The Cauchy problem for a molecular beam epitaxy model with slope selection

by

Magzhan Biyar

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Author . . . .
Department of Mathematics
May 06, 2021

Certified by . . . .
Daniel Oliveira da Silva
Assistant Professor
Thesis Supervisor

Accepted by . . . .
Daniel Pugh
Dean, School of Sciences and Humanities

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Abstract

This thesis is devoted to the study of a nonlinear diffusion equation. We have to prove that the Cauchy problems for a molecular beam epitaxy (MBE) equation with slope selection is locally well-posed for initial data in $W^{1,3}(\mathbb{R}^n)$, $n \leq 2$.

Key words: Cauchy problem, molecular beam epitaxy equation.

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Contents

1 Introduction

Epitaxy is a type of crystal growth or material deposition in which new crystalline layers are formed with well-defined orientations relative to a crystalline substrate. Each layer is called an *epitaxial layer*, or *epitaxial film*. One method of growing epitaxial films is called *molecular beam epitaxy* (MBE), where the layers are created by beams of particles. The process was first discovered at Bell Telephone Laboratories by J. R. Arthur and J. J. LePore [1]. A more detailed description was given by Alfred Y. Cho [5]. MBE is an important tool in the manufacturing of semiconductor devices.

Molecular beam epitaxy is important topic in material science with challenging nature. In this project, focus is well-known MBE model in continuum. In particular, MBE model with slope selection.

1.1 Background

Recently, the epitaxial growth of thin films has received increasing interest. One of the outstanding challenges is to understand qualitatively and quantitatively the epitaxial growth of nanoscale films that are expected to be high-temperature super-conducting, such as $YBa_2Cu_3O_{7-\delta}$ (Yttrium Barium Copper Oxide, YBCO) (Stein [24], 2001), and could be used in the design of semi-conductors. The complex process of building up a thin film layer on a substrate by chemical vapor deposition has now given rise to several descriptions and simulations by atomistic as well as by continuum models (Ortiz et al. [20], 1999; Schulze et al. [21], 1999).

Consider the following two nonlinear diffusion equations — growth equations

$$\partial_t u = -\nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2} + \delta \nabla \Delta u \right), \tag{1.1.1}$$

$$\partial_t u = \nabla \cdot \left[(|\nabla u|^2 - 1) \nabla u \right] - \delta \Delta^2 u. \tag{1.1.2}$$

The spatial derivatives of (1.1.2) have the following physical interpretations:

Δu: diffusion due to evaporation-condensation (Edwards et al. [7], 1992; Mullins

[19], 1957),

$\Delta^2 u$: capillarity-driven surface diffusion (Herring [11], 1951; Mullins [19], 1957),

$\nabla \cdot \left(|\nabla u|^2 \nabla u\right)$: (upward) hopping of atoms (Das Sarma [6], 1992).

In (1.1.1) and (1.1.2), the fourth-order term models surface diffusion, and the nonlinear second-order term models the Ehrlich-Schwoebel effect (Ehrlich et al. [8], 1966; Schwoebel et al. [23], 1966; Schwoebel [22], 1969).

The equation (1.1.2) can be regarded as a gradient flow with respect to the $L^2(\Omega)$ inner product of energy functional

$$E(u) = \int_\Omega \left[\frac{1}{4}(|\nabla u|^2 - 1)^2 + \frac{\delta}{2}|\Delta u|^2\right] dx,$$

it is easy to check that

$$\frac{d}{dt}E(u) = -\|\partial_t u\|_2^2.$$

The energy $E(u)$ often appears in several areas of material modeling. For instance, it serves as a variational model in the theory of liquid crystal (Aviles et al. [2], 1987). It is an example of elastic energy functional of scalar deformations u in the strain-gradient theory for structural phase transitions in solids (Ball et al. [3], 1992; Kohn et al. [14], 1994). It is also a simplified and rescaled folding energy for an out-of-plane displacement u modeling the folding pattern of a blister formed in the buckling-driven delamination of thin films (Gioia et al. [10], 1997; Jin et al. [12], 2000). In the context of thin film epitaxy, the first term in $E(u)$ selects the slope of film surface (Li Bo et al. [4], 2003). It forces the slope of the thin film $|\nabla u| \approx 1$. For this reason equation (1.1.2) is often called the growth equation with slope selection. In the literature there are also models "without slope selection", such as (1.1.1). With slope selection, (1.1.2) predicts that mound-like or pyramid structures in the surface profile tend to have a uniform, constant mound slope (Moldovan et al. [18], 2000; Ortiz et al. [20], 1999).

The nonlinear diffusion equations for thin film epitaxy, with or without slope selection model have been studied recently by many authors [13], [4], [15]. In 2003 authors Belinda B. King, Oliver Stein and Michael Winkler [13] considered the nonlinear parabolic problem

$$\partial_t u + \Delta^2 u - \nabla \cdot (f(\nabla u)) = g, \text{ in } \Omega \times (0,T),$$

$$\partial_N u|_{\partial\Omega} = \partial_N \Delta u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^{4+\kappa}$ for some $\kappa > 0$, $g \in L^2(\Omega \times (0,T))$, and $u_0 \in L^2(\Omega)$. They showed existence, uniqueness and regularity of solutions in a bounded domain with the Neumann boundary conditions of first and third order.

In 2003 authors Li Bo and Jian-Guo Liu [4] considered thin film epitaxy without or with slope selection (1.1.1) –(1.1.2), that model epitaxial growth of thin films, where $u = u(x,t)$ with $x = (x1, x2)$ is a scaled height function of thin film in a co-moving frame and the diffusion coefficient δ is a positive constant. Initial-boundary-value problems in $\Omega \times (0,T]$, for both equations are proven to be well-posed, and the solution regularity is also obtained.

In 2017 authors Dong Li, Zhonghua Qiao, Tao Tang [15] considered with slope selection (1.1.2) which models the epitaxial growth of thin films. In this work authors considered the 2π–periodic case $\Omega = \mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. The function $u = u(t,x)$: $\mathbb{R} \times \Omega \to \mathbb{R}$ represents the scaled height of a thin film and δ is positive parameter which is sometimes called the diffusion coefficient. They established optimal local and global well-posedness for $n \leq 3$ on the 2π–periodic torus with $\delta > 0$.

1.2 Problem

We consider the following initial value problem in $x \in \mathbb{R}^n$, $t > 0$. We prove that the Cauchy problems for molecular beam epitaxy equation with slope selection

$$\begin{aligned} &\partial_t u + \Delta^2 u + \Delta u = \nabla \cdot \left(|\nabla u|^2 \nabla u \right), \\ &u(x,0) = u_0(x), \end{aligned} \tag{1.2.1}$$

is locally well-posed for initial data in $W^{1,3}(\mathbb{R}^n)$, $n \leq 2$.

2 Preliminary notions

In this chapter, we give a review of some definitions and important inequalities. The main references for this chapter are the books by Gilbarg and Trudinger [9], Wheeden and Zygmund [26], and Tao [25] for the definitions and proofs of the theorems. In subsection 2.3 are given the main tools from the work of Medveď [16].

2.1 Notations and some inequalities

We introduce some notations and definitions.

Let $\mathbb{R}^n$ be *n-dimensional Euclidean space.* The Euclidean norm of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 1$ is denoted by

$$|x| = (x \cdot x)^{1/2} = \Big(\sum_{i=1}^{n} x_i^2 \Big)^{1/2}$$

and the *inner (dot) product* of vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

$\mathbb{R}^n_+$ is open upper half-space in $\mathbb{R}^n$, that is, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$.

Let Ω be an open subset in $\mathbb{R}^n$. If $u : \Omega \to \mathbb{R}$, we write as $u(x) = u(x_1, x_2, \ldots, x_n)$, $x \in \Omega$. We denote the space of *continuous functions* $u : \Omega \to \mathbb{R}$ by $C(\Omega)$. $C^k(\Omega)$ is the set of functions having all derivatives of order less than or equal to k continuous in Ω ($k =$ integer ≥ 0 or $k = \infty$).

We use the *Lebesgue norms*

$$\|u\|_{L^q(\Omega)} = \left(\int_\Omega |u(x)|^q dx \right)^{1/q},$$

for $1 \leq q < \infty$ and

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|,$$

for $q = \infty$.

Definition 2.1. A function f is said to be *rapidly decreasing* if for every integer $N \geq 0$ there exists a constant C_N such that

$$|x|^N |f(x)| \leq C_N, \quad \text{for all } x \in \mathbb{R}^n.$$

Definition 2.2. The *Schwartz class* $S(\mathbb{R}^n)$ is the set of all functions $f \in C^\infty(\mathbb{R}^n)$ such that f and all of its derivatives are rapidly decreasing functions on $\mathbb{R}^n$.

For example, the function $f(x) = e^{-|x|^2}$ belongs to $S(\mathbb{R}^n)$, but the function

$$f(x) = \frac{1}{(1+|x|^2)^k}$$

does not belongs to $S(\mathbb{R}^n)$ for any $k \in \mathbb{N}$ since $|x|^{2k} f(x)$ does not decrease to zero as $|x| \to \infty$.

Definition 2.3. The dual of $S(\mathbb{R}^n)$ is called the space of *tempered distributions* and is denoted $S'(\mathbb{R}^n)$.

Definition 2.4. Let f and g be two functions that are measurable in $\mathbb{R}^n$. The *convolution* of f and g, denoted by $f * g$, is the function on $x \geq 0$ defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(s) g(x-s)\, ds, \;\; x \in \mathbb{R}^n,$$

provided the integral exists.

Theorem 2.5. *For all piecewise continuous functions f, g, and h, the following properties hold:*

(1) *Commutativity:* $f * g = g * f$;

(2) *Associativity:* $f * (g * h) = (f * g) * h$;

(3) *Distributivity:* $f * (g + h) = f * g + f * h$.

Any mathematical entity that changes one function into another function is called either an operator or a transform. One particularly important operation in the Schwartz class $S(\mathbb{R}^n)$ is the *Fourier transform* $f \to \widehat{f}$. The Fourier transform of

f, denoted by $\widehat{f}$ or $\mathcal{F}$, is given by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-i\xi x}\, dx.$$

Here, $i = \sqrt{-1}$, and ξ is the frequency variable. The inverse Fourier transform of g, denoted by $\check{g}$ or $\mathcal{F}^{-1}$, is defined by

$$(\mathcal{F}^{-1}g)(x) = \check{g}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} g(\xi)e^{i\xi x}\, d\xi,$$

by ξx usually denotes $\xi x = (\xi \cdot x)$ the inner product in $\mathbb{R}^n$.

Theorem 2.6 (Minkowski's inequality)**.** *Let $f,\, g \in L^p(\mathbb{R}^n)$, where $1 \leq p \leq \infty$. Then we have*

$$\|f + g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)}.$$

Theorem 2.7 (Hölder inequality)**.** *Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 \leq p, q < \infty$ with $1/p + 1/q = 1$. Then the product $fg \in L^1(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} \big|f(x)g(x)\big|\, dx \leq \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}.$$

Theorem 2.8 (Young's inequality or convolution Theorem)**.** *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $1 \leq p, q < \infty$, such that $1/p + 1/q \geq 1$. Then the convolution $f * g \in L^r(\mathbb{R}^n)$, where $1/r = 1/p + 1/q - 1$, and*

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}.$$

We denote the support of a continuous function $u : \Omega \to \mathbb{R}$ by

$$\operatorname{supp} u = \overline{\{x \in \Omega :\, u(x) \neq 0\}}.$$

The function u is *compactly supported* if supp u is compact. The subset of $C^k(\Omega)$ functions with compact support is denoted $C_c^k(\Omega)$. $C^\infty(\Omega)$ is the class of infinitely differentiable functions, and $C_c^\infty(\Omega)$ is the corresponding subset of functions with compact support.

Let V, Ω be an open subsets in $\mathbb{R}^n$. We define $L^p_{loc}(\Omega)$, for $1 \le p \le \infty$, as

$$L^p_{loc}(\Omega) = \{u : \Omega \to \mathbb{R} |\ u \in L^p(V) \ \text{ for every } \ V \subset\subset \Omega\},$$

where $V \subset\subset \Omega$ means the closure $\overline{V} \subset \Omega$ and $\overline{V}$ is compact, that is, V is *compactly contained* in Ω.

As it was stated in Zygmund's book [26], $L^p_{loc}(\Omega)$ is the class of *locally integrable* real-valued functions u on Ω; as usual, we say u is locally integrable on Ω if it is integrable on every compact subset of Ω. By Hölder inequality 2.7 one can show that the space of locally integrable functions $L^1_{loc}(\Omega)$ is the largest space and contains $L^p(\Omega)$ for all $1 \le p \le \infty$, if Ω is a bounded set.

Definition 2.9. Let $u \in L^1_{loc}(\Omega)$. We say $g_i \in L^1_{loc}(\Omega),\ 1 \le i \le n$ is the *weak partial derivative* of $u(x)$ in Ω with respect to x_i if

$$\int_\Omega u(x)\frac{\partial \phi}{\partial x_i}(x)\, dx = -\int_\Omega g_i(x)\phi(x)\, dx$$

for every $\phi \in C_c^\infty(\Omega)$.

It is easy to verify that the weak partial derivative with respect to x_i, if it exists, is uniquely defined by the following: if $g \in L^1_{loc}(\Omega)$ and

$$\int_\Omega g(x)\phi(x)\, dx = 0$$

for every $\phi \in C_c^\infty(\Omega)$, then $g = 0$ almost everywhere. Hence, we write the weak partial derivative of u is

$$\frac{\partial u}{\partial x_i} = g_i, \quad i = (1, 2, \dots, n).$$

We also use

$$\nabla u = \Big(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\Big)$$

and call this vector the *weak gradient* of u in Ω. The weak gradient coincides with the classical gradient ∇u when u is smooth. The notation $|\nabla u|$ means

$$|\nabla u| = \left(\sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{1/2}.$$

Definition 2.10 (Laplacian or the Laplace operator)**.**

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

Sobolev spaces consist of functions from $L^p(\Omega)$ whose weak derivatives also belong to $L^p(\Omega)$:

Definition 2.11 (Sobolev space)**.** Let Ω be an open subset in $\mathbb{R}^n$ and $1 \leq p \leq \infty$, then $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \big\{ u \in L^p(\Omega) : \ u \text{ has a weak gradient in } \Omega \text{ satisfying } \ |\nabla u| \in L^p(\Omega) \big\}.$$

Theorem 2.12. *Let Ω be an open subset in $\mathbb{R}^n$ and $1 \leq p \leq \infty$. The space $W^{1,p}(\Omega)$ is a Banach space, i.e., it is a complete space with respect to the norm*

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)},$$

or for $1 \leq p < \infty$, the equivalent norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|^p_{L^p(\Omega)} \right)^{1/p}.$$

Using second norm when $p = 2$ makes $W^{1,2}(\Omega)$ a Hilbert space with inner product

$$(u, v) = \int_\Omega uv \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx,$$

and denotes by $H^1(\Omega)$. In this work, another equivalent norm will be used:

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left\| \left(\sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{1/2} \right\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

2.2 Methods

Consider an initial value problem for a linear partial differential equation of the form

$$\begin{aligned} u_t - Lu &= F(x,t), \\ u(x,0) &= u_0(x), \end{aligned}$$

where L is some linear differential operator. The solution to this problem is given by *Duhamel's principle*, which states that

$$u(x,t) = e^{tL}u_0(x) + \int_0^t e^{(t-s)L} F(x,s)\, ds.$$

If we want to consider a nonlinear equation of the form

$$\begin{aligned} u_t - Lu &= F(u(x,t)), \\ u(x,0) &= u_0(x), \end{aligned} \tag{2.2.1}$$

then the solution is

$$u(x,t) = e^{tL}u_0(x) + \int_0^t e^{(t-s)L} F(u(x,t))\, ds. \tag{2.2.2}$$

We consider a Banach space $\mathbf{U}$ of functions defined on $\mathbb{R}^n$ and a Banach space $\mathbf{X}_T = C([0,T]; \mathbf{U}(\mathbb{R}^n))$.

Definition 2.13 (Well-posedness)**.** We say that the problem (2.2.1) is *locally well-posed* in $\mathbf{X}_T$ if there exists a time $T > 0$ such that for each $u_0 \in \mathbf{U}(\mathbb{R}^n)$ there exists a strong unique solution $u \in \mathbf{X}_T$ to the integral equation (2.2.2), and furthermore the map $u_0 \mapsto u$ is continuous from $\mathbf{U}$ to $\mathbf{X}_T$. If we can take T arbitrarily large we say the well-posedness is *global* rather than local.

Definition 2.14 (Contraction)**.** A mapping $\mathcal{M}$ from a normed linear space $\mathbb{X}$ into itself is called a *contraction* mapping if there exists a number $0 < \alpha < 1$ such that

$$\|\mathcal{M}u - \mathcal{M}v\| \leq \alpha \|u - v\|$$

for all $u, v \in \mathbb{X}$.

This definition lets us state the main tool in the proof of Theorem 3.4. We say the contraction mapping principle or what is often also called the Banach fixed point theorem.

Theorem 2.15 (Banach Fixed-Point Theorem)**.** *A contraction mapping $\mathcal{M}$ in a Banach space $\mathbb{B}$ has a unique fixed point, that is there exists a unique solution $x \in \mathbb{B}$ of the equation $\mathcal{M}x = x$.*

2.3 Main tools

Before proceeding further, we need following

Definition 2.16 (Medveď [16, p. 350])**.** Let $q > 0$ be a real number and $0 < T \leq \infty$. We say that a function $w : \mathbb{R}^+ \to \mathbb{R}$ ($R^+ = [0, \infty)$) satisfies a condition (q), if

$$e^{-qt}[w(u)]^q \leq R(t) w(e^{-qt}u^q) \text{ for all } u \in \mathbb{R}^+, \; t \in [0, T), \tag{q}$$

where $R(t)$ is a continuous, nonnegative function.

Remark 2.17 (Medveď [16, p. 350]). If $w(u) = u^m, \; m > 0$ then

$$e^{-qt}[w(u)]^q = e^{(m-1)qt} w(e^{-qt}u^q)$$

for any $q > 1$, i.e., the condition (q) is satisfied with $R(t) = e^{(m-1)qt}$.

Let $w(u) = u + au^m$, where $0 \leq q \leq 1, \; m \geq 1$. Then w satisfies the condition (q) with $R(t) = 2^{q-1}e^{qmt}$ (see [16, p. 351]).

Theorem 2.18 (Medveď, Theorem 1 [16, p. 351]). *Let $a(t)$ be a nondecreasing, nonnegative C^1-function on $[0,T)$, $F(t)$ be a continuous, nonnegative function on $[0,T)$, $w:\mathbb{R}^+\to\mathbb{R}$ be a continuous, nondecreasing function, $w(0)=0$, $w(u)>0$ on $(0,T)$, and $u(t)$ be a continuous, nonnegative function on $[0,T)$ with*

$$u(t)\le a(t)+\int_0^t (t-s)^{\beta-1}F(s)w(u(s))ds,\quad t\in[0,T),$$

where $\beta>0$. Then the following assertions hold:

(i) Suppose $\beta>1/2$ and w satisfies the condition (q) with $q=2$. Then

$$u(t)\le e^t\big\{\Omega^{-1}\left[\Omega(2a(t)^2)+g_1(t)\right]\big\}^{1/2},\quad t\in[0,T_1],$$

where

$$g_1(t)=\frac{\Gamma(2\beta-1)}{4^{\beta-1}}\int_0^t R(s)F(s)^2ds,$$

where Γ is the gamma function, $\Omega(v)=\int_{v_0}^v (dy/w(y))$, $v_0>0$, Ω^{-1} is the inverse of Ω, and $T_1\in\mathbb{R}^+$ is such that $\Omega(2a(t)^2)+g_1(t)\in Dom(\Omega^{-1})$ for all $t\in[0,T_1]$.

(ii) Let $\beta\in(0,1/2]$ and w satisfies the condition (q) with $q=z+2$, where $z=(1-\beta)/\beta$ (i.e., $\beta=1/(z+1)$). Then

$$u(t)\le e^t\big\{\Omega^{-1}\left[\Omega(2^{q-1}a(t)^q)+g_2(t)\right]\big\}^{1/q},\quad t\in[0,T_1],$$

where

$$g_2(t)=2^{q-1}K_z^q\int_0^t F(s)^qR(s)\,ds,$$

$$K_z=\left[\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}\right]^{1/p},\qquad \alpha=\frac{z}{z+1},\ \ p=\frac{z+2}{z+1},$$

$T_1\in\mathbb{R}^+$ is such that $\Omega(2^{q-1}a(t)^q)+g_2(t)\in Dom(\Omega^{-1})$ for all $t\in[0,T_1]$.

3 The Cauchy problem for a molecular beam epitaxy model with slope selection

In this chapter, we will present our results of the Cauchy problem for a molecular beam epitaxy model with slope selection that the problem (1.2.1) discussed in section 1.2 of chapter 1 is locally well-posed. We rewrite the initial value problem (1.2.1) in the following way

$$\begin{aligned} &\partial_t u + \Delta^2 u = \nabla \cdot ((|\nabla u|^2 - 1)\nabla u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{3.0.1}$$

Equation (3.0.1) is often called the growth equation with slope selection.

3.1 Bounds for the linear problem

Consider the Cauchy problem for the linear homogeneous equation of (3.0.1)

$$\begin{aligned} &\partial_t u + \Delta^2 u = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{3.1.1}$$

By the Fourier transform the solution of the problem (3.1.1) can be written as

$$u(t,x) = \mathcal{F}^{-1}(e^{-|\xi|^4 t}\mathcal{F}u_0(\xi)) = \mathcal{F}^{-1}(e^{-|\xi|^4 t}) * u_0(x) \equiv \mathcal{K}_t(x) * u_0(x), \tag{3.1.2}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and inverse Fourier transform, respectively, the symbol asterisk denotes the convolution with respect to $x \in \mathbb{R}^n$, and $\mathcal{K}_t$ is the kernel function

$$\mathcal{K}_t(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-|\xi|^4 t} e^{ix\xi}\, d\xi.$$

The kernel function $\mathcal{K}_t \in L^p(\mathbb{R}^n)$, $0 < t < \infty$ for any $1 \leq p \leq \infty$ (see [17, Lemma 2.1]). We denote the action of $(-\Delta)^{\nu/2}$ to the kernel $\mathcal{K}(x)$ by

$$\mathcal{K}^\nu(x) = (-\Delta)^{\nu/2}\mathcal{K}(x), \quad \mathcal{K}_t^\nu(x) = (-\Delta)^{\nu/2}\mathcal{K}_t(x).$$

It is known that (see [17, Lemma 2.2])

$$\mathcal{K}_t^\nu \in L^p(\mathbb{R}^n), \ \ 0 < t < \infty, \ \ \text{for } \nu > 0, \ \ \text{for any } \ 1 \leq p \leq \infty.$$

Using Remark 2.1 (see [17, p. 465]) we have

$$\nabla\mathcal{K}_t \in L^p(\mathbb{R}^n), \ \ 0 < t < \infty.$$

By kernel function

$$\mathcal{K}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-|\xi|^4} e^{ix\xi} \, d\xi,$$

we have

$$\begin{aligned} \mathcal{K}_t(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-|\xi|^4 t} e^{ix\xi} \, d\xi \\ &= (2\pi)^{-n/2} t^{-n/4} \int_{\mathbb{R}^n} e^{-|\eta|^4} e^{it^{-1/4}x\eta} \, d\eta = t^{-n/4}\mathcal{K}(t^{-1/4}x). \end{aligned} \tag{3.1.3}$$

Similar to (3.1.3) the kernel function $\mathcal{K}_t^\nu(x)$ satisfies the following

$$\mathcal{K}_t^\nu(x) = t^{-\nu/4} t^{-n/4} \mathcal{K}^\nu(t^{-1/4}x).$$

We will use the following notation for the constants

$$\begin{aligned} M_p &= \|\mathcal{K}\|_{L^p(\mathbb{R}^n)} < +\infty, \ \ 1 \leq p \leq \infty, \\ M_p^2 &= \|\mathcal{K}^2\|_{L^p(\mathbb{R}^n)} < +\infty, \ \ 1 \leq p \leq \infty, \\ M_p^1 &= \|\nabla\mathcal{K}\|_{L^p(\mathbb{R}^n)} < +\infty, \ \ 1 \leq p \leq \infty. \end{aligned}$$

Then, we have

Proposition 3.1. *For the kernel function the following equality holds*

$$
\begin{aligned}
&(1) \quad \|\mathcal{K}_t(x)\|_{L^1(\mathbb{R}^n)} = \|\mathcal{K}(x)\|_{L^1(\mathbb{R}^n)} = M_1,\\
&(2) \quad \|\Delta\mathcal{K}_{t-s}\|_{L^3(\mathbb{R}^n)} = M_3^2\,(t-s)^{-(n+3)/6},\\
&(3) \quad \|\Delta\mathcal{K}_{t-s}\|_{L^1(\mathbb{R}^n)} = M_1^2\,(t-s)^{-1/2},\\
&(4) \quad \|\nabla\mathcal{K}_{t-s}\|_{L^3(\mathbb{R}^n)} = M_3^1\,(t-s)^{-(2n+3)/12},\\
&(5) \quad \|\nabla\mathcal{K}_{t-s}\|_{L^1(\mathbb{R}^n)} = M_1^1\,(t-s)^{-1/4},
\end{aligned}
$$

where $0 \le s \le t$.

Proof. Let us prove case (1). We make the change of variables $\xi = t^{-1/4}\eta, \quad x = t^{1/4}z$, we have

$$
\begin{aligned}
\|\mathcal{K}_t(x)\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \Big|(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-|\xi|^4 t} e^{ix\xi}\,d\xi\Big|\,dx\\
&= (2\pi)^{-n/2}\Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} e^{-|\eta|^4} e^{iz\eta} t^{-n/4}\,d\eta\Big| t^{n/4}\,dz\Big)\\
&= \|\mathcal{K}(x)\|_{L^1(\mathbb{R}^n)} = M_1,
\end{aligned}
$$

In case (2) we make the change of variables $\xi = (t-s)^{-1/4}\eta, \quad x = (t-s)^{1/4}z$, we get

$$
\begin{aligned}
\|\Delta\mathcal{K}_{t-s}\|_{L^3(\mathbb{R}^n)} &= \Big(\int_{\mathbb{R}^n}\Big|(2\pi)^{-n/2}\int_{\mathbb{R}^n} |\xi|^2 e^{-|\xi|^4(t-s)} e^{ix\xi}\,d\xi\Big|^3 dx\Big)^{1/3}\\
&= (2\pi)^{-n/2}\Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} (t-s)^{-2/4}|\eta|^2 e^{-|\eta|^4} e^{iz\eta}(t-s)^{-n/4}\,d\eta\Big|^3 (t-s)^{n/4}\,dz\Big)^{1/3}\\
&= (2\pi)^{-n/2}\Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} (t-s)^{-(n+2)/4}|\eta|^2 e^{-|\eta|^4} e^{iz\eta}\,d\eta\Big|^3 (t-s)^{n/4}\,dz\Big)^{1/3}\\
&= (2\pi)^{-n/2}(t-s)^{-(n+3)/6}\Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} |\eta|^2 e^{-|\eta|^4} e^{iz\eta}\,d\eta\Big|^3 dz\Big)^{1/3}\\
&= \|\Delta\mathcal{K}\|_{L^3(\mathbb{R}^n)}\,(t-s)^{-(n+3)/6} = \|\mathcal{K}^2\|_{L^3(\mathbb{R}^n)}\,(t-s)^{-(n+3)/6}\\
&= M_3^2\,(t-s)^{-(n+3)/6},
\end{aligned}
$$

where $0 \le s \le t$. Let us prove case (5), the rest can be easily obtained by similar calculations. Here we make the change of variables $\xi = (t-s)^{-1/4}\eta, \quad x = (t-s)^{1/4}z$, we obtain

$$\begin{aligned}
\|\nabla\mathcal{K}_{t-s}\|_{L^1(\mathbb{R}^n)} &= \Big(\int_{\mathbb{R}^n}\Big|(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-|\xi|^4(t-s)}\nabla_x e^{ix\xi}\,d\xi\Big|\,dx\Big)\\
&= (2\pi)^{-n/2}\Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} e^{-|\eta|^4}(t-s)^{-1/4}\nabla_z e^{iz\eta}(t-s)^{-n/4}\,d\eta\Big|(t-s)^{n/4}\,dz\Big)\\
&= (2\pi)^{-n/2}(t-s)^{-1/4}\Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} e^{-|\eta|^4}\nabla_z e^{iz\eta}\,d\eta\Big|\,dz\Big)\\
&= \|\nabla\mathcal{K}\|_{L^1(\mathbb{R}^n)}\,(t-s)^{-1/4} = M_1^1\,(t-s)^{-1/4},
\end{aligned}$$

where $0 \le s \le t$. □

Proposition 3.2. *Let u be a solution of the problem (3.1.1) with $u_0 \in W^{1,3}(\mathbb{R}^n)$. Then, we have*

$$\|u\|_{C([0,T];\,W^{1,3}(\mathbb{R}^n))} \le M_1\|u_0\|_{W^{1,3}(\mathbb{R}^n)}, \tag{3.1.4}$$

where $0 < T < \infty$.

Proof. By (3.1.2) the problem (3.1.1) can be written

$$\begin{aligned}
u(t,x) &= \mathcal{F}^{-1}(e^{-|\xi|^4 t}) * u_0(x) = \mathcal{K}_t(x) * u_0(x)\\
&= \Big((2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-|\xi|^4 t}e^{ix\xi}\,d\xi\Big) * u_0(x).
\end{aligned}$$

Then by using Young's inequality and Proposition 3.1 we obtain

$$\begin{aligned}
\|u\|_{W^{1,3}(\mathbb{R}^n)} &= \|u\|_{L^3(\mathbb{R}^n)} + \|\nabla u\|_{L^3(\mathbb{R}^n)}\\
&\le \|\mathcal{K}_t\|_{L^1(\mathbb{R}^n)}\|u_0\|_{L^3(\mathbb{R}^n)} + \|\mathcal{K}_t * \nabla u_0\|_{L^3(\mathbb{R}^n)}\\
&\le \|\mathcal{K}_t\|_{L^1(\mathbb{R}^n)}\|u_0\|_{L^3(\mathbb{R}^n)} + \|\mathcal{K}_t\|_{L^1(\mathbb{R}^n)}\|\nabla u_0\|_{L^3(\mathbb{R}^n)}\\
&= \|\mathcal{K}_t\|_{L^1(\mathbb{R}^n)}\big(\|u_0\|_{L^3(\mathbb{R}^n)} + \|\nabla u_0\|_{L^3(\mathbb{R}^n)}\big)\\
&= M_1\,\|u_0\|_{W^{1,3}(\mathbb{R}^n)}, \quad t > 0,
\end{aligned}$$

which implies (3.1.4). □

3.2 Bounds for the solution of the nonlinear problem

Lemma 3.3. *Let u be a solution of the problem* (3.0.1) *with $u_0 \in W^{1,3}(\mathbb{R}^n)$, $n \leq 2$. Then we have*

$$\|u\|_{C([0,T_1];\, W^{1,3}(\mathbb{R}^n))} \leq C\|u_0\|_{W^{1,3}(\mathbb{R}^n)},$$

where the constants

$$C = \varepsilon^{-\frac{1}{2q}} 2^{\frac{q-1}{q}} M_1 e^{T_1 + \frac{\gamma_n}{q}(e^{3qT_1}-1)}, \quad T_1 = \frac{1}{3q} \ln\Big(1 + \frac{1}{2\gamma_n} \ln\Big(1 + \frac{1-\varepsilon}{2^{2(q-1)}A^{2q}}\Big)\Big),$$

$$\varepsilon \in (0,1) \ \textit{such that} \ T_1 \leq 1, \quad A = M_1\|u_0\|_{W^{1,3}(\mathbb{R}^n)}, \quad q = \frac{9-n}{3-n},$$

$$\gamma_n = \tfrac{1}{3q} 2^{2(q-1)} (2M)^q K_z^q, \quad M = \max\{M_3^1,\, M_1^1,\, M_3^2,\, M_1^2\},$$

$$K_z = \left[\frac{\Gamma(1 - \frac{(n+3)(9-n)}{36})}{[(9-n)/6]^{1-\frac{(n+3)(9-n)}{36}}}\right]^{6/(9-n)}.$$

Proof. By the Duhamel's principle the solution of the Cauchy problem (3.0.1) can be written in the following integral equation

$$u(x,t) = (\mathcal{K}_t * u_0)(x,t) + \int_0^t \mathcal{K}_{t-s} * \nabla \cdot ((|\nabla u|^2 - 1)\nabla u)(s)\, ds. \tag{3.2.1}$$

Then we have

$$\begin{aligned} u(x,t) &= (\mathcal{K}_t * u_0)(x,t) + \int_0^t \mathcal{K}_{t-s} * \nabla \cdot ((|\nabla u|^2 - 1)\nabla u)(s)\, ds \\ &= (\mathcal{K}_t * u_0)(x,t) + \int_0^t \nabla\mathcal{K}_{t-s} * ((|\nabla u|^2 - 1)\nabla u)(s)\, ds, \end{aligned}$$

and

$$\nabla u(x,t) = (\mathcal{K}_t * \nabla u_0)(x,t) + \int_0^t \Delta\mathcal{K}_{t-s} * ((|\nabla u|^2 - 1)\nabla u)(s)\, ds.$$

By using Young's inequality, Proposition 3.1, and Proposition 3.2 we arrive at

$$
\begin{aligned}
\|u\|_{W^{1,3}(\mathbb{R}^n)} &= \|u\|_{L^3(\mathbb{R}^n)} + \|\nabla u\|_{L^3(\mathbb{R}^n)} \\
&\leq \|\mathcal{K}_t * u_0\|_{L^3(\mathbb{R}^n)} + \Big\| \int_0^t \nabla \mathcal{K}_{t-s} * ((|\nabla u|^2 - 1)\nabla u)(s)\, ds \Big\|_{L^3(\mathbb{R}^n)} \\
&\quad + \|\mathcal{K}_t * \nabla u_0\|_{L^3(\mathbb{R}^n)} + \Big\| \int_0^t \Delta \mathcal{K}_{t-s} * ((|\nabla u|^2 - 1)\nabla u)(s)\, ds \Big\|_{L^3(\mathbb{R}^n)} \\
&\leq \|\mathcal{K}_t * u_0\|_{L^3(\mathbb{R}^n)} + \Big\| \int_0^t \nabla \mathcal{K}_{t-s} * (|\nabla u|^2 \nabla u)(s)\, ds \Big\|_{L^3(\mathbb{R}^n)} \\
&\quad + \Big\| \int_0^t \nabla \mathcal{K}_{t-s} * (\nabla u)(s)\, ds \Big\|_{L^3(\mathbb{R}^n)} \\
&\quad + \|\mathcal{K}_t * \nabla u_0\|_{L^3(\mathbb{R}^n)} + \Big\| \int_0^t \Delta \mathcal{K}_{t-s} * (|\nabla u|^2 \nabla u)(s)\, ds \Big\|_{L^3(\mathbb{R}^n)} \\
&\quad + \Big\| \int_0^t \Delta \mathcal{K}_{t-s} * (\nabla u)(s)\, ds \Big\|_{L^3(\mathbb{R}^n)} \\
&\leq \|\mathcal{K}_t\|_{L^1(\mathbb{R}^n)} \|u_0\|_{L^3(\mathbb{R}^n)} + \int_0^t \big\|\nabla \mathcal{K}_{t-s}\big\|_{L^3(\mathbb{R}^n)} \big\||\nabla u|^3\big\|_{L^1(\mathbb{R}^n)}(s)\, ds \\
&\quad + \int_0^t \big\|\nabla \mathcal{K}_{t-s}\big\|_{L^1(\mathbb{R}^n)} \big\|\nabla u\big\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\quad + \|\mathcal{K}_t\|_{L^1(\mathbb{R}^n)} \|\nabla u_0\|_{L^3(\mathbb{R}^n)} + \int_0^t \big\|\Delta \mathcal{K}_{t-s}\big\|_{L^3(\mathbb{R}^n)} \big\||\nabla u|^3\big\|_{L^1(\mathbb{R}^n)}(s)\, ds \\
&\quad + \int_0^t \big\|\Delta \mathcal{K}_{t-s}\big\|_{L^1(\mathbb{R}^n)} \big\|\nabla u\big\|_{L^3(\mathbb{R}^n)}(s)\, ds
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & M_1 \|u_0\|_{W^{1,3}(\mathbb{R}^n)} + M_3^1 \int_0^t (t-s)^{-\frac{2n+3}{12}} \big\|\nabla u\big\|^3_{L^3(\mathbb{R}^n)}(s)\, ds \\
& + M_1^1 \int_0^t (t-s)^{-\frac{1}{4}} \big\|\nabla u\big\|_{L^3(\mathbb{R}^n)}(s)\, ds + M_3^2 \int_0^t (t-s)^{-\frac{n+3}{6}} \big\|\nabla u\big\|^3_{L^3(\mathbb{R}^n)}(s)\, ds \\
& + M_1^2 \int_0^t (t-s)^{-\frac{1}{2}} \big\|\nabla u\big\|_{L^3(\mathbb{R}^n)}(s)\, ds.
\end{aligned}
$$

Hence it follows that

$$
\begin{aligned}
\|u\|_{W^{1,3}(\mathbb{R}^n)} \leq M_1 \|u_0\|_{W^{1,3}(\mathbb{R}^n)} &+ M_3^1 \int_0^t (t-s)^{-\frac{2n+3}{12}} \|u\|^3_{W^{1,3}(\mathbb{R}^n)}(s)\, ds \\
&+ M_1^1 \int_0^t (t-s)^{-\frac{1}{4}} \|u\|_{W^{1,3}(\mathbb{R}^n)}(s)\, ds + M_3^2 \int_0^t (t-s)^{-\frac{n+3}{6}} \|u\|^3_{W^{1,3}(\mathbb{R}^n)}(s)\, ds \\
&+ M_1^2 \int_0^t (t-s)^{-\frac{1}{2}} \|u\|_{W^{1,3}(\mathbb{R}^n)}(s)\, ds.
\end{aligned}
\tag{3.2.2}
$$

For $0 < t \leq 1$ we have in the case $n = 1$

$$
(t-s)^{-\frac{n+3}{6}} \geq (t-s)^{-\frac{1}{2}} \geq (t-s)^{-\frac{2n+3}{12}} \geq (t-s)^{-\frac{1}{4}},
$$

and in the case $n = 2$

$$
(t-s)^{-\frac{n+3}{6}} \geq (t-s)^{-\frac{2n+3}{12}} \geq (t-s)^{-\frac{1}{2}} \geq (t-s)^{-\frac{1}{4}}.
$$

Denoting by $M = \max\{M_3^1,\, M_1^1,\, M_3^2,\, M_1^2\}$, from (3.2.2), we get

$$
\begin{aligned}
\|u\|_{W^{1,3}(\mathbb{R}^n)} \leq & M_1 \|u_0\|_{W^{1,3}(\mathbb{R}^n)} \\
& + 2M \int_0^t (t-s)^{-\frac{n+3}{6}} \big(\|u\|^3_{W^{1,3}(\mathbb{R}^n)}(s) + \|u\|_{W^{1,3}(\mathbb{R}^n)}(s) \big)\, ds,
\end{aligned}
\tag{3.2.3}
$$

where $0 \leq t \leq 1$. Next we will apply Theorem 2.18 to (3.2.3). Note that

$$
\begin{aligned}
& w(y) = y^3 + y, \ \text{ for } \ n = 1, 2, \quad \beta = \frac{3-n}{6} \in (0, 1/2], \quad z = \frac{1-\beta}{\beta} = \frac{n+3}{3-n}, \\
& q = z + 2 = \frac{9-n}{3-n}, \quad \alpha = \frac{z}{z+1} = \frac{n+3}{6}, \quad p = \frac{z+2}{z+1} = \frac{9-n}{6}, \\
& R(t) = 2^{q-1} e^{qmt} = 2^{\frac{6}{3-n}} e^{\frac{3(9-n)}{3-n} t}, \quad \Omega(v) = \int_{v_0}^{v} \frac{dy}{y^3 + y} = \frac{1}{2} \ln \Big(\frac{v^2}{1+v^2} \cdot \delta \Big), \\
& \text{where } \ \delta = \frac{1+v_0^2}{v_0^2}, \quad v(z) = \Omega^{-1}(z) = e^z / \sqrt{\delta - e^{2z}}, \quad v_0 > 0, \\
& K_z = \left[\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}} \right]^{1/p} = \left[\frac{\Gamma(1 - \frac{(n+3)(9-n)}{36})}{[(9-n)/6]^{1-\frac{(n+3)(9-n)}{36}}} \right]^{6/(9-n)}, \quad F(s) = 2M,
\end{aligned}
$$

$$
\begin{aligned}
g_2(t) &= 2^{q-1}K_z^q\int_0^t F(s)^q R(s)\,ds = \\
&= 2^{\frac{6}{3-n}}K_z^{\frac{9-n}{3-n}}\big(2M\big)^{\frac{9-n}{3-n}}\int_0^t 2^{\frac{6}{3-n}}e^{\frac{3(9-n)}{3-n}t}\,ds \\
&= 2^{\frac{12}{3-n}}\big(2M\big)^{\frac{9-n}{3-n}}K_z^{\frac{9-n}{3-n}}\frac{3-n}{3(9-n)}\Big(e^{\frac{3(9-n)}{3-n}t}-1\Big) \\
&= \gamma_n\Big(e^{\frac{3(9-n)}{3-n}t}-1\Big) = \gamma_n\big(e^{3qt}-1\big),
\end{aligned}
$$

where

$$
\gamma_n = 2^{\frac{12}{3-n}}\big(2M\big)^{\frac{9-n}{3-n}}K_z^{\frac{9-n}{3-n}}\frac{3-n}{3(9-n)} = \frac{1}{3q}2^{2(q-1)}(2M)^qK_z^q.
$$

Then, denoting by $A = M_1\|u_0\|_{W^{1,3}(\mathbb{R}^n)}$ we get

$$
\Omega(2^{q-1}(a(t))^q) = \Omega(2^{q-1}A^q) = \frac{1}{2}\ln\left(\frac{2^{2(q-1)}A^{2q}\delta}{1+2^{2(q-1)}A^{2q}}\right),
$$

$$
\Omega^{-1}\left[\Omega(2^{q-1}a(t)^q)+g_2(t)\right] = \Omega^{-1}\left[\frac{1}{2}\ln\left(\frac{2^{2(q-1)}A^{2q}\delta}{1+2^{2(q-1)}A^{2q}}\right)+\gamma_n(e^{3qt}-1)\right]
$$

$$
\Big/\ \Omega^{-1}(z) = e^z/\sqrt{\delta-e^{2z}}\ \Big/
$$

$$
= \frac{\frac{2^{q-1}A^q\sqrt{\delta}}{\sqrt{1+2^{2(q-1)}A^{2q}}}\cdot e^{\gamma_n(e^{3qt}-1)}}{\sqrt{\delta-\frac{2^{2(q-1)}A^{2q}\delta}{1+2^{2(q-1)}A^{2q}}\cdot e^{2\gamma_n(e^{3qt}-1)}}} = \frac{2^{q-1}A^q}{\sqrt{\frac{1+2^{2(q-1)}A^{2q}}{e^{2\gamma_n(e^{3qt}-1)}}-2^{2(q-1)}A^{2q}}}.
$$

Therefore, we have

$$
\begin{aligned}
\|u\|_{W^{1,3}(\mathbb{R}^n)}(t) &\le e^t\Big\{\Omega^{-1}\left[\Omega(2^{q-1}a(t)^q)+g_2(t)\right]\Big\}^{1/q} \\
&= e^t\left\{\frac{2^{q-1}A^q}{\sqrt{\frac{1+2^{2(q-1)}A^{2q}}{e^{2\gamma_n(e^{3qt}-1)}}-2^{2(q-1)}A^{2q}}}\right\}^{1/q}, \qquad 0\le t\le T_1,
\end{aligned}
\tag{3.2.4}
$$

where

$$
T_1 = \frac{1}{3q}\ln\Big(1+\frac{1}{2\gamma_n}\ln\Big(1+\frac{1-\varepsilon}{2^{2(q-1)}A^{2q}}\Big)\Big) < \frac{1}{3q}\ln\Big(1+\frac{1}{2\gamma_n}\ln\Big(1+\frac{1}{2^{2(q-1)}A^{2q}}\Big)\Big),
$$

for $0<\varepsilon<1$, since

$$\Omega(2^{q-1}A^q)+\gamma_n(e^{3qt}-1)\in\ \mathrm{Dom}(\Omega^{-1}).$$

We will choose ε such that $T_1\leq 1$. For this, it is sufficient that ε satisfies the following inequality

$$2^{2(q-1)}M_1^{2q}\|u_0\|^{2q}_{W^{1,3}(\mathbb{R}^n)}(e^{2\gamma_n(e^{3q}-1)}-1)\geq 1-\varepsilon.$$

Then taking into account that

$$2^{2(q-1)}A^{2q}=(1-\varepsilon)/(e^{2\gamma_n(e^{3qT_1}-1)}-1),$$

we have

$$\begin{aligned}\|u\|_{W^{1,3}(\mathbb{R}^n)}(T_1) &\leq \frac{2^{(q-1)/q}Ae^{T_1}}{\left(\frac{1+2^{2(q-1)}A^{2q}}{e^{2\gamma_n(e^{3qT_1}-1)}}-2^{2(q-1)}A^{2q}\right)^{\frac{1}{2q}}}\\ &= \frac{2^{(q-1)/q}Ae^{T_1}}{\left(\frac{1+(1-\varepsilon)/(e^{2\gamma_n(e^{3qT_1}-1)}-1)}{e^{2\gamma_n(e^{3qT_1}-1)}}-\frac{1-\varepsilon}{e^{2\gamma_n(e^{3qT_1}-1)}-1}\right)^{\frac{1}{2q}}}\\ &= \varepsilon^{-\frac{1}{2q}}2^{\frac{q-1}{q}}M_1e^{T_1+\frac{\gamma_n}{q}(e^{3qT_1}-1)}\|u_0\|_{W^{1,3}(\mathbb{R}^n)}=C\|u_0\|_{W^{1,3}(\mathbb{R}^n)},\end{aligned}$$

where $C=\varepsilon^{-\frac{1}{2q}}2^{\frac{q-1}{q}}M_1e^{T_1+\frac{\gamma_n}{q}(e^{3qT_1}-1)}$, $\quad 0\leq t\leq T_1$, $\quad 0<\varepsilon<1$, thereby we obtain that

$$\|u\|_{C([0,\,T_1];\,W^{1,3}(\mathbb{R}^n))}\leq C\|u_0\|_{W^{1,3}(\mathbb{R}^n)}=\tilde{C}<\infty,$$

which proves our claim. □

3.3 Local well-posedness

In this subsection we prove the local well-posedness of the Cauchy problem (3.0.1) in $C([0,\,T];\,W^{1,3}(\mathbb{R}^n))$.

Theorem 3.4. *Let $u_0\in W^{1,3}(\mathbb{R}^n)$, $n\leq 2$. Then for some $T>0$ there exists a unique* *solution* $u \in C([0,\, T];\, W^{1,3}(\mathbb{R}^n))$ *to the integral equation* (3.2.1)*, with*

$$T < \min\{t_1,\, t_2\}, \quad t_1 = \Big(\frac{3-n}{24M(4R^2+1)}\Big)^{\frac{6}{3-n}},$$
$$t_2 = \Big(\frac{3-n}{12M(12R^2+1)}\Big)^{\frac{6}{3-n}}, \quad R = M_1 \|u_0\|_{W^{1,3}(\mathbb{R}^n)}.$$

Proof. We prove Theorem 3.4 using a Banach fixed-point theorem. Without loss of generality, we can assume that $T \leq 1$. We apply the contraction mapping principle in a ball of a radius $2R$ in a Banach space $C([0,\, T];\, W^{1,3}(\mathbb{R}^n))$

$$\mathcal{B}_{2R} := \big\{u \in C([0,\, T];\, W^{1,3}(\mathbb{R}^n)) : \; \|u\|_{C([0,T_1];\, W^{1,3}(\mathbb{R}^n))} \leq 2R\big\}.$$

where $R = M_1 \|u_0\|_{W^{1,3}(\mathbb{R}^n))}$. For $u \in \mathcal{B}_{2R}$ we define the mapping or the operator $\Phi(u)$ by

$$\Phi(u) = (\mathcal{K}_t * u_0)(x,t) + \int_0^t \mathcal{K}_{t-s} * \nabla \cdot ((|\nabla u|^2 - 1)\nabla u)(s)\, ds.$$

Step 1. First we prove that

$$\|\Phi(u)\|_{C([0,T];\, W^{1,3}(\mathbb{R}^n))} \leq 2R, \tag{3.3.1}$$

where $u \in \mathcal{B}_{2R}$. By using the Young's inequality and Proposition 3.1 we obtain the inequality same as (3.2.3) for Φ, we get

$$\begin{aligned}
\|\Phi(u)\|_{W^{1,3}(\mathbb{R}^n)}(t) &\leq M_1 \|u_0\|_{W^{1,3}(\mathbb{R}^n)} \\
&\quad + 2M \int_0^t (t-s)^{-\frac{n+3}{6}} \big(\|u\|^3_{W^{1,3}(\mathbb{R}^n)}(s) + \|u\|_{W^{1,3}(\mathbb{R}^n)}(s)\big)\, ds \\
&\leq R + \frac{24MR}{3-n}(4R^2+1)t^{\frac{3-n}{6}} \\
&\leq R + \frac{24MR}{3-n}(4R^2+1)T^{\frac{3-n}{6}} \leq 2R, \quad 0 \leq t \leq T,
\end{aligned}$$

since $T < \min\{t_1,\, t_2\}$, where

$$t_1 = \Big(\frac{3-n}{24M(4R^2+1)}\Big)^{\frac{6}{3-n}}.$$

Hence we get 3.3.1. Therefore the mapping Φ transforms a ball of a radius $2R$ into itself in the space $C([0,\, T];\, W^{1,3}(\mathbb{R}^n))$.

Step 2. Next we show that Φ is a contraction mapping in $C([0,\, T];\, W^{1,3}(\mathbb{R}^n))$. Let $u,\, v\ \in \mathcal{B}_{2R}$. Then by using the Young's inequality, we obtain

$$\begin{aligned}
&\|\Phi(u)-\Phi(v)\|_{W^{1,3}(\mathbb{R}^n)}(t) \\
&\quad= \Big\|\int_0^t \mathcal{K}_{t-s} * \Big[\nabla\cdot(|\nabla u|^2-1)\nabla u - \nabla\cdot(|\nabla v|^2-1)\nabla v\Big](s)\, ds\Big\|_{W^{1,3}(\mathbb{R}^n)} \\
&\quad\le \int_0^t \Big\|\nabla\mathcal{K}_{t-s} * \Big[(|\nabla u|^2-1)\nabla u - (|\nabla v|^2-1)\nabla v\Big](s)\Big\|_{W^{1,3}(\mathbb{R}^n)}\, ds \\
&\quad\le \int_0^t \Big\|\nabla\mathcal{K}_{t-s} * \Big[(|\nabla u|^2\nabla u - |\nabla v|^2\nabla v - \nabla u + \nabla v\Big](s)\Big\|_{W^{1,3}(\mathbb{R}^n)}\, ds \\
&\quad\le \int_0^t \Big\|\nabla\mathcal{K}_{t-s} * \Big[(|\nabla u|^2\nabla u - |\nabla v|^2\nabla v\Big]\Big\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\qquad+ \int_0^t \Big\|\nabla\mathcal{K}_{t-s} * \big[\nabla u - \nabla v\big]\Big\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\qquad+ \int_0^t \Big\|\Delta\mathcal{K}_{t-s} * \Big[(|\nabla u|^2\nabla u - |\nabla v|^2\nabla v\Big]\Big\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\qquad+ \int_0^t \Big\|\Delta\mathcal{K}_{t-s} * \big[\nabla u - \nabla v\big]\Big\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\quad\le \int_0^t \big\|\nabla\mathcal{K}_{t-s}\big\|_{L^3(\mathbb{R}^n)}\big\||\nabla u|^2\nabla u - |\nabla v|^2\nabla v\big\|_{L^1(\mathbb{R}^n)}(s)\, ds \\
&\qquad+ \int_0^t \big\|\nabla\mathcal{K}_{t-s}\big\|_{L^1(\mathbb{R}^n)}\big\|\nabla u - \nabla v\big\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\qquad+ \int_0^t \big\|\Delta\mathcal{K}_{t-s}\big\|_{L^3(\mathbb{R}^n)}\big\||\nabla u|^2\nabla u - |\nabla v|^2\nabla v\big\|_{L^1(\mathbb{R}^n)}(s)\, ds \\
&\qquad+ \int_0^t \big\|\Delta\mathcal{K}_{t-s}\big\|_{L^1(\mathbb{R}^n)}\big\|\nabla u - \nabla v\big\|_{L^3(\mathbb{R}^n)}(s)\, ds.
\end{aligned} \tag{3.3.2}$$

Now let us show the following inequality

$$
\begin{aligned}
\big||\nabla u|^2\nabla u - |\nabla v|^2\nabla v\big| &= \big||\nabla u|^2\nabla u - |\nabla u|^2\nabla v + |\nabla u|^2\nabla v - |\nabla v|^2\nabla v\big| \\
&= \big||\nabla u|^2(\nabla u - \nabla v) + (|\nabla u|^2 - |\nabla v|^2)\nabla v\big| \\
&= \big||\nabla u|^2(\nabla u - \nabla v) + (|\nabla u| - |\nabla v|)(|\nabla u| + |\nabla v|)\nabla v\big| \\
&\leq |\nabla u|^2|\nabla u - \nabla v| + (|\nabla u - \nabla v|)(|\nabla u| + |\nabla v|)|\nabla v| \\
&= (|\nabla u|^2 + (|\nabla u| + |\nabla v|)|\nabla v|)|\nabla u - \nabla v| \\
&= (|\nabla u|^2 + |\nabla u||\nabla v| + |\nabla v|^2)|\nabla u - \nabla v|.
\end{aligned}
$$

Then from (3.3.2), by using the Hölder inequality, and by Proposition 3.1 we have

$$
\begin{aligned}
&\|\Phi(u) - \Phi(v)\|_{W^{1,3}(\mathbb{R}^n)}(t) \\
&\leq M_3^1 \int_0^t (t-s)^{-\frac{2n+3}{12}} \big\|\big(|\nabla u|^2 + |\nabla u||\nabla v| + |\nabla v|^2\big)|\nabla u - \nabla v|\big\|_{L^1(\mathbb{R}^n)}(s)\, ds \\
&\quad + M_1^1 \int_0^t (t-s)^{-\frac{1}{4}} \big\|\nabla(u-v)\big\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\quad + M_3^2 \int_0^t (t-s)^{-\frac{n+3}{6}} \big\|\big(|\nabla u|^2 + |\nabla u||\nabla v| + |\nabla v|^2\big)|\nabla u - \nabla v|\big\|_{L^1(\mathbb{R}^n)}(s)\, ds \\
&\quad + M_1^2 \int_0^t (t-s)^{-\frac{1}{2}} \big\|\nabla(u-v)\big\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\leq M_3^1 \int_0^t (t-s)^{-\frac{2n+3}{12}} \big(\|\nabla u\|_{L^3(\mathbb{R}^n)}^2 + \|\nabla u\|_{L^3(\mathbb{R}^n)}\|\nabla v\|_{L^3(\mathbb{R}^n)} + \|\nabla v\|_{L^3(\mathbb{R}^n)}^2\big) \\
&\qquad\qquad \times \|\nabla u - \nabla v\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\quad + M_1^1 \int_0^t (t-s)^{-\frac{1}{4}} \big\|\nabla(u-v)\big\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\quad + M_3^2 \int_0^t (t-s)^{-\frac{n+3}{6}} \big(\|\nabla u\|_{L^3(\mathbb{R}^n)}^2 + \|\nabla u\|_{L^3(\mathbb{R}^n)}\|\nabla v\|_{L^3(\mathbb{R}^n)} + \|\nabla v\|_{L^3(\mathbb{R}^n)}^2\big) \\
&\qquad\qquad \times \|\nabla u - \nabla v\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\quad + M_1^2 \int_0^t (t-s)^{-\frac{1}{2}} \big\|\nabla(u-v)\big\|_{L^3(\mathbb{R}^n)}(s)\, ds \\
&\leq 2M \int_0^t (t-s)^{-\frac{n+3}{6}} \big(\|\nabla u\|_{L^3(\mathbb{R}^n)}^2 + \|\nabla u\|_{L^3(\mathbb{R}^n)}\|\nabla v\|_{L^3(\mathbb{R}^n)} + \|\nabla v\|_{L^3(\mathbb{R}^n)}^2 + 1\big) \\
&\qquad\qquad \times \|\nabla(u-v)\|_{L^3(\mathbb{R}^n)}(s)\, ds
\end{aligned}
$$

$$\le 2M\int_0^t (t-s)^{-\frac{n+3}{6}}\big(\|u\|^2_{W^{1,3}(\mathbb{R}^n)} + \|u\|_{W^{1,3}(\mathbb{R}^n)}\|v\|_{W^{1,3}(\mathbb{R}^n)} + \|v\|^2_{W^{1,3}(\mathbb{R}^n)} + 1\big)$$

$$\times \|u-v\|_{W^{1,3}(\mathbb{R}^n)}(s)\, ds$$

$$\le 2M(12R^2+1)\|u-v\|_{C([0,T];\, W^{1,3}(\mathbb{R}^n))}\int_0^t (t-s)^{-\frac{n+3}{6}}\, ds$$
$$= \frac{12M}{3-n}(12R^2+1)t^{\frac{3-n}{6}}\|u-v\|_{C([0,T];\, W^{1,3}(\mathbb{R}^n))}$$

$$\le \frac{12M}{3-n}(12R^2+1)T^{\frac{3-n}{6}}\|u-v\|_{C([0,T];\, W^{1,3}(\mathbb{R}^n))} \le C_1\|u-v\|_{C([0,T];\, W^{1,3}(\mathbb{R}^n))},$$

where $C_1 = \dfrac{12M}{3-n}(12R^2+1)T^{\frac{3-n}{6}} < 1$, since

$$T < \min\{t_1,\, t_2\}, \quad t_2 = \Big(\frac{3-n}{12M(12R^2+1)}\Big)^{\frac{6}{3-n}}.$$

Then

$$\|\Phi(u)-\Phi(v)\|_{C([0,T];\, W^{1,3}(\mathbb{R}^n))} \le C_1\|u-v\|_{C([0,T];\, W^{1,3}(\mathbb{R}^n))}, \;\; C_1 < 1.$$

Thus Φ is a contraction mapping in $\mathcal{B}_{2R}$. Therefore there exists a unique solution $u \in C([0,\, T];\, W^{1,3}(\mathbb{R}^n))$ to the Cauchy problem (3.0.1). This completes the proof of Theorem 3.4. □

The final thing we must show is that the solution map $u_0 \mapsto u$ is continuous from $W^{1,3}(\mathbb{R}^n)$ to $C([0,\, T];\, W^{1,3}(\mathbb{R}^n))$. Suppose for two initial conditions $u_0,\, v_0 \in W^{1,3}(\mathbb{R}^n)$, we have the corresponding solutions u and v, respectively. Proceeding exactly as above, it is possible to show that

$$\|u-v\|_{C([0,T];\, W^{1,3}(\mathbb{R}^n)} \le M_1\|u_0-v_0\|_{W^{1,3}(\mathbb{R}^n))} + C_1\|u-v\|_{C([0,T];\, W^{1,3}(\mathbb{R}^n)}, \;\; C_1 < 1,$$

hence we have

$$\|u-v\|_{C([0,T];\, W^{1,3}(\mathbb{R}^n)} \le \frac{M_1}{1-C_1}\|u_0-v_0\|_{W^{1,3}(\mathbb{R}^n))}, \;\; C_1 < 1.$$

Therefore we have proven the local well-posedness of the Cauchy problem (3.0.1) for MBE equation with slope selection for initial data in $W^{1,3}(\mathbb{R}^n)$, $n \leq 2$.

4 Conclusion

In this work, we have proven that for some time interval $T > 0$ there exists a unique solution to the Cauchy problem (1.2.1). So, we have proven the local well-posedness of the Cauchy problem for molecular beam epitaxy equation with slope selection

$$\partial_t u + \Delta^2 u + \Delta u = \nabla \cdot \left(|\nabla u|^2 \nabla u \right),$$
$$u(x,0) = u_0(x),$$

for initial data in $W^{1,3}(\mathbb{R}^n)$, $n \leq 2$.

The contraction mapping principle was used for this problem. Initially, the problem was reduced to an integral form using a standard method. Then, with the help of Young inequality, an integral inequality was obtained for the norms of the desired functions. Since the inequality is nonlinear and has a weak singularity in the variable t, we applied the Medveď method to obtain an estimate of the solution in the space under consideration.

The issues of the global well-posedness of the Cauchy problem (1.2.1) and research in $H^l(\mathbb{R}^n)$ for $l \geq 1$, remains outside the scope of my work. Future research will be dedicated to the global existence of this problem.

The method used in this work passes for the space $H^2(\mathbb{R}^n)$, $n \leq 3$. But for the case $H^1(\mathbb{R}^n)$ it does not.

Bibliography

[1] Arthur, J.R. and LePore, J. J., *GaAs, GaP, and $GaAs_x$ P_{1x} Epitaxial Films Grown by Molecular Beam Deposition*, Journal of Vacuum Science and Technology, **6** (1969), 545.

[2] Aviles, P. and Giga, Y., *A mathematical problem related to the physical theory of liquid crystal configurations*, Proc. Centre Math. Anal. Austral. Nat. Univ., **12** (1987), 1–16.

[3] Ball, J.M. and James, R. D., *Proposed experimental tests of a theory of fine microstructure and the two-well problem*, Phil. Trans. R. Soc. Lond. A, **338** (1992), 389–450.

[4] Bo, Li and Liu, Jian-Guo, *Thin film epitaxy with or without slope selection*, European Journal of Applied Mathematics, **14:6** (2003), 713–743.

[5] Cho, A.Y., Arthur, J.R., *Molecular beam epitaxy*, Progress in Solid State Chemistry, **10** (1975), 157–191.

[6] Das Sarma, S., Ghaisas, S.V., *Solid-on-solid rules and models for nonequilibrium growth in 2 + 1 dimensions*, Phys. Rev. Lett. **69** (1992), 3762–3765.

[7] Edwards, S.F., Wilkinson, D.R., *The surface statistics of a granular aggregate*, Proc. Roy. Soc. London Ser. A, **381** (1982), 17–31.

[8] Ehrlich, G. and Hudda, F.G., *Atomic view of surface diffusion: tungsten on tungsten*, J. Chem. Physc. **44** (1966), 1036.

[9] Gilbarg, D., Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 2001.

[10] Gioia, G. and Ortiz, M., *Delamination of compressed thin films*, Adv. Appl. Mech., **33** (1997), 119–192.

[11] Herring, C., *Surface tension as a motivation for sintering*, in: W.E. Kingston (Ed.), The Physics of Powder Metallurgy, McGraw-Hill, New York, 1951.

[12] Jin, W. and Kohn, R.V., *Singular perturbation and the energy of folds*, J. Nonlinear Sci., **10** (2000), 355–390.

[13] King, B.B., Stein, O. and Winkler, M., *A fourth-order parabolic equation modeling epitaxial thin film growth*, J. Math. Anal. Appl. **286** (2003,) 459–490.

[14] Kohn, R. and Müller, S., Surface energy and microstructure in coherent phase transitions. Comm. Pure Appl. Math., **47** (1994), 405–435.

[15] Li, Dong, Qiao, Zhonghua, and Tang, Tao, *Gradient bounds for a thin film epitaxy equation*, J. Differential Equations, **262** (2017), 1720–1746.

[16] Medveď, M., *A new approach to an analysis of Henry type integral inequalities and their Bihari type versions*, J. Math. Anal. Appl. **214** (1997), 349–366.

[17] Miao, C., Yuan, B., and Zhang, B., *Well-posedness of the Cauchy problem for the fractional power dissipative equations*, Nonlinear Analysis: Theory, Methods & Applications, **68**(3) (2008), 461–484.

[18] Moldovan, D. and Golubović, L., *Interfacial coarsening dynamics in epitaxial growth with slope selection*, Phys. Rev. E, **61(6)** (2000), 6190–6214.

[19] Mullins, W.W., *Theory of thermal grooving*, J. Appl. Phys. **28** (1957), 333–339.

[20] Ortiz, M., Repetto, E.A., Si, H., *A continuum model of kinetic roughening and coarsening in thin films*, J. Mech. Phys. Solids, **47** (1999), 697–730.

[21] Schulze, T.P., Kohn, R.V., *A geometric model for coarsening during spiral-mode growth of thin films*, Physica D., **132** (1999), 520–542.

[22] Schwoebel, R.L., *Step motion on crystal surfaces II*, J. Appl. Phys. **40** (1969), 614.

[23] Schwoebel, R.L. and Shipsey, E.J., *Step motion on crystal surfaces*, J. Appl. Phys. **37** (1966), 3682.

[24] Stein, O., *The role of the transition function in a continuum model for kinetic roughening and coarsening in thin films*, Materials science in semiconductor processing, **4** (2001), 405–416.

[25] Tao, T., *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathe- matical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006, Local and global analysis. MR 2233925

[26] Wheeden, R. L. and Zygmund, A., *Measure and Integral: An Introduction to Real Analysis*, Chapman and Hall/CRC; 2nd edition, New York, 2015.