

# COSMOLOGICAL PERTURBATION THEORY IN MARKOV MUKHANOV FRAMEWORK

by

Yerkhan Nagmet

---

A Thesis Submitted to the Faculty of the

DEPARTMENT OF PHYSICS

In Partial Fulfillment of the Requirements

For the Degree of

BACHELOR OF SCIENCE

In the School of Sciences and Humanities

NAZARBAYEV UNIVERSITY

2026

NAZARBAYEV UNIVERSITY, SCHOOL OF SCIENCES AND HUMANITIES

As members of the thesis committee, we certify that we have read the thesis prepared by Yerkhan Nagmet entitled

COSMOLOGICAL PERTURBATION THEORY IN MARKOV MUKHANOV  
FRAMEWORK

and recommend that it be accepted as fulfilling the thesis requirement for the degree of Bachelor of Science.

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to the Department of Physics.

I hereby certify that I have read this thesis prepared under my direction and recommend that it be accepted as fulfilling the thesis requirement.

\_\_\_\_\_  
Thesis Director: Prof. Daniele Malafarina

Date: 06.05, 2026

# TABLE OF CONTENTS

|                                                          |           |
|----------------------------------------------------------|-----------|
| <b>ABSTRACT</b>                                          | <b>5</b>  |
| <b>1 Introduction</b>                                    | <b>6</b>  |
| <b>2 Newtonian Perturbation Theory</b>                   | <b>8</b>  |
| 1 Newtonian fluids . . . . .                             | 8         |
| 1.1 Fluid Equations . . . . .                            | 8         |
| 1.2 Expanding Background . . . . .                       | 9         |
| 1.3 Background Solution . . . . .                        | 10        |
| 1.4 Linear Perturbations . . . . .                       | 10        |
| 1.5 Growth of Density Perturbations . . . . .            | 10        |
| 1.6 Fourier Space and Jeans Scale . . . . .              | 11        |
| 1.7 Limitations of the Newtonian Approximation . . . . . | 11        |
| <b>3 General Relativity</b>                              | <b>12</b> |
| 1 Geometry of Spacetime . . . . .                        | 12        |
| 1.1 Metric and Line Element . . . . .                    | 12        |
| 1.2 Minkowski and Curved Spacetime . . . . .             | 13        |
| 1.3 Index Notation . . . . .                             | 13        |
| 1.4 Relevance for Cosmological Perturbations . . . . .   | 13        |
| 2 Energy-Momentum Tensor . . . . .                       | 13        |
| <b>4 Relativistic Cosmology</b>                          | <b>15</b> |
| 1 Background Cosmology . . . . .                         | 15        |
| 2 Relativistic Perturbation Theory . . . . .             | 16        |
| 2.1 Perturbed Metric and Gauge Choice . . . . .          | 17        |
| 2.2 Relativistic Fluid Equations . . . . .               | 17        |
| 3 Einstein Equations . . . . .                           | 19        |
| 3.1 Computation of the Einstein Tensor . . . . .         | 19        |
| 3.2 Perturbation Equations . . . . .                     | 20        |
| 3.3 Relativistic Poisson Equation . . . . .              | 21        |
| <b>5 Modified Gravity Framework</b>                      | <b>22</b> |

|          |                                                                |           |
|----------|----------------------------------------------------------------|-----------|
| 1        | Action and Field Equations . . . . .                           | 22        |
| 2        | Effective Fluid Description . . . . .                          | 23        |
| 2.1      | Linear Perturbations . . . . .                                 | 24        |
| 3        | Numerical Calculation of the Gravitational Potential . . . . . | 24        |
| 3.1      | Background Evolution . . . . .                                 | 24        |
| 3.2      | Choice of Coupling Function . . . . .                          | 25        |
| 3.3      | Effective Equation of State and Sound Speed . . . . .          | 25        |
| 3.4      | Evolution Equation for the Gravitational Potential . . . . .   | 26        |
| 3.5      | Numerical Implementation . . . . .                             | 26        |
| 3.6      | Evolution of the Gravitational Potential . . . . .             | 27        |
| 3.7      | Dependence on the Coupling Parameters . . . . .                | 28        |
| <b>6</b> | <b>Conclusion</b>                                              | <b>31</b> |

## Abstract

This thesis investigates modified gravity models and their implications for relativistic cosmology. We study the evolution of cosmological perturbations in theories with non-minimal matter-gravity coupling and analyze the behavior of the gravitational potential for different coupling parameters. The theoretical framework is developed starting from Newtonian fluids and General Relativity, followed by the formulation of relativistic cosmology and modified gravity models. Numerical results are obtained for several parameter choices, and their cosmological implications are discussed.

# CHAPTER 1

## Introduction

Modern cosmology provides an accurate description of the large-scale structure and evolution of the Universe, based on General Relativity and the standard  $\Lambda$ CDM model, which describes the Universe as composed of baryonic matter, cold dark matter, and a cosmological constant ( $\Lambda$ ) responsible for the accelerated expansion, in agreement with observations [1, 2]. This framework successfully explains a wide range of observations, including the cosmic microwave background, the distribution of galaxies, and the expansion of the Universe [1, 3, 4].

Despite its success, the standard cosmological model relies on the existence of two components whose fundamental nature remains unknown: dark matter and dark energy. In particular, dark energy is introduced to explain the observed late-time accelerated expansion, while an inflationary phase is used to account for the initial conditions of the Universe [2, 5].

These two periods of accelerated expansion occur at vastly different energy scales and are typically treated independently [5]. At the same time, despite the remarkable success of the standard  $\Lambda$ CDM cosmological model, several open problems remain, including the nature of dark energy and dark matter, the cosmological constant problem, and tensions between different cosmological observations. This raises the question of whether a unified theoretical description can account for both phenomena within a single framework [6].

Recent observational analyses have explored the possibility that dark energy is not a strict cosmological constant, but may exhibit dynamical behavior. In phenomenological terms, this is often described by a time-dependent equation-of-state parameter  $w(z) \neq -1$ . While the standard  $\Lambda$ CDM model remains broadly consistent with current data, recent results from DESI have reported mild preferences for deviations from  $w = -1$  when combining large-scale structure measurements with other cosmological probes [7, 8].

One possible approach is to consider modifications of gravity [9]. In particular, theories with non-minimal coupling between matter and geometry provide a natural generalization of General Relativity [10]. In such models, the gravitational interaction can depend explicitly on the properties of matter, such as the energy density.

In this work, we investigate a class of modified gravity models in which the matter–

gravity coupling depends on the energy density, following the approach proposed by Markov and Mukhanov [11]. This leads to an effective energy–momentum tensor and introduces corrections to both the background evolution and the dynamics of cosmological perturbations.

The main objective of this thesis is to develop a consistent perturbation formalism within this framework and to derive the modified evolution equations that govern the formation of structures. In particular, we obtain the evolution equation for the gravitational potential and analyze how it is affected by the energy-dependent coupling.

The thesis is organized as follows. Chapter 2 reviews the Newtonian theory of cosmological perturbations. Chapter 3 introduces the foundations of General Relativity relevant for cosmology. Chapter 4 develops relativistic cosmological perturbation theory and the evolution of density perturbations. Chapter 5 presents the modified gravity framework together with the numerical analysis of the evolution of the gravitational potential. Finally, Chapter 6 summarizes the main results and conclusions.

# CHAPTER 2

## Newtonian Perturbation Theory

### 1 Newtonian fluids

Before developing a fully relativistic description of cosmological perturbations, it is useful to analyze the Newtonian approximation [5]. It explains how structures grow by balancing two effects: gravity trying to collapse matter and pressure trying to resist that collapse.

The Newtonian treatment applies on scales much smaller than the Hubble radius, which characterizes the scale over which cosmic expansion becomes dynamically important. This scale is often associated with the cosmological horizon, separating sub-horizon modes, which are in causal contact, from super-horizon modes, which lie beyond the region over which signals can propagate within a Hubble time. On sub-horizon scales, regions are effectively causally connected, and the impact of the global expansion is negligible relative to the local gravitational evolution of matter. The approximation also assumes that matter is non-relativistic, with velocities much smaller than the speed of light. Under these conditions, the evolution of matter can be accurately described by classical fluid equations coupled to Newtonian gravity.

This approximation relies on three key assumptions:

- weak gravitational fields,  $|\Phi| \ll 1$ ,
- non-relativistic velocities,  $|\mathbf{v}| \ll c$ ,
- scales well inside the Hubble radius,  $k \gg \mathcal{H}$ .

#### 1.1 Fluid Equations

We consider a non-relativistic fluid characterized by energy density  $\varepsilon$ , pressure  $p \ll \varepsilon$ , and velocity field  $\mathbf{u}$ . The evolution of the fluid is governed by the continuity and Euler equations,

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon \mathbf{u}) = 0, \tag{2.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\varepsilon} \nabla p - \nabla \Phi, \quad (2.2)$$

together with the Poisson equation,

$$\nabla^2 \Phi = 4\pi G \varepsilon. \quad (2.3)$$

These equations express local conservation of mass and momentum, while the Poisson equation determines the gravitational potential generated by the matter distribution. Taken together, they describe the evolution of a self-gravitating fluid and form a closed system once supplemented by an equation of state. In the present work, we adopt a linear barotropic equation of state,

$$p = w\varepsilon, \quad (2.4)$$

where  $w$  is the equation-of-state parameter.

## 1.2 Expanding Background

To account for the expansion of the Universe, we introduce comoving coordinates  $\mathbf{x}$  related to physical coordinates  $\mathbf{r}$  by

$$\mathbf{r}(t) = a(t)\mathbf{x}. \quad (2.5)$$

where  $a(t)$  is the scale factor which characterizes the relative expansion of spatial distances as a function of time.

The velocity field can then be decomposed into the Hubble flow and a peculiar velocity,

$$\mathbf{u} = H\mathbf{r} + \mathbf{v}, \quad (2.6)$$

where  $H = \dot{a}/a$  is the Hubble parameter.

The first term represents the uniform Hubble expansion, while  $\mathbf{v}$  describes peculiar motions arising from local inhomogeneities. Structure formation is entirely driven by these peculiar velocities.

In terms of comoving coordinates, the fluid equations become

$$\frac{\partial \varepsilon}{\partial t} + 3H\varepsilon + \frac{1}{a} \nabla \cdot (\varepsilon \mathbf{v}) = 0, \quad (2.7)$$

$$\frac{\partial \mathbf{v}}{\partial t} + H\mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{a} \frac{\nabla p}{\varepsilon} - \frac{1}{a} \nabla \Phi, \quad (2.8)$$

$$\nabla^2 \Phi = 4\pi G a^2 \varepsilon. \quad (2.9)$$

### 1.3 Background Solution

For a homogeneous fluid with  $\mathbf{v} = 0$ , the continuity equation reduces to

$$\dot{\varepsilon} + 3H\varepsilon = 0, \quad (2.10)$$

which implies that the background density evolves as

$$\varepsilon \propto a^{-3}. \quad (2.11)$$

This scaling reflects the dilution of matter due to the expansion of the Universe, with no additional energy exchange or pressure effects in the non-relativistic limit.

### 1.4 Linear Perturbations

We now introduce small perturbations around the background solution,

$$\varepsilon = \bar{\varepsilon}(1 + \delta), \quad p = \bar{p} + \delta p, \quad \Phi = \bar{\Phi} + \delta\Phi, \quad (2.12)$$

where  $\delta = \delta\varepsilon/\bar{\varepsilon}$  is the density contrast. We assume  $|\delta| \ll 1$ , allowing us to retain only first-order terms in perturbations and neglect higher-order nonlinear effects.

Linearizing the continuity equation yields

$$\dot{\delta} = -\frac{1}{a}\nabla \cdot \mathbf{v}, \quad (2.13)$$

while the Euler equation becomes

$$\dot{\mathbf{v}} + H\mathbf{v} = -\frac{1}{a}\nabla\delta\Phi + \frac{1}{a\bar{\varepsilon}}\nabla\delta p. \quad (2.14)$$

The Poisson equation for perturbations is

$$\nabla^2\delta\Phi = 4\pi G a^2 \bar{\varepsilon} \delta. \quad (2.15)$$

### 1.5 Growth of Density Perturbations

Combining the above equations leads to a second-order differential equation for the density contrast. Taking the time derivative of the continuity equation and substituting the Euler equation gives

$$\ddot{\delta} + 2H\dot{\delta} + \frac{1}{a^2} \left( \frac{1}{\bar{\varepsilon}} \nabla^2 \delta p - \nabla^2 \delta\Phi \right) = 0. \quad (2.16)$$

For a barotropic fluid, where the pressure perturbation satisfies  $\delta p = c_s^2 \delta\varepsilon$ , with  $c_s^2$  denoting the sound speed squared in the medium, this expression reduces to

$$\ddot{\delta} + 2H\dot{\delta} + \left( \frac{c_s^2}{a^2} \nabla^2 - 4\pi G \bar{\varepsilon} \right) \delta = 0. \quad (2.17)$$

The second term represents Hubble friction due to cosmic expansion, which suppresses the growth of perturbations. The pressure term acts against gravitational collapse on small scales, while the gravitational term drives the amplification of density fluctuations. The competition between these effects determines the evolution of structure.

## 1.6 Fourier Space and Jeans Scale

In Fourier space, where  $\nabla^2 \rightarrow -k^2$ , the evolution equation for each Fourier mode  $\delta_k(t)$  becomes

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \left( \frac{c_s^2 k^2}{a^2} - 4\pi G\bar{\epsilon} \right) \delta_k = 0, \quad (2.18)$$

where  $k \equiv |\mathbf{k}|$  is the comoving wavenumber associated with a given perturbation mode, and  $\delta_k(t)$  is the Fourier transform of the density contrast  $\delta(\mathbf{x}, t)$ . This equation defines the Jeans scale,

$$k_J^2 = \frac{4\pi G\bar{\epsilon}a^2}{c_s^2}. \quad (2.19)$$

Physically, the Jeans scale corresponds to the critical length at which pressure support balances gravitational collapse. It can be interpreted as the scale for which the sound-crossing time equals the gravitational free-fall time.

In an expanding Universe, the Jeans scale evolves with time, implying that a given mode can transition between stable and unstable regimes during cosmic evolution.

For  $k \gg k_J$ , pressure dominates and perturbations oscillate as sound waves. For  $k \ll k_J$ , gravity dominates and perturbations grow, leading to gravitational instability.

## 1.7 Limitations of the Newtonian Approximation

Although the Newtonian treatment successfully captures the essential physics of structure formation on small scales, it breaks down on horizon scales and in the early Universe, where relativistic effects become significant. In these regimes, the coupling between matter and spacetime geometry must be described within the framework of General Relativity. This motivates the development of relativistic perturbation theory in the following sections.

# CHAPTER 3

## General Relativity

### 1 Geometry of Spacetime

In General Relativity, gravity is not treated as a force but rather as a manifestation of the curvature of spacetime [5, 12]. The central object that encodes this geometry is the spacetime metric, which determines how distances and time intervals are measured.

#### 1.1 Metric and Line Element

The metric tensor relates coordinate separations to physical distances. In three-dimensional Euclidean space, the infinitesimal distance between two points is given by

$$d\ell^2 = dx^2 + dy^2 + dz^2 = \delta_{ij} dx^i dx^j, \quad (3.1)$$

where  $\delta_{ij} = \text{diag}(1, 1, 1)$  is the Kronecker delta. In more general coordinate systems, such as spherical coordinates  $(r, \theta, \phi)$ , the line element takes the form

$$d\ell^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 = g_{ij} dx^i dx^j, \quad (3.2)$$

where the spatial metric is no longer trivial,  $g_{ij} = \text{diag}(1, r^2, r^2 \sin^2 \theta)$ . Although coordinate distances depend on the chosen coordinate system, the physical distance  $d\ell$  is invariant.

In relativistic physics, this concept is generalized to spacetime. Coordinates are written as  $x^\mu = (ct, x^i)$ , and the spacetime interval is defined as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (3.3)$$

where  $g_{\mu\nu}$  is the spacetime metric. The interval  $ds^2$  is invariant under coordinate transformations and encodes the causal structure of spacetime, distinguishing between timelike, spacelike, and null separations.

## 1.2 Minkowski and Curved Spacetime

In special relativity, spacetime is flat and described by the Minkowski metric,

$$ds^2 = -c^2 dt^2 + \delta_{ij} dx^i dx^j, \quad (3.4)$$

so that  $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ . This spacetime has zero curvature and serves as the local approximation to any spacetime in General Relativity.

In General Relativity, the metric becomes a function of spacetime,  $g_{\mu\nu}(t, \mathbf{x})$ , allowing for curvature. The presence of matter and energy determines how the metric evolves through Einstein's equations, and this curvature governs the motion of particles and light. Thus, gravitational dynamics are entirely encoded in the geometry of spacetime.

## 1.3 Index Notation

The metric is used to raise and lower indices on tensors. For a four-vector  $A^\mu$ , one defines

$$A_\mu = g_{\mu\nu} A^\nu, \quad A^\mu = g^{\mu\nu} A_\nu, \quad (3.5)$$

where  $g^{\mu\nu}$  is the inverse metric satisfying

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu. \quad (3.6)$$

The contraction of two vectors produces an invariant scalar,

$$A_\mu B^\mu = g_{\mu\nu} A^\mu B^\nu. \quad (3.7)$$

Such invariants play a central role in relativistic physics, as they represent physically measurable quantities independent of the choice of coordinates.

## 1.4 Relevance for Cosmological Perturbations

The metric provides the fundamental link between matter and geometry. In cosmology, perturbations in the matter distribution are accompanied by perturbations of the space-time metric, and these two sectors are dynamically coupled through Einstein's equations. As a result, the study of structure formation requires analyzing fluctuations of both the energy-momentum tensor and the metric itself.

# 2 Energy-Momentum Tensor

To describe the matter content of the Universe in a coordinate-independent way, we introduce the energy-momentum tensor. This tensor encodes the local density and flux

of energy and momentum and acts as the source term in Einstein's equations, so it links matter to the geometry of spacetime.

For a general fluid, the energy-momentum tensor can be written as

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu + pg_{\mu\nu} + \Pi_{\mu\nu}, \quad (3.8)$$

where  $\varepsilon$  is the energy density,  $p$  is the isotropic pressure,  $u^\mu$  is the four-velocity of the fluid satisfying  $u^\mu u_\mu = -1$ , and  $\Pi_{\mu\nu}$  is the anisotropic stress tensor. The latter is symmetric, traceless, and orthogonal to the four-velocity,  $u^\mu \Pi_{\mu\nu} = 0$ , and describes directional stresses such as shear.

In the absence of anisotropic stress,  $\Pi_{\mu\nu} = 0$ , the fluid is referred to as a perfect fluid, and the energy-momentum tensor reduces to

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu + pg_{\mu\nu}. \quad (3.9)$$

This form provides an accurate description of the dominant components of the Universe on large scales. In particular:

- Non-relativistic matter (dust) is characterized by  $p = 0$ ,
- Radiation satisfies  $p = \varepsilon/3$ ,
- A cosmological constant ( $\Lambda$ ), representing dark energy, is characterized by the equation of state  $p = -\varepsilon$ .

The conservation of the energy-momentum tensor,

$$\nabla_\mu T^\mu{}_\nu = 0, \quad (3.10)$$

leads to the relativistic counterparts of the continuity and Euler equations which dictate the evolution of the background energy density. Combined with Einstein's equations, this relation determines the dynamics of the cosmological background.

# CHAPTER 4

## Relativistic Cosmology

### 1 Background Cosmology

We begin by describing the homogeneous and isotropic background Universe, which provides the zeroth-order approximation for cosmological dynamics. On sufficiently large scales, the Universe is well described by the Friedmann–Robertson–Walker (FRW) metric [5]. In this work, we restrict to a spatially flat Universe ( $k = 0$ ), consistent with observations [1], so that the metric takes the form

$$ds^2 = -c^2 dt^2 + a^2(t) [dr^2 + r^2 d\Omega^2], \quad (4.1)$$

where  $a(t)$  is the scale factor, which encodes the expansion history of the Universe. The assumption of homogeneity and isotropy is strongly supported by observations [1]. The dynamics of the scale factor are governed by the Friedmann equations [5],

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varepsilon, \quad (4.2)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varepsilon + 3p), \quad (4.3)$$

where  $H = \dot{a}/a$  is the Hubble parameter, and  $\varepsilon$  and  $p$  are the total energy density and pressure of the cosmic fluid. The first equation determines the expansion rate, while the second describes the acceleration or deceleration of the expansion. In particular, a negative effective pressure ( $p < -\varepsilon/3$ ) leads to accelerated expansion.

In the standard  $\Lambda$ CDM cosmological model [1], the total energy density is composed of three main components,

$$\varepsilon = \varepsilon_r + \varepsilon_m + \varepsilon_\Lambda, \quad (4.4)$$

corresponding to radiation ( $r$ ), non-relativistic matter ( $m$ ), and dark energy ( $\Lambda$ ), respectively. Each component evolves differently with the scale factor due to their nature,

$$\varepsilon_r \propto a^{-4}, \quad \varepsilon_m \propto a^{-3}, \quad \varepsilon_\Lambda = \text{const.} \quad (4.5)$$

Radiation redshifts both due to the expansion of volume and the redshift of photon energies, while matter dilutes only due to volume expansion. Dark energy, modeled as a cosmological constant, remains constant in time and eventually dominates the expansion at late times.

The relative contributions of these components determine the different epochs of cosmic evolution: a radiation-dominated era at early times, followed by matter domination, and finally a late-time accelerated phase driven by dark energy.

This homogeneous and isotropic background provides the foundation for the study of cosmological perturbations. Small deviations from this background, arising from primordial fluctuations, grow over time and lead to the formation of large-scale structure in the Universe [5].

## 2 Relativistic Perturbation Theory

A consistent description of cosmological perturbations requires a relativistic framework, since gravity is inherently described by General Relativity and the Newtonian approximation breaks down on horizon scales and in the early Universe. In this approach, both the metric and the energy-momentum tensor are perturbed around a homogeneous and isotropic background [13],

$$g_{\mu\nu}(t, \mathbf{x}) = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}(t, \mathbf{x}), \quad T_{\mu\nu}(t, \mathbf{x}) = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}(t, \mathbf{x}) \quad (4.6)$$

where barred quantities denote background values and  $\delta$  denotes first-order perturbations. These perturbations are dynamically coupled through Einstein's equations and give rise to the evolution of cosmological inhomogeneities. In particular, fluctuations in the energy-momentum tensor act as sources for metric perturbations, which in turn influence the motion of matter. This interplay forms the basis of relativistic perturbation theory and underlies the formation of large-scale structure in the Universe.

The perturbed components of the energy-momentum tensor are

$$\begin{aligned} T_0^0 &= -(\bar{\varepsilon} + \delta\varepsilon), \\ T_i^0 &= (\bar{\varepsilon} + \bar{p}) v_i, \\ T_0^i &= -(\bar{\varepsilon} + \bar{p}) v^i, \\ T_j^i &= (\bar{p} + \delta p) \delta_j^i + \Pi_j^i, \end{aligned} \quad (4.7)$$

where  $v^i$  is the bulk velocity and  $\Pi_j^i$  is the anisotropic stress tensor satisfying  $\Pi^i_i = 0$ .

## 2.1 Perturbed Metric and Gauge Choice

To study cosmological perturbations, it is convenient to rewrite the background FRW metric introduced in Eq. (4.1) in terms of conformal time  $\eta$ , defined by

$$d\eta = \frac{dt}{a(t)}. \quad (4.8)$$

In these coordinates, the metric takes the form

$$ds^2 = a^2(\eta) [-c^2 d\eta^2 + dr^2 + r^2 d\Omega^2]. \quad (4.9)$$

In the following, primes will denote derivatives with respect to conformal time  $\eta$ . The perturbed metric can then be written in the general form

$$ds^2 = a^2(\eta) [-(1 + 2A)d\eta^2 + 2B_i dx^i d\eta + (\delta_{ij} + 2E_{ij})dx^i dx^j], \quad (4.10)$$

where  $A$ ,  $B_i$ , and  $E_{ij}$  represent scalar, vector, and tensor perturbations, respectively. In linear theory, these modes evolve independently, and scalar modes are primarily responsible for the formation of large-scale structure [5, 9].

The presence of coordinate freedom (gauge freedom) implies that not all perturbations are physically meaningful. To eliminate spurious degrees of freedom, one must fix a gauge. In this work, we adopt the Newtonian gauge [5, 9], which provides a direct physical interpretation of metric perturbations and simplifies the equations.

In the Newtonian gauge, the perturbed metric reduces to

$$ds^2 = a^2(\eta) [-(1 + 2\Psi)d\eta^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j], \quad (4.11)$$

where  $\Psi$  and  $\Phi$  are scalar potentials. In the absence of anisotropic stress, these potentials are equal and can be interpreted as the relativistic generalization of the Newtonian gravitational potential.

## 2.2 Relativistic Fluid Equations

The evolution of perturbations is determined by the conservation of the energy-momentum tensor,

$$\nabla_\mu T^\mu_\nu = 0, \quad (4.12)$$

Expanding the covariant derivative,

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha_\nu - \Gamma^\alpha_{\mu\nu} T^\mu_\alpha, \quad (4.13)$$

The covariant derivative involves the connection coefficients, known as the Christoffel

symbols, which are determined entirely by the metric. They are given by

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\lambda\sigma}(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}). \quad (4.14)$$

where  $\partial_{\mu} \equiv \partial/\partial x^{\mu}$  denotes differentiation with respect to the spacetime coordinates  $x^{\mu} = (\eta, \mathbf{x})$ .

These coefficients encode how vectors change under parallel transport in curved space-time and play a central role in defining covariant derivatives and the equations of motion. The relevant Christoffel symbols derived from Eq. (4.11) are

$$\begin{aligned} \Gamma_{00}^0 &= \mathcal{H} + \Psi' \\ \Gamma_{i0}^0 &= \partial_i \Psi \\ \Gamma_{00}^i &= \delta^{ij} \partial_j \Psi \\ \Gamma_{ij}^0 &= \mathcal{H} \delta_{ij} - (\Phi' + 2\mathcal{H}(\Phi + \Psi)) \delta_{ij} \\ \Gamma_{j0}^i &= (\mathcal{H} - \Phi') \delta_j^i \\ \Gamma_{jk}^i &= -2\delta_{(j}^i \partial_{k)} \Phi + \delta_{jk} \delta^{il} \partial_l \Phi \end{aligned} \quad (4.15)$$

where  $\mathcal{H} \equiv a'/a$  is the conformal Hubble parameter, primes denote derivatives with respect to conformal time  $\eta$ . In addition,

$$\delta_{(j}^i \partial_{k)} \Phi = \frac{1}{2} (\delta_j^i \partial_k \Phi + \delta_k^i \partial_j \Phi).$$

At the background level, Eq. (4.12) yields the continuity equation

$$\bar{\varepsilon}' + 3\mathcal{H}(\bar{\varepsilon} + \bar{p}) = 0, \quad (4.16)$$

while the spatial components, corresponding to the Euler equation, vanish identically because of the homogeneity and isotropy of the FRW background.

To study the evolution of inhomogeneities, we perturb the conservation equations to first order around the background solution. Substituting Eqs. (4.7) and (4.15) into Eq. (4.12), we obtain the relativistic continuity and Euler equations governing the evolution of density and velocity perturbations

Substituting Eqs. (4.7) and (4.15) into Eq. (4.12), we obtain the relativistic continuity and Euler equations for the perturbations,

$$\begin{aligned} \delta' &= -\left(1 + \frac{\bar{p}}{\bar{\varepsilon}}\right) (\nabla \cdot \vec{v} - 3\Phi') - 3\mathcal{H} \left( \frac{\delta p}{\delta \varepsilon} - \frac{\bar{p}}{\bar{\varepsilon}} \right) \delta \\ \vec{v}' &= -\left( \mathcal{H} + \frac{\bar{p}'}{\bar{\varepsilon} + \bar{p}} \right) \vec{v} - \frac{1}{\bar{\varepsilon} + \bar{p}} (\nabla \delta p + \nabla \cdot \Pi) - \nabla \Psi \end{aligned} \quad (4.17)$$

These equations describe the evolution of density and velocity perturbations in an expanding Universe. The first equation represents energy conservation, including the effect of metric perturbations through  $\Phi'$ , while the second equation generalizes the Euler

equation by incorporating gravitational forces via  $\Psi$  and pressure gradients.

This system provides the fundamental dynamical equations for the study of cosmological perturbations and forms the basis for the analysis of structure formation in both standard and modified gravity scenarios.

### 3 Einstein Equations

The evolution of metric perturbations is governed by Einstein's field equations,

$$G^\mu{}_\nu = 8\pi G T^\mu{}_\nu \quad (4.18)$$

which relate the geometry of spacetime, encoded in the Einstein tensor  $G^\mu{}_\nu$ , to the energy-momentum tensor  $T^\mu{}_\nu$ . To study perturbations, we expand both sides around the background solution,  $G^\mu{}_\nu = \bar{G}^\mu{}_\nu + \delta G^\mu{}_\nu$  and  $T^\mu{}_\nu = \bar{T}^\mu{}_\nu + \delta T^\mu{}_\nu$ . The background quantities satisfy the Friedmann equations, while the first-order perturbations obey

$$\delta G^\mu{}_\nu = 8\pi G \delta T^\mu{}_\nu \quad (4.19)$$

Thus, each component of Einstein's equations provides a relation between metric perturbations  $(\Phi, \Psi)$  and matter perturbations  $(\delta\varepsilon, \delta p, v, \Pi)$ .

#### 3.1 Computation of the Einstein Tensor

To obtain the perturbation equations, one must compute the Einstein tensor  $G_{\mu\nu}$  from the metric. This proceeds through a sequence of geometric quantities constructed from the metric. The starting point is the Christoffel symbols, from which one constructs the Riemann curvature tensor,

$$R^\varepsilon{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\varepsilon_{\nu\sigma} - \partial_\nu \Gamma^\varepsilon_{\mu\sigma} + \Gamma^\varepsilon_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\varepsilon_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \quad (4.20)$$

This tensor measures the intrinsic curvature of spacetime and quantifies the failure of a vector to return to its original direction after parallel transport around a closed loop.

Contracting the Riemann tensor yields the Ricci tensor,

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \quad (4.21)$$

which encodes the part of curvature relevant for gravitational dynamics. A further contraction gives the Ricci scalar,

$$R = g^{\mu\nu} R_{\mu\nu}. \quad (4.22)$$

Finally, the Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \quad (4.23)$$

The Einstein tensor has the important property of being divergence-free,  $\nabla_\mu G^\mu{}_\nu = 0$ , ensuring consistency with the conservation of the energy-momentum tensor.

### 3.2 Perturbation Equations

Using the perturbed metric in Newtonian gauge and matching each component of  $\delta G^\mu{}_\nu$  to the corresponding component of  $\delta T^\mu{}_\nu$  yields the following equations.

#### (0-0) Component: Energy Constraint

The time-time component gives

$$\nabla^2 \Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Psi) = 4\pi G a^2 \delta\varepsilon. \quad (4.24)$$

This equation generalizes the Newtonian Poisson equation. The first term represents spatial curvature of the gravitational potential, while the second term arises from the expansion of the Universe. On sub-horizon scales ( $k \gg \mathcal{H}$ ), the second term becomes negligible and the equation reduces to its Newtonian form.

#### (0-i) Component: Momentum Constraint

The time-space component yields

$$-(\Phi' + \mathcal{H}\Psi) = 4\pi G a^2 q, \quad (4.25)$$

where  $q = (\bar{\varepsilon} + \bar{p})v$  is the momentum density. This equation relates the time variation of the gravitational potential to the velocity field of the fluid and represents momentum conservation in curved spacetime.

#### (i-i) Trace Component: Dynamical Equation

The trace of the spatial components leads to

$$\Phi'' + \mathcal{H}\Psi' + 2\mathcal{H}\Phi' + \frac{1}{3}\nabla^2(\Psi - \Phi) + (2\mathcal{H}' + \mathcal{H}^2)\Psi = 4\pi G a^2 \delta p. \quad (4.26)$$

This equation governs the time evolution of the gravitational potential and includes contributions from pressure perturbations.

#### (i-j) Off-diagonal Component: Anisotropic Stress

The traceless spatial part gives

$$\Phi - \Psi = 8\pi G a^2 \Pi, \quad (4.27)$$

where  $\Pi$  is the anisotropic stress. This equation shows that any difference between the two gravitational potentials is sourced by anisotropic stresses. In the absence of anisotropic stress ( $\Pi = 0$ ), Eq. (4.27) implies  $\Psi = \Phi$ . Substituting this into Eq. (4.26), we obtain a simplified evolution equation,

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta p. \quad (4.28)$$

This equation shows that pressure perturbations can source the time evolution of the gravitational potential, while cosmic expansion introduces damping through the  $\mathcal{H}$  terms.

### 3.3 Relativistic Poisson Equation

Combining Eqs. (4.24) and (4.25) allows one to eliminate time derivatives of the potentials and obtain a generalized Poisson equation,

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\varepsilon} \Delta, \quad (4.29)$$

where the gauge-invariant quantity  $\bar{\varepsilon} \Delta \equiv \delta\varepsilon + \bar{\varepsilon}'(v + B)$  defines the comoving density contrast.

This equation has the same structure as the Newtonian Poisson equation, but remains valid on all scales, including super-horizon modes. It therefore provides the key link between matter perturbations and the gravitational potential in a fully relativistic framework.

# CHAPTER 5

## Modified Gravity Framework

In the previous chapter, we developed the standard framework of cosmological perturbation theory within General Relativity. Although General Relativity provides an excellent description of gravitational phenomena on a wide range of scales, numerous modified theories of gravity have been proposed to address open problems in cosmology, such as the nature of dark energy, dark matter, and the dynamics of the early Universe [2]. In this chapter, we extend the standard cosmological description by considering a modified gravity framework aimed at providing a unified description of dark energy.

The construction of the model is based on two main ingredients:

- (a) A non-minimal matter–gravity coupling depending on the energy scale, as proposed by Markov and Mukhanov [11],
- (b) A coupling function that can be expanded as a series of higher-order corrections in the matter sector [14, 15].

### 1 Action and Field Equations

We begin with the action [11]

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{8\pi G_N} + 2\chi(\varepsilon)\mathcal{L}_m \right), \quad (5.1)$$

where  $R$  is the Ricci scalar and  $\mathcal{L}_m$  is the matter Lagrangian. We adopt a perfect fluid description with the matter Lagrangian given by

$$\mathcal{L}_m = \varepsilon,$$

The modification to gravity is encoded in the scalar function  $\chi(\varepsilon)$ , which introduces an explicit dependence of the coupling on the energy scale.

Varying the action with respect to the metric yields the modified Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N \tilde{T}_{\mu\nu}, \quad (5.2)$$

where the effective energy-momentum tensor is given by

$$\tilde{T}_{\mu\nu} = (\varepsilon\chi)_{,\varepsilon} T_{\mu\nu} + (\varepsilon^2\chi_{,\varepsilon}) g_{\mu\nu}. \quad (5.3)$$

The standard perfect fluid energy-momentum tensor is

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu + pg_{\mu\nu}. \quad (5.4)$$

From Eqs. (5.2)–(5.3), one can identify an effective gravitational coupling and cosmological term,

$$G(\varepsilon) = G_N(\chi\varepsilon)_{,\varepsilon}, \quad \Lambda(\varepsilon) = -8\pi G_N \varepsilon^2 \chi_{,\varepsilon}. \quad (5.5)$$

where  $G_N$  is Newton's Gravitational constant

To ensure consistency with General Relativity without cosmological constant in the low-density regime, we impose the condition

$$\chi(\varepsilon) \rightarrow 1 \quad \text{as} \quad \varepsilon \rightarrow 0,$$

which guarantees that  $G(\varepsilon) \rightarrow G_N$  and  $\Lambda(\varepsilon) \rightarrow 0$ .

## 2 Effective Fluid Description

Since the field equations are written in terms of an effective energy-momentum tensor, we impose its conservation,

$$\nabla_\mu \tilde{T}^\mu{}_\nu = 0. \quad (5.6)$$

Projecting along the four-velocity  $u^\mu$ , we obtain the effective continuity equation for effective fluid,

$$\nabla_\mu (\tilde{\varepsilon} u^\mu) + \tilde{p} \nabla_\mu u^\mu = 0. \quad (5.7)$$

The effective energy density and pressure are given by

$$\tilde{\varepsilon} = \varepsilon\chi(\varepsilon), \quad \tilde{p} = (\varepsilon\chi)_{,\varepsilon} p + \varepsilon^2 \chi_{,\varepsilon}. \quad (5.8)$$

This leads to an effective equation of state,

$$\tilde{w} = \frac{\tilde{p}}{\tilde{\varepsilon}} = -1 + \frac{(\varepsilon\chi)_{,\varepsilon}}{\chi} \left(1 + \frac{p}{\varepsilon}\right). \quad (5.9)$$

In the limit  $\chi = 1$ , standard General Relativity is recovered,

$$G(\varepsilon) = G_N, \quad \Lambda(\varepsilon) = 0, \quad \tilde{w} = w = \frac{p}{\varepsilon}.$$

## 2.1 Linear Perturbations

To study the growth of structure, we introduce first-order perturbations of the fundamental quantities, namely the energy density and pressure. We write

$$\varepsilon = \bar{\varepsilon} + \delta\varepsilon, \quad p = \bar{p} + \delta p, \quad (5.10)$$

where overbars denote background quantities and  $\delta\varepsilon$ ,  $\delta p$  are their perturbations.

In our model, the effective energy density and pressure depend on these quantities through a coupling function. As a result, perturbations of the effective quantities are induced by perturbations of  $\varepsilon$  and  $p$ . Expanding to first order, we obtain

$$\delta\tilde{\varepsilon} = (\varepsilon\chi)_{,\varepsilon} \delta\varepsilon, \quad (5.11)$$

$$\delta\tilde{p} = (\varepsilon\chi)_{,\varepsilon} \delta p + (\varepsilon + p)(\varepsilon\chi)_{,\varepsilon\varepsilon} \delta\varepsilon. \quad (5.12)$$

These expressions show that perturbations of the effective quantities receive additional contributions proportional to derivatives of the coupling function. Consequently, fluctuations in the energy density are influenced not only by  $\delta\varepsilon$  itself, but also by variations of the coupling.

## 3 Numerical Calculation of the Gravitational Potential

In this section, we describe the numerical procedure used to compute the evolution of the gravitational potential  $\Phi$  within the modified gravity framework. The goal is to solve the perturbation equations derived in the previous sections and analyze their behavior for different scales and coupling models.

### 3.1 Background Evolution

We consider a spatially flat Universe composed of radiation and non-relativistic matter. The total energy density is given by

$$\varepsilon(a) = \Omega_r a^{-4} + \Omega_m a^{-3}, \quad (5.13)$$

where  $\Omega_r$  and  $\Omega_m$  are the present-day density parameters. The conformal Hubble parameter is defined as

$$\mathcal{H}(a) = aH(a), \quad H(a) = H_0 \sqrt{(\Omega_r a^{-4} + \Omega_m a^{-3}) \chi(\varepsilon)}, \quad (5.14)$$

where  $H_0$  is the present-day Hubble parameter and  $\chi(\varepsilon)$  is the coupling function.

The scale factor at radiation–matter equality is

$$a_{\text{eq}} = \frac{\Omega_r}{\Omega_m}. \quad (5.15)$$

### 3.2 Choice of Coupling Function

The modification of gravity is encoded in the function  $\chi(\varepsilon)$ . We consider parametrizations motivated by an expansion in the energy density:

$$\chi(\varepsilon) = 1 + f(\varepsilon). \quad (5.16)$$

In this work, we study three representative models:

- $\chi_1(\varepsilon) = 1$  (standard General Relativity without  $\Lambda$ ),
- $\chi_2(\varepsilon) = 1 + \alpha\varepsilon$ ,
- $\chi_3(\varepsilon) = 1 + \frac{\alpha}{\varepsilon}(1 - \frac{\beta}{\varepsilon})$ .

These functional forms allow us to probe different cosmological regimes. Terms that grow with increasing  $\varepsilon$  dominate in the early, high-density Universe, while inverse powers of  $\varepsilon$  become relevant at late times. This behavior can be understood from the effective cosmological term

$$\Lambda(\varepsilon) = -8\pi G_N \varepsilon^2 \chi_{,\varepsilon}, \quad (5.17)$$

introduced in Eq. (5.5). In particular, a coupling of the form  $\chi \propto 1/\varepsilon$  generates a constant cosmological term, reproducing a  $\Lambda$ CDM behavior, whereas higher inverse powers such as  $\chi \propto 1/\varepsilon^2$  lead to a dynamical effective cosmological term that evolves with the energy density.

### 3.3 Effective Equation of State and Sound Speed

The effective equation of state is given by

$$\tilde{w}(a) = w(a) + (1 + w(a)) \frac{\varepsilon \chi_{,\varepsilon}}{\chi}, \quad (5.18)$$

where  $w(a)$  is the standard background equation of state. Assuming adiabatic perturbations [5], the effective sound speed is defined as

$$c_s^2 \equiv \frac{\delta \tilde{p}}{\delta \tilde{\varepsilon}}. \quad (5.19)$$

For adiabatic fluctuations, this can be expressed in terms of background quantities as

$$c_s^2 = \frac{\tilde{p}'}{\tilde{\varepsilon}'}. \quad (5.20)$$

Using the definition  $\tilde{p} = \tilde{w}\tilde{\varepsilon}$  together with the conservation equation

$$\tilde{\varepsilon}' = -3\mathcal{H}(1 + \tilde{w})\tilde{\varepsilon}, \quad (5.21)$$

one obtains

$$c_s^2 = \tilde{w} - \frac{\tilde{w}'}{3\mathcal{H}(1 + \tilde{w})}. \quad (5.22)$$

Expressing derivatives with respect to the scale factor yields the final form

$$c_s^2 = \tilde{w} - \frac{a}{3(1 + \tilde{w})} \frac{d\tilde{w}}{da}. \quad (5.23)$$

### 3.4 Evolution Equation for the Gravitational Potential

The evolution of the gravitational potential is derived from the linearized Einstein equations in Newtonian gauge. In the absence of anisotropic stress, we set  $\Psi = \Phi$ , and obtain

$$\nabla^2 \Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \delta\tilde{\varepsilon}, \quad (5.24)$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta\tilde{p}. \quad (5.25)$$

Combining these equations and introducing the effective sound speed, we obtain the standard form

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' + [2\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2]\Phi - c_s^2\nabla^2\Phi = 0. \quad (5.26)$$

As in the previous sections, we work in Fourier space, where the perturbation equations decouple into independent modes labeled by the comoving wavenumber  $k$ , related to the inverse wavelength of the perturbation. In this representation, spatial derivatives are replaced according to  $\nabla^2 \rightarrow -k^2$ , reducing the perturbation equations to ordinary differential equations in conformal time.

In the modified gravity framework, the perturbation equations acquire an additional source term arising from the non-minimal matter-gravity coupling. The evolution equation for the gravitational potential then takes the form

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' + [2\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2]\Phi + c_s^2 k^2 \Phi = \varepsilon(1 + \tilde{w}) \frac{(\varepsilon\chi)_{,\varepsilon\varepsilon}}{(\varepsilon\chi)_{,\varepsilon}} [-k^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi)]. \quad (5.27)$$

### 3.5 Numerical Implementation

The system is solved over the range

$$a \in [10^{-7}, 1], \quad (5.28)$$

where  $a$  is the scale factor, normalized such that  $a = 1$  corresponds to the present-day Universe. The initial value  $a = 10^{-7}$  lies deep in the early radiation-dominated epoch, allowing the evolution of the perturbations to be followed from very early times up to the present era.

The initial conditions are chosen as

$$\Phi(a_{\text{ini}}) = 1, \quad \Phi'(a_{\text{ini}}) = 0, \quad (5.29)$$

corresponding to an initially constant gravitational potential.

We consider three characteristic perturbation modes defined relative to the horizon scale at matter–radiation equality,

$$k = 0.03k_{\text{eq}}, \quad k = k_{\text{eq}}, \quad k = 50k_{\text{eq}}, \quad (5.30)$$

where

$$k_{\text{eq}} = \mathcal{H}(a_{\text{eq}}). \quad (5.31)$$

These choices correspond respectively to super-horizon, horizon-scale, and sub-horizon perturbation modes at the epoch of matter–radiation equality.

### 3.6 Evolution of the Gravitational Potential

The numerical evolution of the gravitational potential for the three coupling models is shown in Fig. 5.1. The quantity displayed is the normalized potential  $\Phi(a)/\Phi_{\text{ini}}$  as a function of the normalized scale factor  $a/a_{\text{eq}}$ .

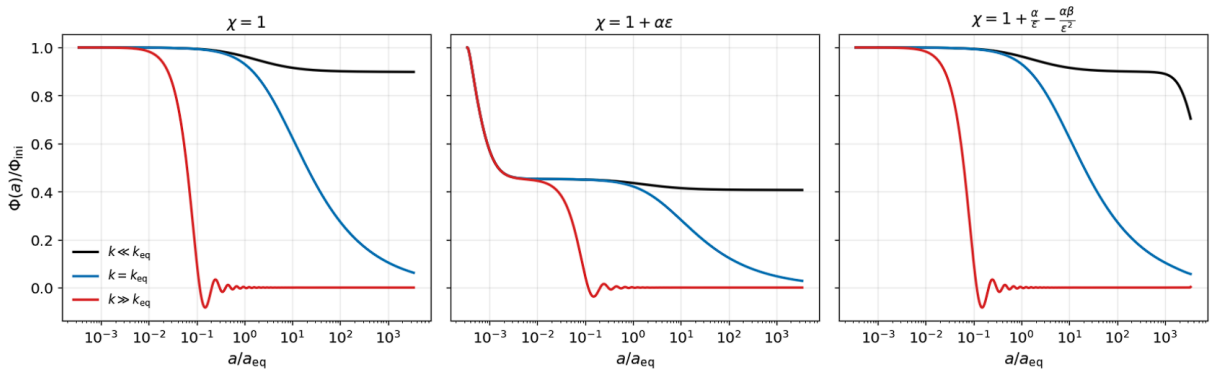


Figure 5.1: Evolution of the gravitational potential for different coupling models. The left panel corresponds to standard General Relativity ( $\chi = 1$ ) without cosmological constant, the middle panel to a linear correction  $\chi = 1 + \alpha\varepsilon$ , and the right panel to the inverse expansion model  $\chi = 1 + \alpha/\varepsilon - \alpha\beta/\varepsilon^2$ . The black, blue, and red curves correspond to super-horizon ( $k \ll k_{\text{eq}}$ ), horizon-scale ( $k = k_{\text{eq}}$ ), and sub-horizon ( $k \gg k_{\text{eq}}$ ) modes, respectively.

Several important features can be observed from the numerical solutions. For modes satisfying  $k \ll k_{\text{eq}}$ , corresponding to wavelengths larger than the horizon at radiation–matter equality, the gravitational potential remains nearly constant throughout most of

the evolution. This behavior reflects the fact that super-horizon perturbations are not significantly affected by causal microphysical processes.

For the intermediate mode  $k = k_{\text{eq}}$ , the potential begins to decay after horizon crossing. This transition occurs because radiation pressure prevents efficient gravitational clustering during the radiation-dominated era. As the Universe evolves toward matter domination, the decay becomes weaker and the potential approaches a nearly constant value.

The strongest evolution is observed for sub-horizon modes ( $k \gg k_{\text{eq}}$ ). In this regime, pressure effects become important, leading to a rapid decay of the gravitational potential accompanied by small oscillations. These oscillations originate from acoustic behavior of the relativistic fluid and are enhanced by the effective sound speed.

Comparing the three panels shows how the coupling function  $\chi(\varepsilon)$  modifies the evolution of perturbations.

In the standard case  $\chi = 1$ , the evolution reproduces the expected behavior of General Relativity without dark energy. Introducing a linear correction  $\chi = 1 + \alpha\varepsilon$  modifies the early-time dynamics and produces a stronger suppression of the potential even for large-scale modes. In contrast, the inverse expansion model  $\chi = 1 + \alpha/\varepsilon - \alpha\beta/\varepsilon^2$  primarily affects the late-time evolution, leading to visible deviations at large values of  $a/a_{\text{eq}}$ .

These results demonstrate that an energy-dependent matter–gravity coupling can significantly alter the evolution of cosmological perturbations, particularly on sub-horizon scales where pressure and sound-speed effects become important. Such modifications may have important implications for the formation and growth of large-scale structures in the Universe. Moreover, models of this type can potentially be tested observationally through cosmological probes, thereby placing constraints on the values of the free parameters.

### 3.7 Dependence on the Coupling Parameters

To further investigate the impact of the modified matter–gravity coupling, we analyze the evolution of the gravitational potential for different values of the coupling parameters. The corresponding numerical solutions are shown in Figs. 5.2, 5.3, and 5.4.

Figure 5.2 shows the effect of the parameter  $\alpha$  in the linear coupling model  $\chi = 1 + \alpha\varepsilon$ . For  $\alpha = 0$ , the standard General Relativity behavior is recovered. Increasing  $\alpha$  produces a stronger suppression of the gravitational potential, particularly for super-horizon and horizon-scale modes. Since positive powers of the energy density mainly affect the high-density regime, the modifications are most significant during the early radiation-dominated epoch.

Figure 5.3 illustrates the evolution of the gravitational potential for the inverse expansion model

$$\chi = 1 + \frac{\alpha}{\varepsilon} \left( 1 - \frac{\beta}{\varepsilon} \right).$$

For  $\beta = 0$ , the model reduces to a coupling proportional to  $1/\varepsilon$ , which generates an

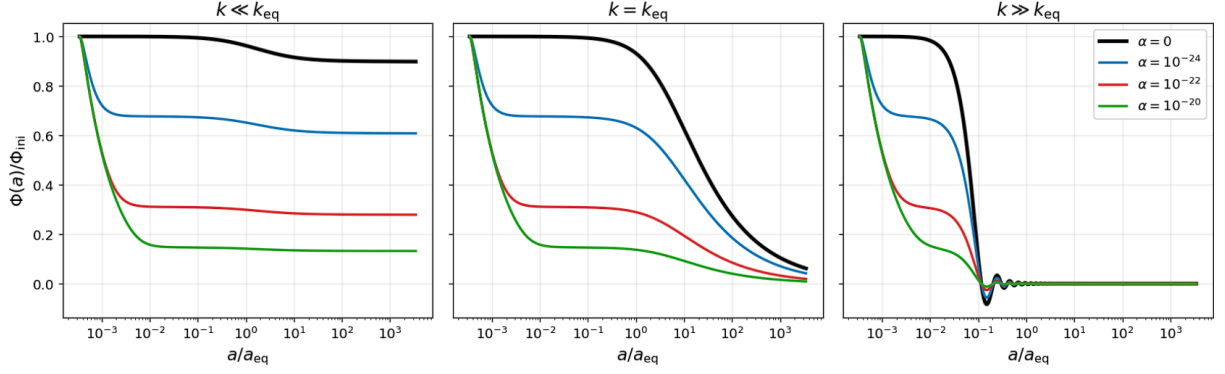


Figure 5.2: Evolution of the gravitational potential for the model  $\chi = 1 + \alpha\varepsilon$  for different values of  $\alpha$ . The left, middle, and right panels correspond to  $k \ll k_{\text{eq}}$ ,  $k = k_{\text{eq}}$ , and  $k \gg k_{\text{eq}}$ , respectively.

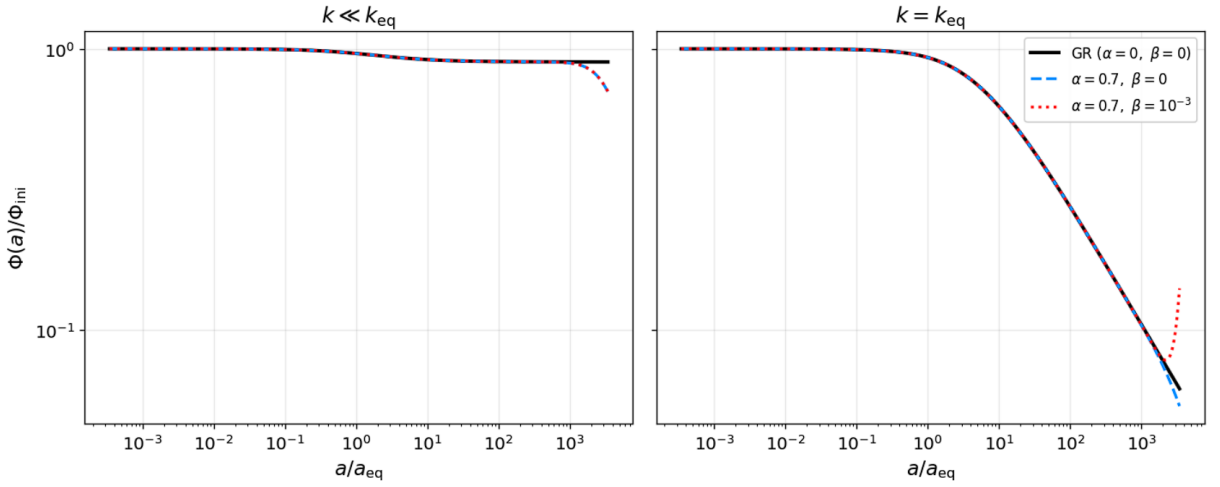


Figure 5.3: Evolution of the gravitational potential for the inverse expansion model  $\chi = 1 + \alpha/\varepsilon - \alpha\beta/\varepsilon^2$  for different values of  $\beta$ . The left and right panels correspond to  $k \ll k_{\text{eq}}$  and  $k = k_{\text{eq}}$ , respectively.

approximately constant effective cosmological term and therefore reproduces a  $\Lambda$ CDM-like behavior. Nonzero values of  $\beta$  introduce an additional inverse quadratic correction, leading to a dynamical dark-energy component that modifies the late-time evolution of perturbations.

This distinction is reflected in the evolution of the gravitational potential shown in Fig. 5.3. In contrast to the linear model, the inverse corrections are strongly suppressed at early times and become important only in the late low-density Universe. Consequently, all solutions initially remain very close to the standard General Relativity evolution.

For the super-horizon mode ( $k \ll k_{\text{eq}}$ ), the gravitational potential remains nearly constant throughout most of the evolution, with only small late-time deviations from the standard case. More noticeable effects appear near the horizon scale ( $k = k_{\text{eq}}$ ), where larger values of  $\beta$  generate a late-time enhancement of the gravitational potential. This behavior indicates that the inverse quadratic correction introduces a dynamical dark-energy component capable of modifying the late-time evolution of scalar perturbations.

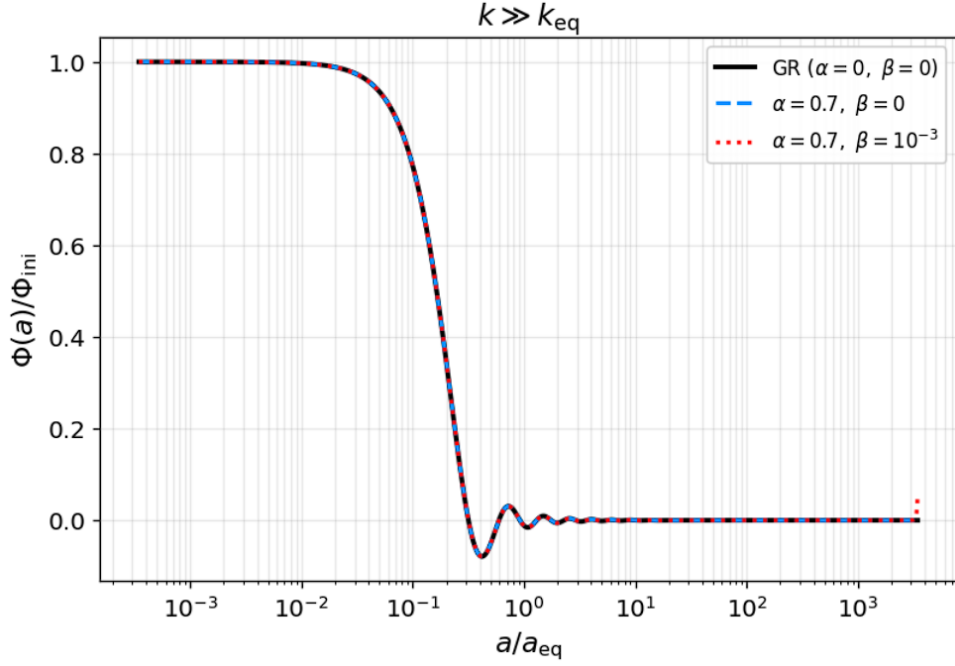


Figure 5.4: Evolution of the gravitational potential for the inverse expansion model  $\chi = 1 + \alpha/\varepsilon - \gamma/\varepsilon^2$  in the sub-horizon regime  $k \gg k_{\text{eq}}$ .

The behavior of the sub-horizon mode ( $k \gg k_{\text{eq}}$ ) is shown in Fig. 5.4 and Fig. 5.4. For most of the cosmological evolution, the solutions remain almost indistinguishable from the standard General Relativity prediction, exhibiting the usual oscillatory decay inside the horizon.

At the same time, the deviations remain relatively small and appear only near the present epoch, making them difficult to distinguish observationally from the standard  $\Lambda$ CDM scenario. Consequently, testing such models requires high-precision measurements of late-time structure formation and the evolution of gravitational potentials.

Overall, these results demonstrate that different parametrizations of the coupling function  $\chi(\varepsilon)$  lead to qualitatively different cosmological behaviors. Positive powers of  $\varepsilon$  primarily modify the early high-density Universe, while inverse powers produce late-time corrections associated with dynamical dark-energy effects.

# CHAPTER 6

## Conclusion

In this thesis, we investigated the evolution of cosmological perturbations within a modified gravity framework based on an energy-dependent matter–gravity coupling. This approach is motivated by the possibility of extending the standard  $\Lambda$ CDM model and providing a unified description of different phases of cosmic acceleration.

After reviewing the Newtonian and relativistic descriptions of cosmological perturbations, we established the theoretical framework needed to describe the evolution of density fluctuations and the gravitational potential.

Building on this framework, we introduced a modified gravity model in which the coupling between matter and geometry depends explicitly on the energy density, following the proposal of Markov and Mukhanov. This leads to an effective energy–momentum tensor and modifies both the background dynamics and the evolution of perturbations.

The main result of this work is the derivation of a modified evolution equation for the gravitational potential, which includes an additional source term arising from the energy-dependent coupling. We analyzed this equation numerically for different forms of the coupling  $\chi(\varepsilon)$  and for perturbation modes on super-horizon, horizon, and sub-horizon scales.

Our results show that such modifications can significantly affect the evolution of perturbations. In particular, models with positive powers of the energy density primarily influence early-time dynamics, while inverse powers introduce deviations at late times.

These findings indicate that non-minimal matter–gravity coupling provides a consistent and flexible framework for exploring deviations from General Relativity. In particular, this framework may offer a useful approach for studying the possible dynamical nature of dark energy and its impact on the evolution of cosmological perturbations. Future work may include confronting these models with observational data, constraining the coupling parameters, and extending the analysis beyond linear perturbation theory.

## REFERENCES

- [1] N. Aghanim *et al.* (Planck Collaboration), *Astron. Astrophys.* **641**, A6 (2020); **652**, C4(E) (2021), arXiv:1807.06209.
- [2] P. J. E. Peebles and B. Ratra, “The cosmological constant and dark energy,” *Rev. Mod. Phys.* **75**, 559 (2003), arXiv:astro-ph/0207347.
- [3] D. J. Eisenstein *et al.* (SDSS Collaboration), “Detection of the baryon acoustic peak in the large-scale correlation function of SDSS luminous red galaxies,” *Astrophys. J.* **633**, 560 (2005), arXiv:astro-ph/0501171.
- [4] A. G. Riess *et al.*, “Observational evidence from supernovae for an accelerating Universe and a cosmological constant,” *Astron. J.* **116**, 1009 (1998), arXiv:astro-ph/9805201.
- [5] D. Baumann, *Cosmology* (University of Amsterdam, Amsterdam, 2022).
- [6] L. Perivolaropoulos and F. Skara, “Challenges for  $\Lambda$ CDM: An update,” *New Astron. Rev.* **95**, 101659 (2022), arXiv:2105.05208.
- [7] A. G. Adame *et al.* (DESI Collaboration), *JCAP* **02**, 021 (2025), arXiv:2404.03002.
- [8] M. Abdul Karim *et al.* (DESI Collaboration), *Phys. Rev. D* **112**, 083515 (2025), arXiv:2503.14738.
- [9] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, “Modified Gravity and Cosmology,” *Phys. Rept.* **513**, 1 (2012), arXiv:1106.2476.
- [10] T. Harko and F. S. N. Lobo, *Eur. Phys. J. C* **70**, 373 (2010), arXiv:1008.4193.
- [11] M. A. Markov and V. F. Mukhanov, *JETP Lett.* **40**, 1 (1984).
- [12] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
- [13] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, “Theory of cosmological perturbations,” *Phys. Rept.* **215**, 203 (1992).
- [14] H. Chakrabarty and D. Malafarina, “A unified model of dark energy and inflation from the Markov–Mukhanov action,” *Eur. Phys. J. C* **85**, 1422 (2025), arXiv:2510.14416.

- [15] A. Zholdasbek, H. Chakrabarty, D. Malafarina, and A. Bonanno, “An emergent cosmological model from running Newton constant,” *Phys. Rev. D* **111**, 103519 (2025), arXiv:2405.02636.