Multigrid method for Mild-Slope equation in Coastal Wave Modelling

by

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Abstract

In this thesis we propose and study an efficient iterative multigrid method for the time-independent modified mild slope equation with and without energy dissipation term. The algorithm relies on a multigrid method preconditioned with shifted-Laplacian preconditioner and solved by Bi-CGSTAB algorithm. Multigrid analysis results are shown by numerical experiments. Numerical experiments are conducted in depth sloped elliptic shoal introduced by Berkhoff et. al [3].

Thesis Supervisor: Yogi Erlangga
Title: Assistant Professor

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A special thanks to my family.

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Chapter 1

Introduction

One of the most important hydrodynamic processes in coastal areas is propagation of water waves. Wave trains travel different depth over water and interact with lateral boundaries such as precipice, breakwaters and beaches, changing its structure, which affects the design of coastal structures and buildings.

Wave transformation can be predicted through experimental, analytical and numerical methods. Collecting experimental data is an expensive and time consuming process. Alternatively, mathematical model is developed to allow a prediction. However, assumptions involved numerical models are based on comparing analytical solutions to approximation. We have analytical solutions for certain special cases under idealized conditions, but they can be computed quick, simple and accurate. Using mathematical description of physical processes we can predict wave attacks on coastal structures and their response. Wave moves from varying depth of water and create phenomena like reflection, refraction, energy dissipation created by sea bed friction and resonance.

Transformations of wave structures can be modelled in various ways. One way to model is applying the mild-slope equation, proposed initially by Berkhoff [3]. The earliest form of the mild-slope equation can however only model transformations of linear type waves over a slowly varying sealed (non permeable) sea bottom. Since then, several modifications have been proposed, to allow the equation model more realistic physical phenomena ([5], see Chamberlain et. al.). They also included energy dissipation term considering energy consumption of water breaks and sea bed friction, analogous to Dingemans (1997)[7]. They considered cases when boundary open, fully absorbing and partially reflecting using the 2^{nd} order parabolic approximation. Using a second-order central difference scheme Silva et al [18] discretized the main equation and boundary conditions. The defined sparse-banded matrix is computed by using estimating method for Gaussian elimination with elements of partial pivoting. Chamberlain and Porter (1995)[5] made implementation into mild-slope equation, which can describe known dispersing effect of singly and doubly periodic ripple beds and permeabilities. From there on, numerous adjusted and extended forms have been proposed to model transformations and effects such as wave nonlinearity, steeper sea-bed slopes, wave-current interaction and bed friction.

1.1 Aim and Scope of the Thesis

The 3D sea waves model can be reduced and approximated by the 2D surface wave model, using the so-called the mild-slope equation, defined on the sea surface,

$$\nabla_h^2 \psi + K_c^2 \psi = 0, \tag{1.1}$$

where $\nabla_h^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and $K_c \in \mathbb{C}$ is a constant that depends on (x, y) and parameters that model the physics of sea waves.

From mathematical point of view, (1.1) is the Hemholtz equations. The objective of the MSc project is to develop an efficient iterative method for the above time-independent model, suitable for large scale modelling. The principle objectives of this research study is:

- develop an efficient iterative method for (1.1).

We shall use mathematical machinery, developed by Erlangga et al.[10], for Helmholtz problem, with the following characteristics:

- solution which is accurate

- work of order $\mathcal{O}(n)$, thus allows increasing size, where n is the size of the matrix u the resultant system of equations

- well suited for parallel computing.

1.2 Review of Related Literature

Numerical solution procedure for (1.1) typically consist of two steps:

1. Discretization of (1.1) and relevant boundary conditions, resulting in a linear system. In this thesis, we focus on finite difference methods.

2. Solving system of linear equations.

One popular approach to solve the system of linear equations associated with (1.1) is originated from the work of Maa et al.[15], which is based on Gaussian elimination. Memory requirements from CPU to hard disk is reduced by row reduction method with elements of partial pivoting in [15]. The method is simple to implement, but computational work increases with the increase in the size of the problems. In general, if $\mathbf{A} \in \mathbb{C}^{n \times n}$, then the cost for solving $\mathbf{Ax} = \mathbf{b}$ is of order $\mathcal{O}(n^2)$.

An alternative to Gaussian elimination procedures is iterative methods. An iterative method is based on an update process of an initial guess of the solution, which can be done by matrix-vector multiplications. A matrix-vector multiplication is an $\mathcal{O}(n)$ work, if $\mathbf{A}$ is sparse, which is the case for $\mathbf{A}$ arising from finite difference approximation of (1.1).

One of the known efficient iterative method is multigrid. The multigrid method was introduced in 1977 by Brandt[4]. Large variety of problems can be solved by this method, but it is however most efficient for equations of the elliptic type (see [20]). Multigrid (MG) methods are for solving a system of linear algebraic equations, based on the use of a sequence of different coarse grids and transition operators from one grid to another. Over the past 30 years, many researchers contributed for development of appropriate multigrid method for Helmholtz problems. Researchers have developed some generalizations of the concepts essential to multigrid methods for problems in different disciplines, but cost-effective direct multigrid solver of non homogeneous problems with high nonlinear wave numbers arising in coastal engineering is still one of the important problems.

In view of the fact that loss of $\mathcal{O}(n)$ complexity and retrogression of convergence are created by problems of smoothing and coarse-grid correction components, multigrid method performance is not efficient for the Helmholtz equation. On a coarser grid some eigenvalues near to the origin can change their signs. If this problem arise, the coarse-grid solution doesn't lead to a convergence acceleration, whereas there can be seen a drastic convergence degradation (or even divergence). Howard Elman et.al. in [8] analyzed and proposed solution to these problematic eigenvalues in coarse grid correction. The efficient remedy that they proposed in [8] is applying multigrid integrated with Krylov subspace iteration methods on coarse and fine grids. Typical multigrid method doesn't work for k^2 -values very close to eigenvalue.

Components of the multigrid method determine its performance. Multigrid techniques include an inventive mixture of the corresponding iterative methods such as coarse-grid correction and smoothing. Simple multigrid methods can't solve the Helmholtz equation because it gets intermediate frequencies [12, 10]. Rather we can use the multigrid for fine grids that maintain desirable error accuracy. During several years, various approaches have been proposed to solve this problem. We can review it in [13]. GMRES [17] or Bi-CGSTAB [22] are known Krylov subspace methods, which acknowledged for their convergence to the solution. If we choose parameter of minimum number of grid points per wavelength in the coarsest grid less than 1 and add more levels of multigrid into problem, we will get large number of iteration. However cost of this iterations for this problem will be small. One most known example of such problem is shifted Laplacian method. In this method multigrid cycle is used as a preconditioner for iterative method (Bi-CGSTAB). The linear system discretized from the Helmholtz equation is usually solved with an effective iterative method Bi-GCSTAB [23].

B.Li and K.Anastasiou [14] used multigrid technique to solve mild-slope equation.

Traditional elliptic and hyperbolic schemes require a significant size of grid points per wavelength, but scheme in [14] needs a small number of points, which will reduce computational cost. Erlangga et al.[11] solved Helmholtz equation with an altered complex wavenumber. They used multigrid as preconditioned and solved by Bi-CGSTAB algorithm. For Poisson type equations, a multigrid (smoothing) method efficiently decrease high frequency elements of errors, whereas a coarse-grid correction technique responsible for the low frequency elements. But for Helmholtz equation it has no problems, until the wavenumber is small in acceptable degree [1, 2].

The smoothing rate of a simple iterative method is crucial than its convergence rate in multigrid method, Jacobi iteration gives needed smoothing effects for the Helmholtz equation.

We will try to use same approach as in [11] to solve the model in this thesis research.

1.3 Outline of the Thesis

The Modified Mild Slope problem is solved by the multigrid method using Bi-CGSTAB with shifted Laplacian preconditioner. The formulation of the Helmholtz and literature review has already been discussed in the previous two sections. In chapter 2 we defined our modified mild slope model with respective governing and boundary equations. Further in subsection 2.3.2, we introduced main tools such as Multigrid, Bi-CGSTAB and shifted Laplacian for solving our problem.

Chapter 2

Modified Mild Slope Model

2.1 Modified Mild Slope Equation

2.1.1 Governing Equation

We suppose irrational flows with constant density over a sealed sea bottom with variable depth $h(x, y)$, where x and y are horizontal Cartesian coordinates. Then there exists velocity potential $\Phi(x, y, z, t)$, which describes fluid motion, satisfying the Laplace equation [3]

$$\nabla_h^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad -h \leq z \leq 0. \tag{2.1}$$

Here z represents the vertical coordinate, measured positively upwards. At $z = 0$, free surface condition is satisfied. $\nabla_h = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ denotes horizontal gradient operator.

Suppose that the potential Φ can be expressed as,

$$\Phi(x, y, z, t) = Re\{\phi(x, y) f(z) e^{-i\omega t}\}, \tag{2.2}$$

where symbol Re represents the real part of 2D complex plane horizontal flow velocity potential ϕ of the water surface, $f(z)$ is depth dependency and is given by the following equation,

$$f(z) = \frac{\cosh\left[k(h+z)\right]}{\cosh(kh)}, \tag{2.3}$$

in which $k \equiv k(x,y)$ indicates local wave number. It can be derived from following linear dispersion relation,

$$\omega^2 = gk\tanh(kh), \tag{2.4}$$

where ω is the angular frequency, satisfying $\omega = \frac{2\pi}{T}$, T the wave period and g gravitational constant.

We can apply Green's second identity to Φ and $f(z)$ [19],

$$\int_{-h}^{0} f\frac{\partial^2\Phi}{\partial z^2}dz - \int_{-h}^{0} \Phi\frac{\partial^2 f}{\partial z^2}dz - \left[f\frac{\partial\Phi}{\partial z} - \Phi\frac{\partial f}{\partial z}\right]_{-h}^{0} = 0. \tag{2.5}$$

As a result of substituting Equations (2.1), (2.2) (2.3) in Equation (2.5), we will get time-independent mild slope equation with modified wave number as,

$$\nabla_h(I_1\nabla\phi) + k^2\phi I_1 + \phi r(h) = 0, \tag{2.6}$$

where

$$r(h) = I_2(h)\nabla_h^2 h + I_3(h)\nabla_h h\cdot\nabla_h h, \tag{2.7}$$

$$I_1(h) = \frac{1}{2k}\tanh(kh)\left[1 + \frac{2kh}{\sinh(2kh)}\right], \tag{2.8}$$

$$I_2(h) = \frac{\operatorname{sech}^2(kh)}{4(2kh+\sinh(2kh))}\left[\sinh(2kh) - 2kh\cosh(2kh)\right], \tag{2.9}$$

$$I_3(h) = \frac{k\operatorname{sech}^2(kh)}{12(2kh+\sinh(2kh))}\Big[(2kh)^4 + 4(2kh)^3\sinh(2kh) - 9\sinh(2kh)\,\sinh(4kh) + 6kh(2kh+2\sinh(2kh))(\cosh^2(2kh) - 2\cosh(2kh)+3)\Big]. \tag{2.10}$$

Equation (2.6) can be modified further to contain an energy dissipation term, which includes dissipation due to wave breaking and bottom friction [18]:

$$\nabla_h(I_1\nabla\phi) + \left[k^2 I_1 + i\omega D I_1 + r(h)\right]\phi = 0, \tag{2.11}$$

where D denotes dissipation factor and is given by sum of bottom friction f_B and wave breaking dissipation f_D

$$D = f_D + f_B. \tag{2.12}$$

Dally et al. [6] derived the breaking wave dissipation factor f_D

$$f_D = \frac{kC_k}{\omega h}\left[1 - \left(\frac{C_G h}{H_B}\right)^2\right], \tag{2.13}$$

where H_B is the height of the wave at the wave breaking point and C_k, C_G are empirical constants, $C_k = 0.15$ and $C_G = 0.4$. The bottom friction dissipation is given by

$$f_B = \frac{4}{3\pi}\frac{C_f a\omega^2}{ng\sinh^3(kh)}, \tag{2.14}$$

where $a = \frac{1}{2}H$ is the local wave amplitude and $H = \frac{2\omega}{g}\sqrt{\phi_1^2 + \phi_2^2}$, C_f is the Darcy-Weishbach friction factor and shoaling factor, and $\phi = \phi_1 + i\phi_2$

$$n = \frac{1}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right). \tag{2.15}$$

Using change of variable

$$\psi = \sqrt{I_1}\phi, \tag{2.16}$$

Equation (2.11) can be cast into

$$\frac{\partial^2\psi}{\partial x^2} + K_c^2\psi = 0, \tag{2.17}$$

with

$$K_c^2 = k^2 + i\omega D + \frac{r(h)}{I_1} - \frac{\nabla^2\sqrt{I_1}}{\sqrt{I_1}}, \tag{2.18}$$

Equation 2.17 is the Helmholtz equation with complex coefficient K_c^2.

Dispersion relation can be modified to increase accuracy in some of it applications. We will use the following dispersion equation that includes nonlinear effects of wave motion,

$$w^2 = gk[1 + (ka)^2 F_1 \tanh^5(kh)] \tanh[kh + (ka)F_2], \tag{2.19}$$

where functions F_1 and F_2 are expressed as,

$$F_1 = \frac{\cosh(4kh) + 8 - 2\tanh^2(kh)}{8\sinh^4(kh)}, \tag{2.20}$$

$$F_2 = \left(\frac{kh}{\sinh(kh)}\right)^4. \tag{2.21}$$

The above dispersion relation however leads to a nonlinear mild-slope equation. In this case, K_c in Eq. (2.17) is now a function of ψ, too.

2.1.2 Boundary Conditions

To solve the PDE (2.17), set of boundary conditions is required. We will define in this work fully reflective, fully absorbing and partially absorbing boundary conditions, except of coastal boundary, with universal form, [24]

$$\frac{\partial \phi}{\partial \eta} = ik\alpha\phi \tag{2.22}$$

where η indicates normal vector that outward-facing to the boundary condition and $\alpha \in [0, 1]$ is reflection coefficient. When $\alpha = 1$ is a fully absorbing boundary with only seaward propagating as the upper boundary shown in Figure 2-1. $\alpha = 0$ implies fully reflective boundary condition, accordingly a vertical sea wall as in Figure 2-1 can fully reflect propagated waves. The incident incoming boundary condition is given by,

$$\frac{\partial \phi}{\partial \eta} = ik(2\phi_{inc} - \phi) \tag{2.23}$$

where $\phi_{inc} = \frac{2kag}{\omega} e^{-kx}$. This boundary condition however depends nonlinearly on the solution.

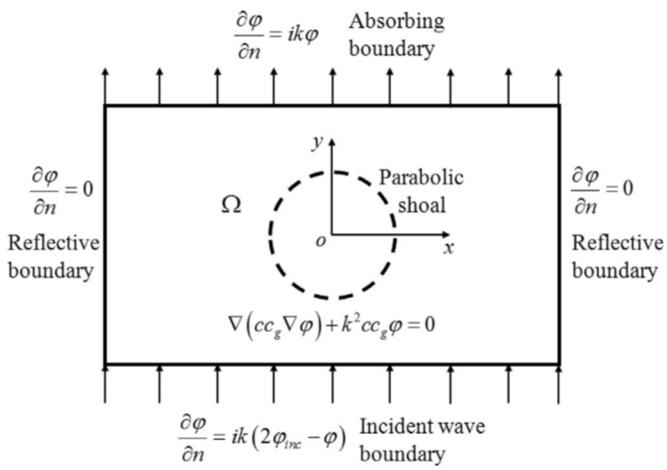

Figure 2-1: Schematic diagram of boundary conditions [24]

The incident boundary condition is generally referred to incorporate the incident wave potential ϕ_{inc} with the effect of reflected waves outgoing from the coastal boundary.

2.2 Finite Difference Method

In Subsection 2.1.1 we defined our model in Helmholtz form in Equation (2.17). Helmholtz problem can be discretized using finite difference method or finite elements method. In this thesis work the finite difference method is applied. Finite difference methods transform a linear (non-linear) partial differential equation into a system of linear (non-linear) equations, which can be solved by various numerical linear algebra techniques.

The Helmholtz Equation (2.17) is discretized using a 2^{nd} order central finite difference scheme. The 2D domain of interest is discretized on an equidistant grid with subintervals L and M respectively in x-direction and y-direction

$$x_l = l\,\Delta x, \qquad l = 0, 1, ..., L \tag{2.24}$$

$$y_m = m\,\Delta y, \quad m = 0, 1, ..., M. \tag{2.25}$$

We can write our governing Equation 2.17 in discretized form using second order standard central difference scheme as,

$$\frac{\psi_{l,m-1} - 2\psi_{l,m} + \psi_{l,m+1}}{\Delta y^2} + \frac{\psi_{l-1,m} - 2\psi_{l,m} + \psi_{l+1,m}}{\Delta x^2} + K_c^2 \psi_{l,m} = 0. \tag{2.26}$$

Discretized equations for interior and boundary used for all grid points in the given domain. The resulting matrix may be written as:

$$A_h[\phi_h]\phi_h = f, \tag{2.27}$$

where A_h is the matrix, which depends in general on ϕ_h, and ϕ_h is the unknown vector with grid point values of the wave potential. Applicable data from discretized boundary conditions is in vector f .

2.3 Numerical Solution Method

2.3.1 The Bi-CGSTAB Method

The biconjugate gradient stabilized (Bi-CGSTAB) algorithm is a transpose-free biorthogonalization method modified from the conjugate gradient squared (CGS) method. The general idea in Bi-CGSTAB is defining the residual vectors as

$$r_i = \psi_i(A)\phi_i(A)r_0 \tag{2.28}$$

$\psi_i(A)$ is a recursive function which defined as

$$\psi_{j+1}(A) = (I - \omega_j A)(\psi_j(A)). \tag{2.29}$$

Appropriate choice of parameter ω_j is to reach a steepest descent step in the direction $\phi_j(A)r_0$, original residual direction which minimizes $||(I - \omega_j A)\phi_j(A)r_0||$. Bi-conjugate gradient stabilized algorithm (Bi-CGSTAB) is well known Krylov subspace algorithm for solving non-Hermitian type. There a lot examples such as in [9, 21] where it is used for Helmholtz problems. Some main advantage of Bi-CGSTAB are its limited memory requirements and as the convergence for Helmholtz problems is usually faster than convergence of GMRES [10]. Using Krylov subspace methods without preconditioner is not common, due to its very slow convergence. The unpreconditioned Bi-CGSTAB algorithm is shown in Figure 2-2

Compute $\boldsymbol{r}_0 = \boldsymbol{b} - A\boldsymbol{x}_0$, choose $\boldsymbol{r}'_0$ such that $\boldsymbol{r}_0 \cdot \boldsymbol{r}'_0 \neq 0$
Set $\boldsymbol{p}_0 = \boldsymbol{r}_0$
for $j = 0, 1, \cdots$ **do**
    $\alpha_j = (\boldsymbol{r}_j \cdot \boldsymbol{r}'_0)/((A\boldsymbol{p}_j) \cdot \boldsymbol{r}'_0)$
    $\boldsymbol{s}_j = \boldsymbol{r}_j - \alpha_j A\boldsymbol{p}_j$
    $\omega_j = ((A\boldsymbol{s}_j) \cdot \boldsymbol{s}_j)/((A\boldsymbol{s}_j) \cdot (A\boldsymbol{s}_j))$
    $\boldsymbol{x}_{j+1} = \boldsymbol{x}_j + \alpha_j \boldsymbol{p}_j + \omega_j \boldsymbol{s}_j$
    $\boldsymbol{r}_{j+1} = \boldsymbol{s}_j - \omega_j A\boldsymbol{s}_j$
    if $||\boldsymbol{r}_{j+1}|| < \varepsilon_0$ **then**
        Break;
    end if
    $\beta_j = (\alpha_j/\omega_j) \times (\boldsymbol{r}_{j+1} \cdot \boldsymbol{r}'_0)/(\boldsymbol{r}_j \cdot \boldsymbol{r}'_0)$
    $\boldsymbol{p}_{j+1} = \boldsymbol{r}_{j+1} + \beta_j(\boldsymbol{p}_j - \omega_j A\boldsymbol{p}_j)$
end for
Set $\boldsymbol{x} = \boldsymbol{x}_{j+1}$

Figure 2-2: Biconjugate Gradient Stabilized(BICGSTAB) algorithm

2.3.2 Multigrid Method

General

We have main components of Multigrid method, which we use to construct complete multigrid iteration.

Multilevel grid. Let's take an m-level grid with coarsening it by a factor of two. Finest grid on the first level will be denoted by Ω_h. Coarsest gird on the last level m will be denoted by Ω_{mh}.

Intergrid Operator. Intergrid operators transfers information in different levels. Function on the fine grid on the k-th level onto the coarse grid on the $(k+1)$-th level is projected by the restriction operator R_k^{k+1}. Information of the coarse grid will be interpolated by the prolongation operator P_k^{k+1}, when the information is returned to the fine grid.

Smoother One multigrid iteration can have pre-smoother and post-smoother. We denote smoothing operator by S. We apply pre-smoothing before restricting our problem to the coarse grid. The post-smoothing applies for prolongated function onto the fine grid. We will denote smoothing steps for pre- and post- smoothing by μ_1 and μ_2. In this thesis, ω-Jacobi iteration is used as smoother.

Multigrid Cycles

In this thesis work we will work with the two-grid problem. We will assume that there are $m+1$ grids, $m \geq 0$, with grid spacing h and the grid coarsened by factor 2, $L = 2^m$.

- Smoother is applied ν^1 times to $A^h u^h = f^h$ with the initial guess ν^h. Denote the result by ν^h.

- Restrict the residual $r^h = f^h - A^h u^h$ to the coarst grid by $r^{2h} = R_h^{2h} r^h$.

  - Smoother will be applied ν^1 times to $A^{2h} e^{2h} = r^{2h}$ with initial guess $\nu^{2h} = 0$. The results denoted by ν^{2h}

  - Restriction $r^{4h} = R_{2h}^{4h} r^{2h} = R_{2h}^{4h}(f^{2h} - A^{2h} u^{2h})$

    ⋮

    Compute $A^{Lh} u^{Lh} = f^{Lh}$.

    ⋮

  - Prolongation and correction $\nu^{2h} := \nu^{2h} + P_{4h}^{2h} \nu^{4h}$.

– Smoother will be applied ν^2 times to $A^{2h}u^{2h} = f^{2h}$ with initial guess ν^{2h}.

- Prolongation and correction $\nu^h := \nu^h + P^h_{2h}\nu^{2h}$.

- Smoother will be applied ν^2 times to $A^h u^h = f^h$ with the initial guess ν^h.

We can see hierarchy of grids for $\gamma = 1$ (V -cycle) and $\gamma = 2$ (W-cycle) in Figure 2-3. We have F-cycle between this cycles.

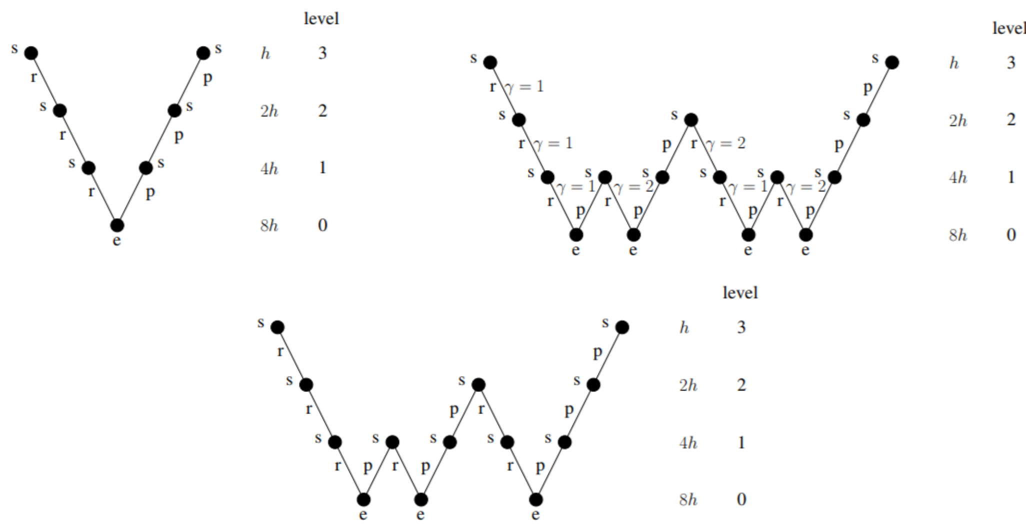

Figure 2-3: V-cycle multigrid,W-cycle multigrid, F-cycle multigrid: r - restriction, p - prolongated, e - exact solver, s - smoothing

We shall use F-cycle (see Figure 2-3) in our numerical tests. In Oosterlee et. al. [16] work, they showed that with preconditioned problem with Bi-CGSTAB algorithm performs too poor in the V - cycles and too expensive on the fine grids for the W-cycle, when wavenumbers are high. In our thesis work we used F-cycle multigrid.

2.3.3 Shifted-Laplacian Preconditioner

We solve with preconditioner $M_h \in \mathbf{C}^{N\times N}$ an equivalent linear system,

$$A_h M_h^{-1}\tilde{\phi}_h = f_h, \qquad \tilde{\phi}_h = M_h\phi_h. \tag{2.30}$$

It is important to find suitable matrix M_h that qualifies as preconditioner matrix, with following basic requirements.

- M_h^{-1} should be efficiently approximated such that it will be inexpensive to solve the linear system

- The preconditioner M_h should be nonsingular and close to matrix A_h to some degree, such that $A_h M_h^{-1}$ is also close to the identity matrix I. With such properties multigrid converges easily.

- In most cases the preconditioning should maintain the symmetry if the original matrix A_h is symmetric.

Same as in [9], a shifted-Laplacian operator considered here as a preconditioner of our problem, where M_h defined as a discretization of

$$\mathcal{M} = -\nabla^2 - k^2(x)(\beta_1 - \beta_2 \hat{i}). \tag{2.31}$$

where $\hat{i}$ the imaginary unit and we can choose parameters (β_1, β_2).

Chapter 3

Numerical Experiments

In previous chapter we defined our model and ingredients of efficient and robust iterative solver. In this chapter we performed series of numerical experiments to solve our (Equation 2.17). Two numerical examples for model with and without energy dissipation term were used to demonstrate the feasibility and accuracy of solution of the given models using our efficient iterative method. Wave propagation is computed over a sloped elliptic shoal. The numerical results obtained by the proposed multigrid method preconditioned with shifted-Laplacian will be shown in section 3.2.

3.1 1D case

For 1D case we visualize the result of the choice of optimal (β_1, β_2) as mentioned in [16], for shifted-Laplacian preconditioner, on the clustering of the eigenvalues of the preconditioned problem. In our case both A and M are the 5-point stencils. Spectra of AM^{-1} for $(\beta_1, \beta_2) = (1, 0.5)$ was presented in Figure 3-1 for different number of grid points.

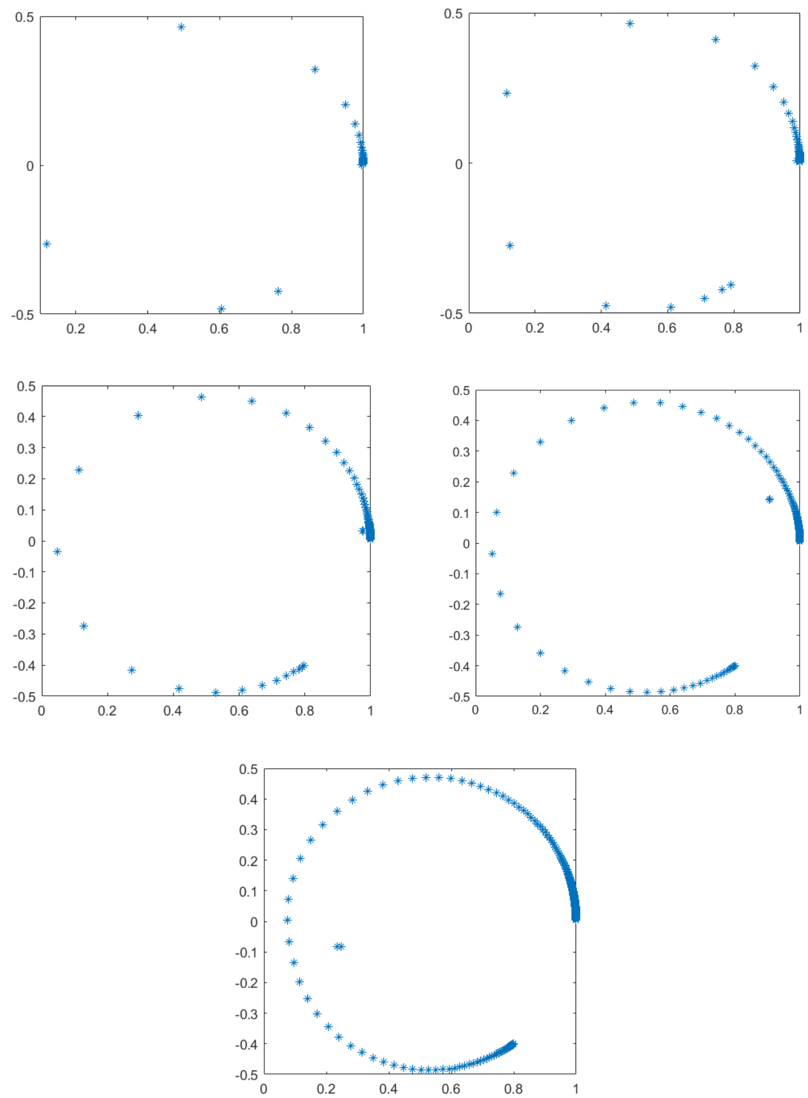

Figure 3-1: Spectral plots of AM^{-1} for $(\beta_1, \beta_2) = (1, 0.5)$ for grid points $n = 65, 129, 257, 513, 1025$. This case for model without dissipation.

Eigenvalues now around the origin on the right half-plane. Oosterlee et. al. in [16] describes this behaviour beneficial for iterative solution methods. We increase number of grid points in order to verify the sensitivity analysis of the preconditioned Bi-CGSTAB solver.

We used a right preconditioner to a Modified Mild slope equation with a complex

Mesh	# iter	Sec	L_2 error
65 x 65	9	0.009249	6.79e-07
129 x 129	19	0.059404	9.00e-07
257 x 257	25	0.251154	8.34e-07
513 x 513	45	1.714066	7.54e-07
1025 x 1025	40	5.888332	2.71e-07

Table 3.1: Bi-CGSTABSL performance for the homogeneous model in terms of number of iterations and CPU time in seconds for different grid sizes.

wavenumber to accelerate convergence rate. The convergence rate is relatively independent than our increasing grid size. Our multigrid scheme only applies 1 iteration of a pre and post smoother.

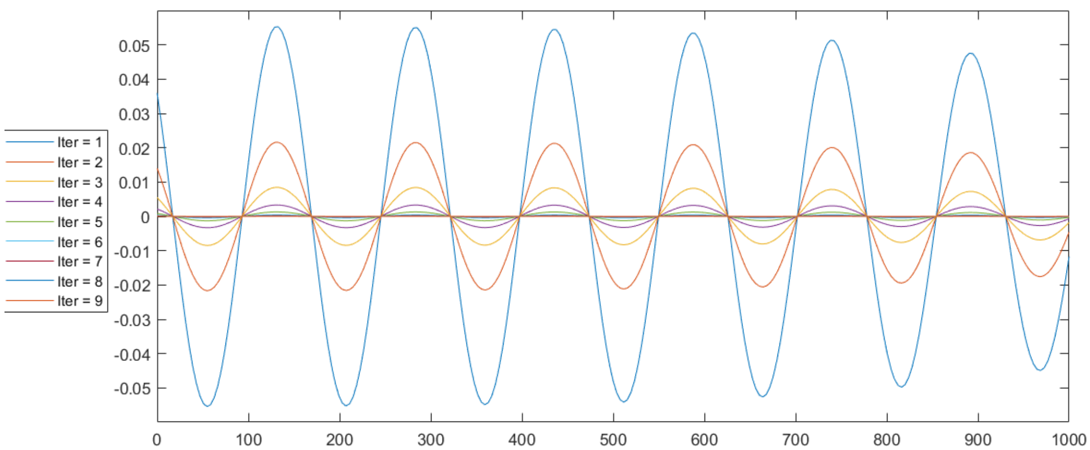

Figure 3-2: Convergence of solution for 1D case

In Figure 3-2 we can see convergence to exact solution.

3.2 2D case

We discretized our respective governing and boundary condition equations using finite difference method as we mentioned in section 2.2. Our matrix structure has the form as in Figure 3-3. We will verify our model in depth sloped elliptic shoal introduced by Berkhoff et. al [3] in Figure 3-4.

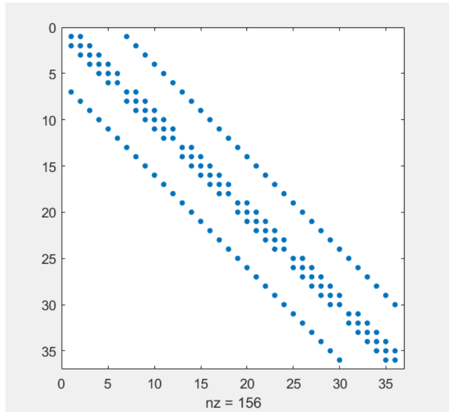

Figure 3-3: Structure of matrix A, 36 x 36

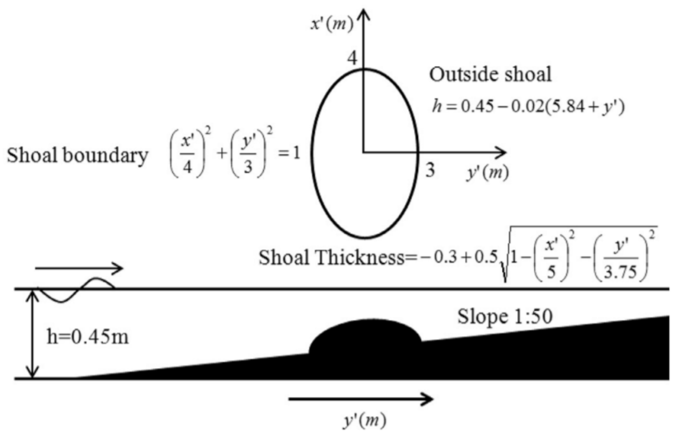

Figure 3-4: Elliptic shoal model (Berkhoff et al.[3])

Computational coordinates and slope oriented coordinates linked with following equations,

$$x' = (x - x_0)\cos 20^\circ + (y - y_0)\sin 20^\circ \tag{3.1}$$

$$y' = (y - y_0)\cos 20^\circ - (x - x_0)\sin 20^\circ \tag{3.2}$$

where x_0, y_0 are coordinates of center of the shoal. The position of the shoal is given by,

$$\left(\frac{x'}{4}\right)^2 + \left(\frac{y'}{3}\right)^2 < 1 \tag{3.3}$$

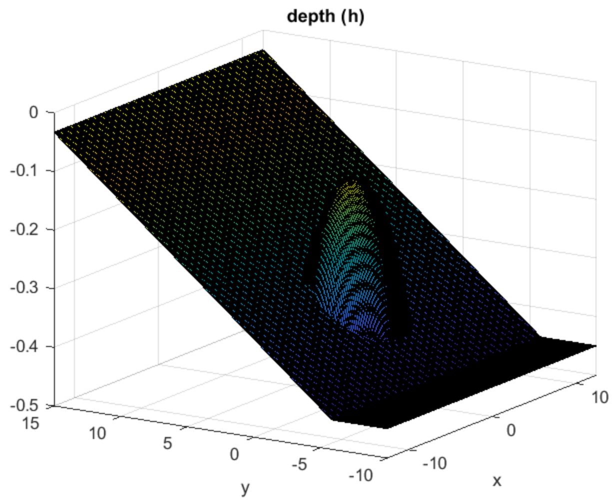

Figure 3-5: The depth of the shoal and the water

The depth of the water out of range of the shoal is set as,

$$h_s = 0.45 - 0.02(5.84 + y') \qquad y' \geq -5.84 \tag{3.4}$$

$$h_s = 0.45 \qquad y' < -5.84 \tag{3.5}$$

and depth over the shoal is computed as,

$$h = h_s + 0.3 - 0.5\sqrt{1 - \left(\frac{x'}{5}\right)^2 - \left(\frac{y'}{3.75}\right)^2} \tag{3.6}$$

The depth of a shoal and the water out of range of the shoal computed as above and plotted in Figure 3-5.

We computed solution describing nonlinear water wave propagation on the surface of a sloping bottom with elliptic shoal.

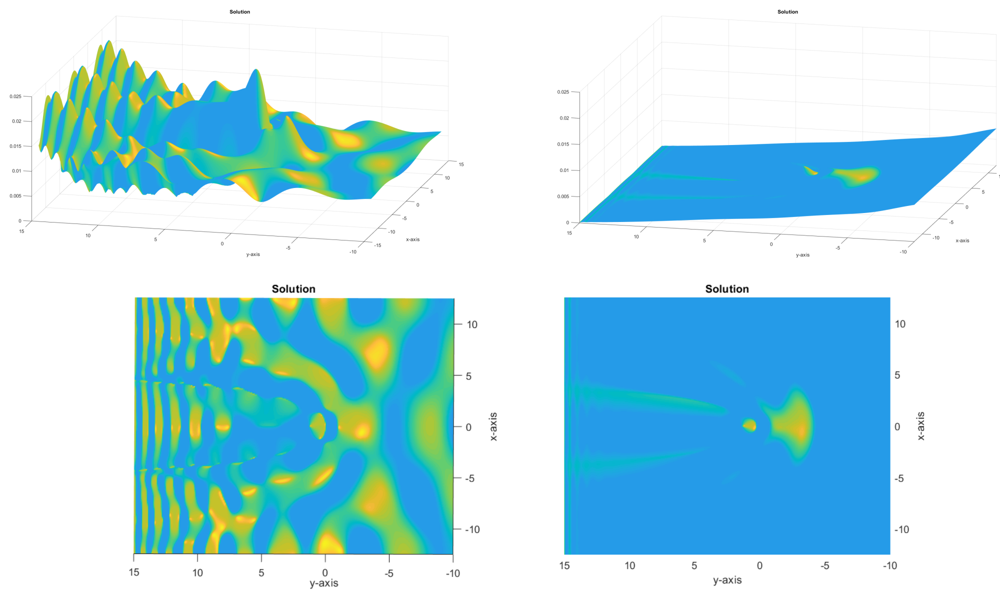

Figure 3-6: 3-D presentation of elliptic shoal model without and with energy dissipation term

We can see that water height increases over the shoal and breaks with increasing monotonic amplitude (Figure 3-6, left side). It shows behaviour of our model for unrealistic ideal case. If we look at plots of our model with energy dissipation term, we can see what happens to wave in real world. Energy dissipation includes bottom friction and wave breaking dissipation terms. Water rays behind the shoal are significant features to analyze our problem (Figure 3-6, 2D plots in the bottom). We showed convergence rate of our method for different mesh sizes.

Mesh	# iter	Sec	L_2 error
65 x 65	176	1.45	5.16e-07
129 x 129	210	7.09	2.99e-07
257 x 257	104	15.35	9.17e-07
513 x 513	56	32.41	7.9e-07

Table 3.2: Bi-CGSTABSL F cycle multigrid with period $T = 2$

As we mentioned before for large mesh sizes Bi-CGSTABSL converges faster Table 3.2. We have less iteration if we increase period of wave Table 3.3. We increased number of grid points to confirm grid independent convergence of BiCGSTABSL solver for a fixed continuum problem. Iterations were terminated when relative residual is less than 10^{-7}.

Mesh	# iter	Sec	L_2 error
65 x 65	21	0.23	4.58e-07
129 x 129	15	0.57	5.26e-07
257 x 257	13	2.003	3.59e-07
513 x 513	12	7.36	7.50e-07

Table 3.3: Bi-CGSTABSL F cycle multigrid with period $T = 4$

In Figure 3-7 we can see convergence rate for 513 x 513 mesh size.

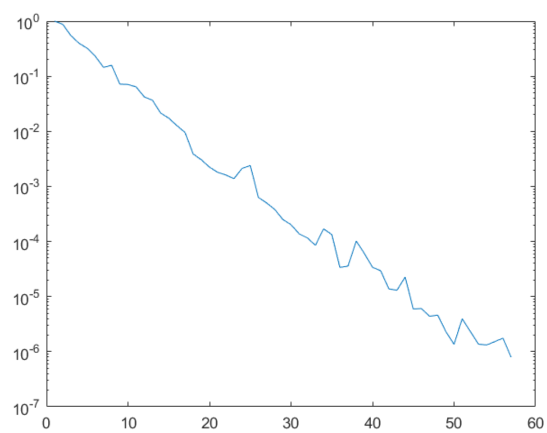

Figure 3-7: Convergence of Bi-CGSTABSL, $T = 2$ and 513 x 513 mesh size

This convergence rate is moderately good for this real-life setting. The performance of the iterative solver relatively good than results obtained in [9]. Most of the parts of numerical implementation on MATLAB is suitable for parallel computation. Main difficulty for solving problem for large grid points is storing elements to sparse matrix. Complexity of storing elements to matrix in loop higher than $\mathcal{O}(n)$.

Chapter 4

Conclusion

We constructed efficient algorithm to solve time-independent modified mild slope equation with energy dissipation term. All numerical experiments are conducted in Matlab. We minimized computational work done by model by parallelizing our code. Convergence rate of Bi-CGSTABSL moderately good. Approximated solution describing nonlinear water wave propagation on the surface of a sloping bottom with elliptic shoal is valid in theoretical point of view.

Appendix A

Figures

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