Comparative Study of Linear and Non-Linear Kriging Approaches for Geostatistical Modeling of Gold Grade in Mine Tailings.

by

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ORIGINALITY STATEMENT

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ABSTRACT

The modern mining industry, particularly resource estimation and mine planning, progressively initiates the software development and their application as a part of technology that concentrates its focus on estimating the high-precision models. These models concern the estimation of reserves, the planning of overall slope angle, and many other aspects of mine design, which take into account many different factors. In this regard, different geological units, complexity of ground water level, or shortage of borehole information pose difficulties in estimating the correct model that will be financially feasible and will be approved for exploitation. Thus, geostatistics was evolved for resolving the uncertainties, or raising the confidence of prediction, over the resource estimation to a larger extent. In terms of functionality, the geostatistical tools might be divided into two categories: deterministic and stochastic. The deterministic geostatistical tools are based on modeling single-prediction maps using mathematical equations on well-known datasets. The most common representatives of this category are IDW, trend surface analysis, and spline interpolation. The IDW stands for inverse distance weighting and works on assigning the value based on the inverse of distance from the known dataset, such as grade value or percentage. Trend surface analysis is based on predicting the value at an unknown region by fitting the polynomial function of low-order into a dataset that provides already collected information. Likewise, the spline interpolation is based on fitting the mathematical equation for creating the smooth surface along which the unknown points along the spatial continuity will be predicted. However, the major limitation of the deterministic approach is that methods do not consider spatial uncertainty using a single-prediction map. Thus, the application of this type of geostatistical tool is only applicable for creating a preliminary review of resource allocation and modeling high-prediction maps exclusively for well-sampled regions. The other category of geostatistical modeling stands for stochastic tools, which constitute simulation techniques. The simulation is commonly represented by SGS, SIS, and turning bands simulation. The SGS is described as a sequential Gaussian simulation and is based on creating numerous prediction maps for delivering the optimal unbiased linearly dependent estimate. The SIS is denoted as sequential indicator simulation and works on the same principle as the above-mentioned techniques except for dividing the dataset on several domains. The turning band simulation also works by the principle of creating several maps but is used for predicting spatial variability for continuous datasets. Another deterministic method with stochastic elements is kriging, which uses various models for predicting local error maps with partially

unknown regions or poorly sampled datasets. As geostatistical tools have been thoroughly discussed, the focus of this paper will be centralized on assessing the kriging techniques, which will be elaborated further and compared for creating higher precision maps based on a univariate dataset.

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1. INTRODUCTION

1.1 Background

In a general sense, resource estimation is the vital step in mining, as it concerns different stages and requires high precision-making for the convenient existence of a mining project. To add a point on this case, resource estimation concerns the economic estimation of mines as it is directly linked to the potential expansion of the pit. Similarly, the resource estimation occupies the mine design, investment, and risk evaluation stages. For instance, the correct allocation of resources is important for delivering professional reports by JORC or other internationally acclaimed standards. Thus, resource estimation is required for controlling not only the potential regions but also already processed facilities, such as tailings.

The tailing facilities play a crucial role in reevaluating the profitability of the mine. Since the tailings might represent the processed ore from decades ago, the technologies or cut-off grades might have drastically changed, which could cause speculations regarding the re-processing of this product. In this case, the geostatistical tools being present in the resource estimation might bring about several advantages. As it has been mentioned in economic aspects, the understanding of tailings composition can give a clear understanding of their storage and utilization, which is the benefit from an environmental perspective. Another point is risk assessment, whereas the critical part stands for mud rush or embankment failure. Consequently, the capital expenditures associated with tailing recovery can be mitigated with in-depth resource estimation being applied. Finally, the resulting malfunction of this storage can leave behind permanent hazards. For example, the leakage of contaminants into local groundwater will result in pollution and acidification of nearby farms and villages. Hence, the importance of resource estimation in tailings is the primary concern for the decent functioning of open pit mines.

At this point, various tools have been evolved in geostatistics for dealing with resource estimation. These tools comprise stochastic and deterministic approaches, which allow us to compute high-precision maps for local-mean prediction. To differentiate between these two approaches, it is necessary to explain their concept individually. The stochastic approach stands for generating multiple local-mean maps and establishing the measure of uncertainty. This approach is commonly known with simulation techniques and has several advantages such as a probabilistic approach, accounting for uncertainties, and realistic depiction of resource estimation. However, the major disadvantages of this approach are computational intensity and large dataset availability for producing numerous maps under the simulation procedure. Other geostatistical tools are represented by a deterministic approach, which is described by mathematical interpolation being applied for generating the single local-mean predication map of best estimate at each assigned location. The advantages of this approach are simplicity and good preliminary estimation. The deterministic approach is not considered as reliable as stochastic tools due to several aspects. First of all, the estimation creates only one map, which is associated with the absence of uncertainty measures. Next, the major obstacle that is derived from the first issue is the significant smoothing effect. Since the smoothing effect is defined as the overestimation of lower-tail estimates and underestimation of upper-tail estimates of the variogram, this results in a biased estimation of the tailing reserves. Thus, the central problem of this study will be concerned with identifying the dependency of the smoothing effect with linear and non-linear kriging methods and LUC and its effect on final histogram results.

1.2 Problem Statement

As various kriging techniques have been established, it is important to differentiate between linear and non-linear techniques, which might pose difficulties in representing the results for unbiased resources estimation. The major obstacle for such estimations is presented in the form of assumed stationarity, where the mathematical interpretation of the dataset is supposed to be consistent for the whole region, when in fact that might be unrealistic. This situation might arise when the presence of various lithologies can disturb the grade distribution of the commodity. Hence, the linear dependency might be inconsistent for spatial continuity. As such, the linear kriging methods, such as ordinary and simple kriging, might exhibit detrimental results, which won't give high-precision maps.

As the grade distribution issue has been explained, this will potentially lead to another case of data skewness. Similarly, it might affect the accuracy of deterministic approaches; however, in the case of MGK, transforming the data using Gaussian anamorphosis, this problem is oblique. The data skewness, in general, does not allow creating linear dependency and will result in misinterpretation of the kriging estimates due to unfitting to the variance map.

The major step in the exploratory data analysis of kriging techniques is establishing the parametric model to the variogram of the sampled data. As it was discussed earlier, the skewness of the data and the absence of stationarity might deteriorate the fitting process of the model. Consequently, reaching a low correlation coefficient will cause an unrepresentative estimation.

Since the fitting of the model takes significantly longer time and effort in comparison to modest stochastic approaches as simulation techniques, the computation might also be considered as an obstacle while considering the limitations of the kriging and LUC.

Lastly, the major issue, which is the product of the above-mentioned problems, is the smoothing effect. In other words, the local-mean map resulting from the kriging technique can squeeze the dataset range and give slightly higher small-scale values and lower high-scale values while comparing the histogram with the original dataset. As such, the smoothing effect is considered a central problem for this paper. Therefore, the aim of this study will be focused on conducting linear and non-linear kriging techniques in order to identify their limitations and ability to reduce smoothing effect presence in the context of the given case study.

1.3 Objectives of the Thesis

1.3.1 Main Objectives

1.3.2 To conduct an in-depth literature review on applied kriging techniques and LUC, in particular, to create a strong association between their linearity/non-linearity and limitations

1.3.3 To generate kriging and variance maps for each method, in particular, derive the histograms of the results and compare them with the original dataset on establishing the smoothing effect

1.3.4 To produce LUC and compare the results of Gaussian anamorphosis on the smoothing of the data and take review with kriging results.

1.3.5 Specific Objectives

1.3.6 Establish parametric variogram models that exceed the 0.7 correlation value after cross-validation.

1.3.7 Use the same dataset and parameters for unique neighborhoods along with depth of the projected maps to establish the difference between methods under identical preliminary conditions.

1.3.8 Use the same color grid between the kriging methods to establish the connection between the results.

1.4 Justification of the R&D

Kazakhstan serves as a new emerging centre of exploration. Nowadays, numerous international companies set their exploration units in Kazakhstan due to its potential reserves of copper in the Shu-Sarysu basin. Likewise, existing mine sites currently require high-precision maps of the measured and indicated reserves following JORC/KAZRC due to its importance for foreign investment and mine development. Hence, resource estimation plays a vital role in sustaining these needs. As kriging techniques are complementary to companies' requirements, the accuracy of these tools must be thoroughly investigated in terms of research and development for industry purposes. As such, this study aims to address the issue of resource estimation for enhancing the valorization of existing mine sites by using kriging and LUC geostatistical tools and mitigating the smoothing effect for the representation of high-precision local-mean maps of the reserves.

1.5 Scope of Work

This thesis includes the estimation of SK, OK, MIK, MGK, and LUC geostatistical tools using ISATIS.NEO and MATLAB software are used for calculation purposes, and theme-associated articles are revised to elaborate on each method and its limitations. Also, the consistent part of work was allocated for drafting the paper, which includes setting up the list of tables, list of figures, reference list, along with major components such as, introduction, results, discussion and conclusion.

2. LITERATURE REVIEW

2.1 Resource Estimation and Spatial Interpolation.

The resource estimation is the fundamental stage in the mining industry, which is related to quantifying the mineralization capacity and setting the limits for extracting potential. In a more specific sense, resource estimation is directly linked with defining geological domains, providing statistical analysis of the collected data, and applying various techniques for classifying the grade concentration at the given region (Glacken and Snowden, 2001). Talking about techniques being present under resource estimation, it is important to mention their complexity and variability, which concerns the approach being used for a particular technique, the linearity of the equations, and the parametric models used for this purpose. The most common application of resource estimation in this context is grade averaging, which might be accomplished with triangulations being formed in the mid-point of the intersections, with the grade value being calculated by estimating the arithmetic mean from the value at the vertices of these triangles (Glacken and Snowden, 2001). In terms of variability, the resource estimation of the grade might take into account the weights that are calculated for adjusting the priority of the data concerning distance, lithology, etc. Other techniques that are used, represent inverse distance weighting or declustering options that serve to correct the estimation. In terms of complexity that is present, resource estimation is concerned with the application of geostatistical tools that have been evolved after the French mathematician Matheron in the 1960s (Glacken and Snowden, 2001). The central idea of these tools is to link the connection between spatial relationships with sampled data, which are based on creating the semi-variograms and providing estimated weights for the unknown point or a block of data. Such a concept was further developed into kriging, which represents different variations. Over the years of resource estimation development, these geostatistical tools have become dependent on several factors, such as correct geological models, high computational capability of computer systems, and validation of results. Consequently, the resource estimation does not only depend on geologists but becomes a complex of work and cooperation with mining, metallurgical, and other departments involved in the mining industry.

2.2 Ordinary Kriging

Ordinary kriging (OK) is extensively authorized as an outstanding geostatistical technique for conducting the spatial interpolation. Nevertheless, mutual concerns being driven, this method is highly associated with smoothing effect, which catalyzes the underestimation and overestimation of the original grade values at the higher and lower bounds of the grid. In this instance, Yamamoto (2005) unequivocally addresses this problem by proposing the procedure consisting of four steps. These steps employ enhancement of the reproduction of experimental data through adjustment of statistical properties. The study constitutes that despite the effectiveness of the ordinary kriging in producing unbiased kriging estimates, the spatial interpolation generally decreases the variance and causes smoothing of the grade distribution while considering the difference along with the original dataset. Consequently, such deviations lead to the inadequate representation of the grade propagation, failing to reproduce the actual variance map that imitates the original sampled points.

Likewise, considering the context of the raised issue, Rezaee, Asghari, and Yamamoto (2012) contemplated another innovative approach to curtail the smoothing effect in ordinary kriging estimation. Their case study, which is focused on mineral deposits, portrayed how the corrections in the kriging parameters substantially mitimage the presence of smoothing by increasing accuracy in local-mean prediction maps, whilst preserving the above-mentioned unbiasedness of this particular method. To add a point on this matter, Yamamoto (2000) accentuates the attention on variance during the interpolation process as a complementary measure for uprising the reliability of the ordinary kriging, since it leads to plausible insights into the estimation precision in comparison to traditional approaches.

Proceeding with the smoothing effect problem, Journel, Kyriakidis, and Mao (2000) offered a spectral post-processing algorithm that is explicitly anticipated for adjusting the smoothing effect, which is highly associated with estimates derived from ordinary kriging. While using the environmental and mineral exploration sampled data, the authors have successfully established the geostatistical method's capability to preserve spatial variability lost during the interpolation of the ordinary kriging. Further elaborating on this issue, Yamamoto (2008) contrasted ordinary kriging with sequential Gaussian simulations, marking that ordinary kriging consistently smooths the resulting estimation, evidently lowering the extreme values present in the dataset. The study provided by this author, which serves as a comparison of kriging and simulated maps, clarified the necessity of additional techniques for reaching out higher precision of resource assessment and decision-making.

2.3 Multiple Indicator Kriging

Multiple indicator kriging (MIK) is frequently applied in resource estimation due to its proficiency in dealing with uncertainty and categorical data. Goovaerts (2004) utilized MIK in environmental remediation to optimize sampling designs. His study highlighted that despite the strength of MIK in handling categorical or threshold-based data, the method inherently smooths probability estimates, potentially underrepresenting areas of extreme contamination or resource grade.

As the ordinary kriging, the primary scope of comparison, is based on the linear regression, and indicator kriging stands for non-linear interpolation by splitting the data into indicators, there are a few more non-linear kriging methods that appertain to the latter sub-group of geostatistics. In 2008, a paper presented by Emery brought about the discussion on Multi-Gaussian kriging as a way for predicting local-mean maps. As it was conducted under comparison with simple and ordinary kriging, three case studies were depicted in this work, where the spatial data had a mean value, no mean value, and addition of random drift scenario was accompanied with outcomes of each situation. To begin, the author started explaining the mathematical formulation of each case under the given conditions. In first, the model assumptions stood with mean being 0, and the variance equal to 1, creating the stationarity of the calculation. When no mean value was present, to avoid the biases of the estimation, the mean value was replaced with a random variable constant over spatial continuity. Lastly, adding up the unknown drift creates the expression, where component M, defined as a random drift, is summed up with component U, which is a stationary standard Gaussian random field, following standard Gaussian distribution. The next step was adjusting the MatLab program with the data we received from the conditions of each case and conducting the cross-validation to ensure the accuracy of the fitness of the chosen semivariogram. As a result, one of the major points that have been derived by the authors is that kriging neighborhoods played an important role in the accuracy of the estimation as an excessively close set neighborhood decreases the spatial variance.

Deraisme compared multiple indicator kriging and uniform conditioning (Deraisme, 2016) in estimating recoverable resources, highlighting the method's ability to provide a probabilistic model of resource distribution. However, smoothing effects were evident, particularly in highly variable ore deposits. The study presented detailed analyses demonstrating MIK's capability to manage uncertainty effectively but warned about possible inaccuracies due to smoothing, urging cautious interpretation.

Liu and Kitanidis (2012) explored constrained multiple indicator kriging using sequential quadratic programming, examining its application to thickness estimation in geological deposits. Their results showed MIK’s effectiveness in capturing major spatial patterns, but they also documented that estimates tend to smooth localized anomalies, reducing accuracy in extreme value areas. They concluded that additional constraints or complementary techniques might be necessary to reduce smoothing and improve local accuracy.

Further reinforcing these findings, Vann, Guibal, and Harley (2000) discussed MIK's suitability for different deposit types. Through case studies in mining geology, they identified specific scenarios where smoothing significantly impacted the reliability of recoverable resource estimations, leading to conservative estimations and potential economic miscalculations. The authors highlighted the importance of understanding these smoothing implications to ensure more informed decision-making.

Lastly, Deraisme (2016) revisited the comparison between indicator kriging and uniform conditioning methods. He emphasized that although MIK provided robust probabilistic estimates, the smoothing effect consistently reduced the representativeness of extreme grades or contamination levels, suggesting a combined approach with conditional simulations to mitigate these limitations effectively.

2.4 Multi-Gaussian Kriging

Multi-Gaussian kriging (MGK) is particularly favored for spatial prediction when data exhibit Gaussian characteristics, facilitating probabilistic predictions and uncertainty quantification. Afzal et al. (2015) conducted an extensive case study in the Dardevey iron ore deposit in southeastern Iran, demonstrating how multi-Gaussian kriging effectively delineated mineralized zones while addressing the smoothing effect. Their findings indicated that while MGK provides strong theoretical advantages, particularly in quantifying uncertainty, it still tended toward smoothing spatial variability, potentially obscuring detailed geological features.

Similarly, Emery (2004) demonstrated MGK's applicability in spatial uncertainty modeling within a porphyry copper deposit. Emery's comprehensive study highlighted the strength of MGK in probabilistically quantifying resource uncertainty; however, the results confirmed the existence of smoothing, leading to conservative grade estimations. He proposed supplementary stochastic simulations to address these limitations, underscoring that MGK alone might not suffice for accurate resource estimation in highly variable geological contexts.

Afzal et al. (2018) expanded MGK's applications to mapping rock quality designation (RQD) in phosphate ore deposits. Their research emphasized the smoothing effect's practical implications, noting significant differences between estimated and actual measured RQD values. Despite this drawback, they underscored MGK's strengths in probabilistic estimation, suggesting its use as a preliminary predictive tool supplemented by field verification to improve reliability.

Kim et al. (2016) explored the temporal dimension by applying multi-Gaussian space-time kriging to map particulate matter (PM10) concentrations in Seoul. The smoothing effect was again noted, particularly in short-term predictions, as high pollution events were frequently underestimated. The study highlighted MGK's potential but recommended it be combined with other predictive techniques or hybrid models to enhance forecast accuracy and reduce inherent smoothing.

Lastly, Afzal et al. (2017) investigated MGK's role in geostatistical analysis of 3D magnetic inversion results at the Darreh-Ziarat Iron Deposit. The authors found that MGK substantially improved the estimation of iron ore grades compared to traditional methods; however, smoothing still occurred, potentially impacting resource classification accuracy. They suggested integrating conditional simulations to capture spatial variability better and reduce uncertainty related to smoothing.

2.5 Localization and Uniform Conditioning

Localized Uniform Conditioning (LUC) is an advanced geostatistical technique designed to enhance the accuracy and detail of mineral resource estimates, particularly in scenarios where sampling data is sparse relative to the desired estimation scale. Developed by Abzalov (2006), LUC directly addresses limitations inherent in traditional Uniform Conditioning (UC), notably the smoothing effect—where spatial interpolation techniques tend to reduce local variability, underestimating extreme values and localized resource richness.

Taking into account all the nuances of kriging techniques being discussed, it is important to consider the relatively modern variation of this method, where block anamorphosis and panel grade estimation play a vital role in correct estimation. Thus, in 2006, the localized uniform conditioning (LUC) method was developed by Abzalov for alternative uniform conditioning that works on grade dissemination over the large mining panels. This approach was further introduced in 2016 by Asghari et al., where the concept and potential limitations of LUC have been investigated. To begin with, localized uniform conditioning is based on partitioning the panel into SMUs (selective mining units), which in this regard serve as a portion of the whole block proportioned by ore and waste, ranking this units into ranks based on grade, in a sense similar to MIK, and localizing the resulted blocks to the grid size of the choice. As the distribution of grades might not follow uniform conditions and symmetry, the block ranking poses a problem when the kriging technique is applied. As such, the application of block anamorphosis serves as a prerequisite for conducting this method. To add a point on a case, the block anamorphosis might be achieved through a discrete Gaussian model (DGM), which is used for transforming the thresholds into normal scores before the ranking. Now, as the foundation of this technique was elaborated, the primary steps that comprise the uniform conditioning and localization include, but not limited to estimation of panel grades using ordinary kriging, fitting the DGM, deriving support correction estimators for blocks, transforming the block grades into normal scores, and calculating the cut-offs along with local mean prediction maps, metal quantities for each panel. In the provided case, the authors illuminated the observed geology of the region, the drillhole data that was collected, and histogram charts on the grade and frequency of copper, molybdenum, and copper oxide as primary commodities of interest. Adjusting the SMUs and interpreting the necessary steps, the technique portrayed affirmative output, where the 3D block models of the metal quantities have been presented. From these models, it is reasonably concluded that the discretization of blocks in the grid creates the support effect, with higher error in variance and lower selectivity in terms of large size blocks. If the assigned block size does not follow the parameters of SMUs, then a smoothing effect takes place in the resource estimation. Another point was made regarding the low-range grades that have been intensively affected by the support effect.

Abzalov's seminal paper (2006) laid the foundational theory of LUC, introducing the methodology as a practical approach for achieving improved localized estimates of small mining blocks, termed Selective Mining Units (SMUs). Unlike traditional UC, which often produces estimates smoothed across larger blocks, LUC incorporates a localization step that mitigates this effect, offering more precise resource estimations. The smoothing issue is critical in resource modeling as it can significantly impact economic evaluation, mine planning, and operational decision-making. Abzalov demonstrated LUC's effectiveness through multiple simulated and real-world datasets, revealing enhanced precision in capturing spatial variability, particularly within high-grade zones, thereby reducing the conservativeness and inaccuracies typically associated with the smoothing phenomenon.

Building upon this work, Abzalov (2014) provided further validation of LUC through a series of practical application case studies across diverse geological settings. These case studies notably included gold deposits in West Africa and nickel laterite deposits in Australia. The studies consistently illustrated that LUC outperformed traditional UC by capturing more realistic distributions of mineralization. Specifically, the case of nickel laterite demonstrated that traditional UC significantly smoothed local variability, leading to systematic underestimation of high-grade zones. Conversely, applying LUC allowed for the accurate delineation of these high-grade zones, improving both short-term planning and long-term resource reliability. This study underscored LUC's ability to offer both precise and practical solutions for resource estimation challenges.

Further advancements were proposed by Deraisme and Assibey-Bonsu (2011), who expanded LUC into multivariate settings through a case study involving a porphyry copper-gold deposit. This research demonstrated the versatility and robustness of LUC, particularly its capability to simultaneously manage multiple variables, such as copper and gold grades. In doing so, the method effectively reduced the smoothing effect seen in single-variable interpolation techniques. The authors highlighted significant improvements in grade-tonnage predictions, enabling better-informed economic evaluations and mine planning decisions. This multivariate application showcased how LUC effectively maintains local grade variability and reduces smoothing-related underestimation risks.

Across all three kriging methods—ordinary kriging, multiple indicator kriging, and multi-Gaussian kriging—the reviewed literature consistently highlights the pervasive challenge of the smoothing effect. Each method offers robust theoretical and practical benefits, but each also inherently smooths spatial variability, potentially underestimating extremes and influencing resource estimation reliability. These studies collectively underscore the importance of supplementary methods, constraints, or application of kriging techniques to mitigate smoothing and enhance estimation accuracy in various applications. Likewise, the articles that served as a reference on application of localization and uniform conditioning (LUC) encompass its strength in reducing the smoothing effect inherent to conventional estimation techniques, providing more accurate, localized, and reliable resource estimates. By preserving spatial variability and effectively managing local extremes, LUC significantly enhances the quality and practicality of resource evaluations across various mineral deposit types. Hence, as the research of this thesis will be concerned with gold commodity, the primary focus will be translated on higher grade values, and potential use of this technique will be considered along with kriging tools.

3. METHODOLOGY

This thesis presents the comparison between linear/non-linear kriging techniques and localized uniform conditioning in mitigating the smoothing effect of the univariate dataset. The methodology starts with explaining each technique individually. The sequence of all methods follows the exploratory data analysis review that constitutes the statistical information gathering and variogram fitting. Next, the geostatistical tools will be elaborated on their transformation of the data, which concerns MIK, MGK, and LUC. Finally, the application of equations for each particular method will be derived before the results section. As all of the presented methods use an ordinary model for neighborhood search and point-based estimation, this part will be excluded from the methodology explanation.

3.1 Ordinary Kriging

1. Exploratory data analysis serves as a primary step in ordinary kriging, which is associated with extracting descriptive statistics, noisy and smooth patterns in concentration distribution through the base map along with defining spatial continuity issues. In terms of descriptive statistics, the EDA provides an opportunity to project the mean, median or skewness, insert the grade dataset into box plot and scatter plot options and bring the estimation for transferring the data into normal score. Besides, initial spatial correlation of the sampled points can be evaluated via various direction

settings, whether omnidirectional or multi-directional with managing the number of regular directions.

2. Variogram analysis is the next step of the methodology, which is related to the direct implementation of variogram parameters for the most precise fitting of the theoretical variogram. In this stage it is required to establish the number of nested structures, lag and maximum distances and decide on the stationarity model. The general equation for construction of the semivariance at this stage is conceptualized with the following formula:

   $$\gamma(h) = 1/2N(h) * \Sigma_{i=1}^{N(h)}[z(x_i) - z(x_i + h)]^2 \quad (1)$$

   Where,

   $\gamma(h)$- variance at the lag distance h.

   $N(h)$ - the number of lags.

   $z(x_i)$- observed value at location.

   $z(x_i + h)$ - observed value at distance h from initially observed point.

3. The subsequent step is related to assessing the geostatistical set being formed at the EDA. In particular, there are numerous statistical indicators, such as mean error (ME), root mean square error (RMSE), and Pearson correlation factor (R^2), which serve for identifying the linear dependency of the original variogram with the theoretical one. For the sake of simplicity, all methods will follow Pearson correlation factor to justify the application of techniques and their fitness. In the case of this study, the Pearson correlation must be achieved to proceed with the kriging estimation. The value of 0.8 is taken as a minimum correlation required for unbiased estimation.

4. In case of successful iteration of the cross-validation stage, the procedure moves towards estimating the kriging weights, which are used in kriging interpolation for local-mean prediction map construction. This step constitutes solving the matrix, and afterwards proceeding with assigning grade values using the equation below:

   $$z^*(x) = \Sigma_{i=1}^{n} \lambda_i z(x_i) \qquad (2)$$

   Where,

   $z^*(x)$ - predicted value at unsampled point,

   λ_i- weight being assigned to the estimated point,

   $z(x_i)$ - measured value of grade at sampled point.

5. As the kriging estimates have been calculated, the local mean prediction maps are formed and displayed in the 3D map in order to visualize the trends, distribution of high grade concentration and to discover the regions with inefficiently sampled data. As the data is not uniformly collected and declustering has not been adjusted, the variance map is included in this stage for inspecting the variability across the grid.
6. Validation of the resulting maps is performed with the application of descriptive statistics, which is derived in the form of histogram and compared with histogram of the original dataset. In such a scenario, the maximum value of the grade in the predicted map will advocate for smoothing effect presence and the degree of its mitigation.

   This methodological framework ensures accurate and reliable spatial predictions, supporting robust resource characterization and informed decision-making.

3.2 The Multiple-Indicator Kriging

1. The EDA commences the MIK process by revising the statistical information, which includes quartiles and median identification, histogram creation, and trend lining the probability and cumulative distribution functions. All of these units comprise the visual elements that serve for evaluation of potential outliers, skewness and variability.

In short, this stage of methodology is commonly affiliated with each interpolation technique.

2. Data transformation into binary indicators based on threshold values (z_k) is essential in MIK. Indicators are defined as:

$$I(x; z_k) = \{ 1, \ if \ z(x) \leq z_k; \ 0, \ Otherwise \quad (3)$$

Where,

$z(x)$ - the value at the sampled location,

z_k - the trimming value of the given indicator,

$I(x; z_k)$ - the indicator's probabilistic output.

3. The spatial continuity of thresholds is trimmed by allowed range of each indicator and the classical variogram for them is based on the following equation:

$$\gamma(h; z_k) = 1/2N(h; z_k) * \Sigma_{i=1}^{N(N)} [z(x_i; z_k) - I(x_i + h; z_k)]^2 \quad (4)$$

where $\gamma(h; z_k)$ denotes semivariance at lag distance for threshold, and N is the number of data pairs separated by distance. Theoretical models (e.g., spherical, exponential) are fitted to the experimental variogram.

4. The cross-validation part, as it was annotated in ordinary kriging section, concerns exclusively the R^2 correlation factor. The prediction of the accuracy for variogram fitness is directly linked with the factor this validation provides. In this regard, higher values will denote the percentage of experimental variogram that matches the theoretical data from the variogram.

5. MIK predicts conditional cumulative distribution functions (CCDFs) at unsampled locations through indicator estimates:

$$I^{*}(x_0; z_k) = \Sigma_{i=1}^{n} \lambda_i I(x_i; z_k) \quad (5)$$

Weights are obtained by solving the kriging system for each indicator threshold:

$$\Sigma_{i=1}^{n} \lambda_i \gamma(x_i - x_j; z_k) + \mu(z_k) = \gamma(x_i - x_0; z_k),\ with\ \Sigma_{i=1}^{n} \lambda_i(z_k) = 1 \quad (6)$$

where $\mu(z_k)$ is the Lagrange multiplier ensuring unbiasedness.

6. The resulting CCDFs at each unsampled location are used to estimate resource quantities, probabilities exceeding given thresholds, and uncertainty maps. The visualization and quantification tools are used to interpret results and derive actionable resource classifications and risk assessments. This stage of methodology is related to the post-processing of the results. Thus, the indicator values of 1 and 0 are converted to grades and provide us with the local mean prediction maps of the deposit.

7. Likewise, the procedure for MIK in terms of validation is the same with OK, whereas the validation process is designated with comparing histograms of original and predicted maps. As the minimum value is not a central concern, from the economical perspective of the resource estimation, the maximum values are exposed for evaluation on smoothing effect.

This methodological approach using Multiple-Indicator Kriging provides comprehensive uncertainty quantification and robust resource estimation, aiding effective exploration and informed decision-making.

3.3 Multi-Gaussian Kriging

The following method to review is multi-Gaussian kriging, which is based on transforming the raw data into a normal score, creating a non-skewed distribution, and performing the unbiased kriging estimates after back-transformation. In general, this method is significantly distinguished from previous techniques because the MGK directly conducts estimation with normal scores and only after that performs back-transformation. To fully elaborate the procedure, the MGK computation consists of the following steps:

1. Convert the raw scores to z* normal scores. The conversion is based on a formula

   $$Y(x) = \phi^{-1}(F_z(Z(x))) \quad (7)$$

   Where,

   $F_z(Z(x)$ = CDF of original scores

   ϕ^{-1} = inverse standard normal CDF

   $Y(x)$ = standard normal variable

   This step is required to avoid skewness of the data, which poses difficulties related to the smoothing effect.
2. Create the experimental variogram by setting the values. The setup proceeded in Matlab software to fill in the coordinates information, columns of normal score data, trimming limits of the data, and directions of the neighborhood search. Finally, this stage of the procedure will give us traditional variogram, madogram, and rodogram graphs.
3. The next step requires us to validate the variogram of order "w". This order "w" is an important parameter in assessing the sensitivity of variograms to extreme values. To add a broader explanation, the standard variogram follows the second-order, which is represented $\gamma(h) = 1/2 * Var[Z(x + h) - Z(x)]$ by rewriting eq 3., while the variance of order w is represented as $\gamma_w(h) = 1/2 * Var[|Z(x + h) - Z(x)|^w]$. In MGK, the variogram of order "w" is widely used to model nonlinear spatial relationships. As in the case of simple and ordinary kriging, the second-order variogram is used, and the cross-validation might result in discrepancies and inability to interpret the dependency between datasets due to low Pearson correlation. In this

study, we used previously derived madogram and rodogram to identify the required order of variogram. Shortly speaking, the order of 1 will be used to test the skewed data, whilst lower orders, such as 0.5, will help us to analyze how data behave in a smoother representation of spatial variability.

4. After assessing the behavior of the dataset under different orders, we proceed with building a theoretical variogram over the experimental variogram that is constructed with Gaussian scores. In this step, the software is required to estimate the number of nested structures, lag distances, and maximum distances, as well as to define parametric models that will be used for reaching the highest correlation.
5. The final step is conducting our kriging estimation. In this part, we need to add information regarding the grid dimensions, block discretization, trimming values of the original data, neighborhood search area and form, the kriging variation, and several realizations for this particular search. Worth noting that the parameters must be identical to the previous techniques to make a realistic comparison between these geostatistical tools. After receiving the kriging estimates, the values must be back-transformed by using the following formula:

$$Z^*(x) = F_Z^{-1}(\phi(Y(x^*))) \quad (8)$$

Thus, the resulting estimates must provide the tables on local intervals, mean, quantiles, recovery, and variance map.

3.4 Localization and Uniform Conditioning

Localization and Uniform Conditioning (LUC) serves as an advanced geostatistical technique that is widely applied in the mining industry for resource estimation. It is designed for a more accurate representation of grade distribution along the panel with the application of the ranking option, which is similar to MIK's threshold limiting. The following steps constitute the methodology that must be applied for formulating the correct estimation of local mean prediction maps.

1. The primary step in LUC is exploratory data analysis that is required for assessing the distribution of the sampled points via basemaps, displaying statistical information on

median, mean, and quartiles of the grade concentration as well as building probability distribution function of the original data. In a general perspective, this step allows adjusting the descriptive statistics, establishing visualization tools, such as histograms or box plots, and tracking the trends with outliers, which are the cause of the smoothing effect- the central problem of this thesis.

2. The next step is setting up the support correction, which is related to creating Gaussian anamorphosis. It is exploited for transforming the raw sampled data that might contain skewness into normal score distribution following the Gaussian distribution. This transformation employs the application of the geostatisical technique to read the distribution of the data, assuming the normality of the data, where the variability follows the value of 1. The Gaussian anamorphosis function $Y = \phi(Z)$ transforms the sampled dataset set Z into a normal score Y.
3. Afterward, we proceeded with variogram analysis, where the semivariogram emerged after applying the following formula:

$$\gamma(h) = 1/2N(h) * \Sigma_{i=1}^{N(h)}[z(x_i) - z(x_i + h)]^2 \quad (9)$$

4. Since the variogram fitting and EDA have been successfully implemented, the subsequent stage is concerned with setting up the QTM, which stands for quantity of metal, tonnages, and metal quantity. These parameters serve as limiting factors while ranking the kriging estimates and constitute uniform conditioning.
5. The localization procedure is the next step when the UC procedure has been established, along with valid QTM variables. It is used for generating the detailed resource maps that describe the prediction locally along the panel. The localization formula is $Z_{SMU} = \phi^{-1}(F_{UC}^{-1}(p))$ (10)

Where,

Z_{SMU} - estimated grade for each small mining unit.

ϕ^{-1} - inverse Gaussian Anamorphosis,

F_{UC}^{-1} - inverse cumulative distribution derived from uniform conditioning.

p - probability associated with SMU's rank.

6. The final stage is conducting the cross-validation of the results to check the precision of the resulting estimates. Similar to previous methods, the Pearson correlation must be derived for assessing the effectiveness of the variogram fit. In this regard, the values above 0.8 will advocate for the correctness of the estimation, where more than 80 percent of the experimental points follow the best-fit line.
7. The vertex of this procedure will be the display of the interpolation results in the 3D map. This map is supposed to visualize the distribution of the grade along the grid and give an understanding of effective resource characterization. Displayed maps beside the interpolation of the grade itself will also represent the QTM maps and variance map of this dataset.
8. Finally, we validate the resulting histogram with the original one to establish the smoothing effect and proximity between them.

4. RESULTS

4.1 Case Study

The above-described methodology is related to the former Haveri tailing storage facility that contained metalliferous commodities. The dataset is based on borehole data with 1,114 sample points composited in 1 meter length that constitute 10 elements: Ag (ppm), As (ppm), Au (ppm), Co (ppm), Cu (ppm), Fe (%), Ni (ppm), Pb (ppm), S (%), and Zn (ppm). In addition to grade concentrations, the dataset contains the coordinates, hole id, the composition length, and the year of deriving the information into the database. The primary elements of this study are gold and copper, whilst gold will be used for univariate geostatistical modeling with the application of OK, MIK, MGK, and LUC techniques. Hence, the other elements will not be tested and presented as an overview of tailing content.

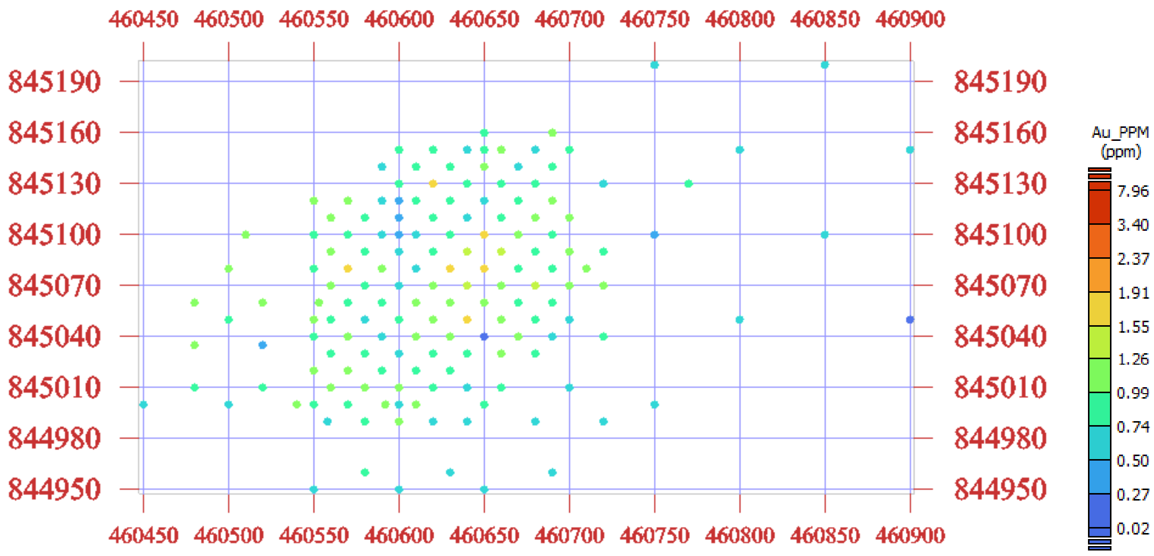


Figure 1. The basemap in 3D projection of the drillhole data of the Au (ppm).

4.2 Ordinary Kriging

The estimation process begins with producing the commonly used ordinary kriging. As such, this section presents the derived geostatistical set for variogram, associated cross-validation, and local prediction maps on variance and kriging mean estimates.

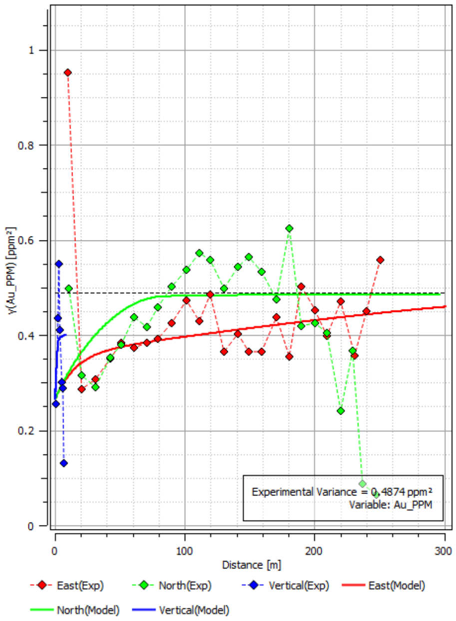


Figure 2. The multi-directional variogram is based on a raw model for ordinary kriging.

The variogram analysis that is present in figure 2 is based on a gold grade dataset associated with 2 regular directions with lag 10m and maximum distance of 255m for horizontal direction and lag 0.5m and maximum distance 8m for vertical direction. The model option in this case is a raw model only with intrinsic stationarity for ordinary kriging purposes. The variogram consists of only one nested structure that is fitted with the exponential model.

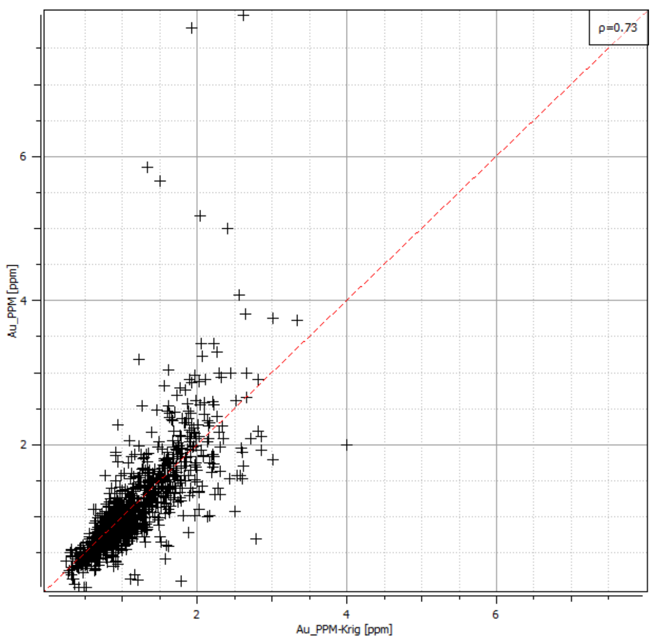


Figure 3. The cross-validation results with Pearson correlation factor for ordinary kriging.

The given cross-validation graph presented in Figure 3 illustrates the correlation of 0.73 for the Pearson factor, where the testing was based on the Au_ppm (Raw)-Ok geostatistical set that was saved after variogram fitting.

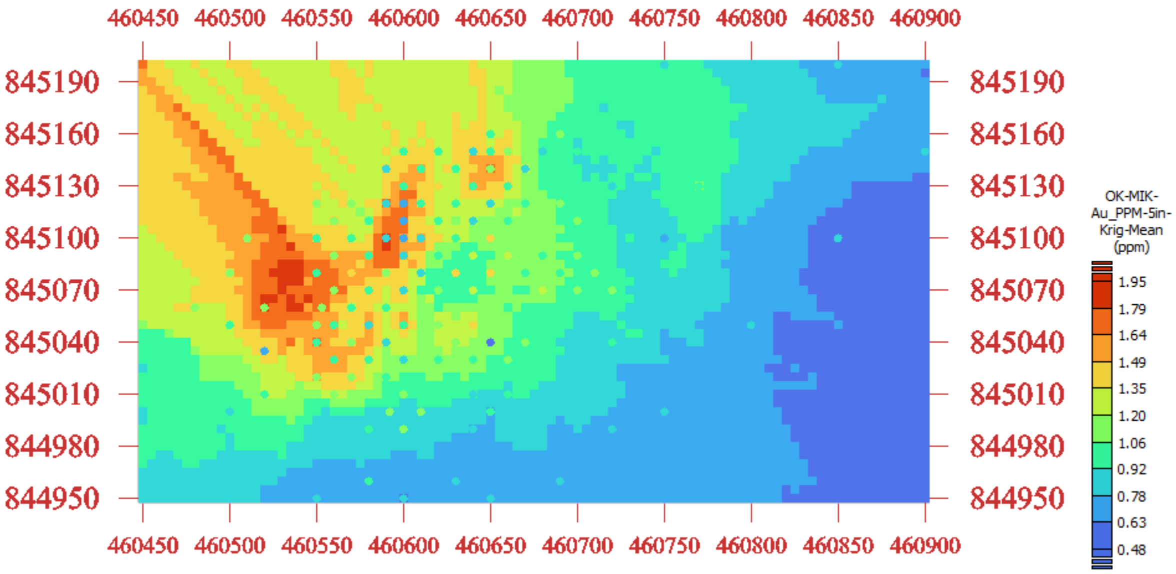


Figure 4. The kriging map for ordinary kriging.

The Figure 4 portrays the grade distribution of the gold commodity along the grid with inserted parameters 5*5*1 mesh size corresponding to x,y, and z dimensions. The adjusted property of this figure stands for the 4th slice being projected along the z direction.

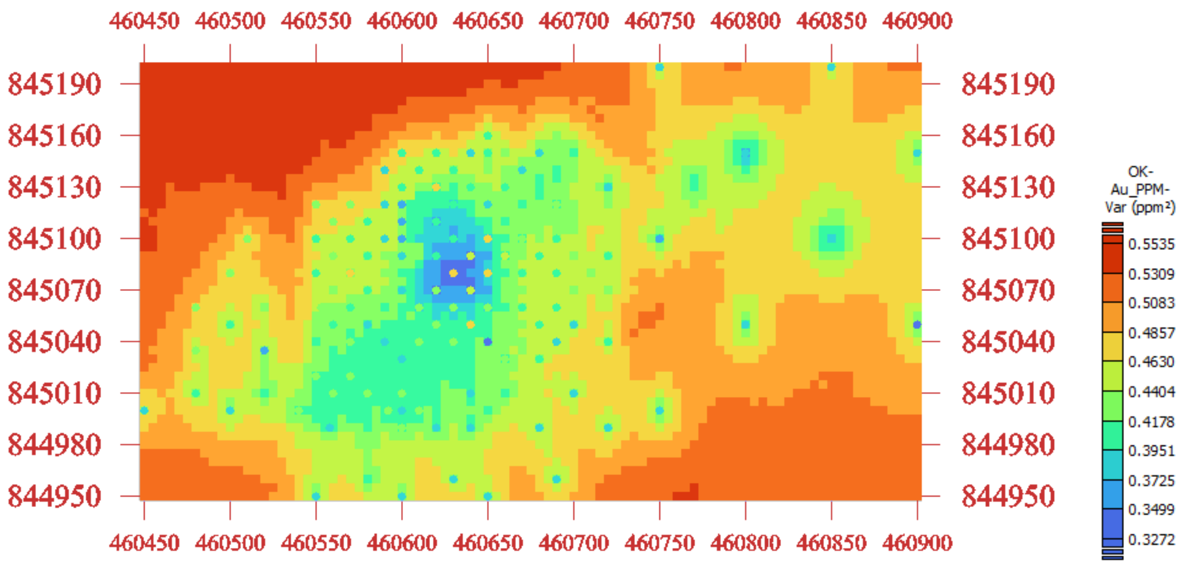


Figure 5. The variance map for ordinary kriging.

Figure 5 gives a visual representation of the variability of gold grades along the above-mentioned grid. This figure has adjusted properties with the 4th slice along the z direction as well as kriging results being presented.

4.3 Multiple Indicator Kriging (MIK)

The study of MIK begins with setting up thresholds before the exploratory data analysis. In this case, we tested 5, 7, and 9 indicators. Figure 6 with cases (a-c) presented below show that indicator values' CDF matches the cumulative distribution of the original dataset.

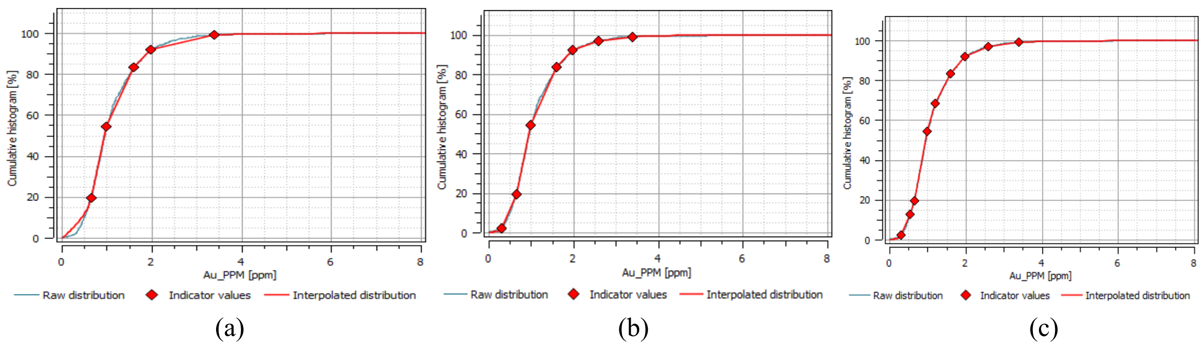


Figure 6. The cumulative distribution function of 5-7-9- threshold number for multiple indicator kriging. (a) 5 indicator CDF, (b) 7 indicator CDF, (c) 9 indicator CDF.

Figure 6 describes the cumulative distribution functions for different numbers of thresholds. In case (a), the function has a 0.65 exponent factor and Beta Power model for lower tail, and 3.4 exponent factor with Hyperbolic model for upper tail. In cases (b) and (c), the function has a 0.31 exponent factor and Beta Power model for lower tail, and 3.4 exponent factor with Hyperbolic model for upper tail. The significance of these exponent factors stands for the minimum and maximum threshold present for the given CDFs. The decision for experimental fitting models was based on the highest proximity to the CDF of the sampled data.

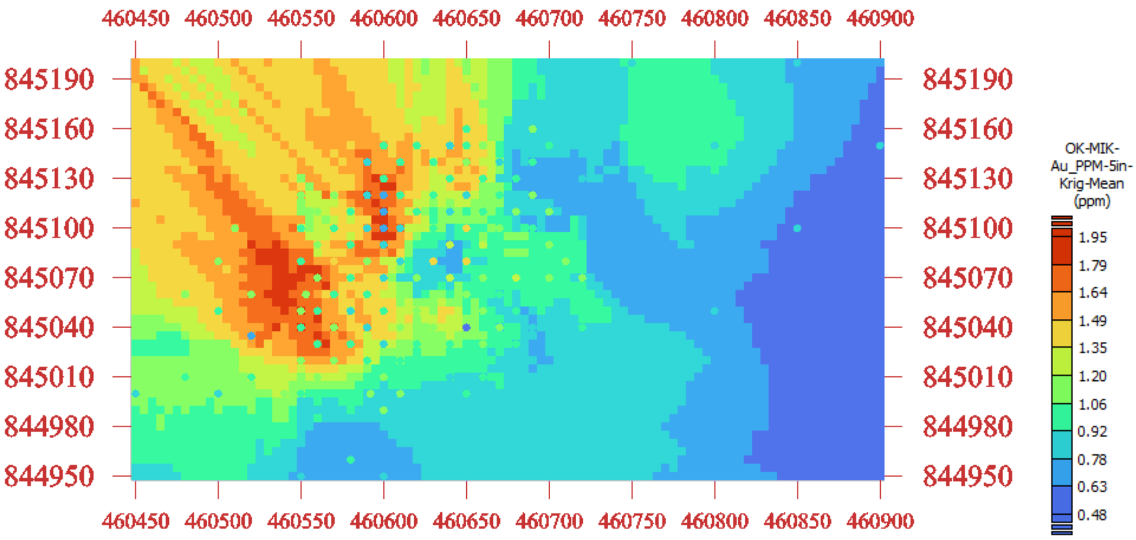


Figure 7. The post-processing results of a local-mean prediction map with 5 indicators.

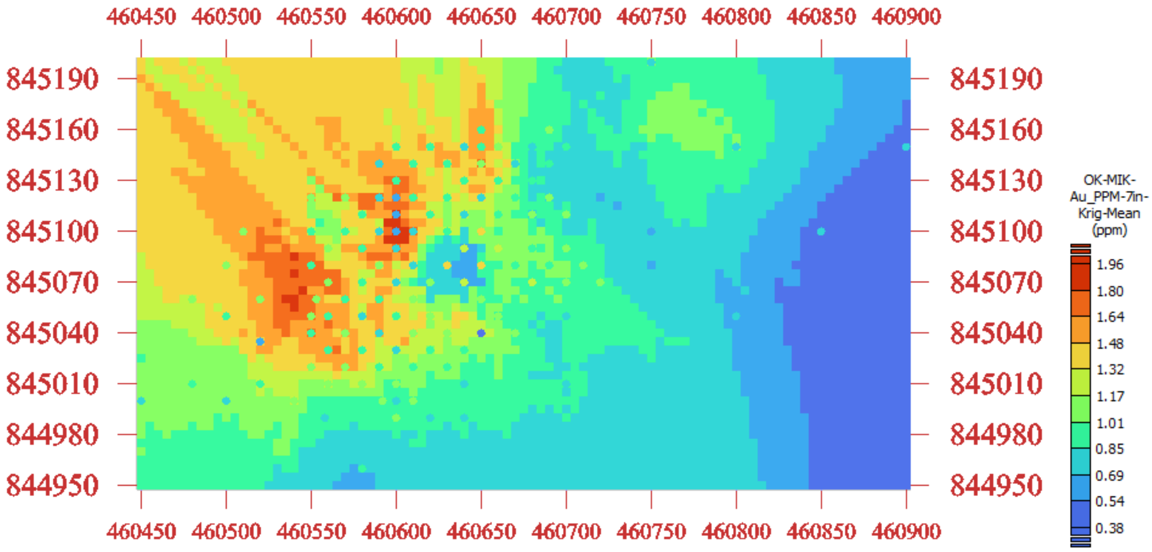


Figure 8. The post-processing results of a local-mean prediction map with 7 indicators.

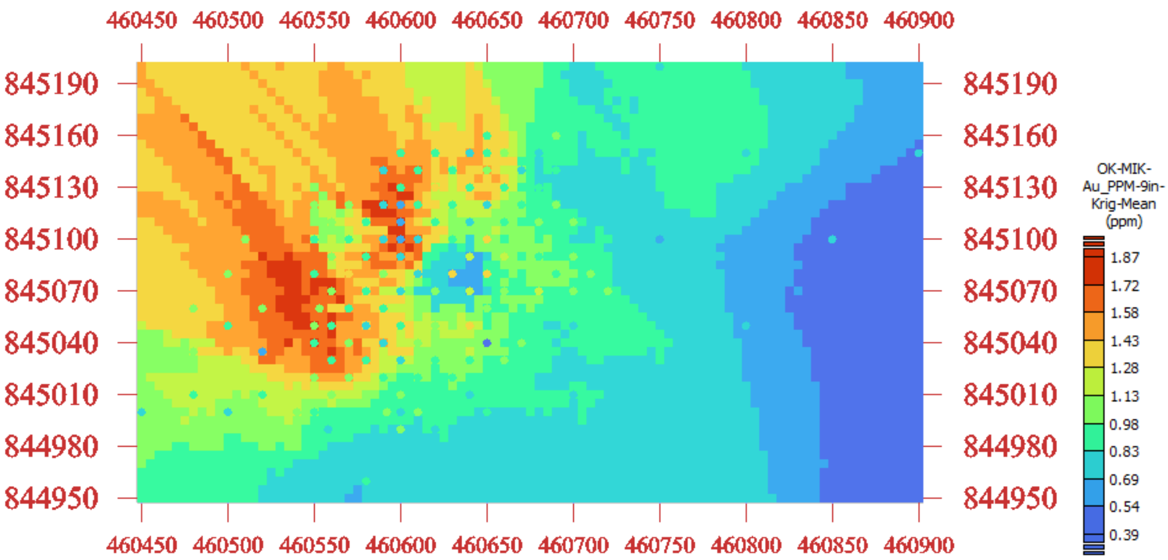


Figure 9. The post-processing results of a local-mean prediction map with 9 indicators.

The above-presented Figures 7-9 illustrate the post-processing of the multiple-indicator kriging, which means the kriging estimation is calculated following the ordinary kriging procedure, where weights were calculated in the pre-processing stage with indicator values and their probability in threshold. All three figures express higher grade distribution in the upper left angle.

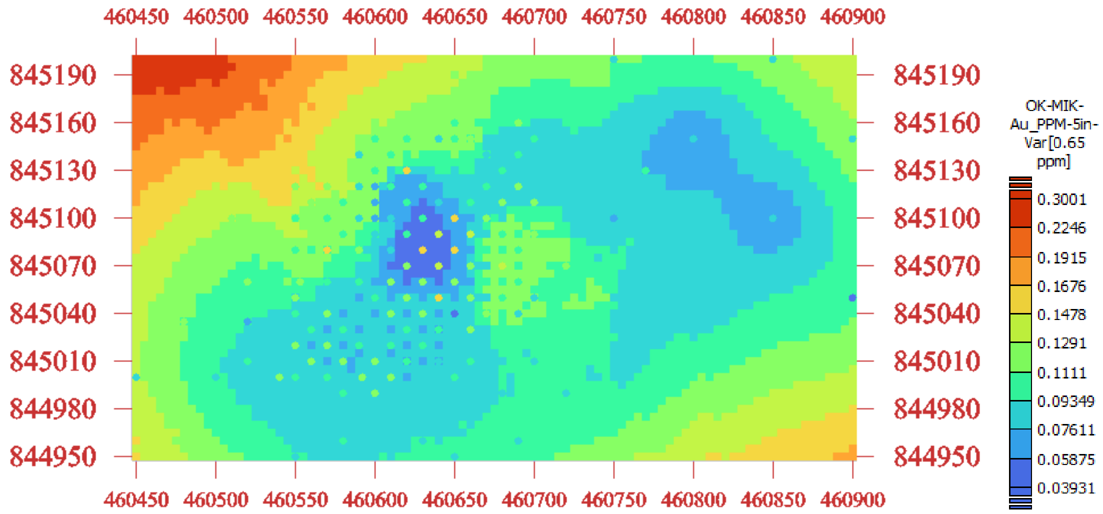


Figure 10. The variance map of the 5-indicator MIK.

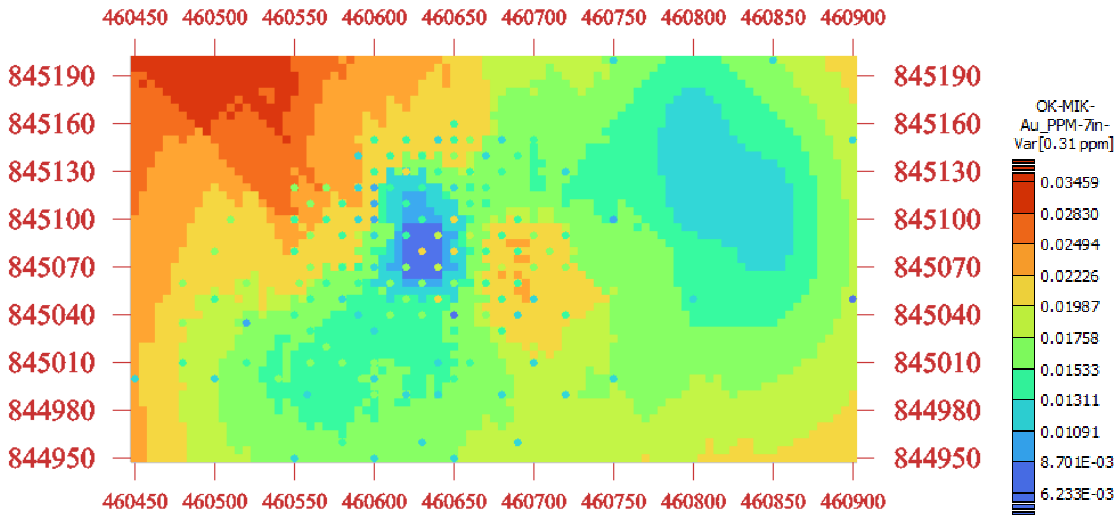


Figure 11. The variance map of the 7-indicator MIK.

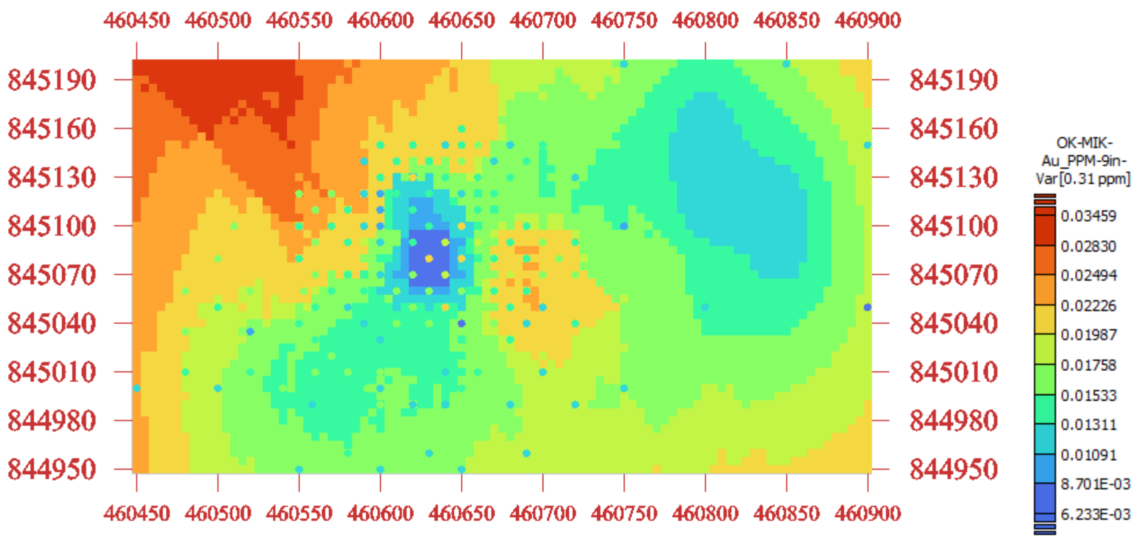
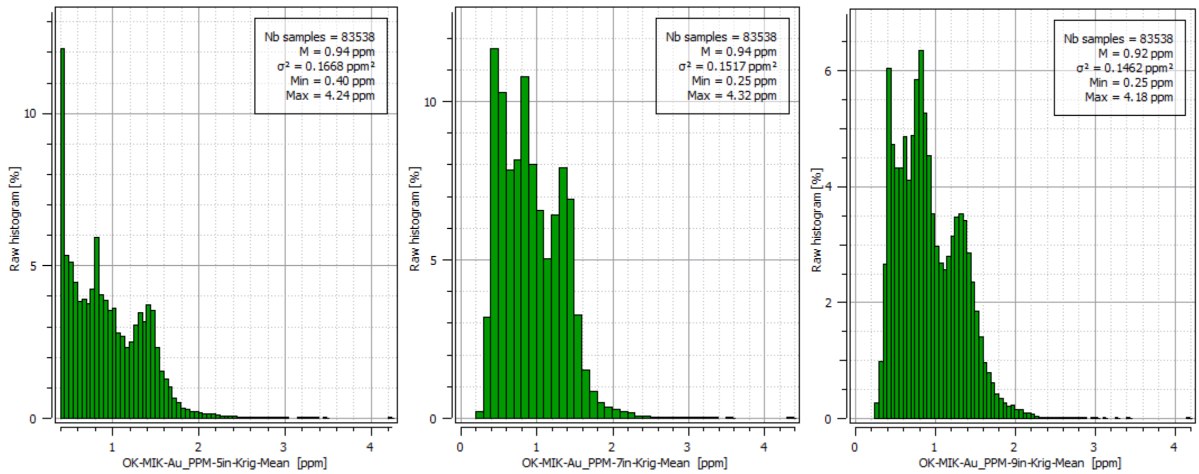


Figure 12. The variance map of the 9-indicator MIK.

The following three maps, namely Figure 10-12, exhibit the variability of grade across the grid. Figure 10 shows the variance map of the gold grade variability with 5 threshold numbers. Likewise, Figure 11 and Figure 12 reproduce variance maps with additional 2-indicator increment in number of indicators with 7 and 9 thresholds, respectively. From these pictures, it is evident that the higher variability is persistent at the corners of the grid, where the sampled points were in scarcity.

From the received post-processing results, we construct the histograms for evaluating the smoothing effect. These are the histograms of the resulting MIK with 5, 7, and 9 indicators:



(a) (b) (c)

Figure 13. The raw histogram results of 5-7-9- indicator MIK. (a) The 5-indicator MIK histogram. (b) the 7 indicator MIK histogram. (c) the 9-indicator MIK histogram.

The portrayed histograms in Figure 13 show the maximum grade prediction along the grid. The 7 threshold kriging possesses the highest value of 4.32 ppm, whereas the highest variance was exhibited by 5 threshold kriging.

4.4 Localization and Uniform Conditioning

The exploratory data analysis of the localization and uniform conditioning requires building the variogram based on raw and Gaussian data. Hence, after creating the probabilistic distribution with Gaussian data, we were able to obtain the following variogram:

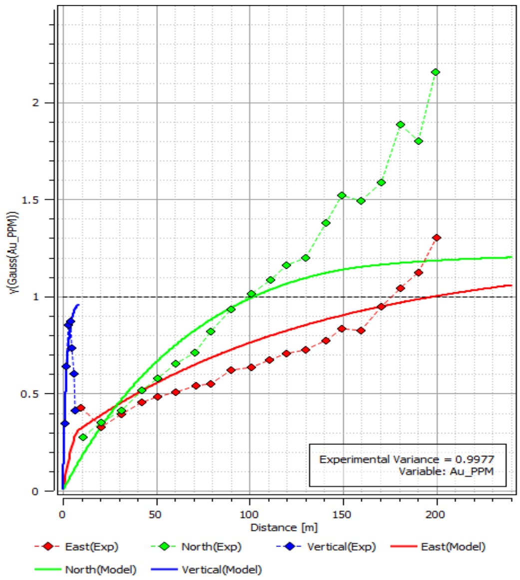


Figure 14. The multi-directional variogram is based on raw and Gaussian models for localization and uniform conditioning.

This variogram in Figure 14 is based on 2 regular directions with lag 9m and maximum distance 200m for horizontal direction and lag 0.5m and maximum distance 8m for vertical direction. This variogram consists of 2 nested structures, which are spherical and exponential models.

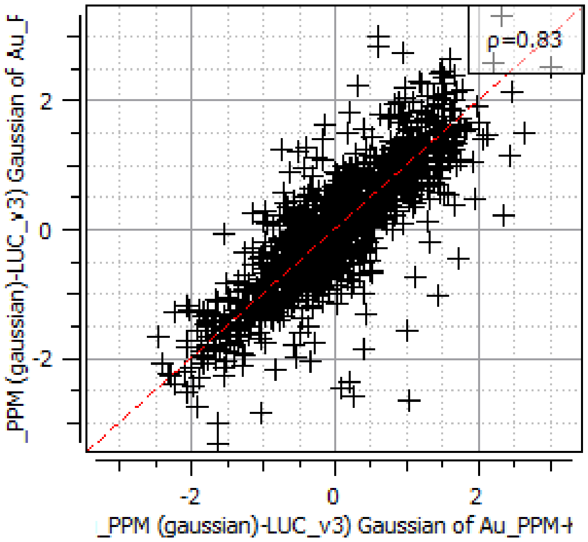
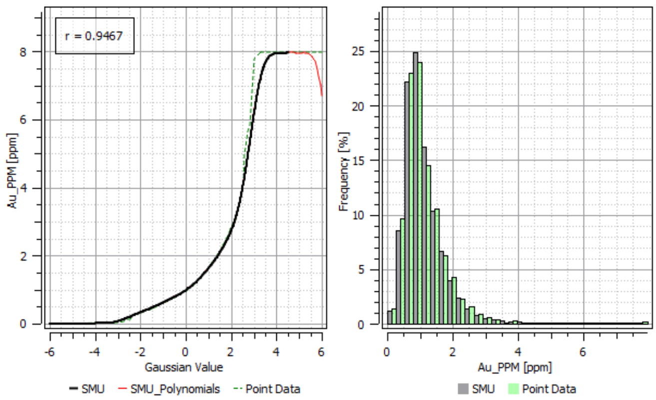
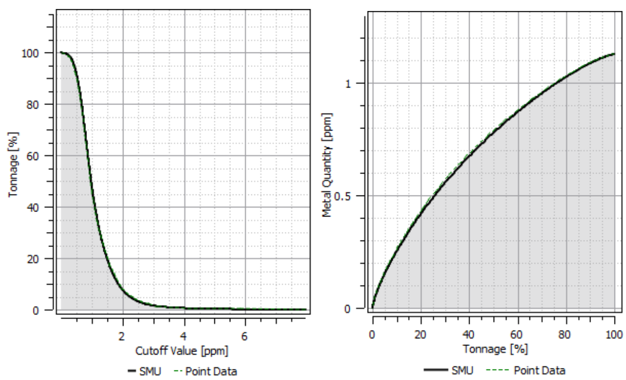


Figure 15. The cross-validation results of the raw-Gaussian variogram with Pearson correlation factor.

Following the cross-validation, we set the support correction for our model as it was described in the methodology section. In this part, the Gaussian anamorphosis will be adjusted with Hermite polynomials, and other parameters, such as the quantity of metal and tonnage, will be projected for subsequent ranking of the SMUs along the panel. These data are presented in the form of Figure 16 below:



(a) (b)



(c) d)

Figure 16. (a) The Gaussian block anamorphosis and Hermite model fitting with support discretization 10*10*1, mean value 1.13 ppm and variance 0.42 ppm^2. (b) The distribution of SMU grades along with the sampled dataset. (c) The Tonnage curve in respect to cut off value of the thresholds. (d) Mean metal quantity graph distribution with regards to tonnage compared with the original dataset.

Figure (16a) shows the above-mentioned Hermite polynomials that were used to adjust the function for Gaussian Anamorphosis. It is worth noting that the r factor is 0.9467, which stands for the ratio of raw data transformed to Gaussian. Figure (16b) represents the histogram of small mining units and original data. Figure (16c) depicts the cutoff value to the tonnage, explaining the distribution of metric tons of metal to grade concentration. Lastly, Figure (16d) portrays the graph of metal quantity to tonnage.

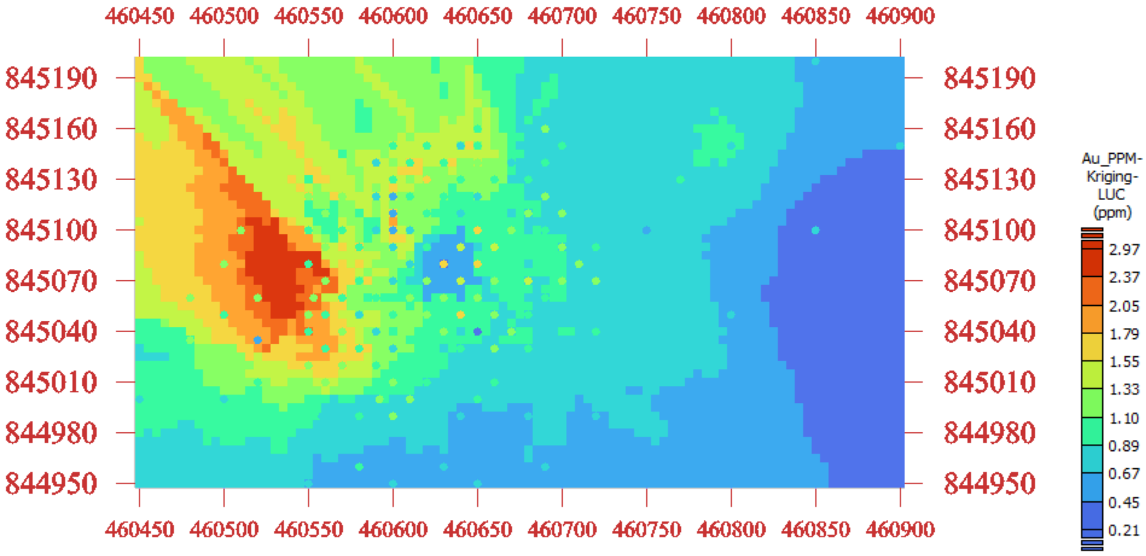


Figure 17. The LUC local-mean prediction map after kriging.

Figure 17 provides visual distribution of the gold grade over the grid. The slicing parameters were preserved for all methods to be identical, hence, this picture was made for the 4th section along the z axis. The resulting map also exhibits a positive trend for high-grade concentration at the left corner.

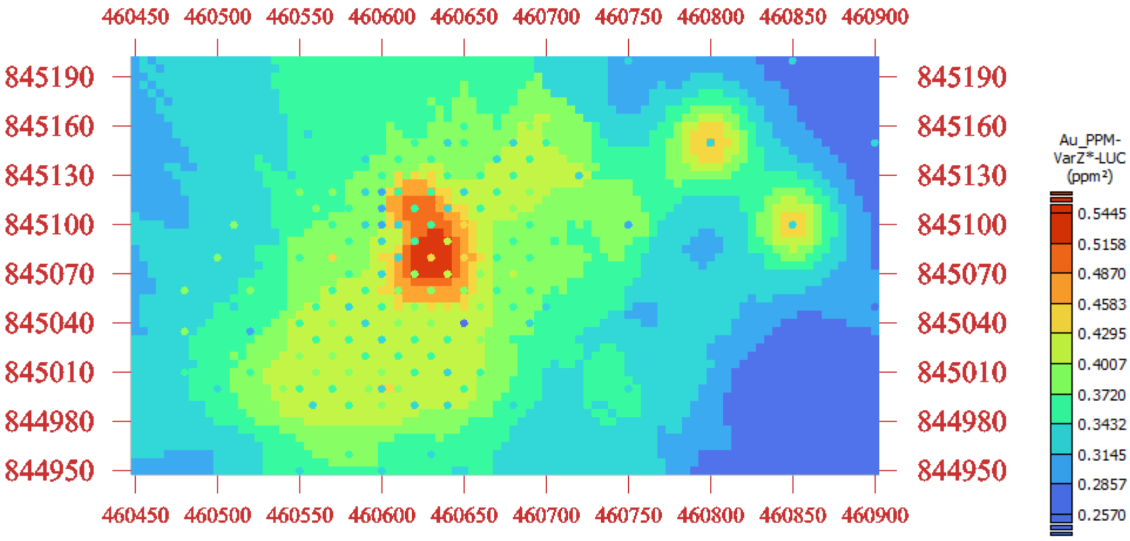


Figure 18. The Variance map for localization and uniform conditioning (LUC).

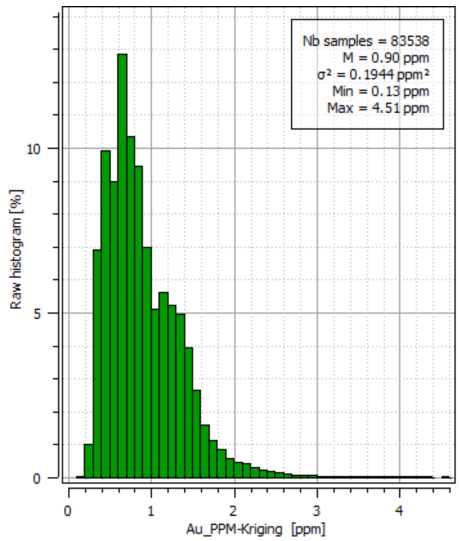


Figure 19. The histogram of LUC kriging results.

Figure 19 presents the resulting histogram after completing the map visualization of Localization and Uniform Conditioning (LUC). The maximum predicted grade for this geostatistical technique stands for 4.51 ppm.

4.5 Multi-Gaussian Kriging

The multi-Gaussian kriging results presented in this section employ the methodology procedure with obtaining the variogram of the original data, and moving towards Gaussian transformation with subsequent kriging estimation and retrieving of the histogram chart of predicted map. To begin with, Figure 20 initiates the resulting part with represented experimental variogram below:

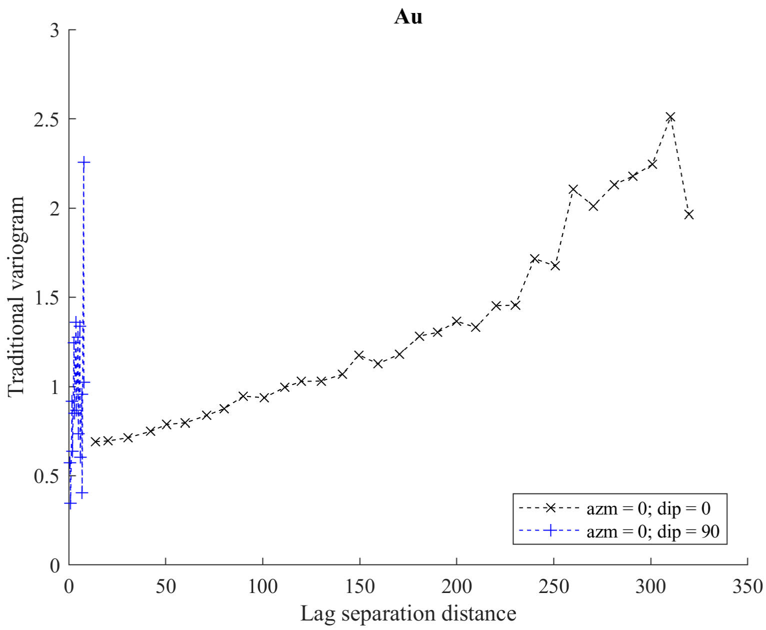


Figure 20. The Experimental Variogram of the sampled points.

Figure 20. shows the experimental variogram of sampled data over 350m distance in horizontal direction. It can be drawn that data has higher variability over larger distances.

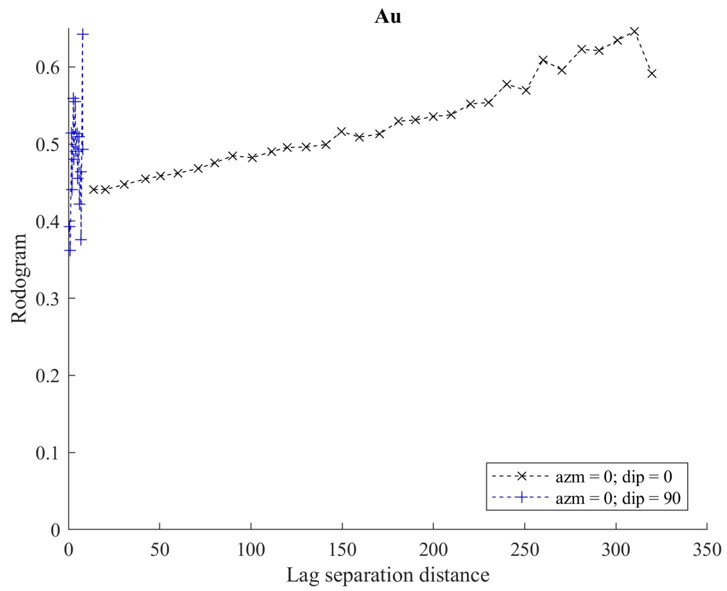


Figure 21. The Experimental Rodogram of the sampled points.

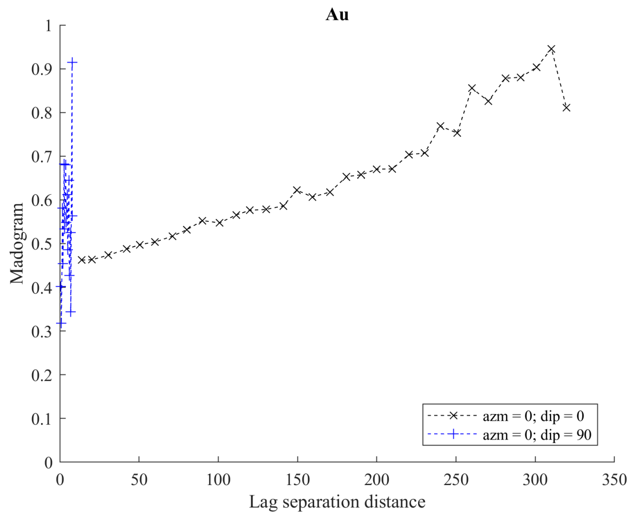


Figure 22. The Experimental Madogram of the sampled points.

The above-illustrated Figure 21 and Figure 22 show the experimental rodogram and madogram graphs. Their application stands for illustrating different orders of W for varogram construction. As regular semi-variogram is based on 2nd order, the madogram in this instance serves for better visualization of median differences between sampled points instead of square means. The rodograms stands for order 0.5 and provides wider perspective on spatial variability of the sampled data, and give a glimpse on sensitivity of the model to changes in quantity of original data.

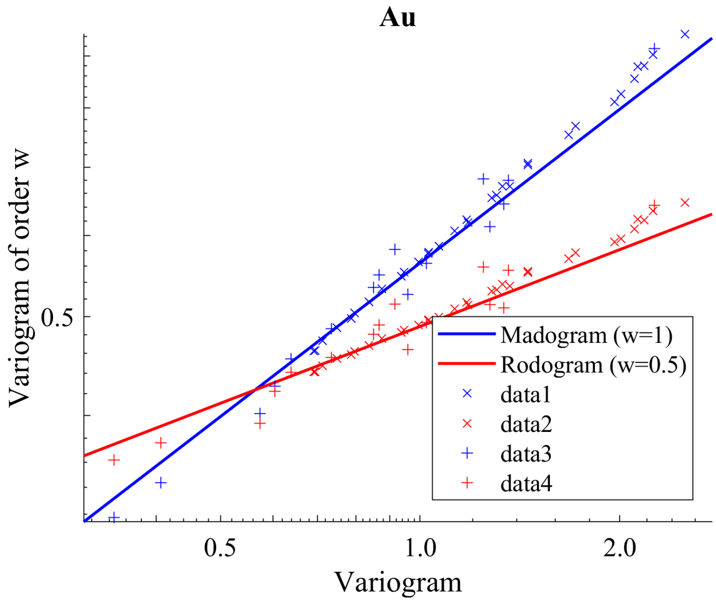


Figure 23. The variogram of order "w" for madogram and rodogram.

The plot of variogram of order w presented above gives specific annotation on effectiveness of classical variogram. The presence of madogram and rodogram is used for the purposes of diagnosing the multi-Gaussian assumptions. In particular, the deviations of the points between different orders of the variogram, depicted as data (1,2,3,4) will give reasoning to the presence of heavy tails, skewness, and local anomalies.

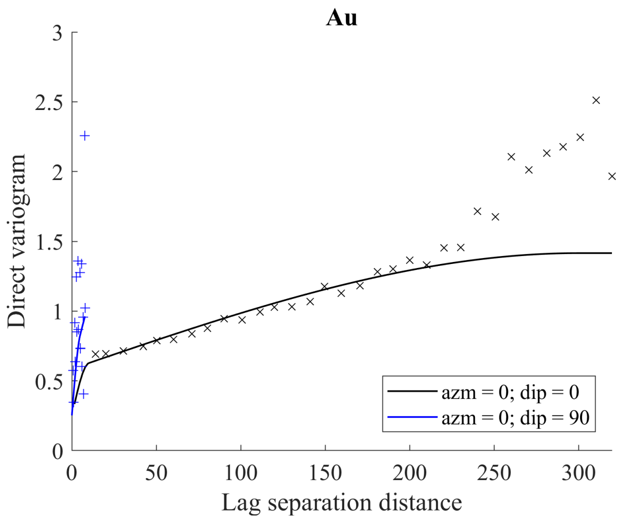


Figure 24. The Theoretical Variogram fitting with exponential and spherical nested structures.

Figure 24. shows the theoretical variogram fitting to the experimental variogram. The experimental variogram possesses 2 regular directions with lag 10m and maximum distance of 300m for horizontal direction and lag 0.5m and maximum distance of 8m for vertical direction. It has 2 nested structures with exponential and spherical models for lower and upper tails, respectively.

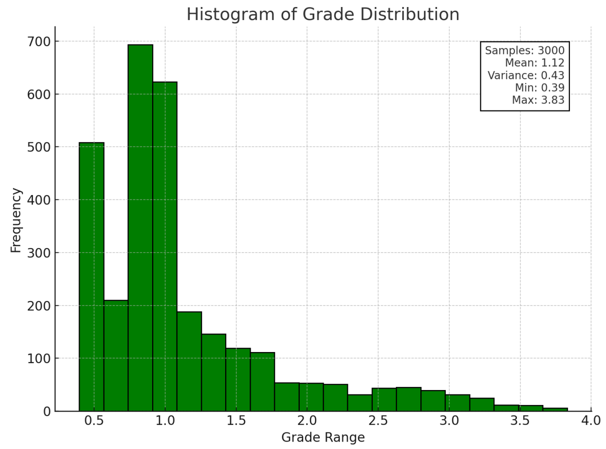


Figure 25. The resulting histogram with grade distribution of the MGK.

This histogram chart delineates the grade distribution to frequency. From this figure it is evident that the maximum values levels at 3.83 ppm, whereas the variance reached 0.43 ppm^2

From the results section, we can derive the information regarding the resulting maximum values of the gold concentration for local-mean prediction maps. To review this information, the table displayed below summarizes the statistical parameters received after iterating each geostatistical method separately.

Method	MIN	MAX	MEAN	VAR
SK	0.34	3.43	1.04	0.054
OK	0.25	3.44	0.92	0.154
MIK, 5-in.	0.4	4.24	0.94	0.1668

MIK, 7-in.	0.25	4.32	0.94	0.152
MIK, 9-in.	0.25	4.18	0.92	0.146
MGK	0.39	3.83	1.12	0.43
LUC	0.13	4.51	0.9	0.194

Table 1. The statistical results of the kriging geostatistical techniques and LUC.

5. DISCUSSION

To begin with, Table 1 presents the results of simple and ordinary kriging to provide a general understanding of how deterministic linear methods underperform in managing the smoothing effect. Additionally, the table helps identify the method that is slightly more effective, given that all subsequent methods are, to some extent, based on a simple or ordinary model in variogram modeling. It is evident from Table 1 that ordinary kriging yields higher precision than simple kriging. However, the cross-validation results depicted in Figure 3 clearly indicate that the correlation between the geostatistical set based on the theoretical variogram and the experimental variogram is inadequate—approximately 23% of the data do not match. These results have been justified by the Yamomoto (2005) who stated that interpolation of ordinary kriging tends to reduce the variance and employ non-representative maps of grade distribution. This discrepancy likely contributes to the underperformance in managing the smoothing effect.

The next method presented in the results section is multiple indicator kriging. The cumulative distribution functions (CDFs) for 5, 7, and 9 indicators reveal that case (a) exhibits a higher exponent factor than cases (b) and (c). This difference is explained by the fact that the lowest indicator, given the number of thresholds, results in a lower exponent, whereas the highest indicator produces an upper exponent. These probabilistic approaches in clustering the dataset into domains have been explicitly delineated in Deraisme (2016) marking the robustness in establishing the binary estimates for the sampled data. The results indicate that partitioning the dataset into 7 thresholds produced the highest resistance to the smoothing effect, with a maximum value leveling at 4.32 ppm. Two primary explanations account for this outcome. First, each threshold requires its own fitted theoretical variogram, and in the case of 7 indicators, there was sufficient data in the upper cut-off indicator to achieve higher precision in estimation. Second, systematic errors may have occurred during the variogram fitting for the upper threshold in the 9-indicator kriging model. Because the geostatistical variogram can only be evaluated as a set within the context of multiple indicator kriging for cross-validation purposes, applying alternative parametric models or adjusting lags might have resulted in improved precision.

Regarding multi-Gaussian kriging (MGK), the results show a maximum value of 3.83 ppm for the local-mean prediction map. MGK is unique in that it transforms and back-transforms the raw dataset into a "z*" score. Therefore, it is reasonable to consider applying an alternative standard score to verify the correctness of the multi-Gaussian kriging process. An additional point of interest is the use of a madogram and rodogram in creating the "w" order variogram analysis, as observed in Figure 23. Orders of 0.5 and 1 were tested to assess the behavior of the experimental variogram in the analysis of spatial variability. In this way, the MGK method emphasizes the identification of appropriate variograms for generating higher-precision local-mean maps. Likewise, Emery (2004) has mentioned the probabilistic nature of revising resource uncertainty that has been revealed in "w" order of variogram test. As shown in Figure 23, the data corresponding to the madogram and rodogram lines (1, 2, 3, 4) do not follow an ideal linear relationship. Consequently, it can be concluded that the system exhibits heavy tails—also evidenced by the resulting histogram—and that local anomalies manifest as outliers. This comparison with the classical variogram underscores the absence of strict Gaussianity and suggests that an alternative model should be applied to minimize the smoothing effect.

Lastly, the locally unbiased covariance (LUC) method produced the highest maximum values among all methods, leveling at 4.51 ppm. The significance of this outcome lies in its unique approach based on support correction, particularly Gaussian anamorphosis. Initially, LUC's exploratory data analysis (EDA) involves constructing a variogram based on raw and Gaussian data. This model fitting is essential for proceeding with support correction via Gaussian anamorphosis. In this context, Figure 16 provides a descriptive explanation of the relationship between SMUs, tonnage, and metal quantity. Following this ranking technique, Abzalov (2014) managed to give a glimpse into practical application of LUC employing these selective mining units, tonnage and metal quantity graphs. As the paper embraced positive aspects of this geostatistical tool they are compliant with results presented in Table 1. These graphs are used to create a ranking that establishes the probabilistic relationship between the SMUs and the estimated kriging weights. Figure 15 shows the cross-validation results, with a Pearson correlation of 0.83, indicating a good correlation between the original dataset and the theoretical variogram output. Although the kriging local-mean map clearly exhibits a grade distribution pattern similar to that obtained from the previously applied kriging methods, the variance map produced erratic results. Specifically, the variance map shows the highest variability at the center of the grid, where the sampling density is high, whereas the corners, with lower sampling density, display lower variability. Several explanations may account for this outcome. First, the model may be mis-specified; the variance model used for variogram fitting in LUC might differ from those used in other geostatistical methods. Second, differences in the treatment of the nugget effect could have adversely affected the variance results. Finally, issues related to matrix inversion may be at play; redundancy or excessive sample concentration in the grid's center can cause instability during inversion, leading to extreme variability values. Since declustering was not applied in this study, this issue might be the primary cause of the unreliable variance map.

6. CONCLUSION AND RECOMMENDATIONS

Application of these geostatistical methods revealed both the limitations and advantages of each technique. For ordinary kriging, a major limitation was the reliance on linear mathematical equations to determine kriging weights without prior data subdivision and the neglect of data skewness when fitting the theoretical variogram. These simplifications—an inherent attribute of ordinary kriging—resulted in poor management of the smoothing effect. Nevertheless, the intrinsic stationarity assumed in other methods based on the ordinary kriging model proved more useful than simple kriging, as evidenced by the maximum values obtained from the histograms in the results section. Thus, ordinary kriging remains a non-optimal option for modeling resource estimation at a tailing facility within the context of spatial interpolation.

Considering data subdivision, multiple-indicator kriging successfully thresholded the data. However, this procedure is unable to create separate variogram geostatistical sets that could enhance interpolation. Moreover, adjusting a sufficient number of indicators poses challenges in terms of time consumption when determining the optimum quantity. An additional obstacle is the uneven distribution of data among the thresholds. Consequently, the highest threshold suffers from inadequate variogram fitting, which still results in a smoothing effect.

In transforming the data and mitigating skewness, multi-Gaussian kriging established a more efficient exploratory data analysis (EDA) procedure. Specifically, converting the raw data into Gaussian normal scores enables the creation of valid variograms and enhances the performance of this technique. Conversely, a limitation of this method arises when there is a lack of linearity between the madogram and rodogram, indicating that the dataset does not follow Gaussianity; in such cases, alternative methods must be applied. Otherwise, this geostatistical tool serves as a reliable estimator for producing local mean prediction maps with higher precision relative to the original dataset.

Finally, the localization and uniform conditioning method produced the most resistant output to the smoothing effect during validation. As evidenced by both histograms, the resulting maximum values—ranging from 4.51 ppm to 7.96 ppm—highlight the potential advantages of this technique over other kriging methods. One major challenge encountered was the creation of the variance map. As discussed, the absence of declustering and the redundancy of data at the center of the grid led to severe matrix inversion problems, thereby yielding questionable variance outcomes.

Overall, all methods were applied in accordance with the established methodology and produced prognostic results. A positive aspect of this research is the identification of each method's limitations and the demonstration of how deterministic tools tend to minimize the effects of underestimation and overestimation of grade concentrations at unsampled locations.

It is worth noting that deterministic tools remain preliminary instruments for resource estimation. The absence of a stochastic approach and the inability to produce multiple maps create gaps in understanding the uncertainty associated with the results, thereby increasing the potential for the smoothing effect. Despite their disadvantages, the adequate application of these techniques—when aligned with the descriptive statistics of the dataset—can enhance interpolation and yield sufficient accuracy for resource estimation.

To conclude, the major contribution was assessment of linear techniques, namely simple and ordinary kriging, to non-linear methods, MIK,MGK and LUC, identification of limitations, and deliberate revision of smoothing effect presence upon adjusting similar parameters across the kriging tools. The thesis revealed that LUC is the best option for deterministic approach employment, which is advocated for highest maximum values that moderately represent the original histogram. The primary recommendations subdivided upon the problems associated with processing these techniques. In case of stationarity it is suggested to use intrinsic stationarity as it creates locally allocated mean instead of global one. For the variogram fitting, the MGK and LUC have exhibited the best fitting since they comprise the normal score transformation and Gaussian anamorphosis, respectively. Lastly, the smoothing effect was the most critical issue, in particular for resource estimation, hence, it is recommended to adjust the uniform conditioning along with localization, which enhances the grade distribution locally for each panel individually.

7. REFERENCES

1. Abzalov, M. Z. (2006). Localised uniform conditioning (LUC): A new approach for direct modelling of small blocks. *Mathematical Geology, 38*(4), 393–411. https://doi.org/10.1007/s11004-005-9024-6
2. Abzalov, M. Z. (2014). Localized uniform conditioning (LUC): Method and application case studies. *Journal of the Southern African Institute of Mining and Metallurgy, 114*(3), 243–253.
3. Afzal, M., Alghalandis, Y. F., Khakzad, A., Moarefvand, P., & Omran, N. R. (2015). Multi-Gaussian kriging: A practice to enhance delineation of mineralized zones by concentration-volume fractal model in Dardevey iron ore deposit, SE Iran. *Journal of Geochemical Exploration, 158*, 10–21. https://doi.org/10.1016/j.gexplo.2015.07.007
4. Afzal, M., Shahbeik, S., & Yasrebi, A. B. (2017). A geostatistical investigation of 3D magnetic inversion results using multi-Gaussian kriging. *Journal of Applied Geophysics, 145*, 1–12. https://doi.org/10.1016/j.jappgeo.2017.07.005
5. Afzal, M., Teymouri, M., Madani, N., & Shahbeik, S. (2018). Spatial mapping of the rock quality designation using multi-Gaussian kriging. *Minerals, 8*(11), 530. https://doi.org/10.3390/min8110530
6. Datamine Software. (n.d.). *About localized uniform conditioning*. Retrieved from https://docs.dataminesoftware.com/StudioRM/Latest/Uniform_Conditioning/About_Localized_Uniform_Conditioning.htm
7. Deraisme, J. (2016). Recoverable resources estimation: Indicator kriging or uniform conditioning? Retrieved from https://www.geovariances.com/wp-content/uploads/2016/09/eage_madrid_deraisme.pdf
8. Deraisme, J., & Assibey-Bonsu, W. (2011). Localised uniform conditioning in the multivariate case—An application to a porphyry copper gold deposit. *35th APCOM Symposium Proceedings*, 131–140.
9. Emery, X. (2004). *Uncertainty modeling and spatial prediction by multi-Gaussian kriging*. Retrieved from https://repositorio.uchile.cl/bitstream/handle/2250/125159/Emery_Xav.pdf?sequence=1
10. Glacken, I. M., Snowden, D. V., & Edwards, A. C. (2001). Mineral resource estimation. *Mineral resource and ore reserve estimation-the Aus. Inst. Mining and Metallurgy guide to good practice*, 189-198.
11. Goovaerts, P. (2004). Multiple indicator cokriging with application to optimal sampling for environmental remediation. *Computers & Geosciences, 30*(5), 465–478. https://doi.org/10.1016/j.cageo.2004.02.006
12. Journel, A. G., & Huijbregts, C. J. (1978). *Mining geostatistics*. Academic Press.
13. Journel, A. G., Kyriakidis, P., & Mao, S. (2000). Correcting the smoothing effect of estimators: A spectral postprocessor. *Mathematical Geology, 32*(7), 787–813. https://doi.org/10.1023/A:1007568503857
14. Kim, H., Kim, H. M., & Lee, Y. G. (2016). Time-series mapping of PM10 concentration using multi-Gaussian space-time kriging. *Advances in Meteorology, 2016*, Article 9452080. https://doi.org/10.1155/2016/9452080

15. Rezaee, H., Asghari, O., & Yamamoto, J. (2012). On the reduction of the ordinary kriging smoothing effect. *Journal of Mining and Environment, 3*(1), 61–72.
16. Yamamoto, J. (2000). An alternative measure of the reliability of ordinary kriging estimates. *Mathematical Geology, 32*(4), 489–509. https://doi.org/10.1023/A:1007556820813
17. Yamamoto, J. (2005). Correcting the smoothing effect of ordinary kriging estimates. *Mathematical Geology, 37*(1), 69–94. https://doi.org/10.1007/s11004-005-8748-7
18. Yamamoto, J. (2008). Estimation or simulation? That is the question. *Computers & Geosciences, 34*(12), 1827–1837. https://doi.org/10.1016/j.cageo.2008.02.011
19. Vann, J., Guibal, D., & Harley, M. (2000). Multiple indicator kriging: Is it suited to my deposit? https://www.srk.com/sites/default/files/file/JVann_IndicatorKriging.pdf