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Mathematical equations for modelling soil-water characteristic curve

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ABSTRACT: Climate change is continuously becoming a vivid global problem that needs to be tackled from every angle for the survival of our planet. Landslide is one of the fore-front phenomena attributed to climate change, claiming lives, and destroying the environment across the globe. To prevent the occurrences of those slides, a good understanding of unsaturated soil is necessary. Soil-water characteristic curve (SWCC) which describes the amount of water retained by a particular soil is an important hydraulic property of the soil, especially related to the soil's pores sizes and spaces connection within it. Modeling water distribution and flow in unsaturated soils requires knowledge of the SWCC which plays a critical role in water management and in prediction of its transport within the soil. There are many attempts to model SWCC mathematically, with the likes of the Satyanaga, Zhai, Fredlund and Xing, Van Genutchen, Brooks and Corey, and several other models for soils with unimodal and bimodal characteristics. Each equation has its own advantage and disadvantage which can model certain type of soil due to the variability of soil properties. It is however important to analyze the available models, study their best applicable scenario and limitations. This paper will therefore study and review several available mathematical equations for modelling SWCC.

1 INTRODUCTION

Soil slopes constitutes an inclined earth face that are either naturally occurring as in hills and cliffs, or artificially made, usually to carry transportation infrastructure such as in road and rail embankments. Those slopes are continuously facing a danger of slides in form of partial or total collapse, and when the slopes fail, it usually involves damage to such infrastructures they supported, deters daily economic activities and often even leads to loss of lives (Bello et al., 2019, Gariano and Guzzetti, 2016). Climate change is the fore-front phenomenon contributing to landslides and slope failures. It causes an escalation of rainfall, and the increases in the rainfall intensity yield a significant impact on groundwater changes and alter those parameters responsible for soil strength, such as suction, thereby causing various geotechnical problems such as excessive soil settlement and slope failures (Satyanaga et al., 2022a, Tamm et al., 2008, Jakob, 2022). As such, the need to quantify effects of climate variables on geohydrogical hazards is therefore becoming more necessary (Gariano and Guzzetti, 2016).

With the incorporation of the unsaturated soil mechanics principles, many of the geotechnical problems including slope instability could be attenuated. This is achieved by proper understanding of the soil's mechanisms and making appropriate designs using actual soil's condition and its variability. Whilst soils suction is highly dependent on soil types and its infiltration settings, understanding its mechanism plays a vital role in stabilising geotechnical structures, as it is a common knowledge that changes in soil's pore-water pressure is the top variable influencing a slope performance or failure (Satyanaga et al., 2022a, Griffiths and Lu, 2005). In fact, the variation in negative pore-water pressure (suction) and water content hugely influences the

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unsaturated soil's shear strength as observed by (Rahardjo, 2009), and the two most distinct hydraulic properties related to unsaturated soil are its water-characteristic curve (SWCC) and permeability function (Satyanaga et al., 2022b).

SWCC signifies a relationship between soil suction (ψ) and its pore water content (either in gravitational or volumetric form). It is believed to be one of the most important soil parameters for geotechnical engineering operation involving unsaturated soil mechanics. Various important soil behaviour such as its shear strength, water volume storage, change in volume, thermal conductivity and specific heat could all be deduced using SWCC (Sillers, 1997, Sillers et al., 2001, Pan et al., 2021). SWCC could be measured directly in the laboratory, but the procedure is highly time consuming, coupled with high cost and results variability, due to its importance, various empirical equations were developed to calculate SWCC. Therefore, in doing that, a suitable mathematical equation that will clearly provide physical definitions of the SWCC variables must be sought and used (Ellithy, 2017, Pan et al., 2021, Satyanaga et al., 2017). The main objective of this paper is therefore to understand some important available mathematical equations for modelling SWCC and provide an informed review accordingly.

2 UNIMODAL SWCC

Plotting SWCC in respect of water content (θ_w) against soil suction (u_a-u_w) was first proposed by Fredlund (2006), the curve always having a sigmoidal shape have four main elements: the saturated volumetric water content (θ_s) residual volumetric water content (θ_r,) air entry value (ψ_a) and water-entry value (ψ_w). The curve are generally plotted on a logarithmic scale for whole abscissa range (Satyanaga et al., 2017).

There are a variety of fitting equations developed to model the relationship between quantity of water retained by a soil and its matric suction (Zhai and Rahardjo, 2013), the following are some of the best fit equations used for unimodal SWCC.

2.1 *Brooks and Corey, (1964)*

Brooks and Corey (1964) provided one of the earliest fitting models for drying SWCC at a suction lager than air entry value of the soil, in the form of equation (1) (Tinjum et al., 1997, Fredlund, 2006, Wang et al., 2021).

$$\Theta_n = \left[\frac{\psi}{\psi_{aeb}}\right]^{\lambda} for\ \psi > \psi_{aeb} \tag{1}$$

Where, Θ_n is dimensionless normalized volumetric water content, and it is equal to $\left[\frac{(\theta-\theta_r)}{(\theta_s-\theta_r)}\right]$
θ *and* θ_r are saturated and volumetric water contents respectively
ψ is soil suction at a point and ψ_{aeb} is air entry value of the soil
λ is a fitting parameter called pore size distribution index

Degree of Saturation (S) could replace the normalised water content Θ_n in the equation, while residual water content θ_r could be established on semi log plot of straight line by trial and error (Fredlund and Xing, 1994).

Brooks and Corey equation has been widely used by many researchers, but its main shortfall is it only describes SWCC between air entry value (AEV) and residual condition. In order to solve this, McKee and Bumb (1987) and Bumb (1987) presents a more fitting model for SWCC approximation in the area before AEV but unfortunately not useful in the high suction range (Gardner et al., 1970, Rogowski, 1971, Williams et al., 1983, Bumb, 1987, McKee and Bumb, 1987).

The McKee and Bumb equation is given by;

$$\Theta_n = \frac{1}{1 + exp[\frac{\psi - a_m}{n_m}]} \tag{2}$$

where a and n are fitting parameters.

2.2 *Van Genuchten (1980)*

Van Genuchten model is another popular relationship between water content and suction that was widely used among researchers.

$$\Theta = \left[\frac{1}{1 + (p\psi)^n}\right]^m \tag{3}$$

p, n and m are soil parameters. The equation provides additional flexibility compared to Brooks and Corey, (1964) (Ellithy, 2017, Pan et al., 2021). However, as noted by Fredlund and Xing, 1994, “in trying to get a closed-form solution for hydraulic conductivity, Van Genuchten (1980) related m and n through the equation m = (1-1/n), and therefore reduced the added flexibility” and as shown by best fit analysis, identical quality fit could be equally be possible by simply taking the variable (m) as 1. (Fredlund and Xing, 1994, Fredlund, 2006).

2.3 *Gardner, (1958)*

Gardner (1958) presents permeability function equation that mirrored SWCC and could be seen as an extension of Van Genuchten model.

$$\Theta = \frac{1}{1 + q\psi^n} \tag{4}$$

q and n are curve fitting parameters related to soil’s AEV and the inflection point’s slope of the SWCC, respectively.

2.4 *Fredlund and Xing, (1994)*

Fredlund and Xing, (1994) proposed a fitting model relating volumetric water content (θ) and soil suction (ψ) which complements the limitations of the previous fitting equations (i.e., this equation is valid for the entire suction range up to 10^6 kPa). The proposed equation was in the form of;

$$\theta = \theta_s \left[\frac{1}{In(e + \psi/a)^n}\right]^m \tag{5}$$

Equation (5) have three fitting parameters *a*, *m* and *n*, such that if *m* and *n* are fixed then *a* will have a value so close to the soil air entry value for albeit for a small value of *m*, otherwise the *a* is generally greater than *aev*.

$$a = \psi i,\ m = 3.6\ In(\theta_s/\theta_r),\ and\ n = \frac{1.31^{m+n}}{m\theta_s} 3.72s\ \psi i$$

and the slope to the tangent line of the curve (s) could be calculated using $s = \frac{\theta_i}{\psi \mathrm{p} - \psi i}$ where ψp is an intercept of the tangent to the suction.

The equation (5) clearly indicated that (θ) equals (θ_s) when (ψ) is zero; as such it will be valid at the lower suction range, but it can be noted that θ becomes (0) when suction approach infinity, therefore still needing modification for fitting the upper limit of suction. As indicated in experimental data by (Croney and Coleman, 1961), that soil suction attains its peak value of 10^6 kPa when $\theta = 0$, thus the upper limit of the suction range needs to be taken care by the equation. In doing so, a correction factor C(ψ) was introduced into the equation (5).

$$\theta(\Psi, a, m, n) = C(\psi)\left[\frac{\theta_s}{In(e + \psi/a)^n}\right]^m \tag{6}$$

$$\text{Where } \mathrm{C}(\psi) = \left[-\frac{In(1+\psi/\psi_r}{In(1+10^6/\psi_r)}\right] + 1$$

It could be seen that at the limiting value of $\psi = 10^6$ kPa, the C will be equals to 0, clearly indicating at that limiting value the corresponding water content (θ) calculated using equation (6) will now be equals to (0) (Fredlund and Xing, 1994). The equation (6) provided by Fredlund and Xing 1994, could safely be used for whole suction range (i.e., 0kPa -10^6 kPa) and has successfully taken care of the limitations of the previous fitting equations.

2.5 *Satyanaga Et Al., (2017)*

Although equation (6) could be used to fit SWCC for the entire suction range, it has one important disadvantage, in the sense that most of its parameters lacked a clear physical meaning, which refers to as "an ability of the equation parameters to clearly represents the variables in the SWCC" (Satyanaga et al., 2017).

So, for the fact that a convenient mathematical equation containing parameters that are able to represents the exact variables in the SWCC is needed, Satyanaga (2017) presents an efficient best fitting model for that;

$$\theta_w = C(\psi)\left[\theta_r + \left\{\theta_s - \theta_r\left(1 - (\beta) efrc\left(\frac{In\left(\frac{\psi_a - \psi}{\psi_a - \psi_m}\right)}{s}\right)\right)\right\}\right] \tag{7}$$

Where;
β is 0 at the suction range at or before aev, and 1 at any suction beyond aev
θ_w is calculated volumetric water content
θ_s is saturated volumetric water content
ψ is matric suction in view (kPa)
ψ_a is soil's air entry value (kPa)
ψ_m is the matric suction at inflection point of the SWCC (kPa)
S is representing standard deviation of the SWCC
θ_r soil's residual volumetric water content
ψ_r is matric suction corresponding to θ_r on the SWCC (kPa)

The good thing about equation (7) is, while it is valid for the entire suction range, its parameters equally have a clear physical meaning to represents the variables of soil water characteristics curve.

3 BIMODAL SWCC

Debris of weathered rocks found in an unsaturated zone are called residual soils, and in most cases, those residuals and colluvial soils are widely gap graded, exhibiting bimodal grain-size distribution (GSD). And usually, colluvial soils with bimodal characteristics GSD brings about bimodal characteristics of SWCC, the bimodal SWCC happens as a result of the soil containing two pore series (Kutílek, 2004, Rahardjo et al., 2004, Zhang and Chen, 2005, Satyanaga et al., 2013).

GSD and SWCC modelling using continuous function is very important, as the SWCC predicted using GSD could be used to establish various soil properties in very short duration, as compared to the time-consuming laboratory SWCC tests. Among many GSD equations developed Fredlund et al., is commonly used for fitting bimodal GSD, although the equation's parameters are unrelated to the soil's physical properties (Satyanaga et al., 2013).

3.1 *Zhang and Cheng (2005)*

Zhang and Cheng (2005) developed a method of predicting bimodal soil water characteristics curve for bimodal soil. They utilized the theoretical basis of estimating SWCC from a pore distribution as provided by Fredlund and Xing, 1994. The SWCC for the bimodal soil was established as the sum of the two components of the soil with large pore series and small pore series. And the bimodal SWCC could be described using either Van Genuchten, 1980 function or Fredlund and Xing, (1994) function as follows.

$$\Theta(\psi) = p_i n_{p1}\left[\frac{1}{1+(a_i\psi)^{ni}}\right]^{mi} + p_s n_{ps}\left[\frac{1}{1+(a_s\psi)^{n}s}\right]^{m_s} \tag{8}$$

where,

a_l, n_l, and m_l are fitting parameters for the large-pore series component, and a_s, n_s, and m_s are fitting parameters for the small-pore series component.

p_l and p_s are respectively the volumetric percentages of the components with the large-pore series and the small-pore series in the soil mass and could be calculated easily based on the density values and the percentages by dry weight of the soil components.

n_{pl} and n_{ps} are porosities of the components with the large-pore series and the small-pore series when they are considered individually

and;

$$\begin{aligned}\theta(\psi) = &\mathrm{p_i n_{pl}}\left[\left[1-\frac{\mathrm{In}(1+\psi/\psi_\mathrm{r}}{\mathrm{In}(1+10^6/\psi_\mathrm{r}}\right]\right]\left\{\frac{1}{\mathrm{In}(\mathrm{e}+\psi/\mathrm{a_l})^{\mathrm{nl}}}\right\}^{\mathrm{m_l}}\\ &+\mathrm{p_s n_{ps}}\left[\left[1-\frac{\mathrm{In}(1+\psi/\psi_\mathrm{r}}{\mathrm{In}(1+10^6/\psi_\mathrm{r})}\right]\right]\left\{\frac{1}{\mathrm{In}(\mathrm{e}+\psi/\mathrm{a_s})^{\mathrm{n_s}}}\right\}^{\mathrm{m_s}}\end{aligned} \tag{9}$$

where,

e is the base of natural logarithm; and all other parameters as explained.

3.2 *Satyanaga Et Al., (2013)*

Although Zhang and Cheng (2005) present a model that could predicts a bimodal soil water characteristics curve for a bimodal soil, the equations where developed based on Van Genuchten 1980 and Fredlund and Xing (1994), whose some of the variables having no physical meaning in relation to the SWCC's parameters. Therefore Satyanaga et al. (2013) provide an equation to mitigate limitation of the Zhang and Cheng (2005) model.

"The equation was developed considering five important factors.

i. The equation parameters could be related to the soil's physical meaning
ii. The soil water content is set to zero as the soil's suction reached 10^6 and the Fredlund and Xing's correction function (C) is utilized
iii. The shape of SWCC is like lognormal distribution. Therefore, Kosugi (1994) equation is used as a basis for the development of the proposed equations.
iv. Water content when soil is fully saturated (saturated water content, w_{sat}) is known.
v. Since the bimodal characteristic of SWCC occurs if the percentage of coarse-grained soil is high, the total volume of soil is assumed to be constant during the SWCC tests (Satyanaga et al., 2013, Kosugi, 1994)."

$$\begin{aligned}\theta_w = &\left(1-\frac{In(1+\psi/\psi_r}{In(1+10^6/\psi_r)}\right)\left[\theta_r + (\theta_{s_1}-\theta_{s_2})\left(1-efrc\left(\frac{In\left(\frac{\psi_{a_1}-\psi}{\psi_{a_1}-\psi_{m_1}}\right)}{s_1}\right)\right)+\right.\\ &\left.(\theta_{s_2}-\theta_r)\left(1-efrc\left(\frac{In\left(\frac{\psi_{a_2}-\psi}{\psi_{a_2}-\psi_{m_2}}\right)}{s_2}\right)\right)\right]\end{aligned} \tag{10}$$

where,

θ_w is the calculated volumetric water content, θ_s the saturated volumetric water content (measured in the laboratory), ψ is the matric suction under consideration, ψ_a represents the air-entry value of soil, and ψ_m represents the matric suction at the inflection point of SWCC and S represents the geometric standard deviation of SWCC (Satyanaga et al., 2013).

4 CONCLUSION

Many SWCC equations were reviewed in this work, such as Brooked and Corey, (1964), Van Genuchten, (1980), Gardener, (1958), Fredlund and Xing, (1994) and Satyanaga et al., (2017) for unimodal SWCC, while Zhang and Cheng, (2015) and Satyanaga et al., (2013) were used for bimodal SWCC. And it could be concluded that Satyanaga, (2017) for unimodal SWCC (equation 7) and Satyanaga et al., (2013) for bimodal SWCC (equation 10) are best fit for the said purpose. This is for their unique ability of been valid for the entire soil suction range (up to 10^6kPa), also containing parameters that have physical meanings that best describe the soil variables.

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