{\frac{n-1}{2}} + r_2^{\frac{n-1}{2}} \right) \left(r_1^{\frac{n}{2}} + r_2^{\frac{n}{2}} \right)} \\ &= \frac{r_1^2 + r_2 - r_1 - 1}{r_1^{\frac{1}{2}} \left(r_1^{\frac{n-1}{2}} + r_2^{\frac{n-1}{2}} \right) \left(r_1^{\frac{n}{2}} + r_2^{\frac{n}{2}} \right)}. \end{aligned}$$

And note that

$$\begin{aligned} r_1^2 + r_2 - r_1 - 1 &= \frac{b^2 - 2 + b\sqrt{b^2 - 4}}{2} - \sqrt{b^2 - 4} - 1 \\ &= \frac{b^2 - 4}{2} + \frac{\sqrt{b^2 - 4}}{2}(b - 2) > 0. \end{aligned}$$

Therefore,

$$\frac{y_{n-1} + by_1}{y_n + y_1} > \frac{y_n + by_1}{y_{n+1} + y_1},$$

which means

$$\max_{n \geq 3} \frac{\gamma_{n-1} + b\gamma_1}{\gamma_n + \gamma_1} = \frac{2\gamma_2}{\gamma_3 + \gamma_1} = \frac{2\gamma_2}{b\gamma_2} = \frac{2}{b}.$$

Also, we note that

$$\frac{\gamma_{n-1} + b\gamma_1}{\gamma_n + \gamma_1} = \frac{r_1^{\frac{n-3}{2}} + r_2^{\frac{n-3}{2}}}{r_1^{\frac{n-1}{2}} + r_2^{\frac{n-1}{2}}} > \frac{r_1^{\frac{n-3}{2}} + r_2^{\frac{n-3}{2}}}{r_1 \left(r_1^{\frac{n-3}{2}} + r_2^{\frac{n-3}{2}} \right)} = \frac{1}{r_1} = r_2.$$

Therefore,

$$\frac{\gamma_{n+1} - \gamma_n - \gamma_1}{\gamma_n + \gamma_1} = b - 1 - \frac{\gamma_{n-1} + b\gamma_1}{\gamma_n + \gamma_1} \geq b - 1 - \frac{2}{b}$$

and

$$\frac{\gamma_{n+1} - \gamma_n - \gamma_1}{\gamma_n + \gamma_1} < b - 1 - r_2 = r_1 - 1.$$

Putting all together, we have

$$b - 2 < b - 1 - \frac{2}{b} \leq \frac{\gamma_{n+1} - \gamma_n - \gamma_1}{\gamma_n + \gamma_1} < r_1 - 1 < b - 1,$$

which can be simplified to obtain the lemma.

B Proof of Lemma 3.2

As we know for $i \geq j$ from (1.9)

$$\tilde{t}_{i,j}^{-1} = t_{i,j}^{-1} - \frac{\beta}{\Delta} [t_{i,1}^{-1}(m_{1,1}t_{1,j}^{-1} - m_{1,2}t_{n,j}^{-1}) + t_{i,n}^{-1}(-m_{1,2}t_{1,j}^{-1} + m_{1,1}t_{n,j}^{-1})],$$

where $t_{i,j}^{-1} = \frac{\gamma_j \gamma_{n+1-i}}{\gamma_i \gamma_{n+1}}$ for $i \geq j$ and $\beta = \tilde{b} - b$ with

$$m_{1,1} = 1 + \beta t_{n,n}^{-1} = 1 + \frac{\beta \gamma_n}{\gamma_{n+1}},$$

$$m_{1,2} = \beta t_{n,1}^{-1} = \frac{\beta \gamma_1}{\gamma_{n+1}}.$$

Now, we will find each piece of $\tilde{t}_{i,j}^{-1}$ and try to write it in a more compact way:

$$\begin{aligned} m_{1,1}t_{1,j}^{-1} - m_{1,2}t_{n,j}^{-1} &= \left(1 + \frac{\beta \gamma_n}{\gamma_{n+1}} \right) \frac{\gamma_{n+1-j}}{\gamma_{n+1}} - \frac{\beta \gamma_1}{\gamma_{n+1}} \frac{\gamma_j}{\gamma_{n+1}} \\ &= \frac{\gamma_{n+1-j}}{\gamma_{n+1}} + \frac{\beta}{\gamma_{n+1}^2} (\gamma_n \gamma_{n+1-j} - \gamma_1 \gamma_j) \\ &= \frac{\gamma_{n+1-j}}{\gamma_{n+1}} + \frac{\beta \gamma_{n-j}}{\gamma_{n+1}}. \end{aligned}$$

The last equality due to

$$\begin{aligned} \gamma_n \gamma_{n+1-j} - \gamma_1 \gamma_j &= r_1^{2n+1-j} - r_1^{j-1} - r_2^{j-1} + r_2^{2n+1-j} - r_1^{j+1} + r_1^{j-1} + r_2^{j-1} - r_2^{j+1} \\ &= r_1^{2n+1-j} - r_1^{j+1} - r_2^{j+1} + r_2^{2n+1-j} \\ &= \gamma_{n+1} \gamma_{n-j}. \end{aligned} \tag{A1}$$

In the same way, we obtain

$$\begin{aligned} -m_{1,2}t_{1,j}^{-1} + m_{1,1}t_{n,j}^{-1} &= -\frac{\beta\gamma_1}{\gamma_{n+1}} \frac{\gamma_{n+1-j}}{\gamma_{n+1}} + \left(1 + \frac{\beta\gamma_n}{\gamma_{n+1}}\right) \frac{\gamma_j}{\gamma_{n+1}} \\ &= \frac{\gamma_j}{\gamma_{n+1}} + \frac{\beta}{\gamma_{n+1}^2} (\gamma_n \gamma_j - \gamma_1 \gamma_{n+1-j}) \\ &= \frac{\gamma_j}{\gamma_{n+1}} + \frac{\beta\gamma_{j-1}}{\gamma_{n+1}}. \end{aligned}$$

The Δ part is

$$\begin{aligned} \Delta &= m_{1,1}^2 - m_{1,2}^2 = \left(1 + \frac{\beta(\gamma_n + \gamma_1)}{\gamma_{n+1}}\right) \left(1 + \frac{\beta(\gamma_n - \gamma_1)}{\gamma_{n+1}}\right) \\ &= \frac{(\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2}{\gamma_{n+1}^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{t}_{i,j}^{-1} &= \frac{\gamma_j \gamma_{n+1-i}}{\gamma_1 \gamma_{n+1}} - \frac{\beta}{\Delta \gamma_{n+1}^2} (\gamma_{n+1-i} (\gamma_{n+1-j} + \beta\gamma_{n-j}) + \gamma_i (\gamma_j + \beta\gamma_{j-1})) \\ &= \frac{\gamma_j \gamma_{n+1-i}}{\gamma_1 \gamma_{n+1}} - \beta \frac{\gamma_{n+1-i} (\gamma_{n+1-j} + \beta\gamma_{n-j}) + \gamma_i (\gamma_j + \beta\gamma_{j-1})}{(\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2}. \end{aligned} \quad (\text{A2})$$

Now, let us consider the following parts of the $\tilde{t}_{i,j}^{-1}$, so we can simply further

$$\begin{aligned} P_{i,j} &= \gamma_j ((\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2) - \beta\gamma_1 \gamma_{n+1} (\gamma_{n+1-j} + \beta\gamma_{n-j}) \quad [\text{using } \gamma_n^2 - \gamma_1^2 = \gamma_{n+1} \gamma_{n-1}] \\ &= \gamma_j \gamma_{n+1}^2 + \beta(2\gamma_j \gamma_n \gamma_{n+1} - \gamma_1 \gamma_{n+1} \gamma_{n+1-j}) + \beta^2 (\gamma_j \gamma_{n+1} \gamma_{n-1} - \gamma_1 \gamma_{n+1} \gamma_{n-j}) \\ &= \gamma_{n+1} [\gamma_j \gamma_{n+1} + \beta(2\gamma_j \gamma_n - \gamma_1 \gamma_{n+1-j}) + \beta^2 (\gamma_j \gamma_{n-1} - \gamma_1 \gamma_{n-j})] \\ &= \gamma_{n+1} [\gamma_j \gamma_{n+1} + \beta(\gamma_j \gamma_n + \gamma_{n+1} \gamma_{j-1}) + \beta^2 \gamma_n \gamma_{j-1}]. \end{aligned}$$

The last equality is due to the previously shown identity $\gamma_n \gamma_{n+1-j} - \gamma_1 \gamma_j = \gamma_{n+1} \gamma_{n-j}$ with $j \rightarrow n+1-j$, so, we obtain $\gamma_j \gamma_n - \gamma_1 \gamma_{n+1-j} = \gamma_{n+1} \gamma_{j-1}$ and substituting $n \rightarrow n-1$ to the new one, we obtain $\gamma_j \gamma_{n-1} - \gamma_1 \gamma_{n-j} = \gamma_n \gamma_{j-1}$.

$$\begin{aligned} Q_{i,j} &= \frac{\gamma_{n+1-i} \cdot P_{i,j}}{\gamma_{n+1}} - \beta\gamma_1 \gamma_i (\gamma_j + \beta\gamma_{j-1}) \\ &= \gamma_{n+1-i} [\gamma_j \gamma_{n+1} + \beta(\gamma_j \gamma_n + \gamma_{n+1} \gamma_{j-1}) + \beta^2 \gamma_n \gamma_{j-1}] - \beta\gamma_1 \gamma_i (\gamma_j + \beta\gamma_{j-1}) \\ &= \gamma_{n+1-i} \gamma_{n+1} (\gamma_j + \beta\gamma_{j-1}) + \beta\gamma_j (\gamma_{n+1-i} \gamma_n - \gamma_1 \gamma_i) + \beta^2 \gamma_{j-1} (\gamma_{n+1-i} \gamma_n - \gamma_1 \gamma_i) \\ &= \gamma_{n+1-i} \gamma_{n+1} (\gamma_j + \beta\gamma_{j-1}) + \beta\gamma_j \gamma_{n+1} \gamma_{n-i} + \beta^2 \gamma_{j-1} \gamma_{n+1} \gamma_{n-i} \quad [\text{by identity}] \\ &= \gamma_{n+1-i} \gamma_{n+1} (\gamma_j + \beta\gamma_{j-1}) + \beta\gamma_{n+1} \gamma_{n-i} (\gamma_j + \beta\gamma_{j-1}) \\ &= \gamma_{n+1} (\gamma_{n+1-i} + \beta\gamma_{n-i}) (\gamma_j + \beta\gamma_{j-1}). \end{aligned}$$

Therefore, for $i \geq j$,

$$\begin{aligned} \tilde{t}_{i,j}^{-1} &= \frac{\gamma_{n+1-i}}{\gamma_1 \gamma_{n+1} ((\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2)} P_{i,j} - \frac{\beta\gamma_i (\gamma_j + \beta\gamma_{j-1})}{(\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2} \\ &= \frac{Q_{i,j}}{\gamma_1 ((\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2)} \\ &= \frac{\gamma_{n+1}}{\gamma_1 ((\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2)} [\gamma_{n+1-i} + \beta\gamma_{n-i}] (\gamma_j + \beta\gamma_{j-1}) \\ &= C (\gamma_{n+1-i} + \beta\gamma_{n-i}) (\gamma_j + \beta\gamma_{j-1}), \end{aligned} \quad (\text{A3})$$

where $C := \frac{\gamma_{n+1}}{\gamma_1 ((\gamma_{n+1} + \beta\gamma_n)^2 - (\beta\gamma_1)^2)}$.

C Proof of Lemma 3.4

With $r_1 r_2 = 1$,

$$\begin{aligned}
\sum_{i=1}^n \{\gamma_i^2 + \gamma_{n+1-i}^2\} &= \sum_{i=1}^n \left\{ r_1^{2(n+1)} \left(\frac{1}{r_1^2} \right)^i + r_2^{2(n+1)} \left(\frac{1}{r_2^2} \right)^i + (r_1^2)^i + (r_2^2)^i - 4 \right\} \\
&= r_1^{2(n+1)} \left\{ \sum_{i=0}^n \left(\frac{1}{r_1^2} \right)^i - 1 \right\} + r_2^{2(n+1)} \left\{ \sum_{i=0}^n \left(\frac{1}{r_2^2} \right)^i - 1 \right\} + \sum_{i=0}^n (r_1^2)^i - 1 + \sum_{i=0}^n (r_2^2)^i - 1 - \sum_{i=1}^n 4 \\
&= r_1^{2(n+1)} \left[\frac{1 - \left(\frac{1}{r_1^2} \right)^{n+1}}{1 - \frac{1}{r_1^2}} - 1 \right] + r_2^{2(n+1)} \left[\frac{1 - \left(\frac{1}{r_2^2} \right)^{n+1}}{1 - \frac{1}{r_2^2}} - 1 \right] + \frac{1 - (r_1^2)^{n+1}}{1 - r_1^2} + \frac{1 - (r_2^2)^{n+1}}{1 - r_2^2} - 4n - 2 \\
&= \frac{r_1^{2(n+1)} - 1}{r_1^2 - 1} (r_1^2 + 1) + \frac{r_2^{2(n+1)} - 1}{r_2^2 - 1} (r_2^2 + 1) - (r_1^{2(n+1)} + r_2^{2(n+1)}) - 4n - 2 \\
&= \frac{(r_1^{2(n+1)} - 1)(r_1^2 + 1)(r_2^2 - 1) + (r_2^{2(n+1)} - 1)(r_2^2 + 1)(r_1^2 - 1)}{(r_1^2 - 1)(r_2^2 - 1)} - (r_1^{2(n+1)} + r_2^{2(n+1)}) - 4n - 2 \\
&= \frac{(r_1^{2(n+1)} - 1)(r_2^2 - r_1^2) + (r_2^{2(n+1)} - 1)(r_1^2 - r_2^2)}{2 - (r_1^2 + r_2^2)} - (r_1^{2(n+1)} + r_2^{2(n+1)}) - 4n - 2 \\
&= -\frac{\gamma_2 (r_1^{2(n+1)} - r_2^{2(n+1)})}{2 - (r_1^2 + r_2^2)} - (r_1^{2(n+1)} + r_2^{2(n+1)}) - 4n - 2,
\end{aligned} \tag{A1}$$

because $\gamma_2 = r_1^2 - r_2^2$. Note that for $k = 1, 2, \dots$,

- $r_1^{2k} + r_2^{2k} = (r_1^k - r_2^k)^2 + 2(r_1 r_2)^k = \gamma_k^2 + 2$, and
- $(r_1^{2k} - r_2^{2k}) = (r_1^k + r_2^k)(r_1^k - r_2^k) = \gamma_k \left(\gamma_k^2 + 2 \right)$.

Using the aforementioned relations, $\gamma_1 = \sqrt{b^2 - 4}$, and $\gamma_2 = b\sqrt{b^2 - 4}$, we obtain the result in the lemma.