Received: 12 April 2023 | Revised: 18 January 2024 | Accepted: 10 April 2024
DOI: 10.1002/mma.10158

RESEARCH ARTICLE

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Traveling waves for a nonlinear Schrödinger system with quadratic interaction in $\mathbb{R}^4$

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Communicated by: C. Miao

Funding information
Nazarbayev University; Faculty Development Competitive Research Grants Program, Grant/Award Number: 20122022FD4121

In this paper, we consider a nonlinear Schrödinger system with quadratic interaction. We extend the recent results of Fukaya et al. (Math. Ann. 2024) and show that the system has a ground state in $\mathbb{R}^4$ when the mass parameter κ is larger than $\frac{1}{2}$.

KEYWORDS

ground state, NLS system, variational method

MSC CLASSIFICATION

35Q40, 35Q55, 35J10, 35A15

1 | INTRODUCTION

In this paper, we study the following system of nonlinear Schrödinger equations with quadratic interaction

$$\begin{cases} \mathrm{i}u_t + \Delta u + 2v\bar{u} = 0, \\ \mathrm{i}v_t + \kappa\Delta v + u^2 = 0, \qquad (x,t) \in \mathbb{R}^4 \times \mathbb{R}, \qquad \kappa > 0, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), \end{cases} \tag{1.1}$$

where u and v are complex scalar functions. The system models the Raman amplification phenomena in plasma and also appears in some phenomena in the nonlinear optics [1]. See also Mathieu et al. [2] for more details. System (1.1) has formally the following conserved quantities:

$$\begin{aligned} \mathbb{E}(\vec{u}) &= \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^4)} + \frac{\kappa}{2}\|\nabla v\|^2_{L^2(\mathbb{R}^4)} - \mathbb{K}(\vec{u}), \\ \mathbb{F}(\vec{u}) &= \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^4)} + \|v\|^2_{L^2(\mathbb{R}^4)}, \qquad P(\vec{u}) = \frac{1}{2}\langle \mathrm{i}\nabla u, u\rangle + \frac{1}{2}\langle \mathrm{i}\nabla v, v\rangle, \end{aligned}$$

where

$$\mathbb{K}(\vec{u}) = \Re\int_{\mathbb{R}^4} u^2\bar{v}\,\mathrm{d}x,$$

and $\langle f, g\rangle = \Re\int_{\mathbb{R}^4} f\bar{g}\,\mathrm{d}\,\mathrm{x}$. We notice that (1.1) can be written as the Hamiltonian system

$$\frac{\mathrm{d}}{\mathrm{d}t}(u,v) = \begin{bmatrix} -\mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix} \mathbb{E}'(u,v).$$

It is also invariant under the scaling

$$(u,v) \mapsto \lambda^2 \left(u(\lambda x, \lambda^2 t), v(\lambda x, \lambda^2 t)\right)$$

and the gauge transform $(u,v) \mapsto (\exp(i\theta)u, \exp(2i\theta)v)$ for $\theta \in \mathbb{R}$.

It was proved in Hayashi et al. [3] that (1.1) is locally well-posed in the energy space $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ and the flow conserves the aforementioned invariants. When $\kappa = \frac{1}{2}$ (called the mass-resonance condition), (1.1) is invariant under the Galilean transformation

$$(u,v) \mapsto \left(e^{\frac{i}{2}c\cdot x - \frac{i}{4}|c|^2 t} u(x-ct,t), e^{ic\cdot x - \frac{i}{2}|c|^2 t} v(x-ct,t)\right), \qquad \forall c \in \mathbb{R}^4.$$

In this case, (1.1) is also invariant under a pseudo-conformal transformation

$$(u,v) \mapsto \left(\frac{1}{A^2(t)} e^{\frac{i\beta|x|^2}{4A(t)}} u\left(\frac{B(t)}{A(t)}, \frac{x}{A(t)}\right), \frac{1}{A^2(t)} e^{\frac{i\beta|x|^2}{2A(t)}} v\left(\frac{B(t)}{A(t)}, \frac{x}{A(t)}\right)\right),$$

where $A(t) = \alpha + \beta t$, $B(t) = \gamma + \delta t$, and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha\delta - \beta\gamma = 1$. Hence, the traveling waves

$$(u,v) = (e^{i\omega t}\phi(x-ct), e^{2i\omega t}\psi(x-ct))$$

are derived via the above symmetries and the traveling waves with zero-speed $c = 0$. No such symmetries exist for the solutions of (1.1) if $\kappa \neq \frac{1}{2}$. But to see the effect of lack of symmetry on the global behavior of solutions of (1.1), it is crucial to find the traveling waves of (1.1) in the nonmass resonance case $\kappa \neq \frac{1}{2}$. It was proved recently in Fukaya et al. [4] that (1.1) possesses a nonzero speed traveling wave if

$$\begin{cases} 1 \le n \le 5,\ \kappa > 0, & \omega > \max\left\{\frac{|c|^2}{4}, \frac{|c|^2}{8\kappa}\right\}, \\ 3 \le n \le 5,\ 0 < \kappa < \frac{1}{2}, & \omega = \frac{|c|^2}{8\kappa}, c \neq 0, \\ n = 5, \kappa > \frac{1}{2}, & \omega = \frac{|c|^2}{4}, c \neq 0. \end{cases}$$

In this paper, we extend the previous result to $n = 4$ which was left open in Fukaya et al. [4]. Indeed, the concentration-compactness argument does not work. We show that this case is considered as a limit case (energy critical case) of an elliptic equation including a nonlocal nonlinear term. To describe our result, we consider traveling waves of (1.1) in the form $u(x,t) = e^{i\omega t}\varphi(x-ct)$ and $v(x,t) = e^{2i\omega t}\psi(x-ct)$, where $(\omega, c) \in \mathbb{R} \times \mathbb{R}^4$. So that (φ, ψ) satisfies

$$\begin{cases} -\Delta\varphi + \omega\varphi + ic\cdot\nabla\varphi - 2\bar{\varphi}\psi = 0, \\ -\kappa\Delta v + 2\omega\psi + ic\cdot\nabla\psi - \varphi^2 = 0. \end{cases} \tag{1.2}$$

As obtained in Fukaya et al. [4], we should assume that $\kappa > \frac{1}{2}$ and $\omega = \frac{|c|^2}{4}$. So the change of variables $\varphi \to e^{i\frac{c}{2}\cdot x}u$ and $\varphi \to e^{ic\cdot x}v$ turns (1.2) into

$$\begin{cases} -\Delta u = 2\bar{u}v, \\ -\kappa\Delta v + \frac{|c|^2}{2}(2\kappa - 1)v - i(2\kappa-1)c\cdot\nabla v = u^2. \end{cases} \tag{1.3}$$

Recall that $D^{1,2}(\mathbb{R}^4;\mathbb{C}) = \left\{u \in L^4(\mathbb{R}^4),\ \nabla u \in L^2(\mathbb{R}^4)\right\}$. By the Sobolev inequality, we have $D^{1,2}(\mathbb{R}^4;\mathbb{C}) = \dot{H}^1(\mathbb{R}^4)$ (with equivalent norms). We denote the set of all nontrivial solutions of (1.3) by

$$\mathfrak{A}_c = \left\{(u,v) \in H = D^{1,2}(\mathbb{R}^4;\mathbb{C}) \times H^1(\mathbb{R}^4;\mathbb{C}) \backslash \{(0,0)\}, S'(u,v) = 0\right\},$$

where $S = \mathbb{E} + \omega\mathbb{F} + c\cdot P$, and the set of all ground states is defined by

$$G_c = \{(u,v) \in \mathfrak{A}_c,\ S(u,v) \le S(f,g),\ \forall (f,g) \in \mathfrak{A}_c\}.$$

Theorem 1.1. *Let $\kappa > \frac{1}{2}$ and $c > 0$. Then there exists a ground state of (1.3), that is, G_c is not empty.*

To prove Theorem 1.1, we rewrite (1.3) in the form of an elliptic equation with a nonlocal nonlinearity $\bar{u}K_{c,\kappa} * u^2$ (see (2.1)). We use the concentration-compactness method to obtain a constrained minimization problem deduced from a general critical Sobolev-type inequality, which leads us to the existence of ground state solutions in the H^1-critical case. Such a problem was studied by Lions [5], but we need to modify the approach due to the presence of the nonlocal nonlinearity, the lack of compactness of the injection $D^{1,2}(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$, and also lack of coordinate-scaling of the nonlinear term. It is worth noting that our solutions of (1.3) with $\kappa > \frac{1}{2}$ are essentially nonradial and complex-valued, which is different from the previous works on zero mass problems in nonlinear elliptic equations (see, e.g., the seminal work [6] and articles cited to it).

2 | EXISTENCE OF GROUND STATES

In this section, we prove the existence of ground states of (1.3). It is clear that if $(u, v) \in H$ is a solution of (1.3), then $u \in D^{1,2}(\mathbb{R}^4; \mathbb{C})$ is a solution of

$$-\Delta u = 2\bar{u}\left(K_{c,\kappa} * u^2\right), \tag{2.1}$$

where

$$\hat{K}_{c,\kappa}(\xi) = \frac{1}{\kappa|\xi|^2 + (2\kappa - 1)c \cdot \xi + \frac{|c|^2}{2}(2\kappa - 1)} > 0.$$

Moreover, if $u \in D^{1,2}(\mathbb{R}^4; \mathbb{C})$ is a solution of (2.1), then $(u, v) \in H$ is a solution of (1.3) provided $v = K_{c,\kappa} * u^2 \in H^1(\mathbb{R}^4)$. To see this fact, we have from $u \in D^{1,2}$ that $u^2 \in L^2(\mathbb{R}^4)$. Since $\langle \xi \rangle \hat{K}_{c,\kappa} \in L^\infty$, we obtain from the Plancherel theorem that $| \langle \xi \rangle \hat{K}_{c,\kappa}\widehat{u^2}| \lesssim |\widehat{u^2}| \in L^2(\mathbb{R}^4)$, from which we conclude $K_{c,\kappa} * u^2 \in H^1(\mathbb{R}^4)$. Therefore, it suffices to investigate (2.1).

To study (2.1), we define the action $\mathfrak{S}(u) = \|\nabla u\|^2_{L^2(\mathbb{R}^4)} - \mathfrak{K}(u)$, where $\mathfrak{K}(u) = \Re \left\langle u^2, K_{c,\kappa} * u^2 \right\rangle$. So, φ is a ground state of (2.1) if $\mathfrak{S}(\varphi)$ minimizes the action $\mathfrak{S}$ among all solutions of (2.1).

Remark 2.1. It is simple to see that any solution of (2.1) satisfies the Pohozaev-type identity

$$\|\nabla u\|^2_{L^2(\mathbb{R}^4)} = 2\mathfrak{K}(u).$$

Moreover, notice that

$$\mathfrak{K}(u) = \left\langle \widehat{u^2}, \hat{K}_{c,\kappa}\widehat{u^2} \right\rangle = \int_{\mathbb{R}^4} \hat{K}_{c,\kappa}|\widehat{u^2}|^2 \,\mathrm{d}\xi = \int_{\mathbb{R}^4} \left|\sqrt{\hat{K}_{c,\kappa}}\widehat{u^2}\right|^2 \mathrm{d}\xi = \int_{\mathbb{R}^4} \left|\mathfrak{H} * u^2\right|^2 \mathrm{d}x \in \mathbb{R},$$

where $\hat{\mathfrak{H}}(\xi) = \sqrt{\hat{K}_{c,\kappa}(\xi)}$,

$$\mathfrak{H}(x) = \frac{a^{\frac{3}{2}}}{\sqrt{\kappa}}\mathrm{e}^{-\mathrm{i}\left(1-\frac{1}{2\kappa}\right)c\cdot x}B_1\left(\sqrt{a}x\right), \qquad a = \frac{(2\kappa - 1)|c|^2}{4\kappa^2},$$

and

$$B_1(x) = C\int_0^\infty \mathrm{e}^{-\frac{|x|^2}{s}}\mathrm{e}^{-s}s^{-\frac{5}{2}}\,\mathrm{d}s.$$

We note that $B_1(x) \sim \exp(-|x|/2)$ if $|x| \geq 2$ and $B_1(x) \sim |x|^{-3}$ if $|x| \leq 2$. So we have from Huang et al. [7, 8] that

$$\|B_1 * f\|_{L^r} \lesssim \|f\|_{L^p}, \tag{2.2}$$

where $1 < p < \infty$, $\frac{4}{p} \leq 1 + \frac{4}{r}$, and $p \leq r < \infty$. Moreover, it is known for $1 \leq p \leq \infty$ that

$$\|B_1 * f\|_{L^p} \lesssim \|f\|_{L^p}.$$

This means that $\mathfrak{K}(u)$ is a real-valued function and positive.

Remark 2.2. It is also known that $u(x) = \frac{8\delta}{\delta+|x-z|^2}$ with $\delta > 0$ and $z \in \mathbb{R}^4$ is the minimizer for S, where S^{-1} is the best constant of

$$\|u\|^2_{L^4(\mathbb{R}^4)} \leq S^{-1}\|\nabla u\|^2_{L^2(\mathbb{R}^4)}.$$

Keeping Remark 2.1 in our mind, we now consider the following minimization problems

$$I = \inf\left\{\|\nabla u\|^2_{L^2(\mathbb{R}^4)},\ u \in D^{1,2}(\mathbb{R}^4),\ \mathfrak{K}(u) = 1\right\}, \tag{2.3}$$

$$I^\infty = \inf\left\{\|\nabla u\|^2_{L^2(\mathbb{R}^4)},\ u \in D^{1,2}(\mathbb{R}^4),\ \gamma\|u\|^4_{L^4(\mathbb{R}^4)} = 1\right\} \tag{2.4}$$

and

$$m_{c,\kappa} = \inf_{u \in D^{1,2}(\mathbb{R}^4)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^4)}}{\mathfrak{K}^{\frac{1}{2}}(u)}, \tag{2.5}$$

where $\gamma > 0$ is the (best) constant in $\mathfrak{K}(u) \leq \gamma\|u\|^4_{L^4(\mathbb{R}^4)}$. It is trivial that $I < I^\infty$.

If u is minimizer of (2.5), then we have for any $h \in C^\infty(\mathbb{R}^4)$ that

$$\mathfrak{R}\langle -\Delta u, h\rangle - \frac{m_{c,\kappa}}{\mathfrak{K}^{\frac{1}{2}}(u)}\mathfrak{R}\left\langle K_{c,\kappa} * u^2, uh\right\rangle = 0;$$

and by changing $h \to -\mathrm{i}h$, we get

$$\mathfrak{I}\langle -\Delta u, h\rangle - \frac{m_{c,\kappa}}{\mathfrak{K}^{\frac{1}{2}}(u)}\mathfrak{I}\left\langle K_{c,\kappa} * u^2, uh\right\rangle = 0;$$

so that u satisfies

$$-\Delta u - \frac{m_{c,\kappa}}{2\mathfrak{K}^{\frac{1}{2}}(u)}2\bar{u}K_{c,\kappa} * u^2 = 0.$$

Taking $\tilde{u} = \sqrt{\frac{m_{c,\kappa}}{\mathfrak{K}^{\frac{1}{2}}(u)}}u$, it is seen that $\tilde{u}$ satisfies (2.1). The constant $m_{c,\kappa}$ actually determines the corresponding interpolation estimate

$$\mathfrak{K}^{\frac{1}{2}}(u) \leq m^{-1}_{c,\kappa}\|\nabla u\|^2_{L^2(\mathbb{R}^4)}.$$

Remark 2.3. If u is a minimizer of (2.3), it is also a minimizer of (2.5). Moreover, $m_{c,\kappa} = 2\mathfrak{K}^{\frac{1}{2}}(u)$. In addition, u is a ground state if and only if u is a minimizer of (2.5).

By using Remark 2.3, we show the existence of a minimizer of (2.3). To this end, we use the following result inspired by the concentration-compactness principle (see Lions [5]) in the limiting case to describe the lack of compactness of the injection $D^{1,2}(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$.

Lemma 2.4. *Let $\{u_k\} \subset D^{1,2}(\mathbb{R}^4)$ be a bounded sequence such that $u_k \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^4)$, $|\nabla u_k|^2 \rightharpoonup \mu$ and $|u_k|^4 \to \nu$, where $\mu, \nu \geq 0$ are measures on $\mathbb{R}^4$. Then, we ave the following:*

(i) There exists an at most countable set J, distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^4$, and nonnegative weights $\{\mu_j, \nu_j\}_{j\in J}$ such that $\mu_j > 0$ and

$$\nu = |u|^4 + \sum_{j\in J}\nu_j\delta_{x_j}, \qquad \mu \geq |\nabla u|^2 + \sum_{j\in J}\mu_j\delta_{x_j}.$$

(ii) It holds for all $j \in J$ that $\nu_j \leq I^{-2}\mu_j^2$, where I is defined in (2.3). Moreover, $\sum_{j\in J}\sqrt{\nu_j} < +\infty$.

The following result is crucial in applying Lemma 2.4.

Lemma 2.5. *Assume that* $u_k \rightharpoonup u$ *in* $D^{1,2}(\mathbb{R}^4)$. *Then*

$$\int_{\mathbb{R}^4} \left| \bar{u}_k^2 K_{c,\kappa} * u^2 - \bar{u}^2 K_{c,\kappa} * u^2 - (\overline{u_k - u})^2 K_{c,\kappa} * (u_k - u)^2 \right| \, \mathrm{d}x \to 0 \qquad \text{as } k \to \infty.$$

Proof. The proof replies one an elementary inequality. For any $a_1, a_2, b_1, b_2 \in \mathbb{C}$, and $\epsilon > 0$, we have from the Young inequality that

$$\begin{aligned} \left|(a_1 + a_2)(b_1 + b_2)^2 - a_1 b_1^2\right| &\le 2\,|a_1 b_1 b_2| + 2\,|a_2 b_1 b_2| + |a_1|\,|b_2|^2 + |a_2|\,|b_1|^2 + |a_2|\,|b_2|^2 \\ &\le \epsilon\left(3|a_1|^2 + 5|b_1|^4 + 3|a_2|^2\right) + 4\epsilon^{-3}|b_2|^4 + \epsilon^{-1}\left(2|b_2|^4 + |a_2|^2\right). \end{aligned}$$

Taking $a_1 = K_{c,\kappa} * (u_k^2 - u^2)$, $a_2 = K_{c,\kappa} * u^2$, $b_1 = \overline{u_k - u}$ and $b_2 = \bar{u}$ in the above inequality, we derive that

$$\begin{aligned} \varpi(\epsilon) = {} & \Big| \bar{u}_k^2 K_{c,\kappa} * u_k^2 - (\bar{u}_k - \bar{u})^2 K_{c,\kappa} * (u_k - u)^2 - \bar{u}^2 K_{c,\kappa} * u^2 \\ & - \epsilon\left(3\left|K_{c,\kappa} * (u_k^2 - u^2)\right|^2 + 5|u_k - u|^4 + 3\left|K_{c,\kappa} * u^2\right|^2\right)\Big| \\ \le {} & \left|\bar{u}^2 K_{c,\kappa} * u^2\right| + 4\epsilon^{-3}|u|^4 + \epsilon^{-1}\left(2|u|^4 + \left|K_{c,\kappa} * u^2\right|^2\right). \end{aligned}$$

The dominated convergence theorem together with the embedding $D^{1,2}(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$ implies that $\varpi(\epsilon) \to 0$ in $L^1(\mathbb{R}^4)$ as $k \to \infty$. This implies by using

$$\begin{aligned} & \left|\bar{u}_k^2 K_{c,\kappa} * u_k^2 - (\bar{u}_k - \bar{u})^2 K_{c,\kappa} * (u_k - u)^2 - \bar{u}^2 K_{c,\kappa} * u^2\right| \\ & \le \varpi(\epsilon) + \epsilon\left(3\left|K_{c,\kappa} * (u_k^2 - u^2)\right|^2 + 5|u_k - u|^4 + 3\left|K_{c,\kappa} * u^2\right|^2\right) \end{aligned}$$

the boundedness of $\{u_k\}$ in $D^{1,2}(\mathbb{R}^4)$ and arbitrariness of ϵ that

$$\int_{\mathbb{R}^4} \left| \bar{u}_k^2 K_{c,\kappa} * u^2 - \bar{u}^2 K_{c,\kappa} * u^2 - (\overline{u_k - u})^2 K_{c,\kappa} * (u_k - u)^2 \right| \, \mathrm{d}x = o(1)$$

as $k \to \infty$. ☐

Now, let $\{u_k\}$ be a minimizing sequence of (2.3). Then $\{u_k\}$ is bounded in $D^{1,2}(\mathbb{R}^4)$ whence in $L^4(\mathbb{R}^4)$. The proof is based on using the concentration-compactness result with the sequence

$$\rho_k = |\nabla u_k|^2 + |u_k|^4 + \left|\bar{u}_k^2 K_{c,\kappa} * u_k^2\right|.$$

Notice that $\left\{\bar{u}^2 K_{c,\kappa} * u^2\right\}$ is bounded in $L^1(\mathbb{R}^4)$. So we can presume that

$$\int_{\mathbb{R}^4} \rho_k \, \mathrm{d}x \to \wp_{c,k} > 0.$$

To show that $\{u_k\}$ is relatively compact in $D^{1,2}(\mathbb{R}^4)$, we rule out the possibility of occurring the evanescence and dichotomy cases in the concentration-compactness principle.

Avoiding the evanescence case. Suppose by contradiction that

$$\sup_{y \in \mathbb{R}^4} \int_{B_R(y)} \rho_k \, \mathrm{d}x \to 0 \tag{2.6}$$

as $k \to \infty$ for some $R > 0$. We conclude if we show that $\mathfrak{K}(u_k) \to 0$ as $k \to \infty$.

Now, we first assume that $\{u_k\} \subset L^\infty(\mathbb{R}^4)$ is bounded. It follows from (2.6) that

$$\sup_{y\in\mathbb{R}^4} \int_{B_R(y)} |u_k|^4 \, \mathrm{d}x \to 0 \tag{2.7}$$

as $k \to \infty$. Recall that $\mathfrak{H}(x) \lesssim \exp(-|x|/2) + |x|^{-3}$. Thus,

$$\begin{aligned} |\mathfrak{K}(u_k)| &\lesssim \left\| \mathrm{e}^{-\frac{|x|}{2}} * u_k^2 \right\|_{L^2(\mathbb{R}^4)} + \left\| \left(|x|^{-3}\chi_{\{|x|\le 2\}}\right) * u_k^2 \right\|_{L^2(\mathbb{R}^4)} \\ &\lesssim \|u_k\|_{L^8(\mathbb{R}^4)} + \|u_k\|^2_{L^{\frac{8}{3}}(\mathbb{R}^4)} . \end{aligned} \tag{2.8}$$

Since $\|u_k\|_{L^4(\mathbb{R}^4)}$ and $\|\nabla u_k\|_{L^2(\mathbb{R}^4)}$ are bounded, it is deduced from (2.7) and an application of Lions [9, Lemma I.1] that $u_k \to 0$ in $L^q(\mathbb{R}^4)$ for any $4 < q < \infty$. The $L^\infty(\mathbb{R}^4)$-boundedness of u_k and (2.8) shows that $\mathfrak{K}(u_k) \to 0$ as $M \to \infty$. In general case, where $\{u_k\}$ is bounded in $D^{1,2}(\mathbb{R}^4)$, we see that the functions

$$v_k = \max\{\min u_k, M, -M\}, \quad \forall M > 0$$

are bounded in $L^\infty(\mathbb{R}^4)$. Then, $\mathfrak{K}(v_k) \to 0$ as $k \to \infty$; whence,

$$\begin{aligned} \mathfrak{K}(u_k) &\le \mathfrak{K}(v_k) + \int_{\mathbb{R}^4} \left| \bar{v}_k^2 K_{c,\kappa} * \left(u_k^2 \mathbf{1}_{\{|u_k|\ge M\}}\right)\right| \, \mathrm{d}x + \int_{\mathbb{R}^4} \left| \left(\bar{u}_k^2 \mathbf{1}_{\{|u_k|\ge M\}}\right) K_{c,\kappa} * u_k^2 \right| \, \mathrm{d}x \\ &\le \mathfrak{K}(v_k) + \|v_k\|^2_{L^4(\mathbb{R}^4)} \|u_k \mathbf{1}_{\{|u_k|\ge M\}}\|^2_{L^4(\mathbb{R}^4)} + \varepsilon_M \|u_k\|^2_{L^4(\mathbb{R}^4)}, \end{aligned}$$

with $\varepsilon_M \to 0$ as $M \to +\infty$. This yields that $\|\nabla u_k\|^2_{L^2(\mathbb{R}^4)} \to I$ and

$$\lim_k \gamma \|u_k\|^4_{L^4(\mathbb{R}^4)} \ge 1$$

contradicting $I < I^\infty = \gamma^{-1/2} I^0$.

Avoiding the dichotomy case.

Define for $\lambda > 0$, the following minimization problem

$$I_\lambda = \left\{ \|\nabla u_\lambda\|^2_{L^2(\mathbb{R}^4)} ; \ u \in D^{1,2}(\mathbb{R}^4), \ \mathfrak{K}(u) = 1 \right\},$$

where $u_\lambda(x) = u(\lambda^{-1/4}x)$. Clearly, $I_\lambda = \sqrt{\lambda} I$ implying

$$I < I_\tau + I_{1-\tau} \qquad \forall \tau \in (0,1). \tag{2.9}$$

Suppose that the dichotomy case occurs, then we can find $\tau \in (0, \wp_{c,\kappa})$ such that for any $\epsilon > 0$, there exist $\{y_k\} \subset \mathbb{R}^4$, $R_0 > 0$ and $\{R_k\} \in (R_0, +\infty)$ such that $R_k \to \infty$,

$$\left| \int_{B_{R_0}(y_k)} \rho_k \, \mathrm{d}x - \tau \right| \le \epsilon, \quad \left| \int_{\mathbb{R}^4 \setminus B_{R_k}(y_k)} \rho_k \, \mathrm{d}x - \wp_{c,\kappa} + \tau \right| \le \epsilon,$$
$$\left| \int_{B_{R_k}(y_k) \setminus B_{R_0}(y_k)} \rho_k \, \mathrm{d}x - \tau \right| \le \epsilon.$$

Define for any $\lambda \ge 1$ and $R > 0$ the mappings

$$T_{\lambda,R}(x) = x \mathbf{1}_{B_R(0)}(x) + \left(\lambda x + (1-\lambda) R \frac{x}{|x|} \right) \mathbf{1}_{B_R^c(0)}(x)$$

and set $u_{k,1}(x) = u_k(T_{\lambda,R}(x))$. Notice that when $|y| \geq R$, then

$$\left| y\lambda + (1-\lambda)R\frac{x}{|x|} \right| \geq \lambda|y| - |1-\lambda|R \geq R.$$

Hence,

$$\begin{aligned}
\left| \mathfrak{K}(u_{k,1}) - \int_{B_R(0)} \bar{u}_k^2 K_{c,\kappa} * u_k^2 \, \mathrm{d}x \right| &\lesssim \left| \int_{B_R(0)} \int_{B_R^c(0)} \bar{u}_k^2(x) K_{c,\kappa}(x-y) u_k^2(T_{\lambda,R}(y)) \, \mathrm{d}y \, \mathrm{d}x \right| \\
&\quad + \left| \int_{B_R^c(0)} \int_{\mathbb{R}^4} \bar{u}_k^2(T_{\lambda,R}(x)) K_{c,\kappa}(x-y) u_{k,1}^2(y) \, \mathrm{d}y \, \mathrm{d}x \right| \\
&\lesssim \left| \int_{B_R(0)} \int_{B_R^c(0)} \bar{u}_k^2(x) K_{c,\kappa}(x-y) u_k^2(T_{\lambda,R}(y)) \, \mathrm{d}y \, \mathrm{d}x \right| \\
&\quad + \left| \int_{B_R^c(0)} \int_{\mathbb{R}^4} \bar{u}_k^2(T_{\lambda,R}(x)) K_{c,\kappa}(x-y) u_{k,1}^2(y) \, \mathrm{d}y \, \mathrm{d}x \right| \\
&\lesssim \left| \int_{B_R(0)} \int_{B_R^c(0)} \bar{u}_k^2(x) K_{c,\kappa}(x - T_{\lambda,R}^{-1}) u_k^2(y) \chi_{\lambda,R}(y) \, \mathrm{d}y \, \mathrm{d}x \right| \\
&\quad + \left| \int_{B_R^c(0)} \int_{\mathbb{R}^4} \bar{u}_k^2(x) K_{c,\kappa}(x-y) u_{k,1}^2(y) \chi_{\lambda,R}(x) \, \mathrm{d}y \, \mathrm{d}x \right|,
\end{aligned}$$

where $\chi_{\lambda,R}^{-1}(x) = \lambda(\lambda + R(1-\lambda)|T_{\lambda,R}^{-1}(x)|^{-1})^3$ for $x \in \mathbb{R}^4 \backslash B_R(0)$. Since $\chi_{\lambda,R}^{-1}(x) \geq \lambda$, we get

$$\left| \mathfrak{K}(u_{k,1}) - \int_{B_R(0)} \bar{u}_k^2 K_{c,\kappa} * u_k^2 \, \mathrm{d}x \right| \lesssim \lambda^{-1}.$$

On the other hand, we have from the definition of T that

$$\int_{\mathbb{R}^4} |\nabla u_{k,1}|^2 \, \mathrm{d}x = \int_{B_R(0)} |\nabla u_k|^2 \, \mathrm{d}x + \int_{B_R(0)^c} |\nabla T \cdot \nabla u_k(T(x)) u_k|^2 \, \mathrm{d}x$$

and $\partial_{x_j}(T_i)(x) = \lambda\delta_{ij} - (1-\lambda)R|x|^{-3}(\delta_{ij}|x|^2 - x_i x_j)$; whence,

$$\begin{aligned}
\int_{B_R^c(0)} |\nabla T \cdot \nabla u_k(T(x))|^2 \, \mathrm{d}x &\lesssim \lambda \int_{B_R^c(0)} |\nabla u_k|^2 \chi_{\lambda,R}(x) \, \mathrm{d}x \\
&\lesssim (\lambda + (1-\lambda)RR_\lambda^{-1})^{-3} + \int_{B_R^c(0) \cap B_{R_\lambda}(0)} |u_k(x)|^2 \, \mathrm{d}x,
\end{aligned}$$

where $R_\lambda \geq R \geq R_0$ and $\lambda \bar{R}_\lambda = R_\lambda + (\lambda - 1)R$. Hence,

$$\left| \|\nabla u_{k,1}\|_{L^2(\mathbb{R}^4)}^2 - \int_{B_R(0)} |\nabla u_k|^2 \, \mathrm{d}x \right| \lesssim \epsilon + \left(\lambda - (\lambda - 1)\frac{R}{\bar{R}_\lambda} \right)^{-3},$$

if $R \in [R_0, R_\lambda]$. If we choose $R = R_0$ and $\lambda \gg 1$, then we get that

$$\left| \mathfrak{K}(u_{k,1}) - \int_{B_{R_0}(0)} \bar{u}_k^2 K_{c,\kappa} * u_k^2 \, \mathrm{d}x \right| \leq \epsilon, \qquad \left| \|u_{k,1}\|_{D^{1,2}(\mathbb{R}^4)}^2 - \int_{B_{R_0}(0)} |\nabla u_k|^2 \, \mathrm{d}x \right| \lesssim \epsilon \tag{2.10}$$

when n is large enough. For the above R_0, we define for any $\lambda \geq 1$ and $R > R_0$ the mapping

$$S_{\lambda,R}(x) = \lambda x \mathbf{1}_{B_R(0)} + \left(x - (1-\lambda)R\frac{x}{|x|} \right) \mathbf{1}_{B_R^c(0)}$$

and $u_{k,2}(x) = u_k(S_{\lambda,R}(x))$. Similar to the above argument and following Lions [5], by choosing $\lambda = \epsilon^{-\frac{1}{2}}$ and $R = R_0$ in the definition of $S_{\lambda,R}$, we obtain that

$$\left|\mathfrak{K}(u_{k,2}) - \int_{B^c_{R_k}(0)} \bar{u}_k^2 K_{c,\kappa} * u_k^2 \, \mathrm{d}x\right| \lesssim \epsilon, \qquad \left|\|u_{k,2}\|^2_{D^{1,2}(\mathbb{R}^4)} - \int_{B^c_{R_0}(0)} |\nabla u_k|^2 \, \mathrm{d}x\right| \lesssim \sqrt{\epsilon}, \tag{2.11}$$

when n is large enough.

Now set $\beta_\epsilon = \lim_k \mathfrak{K}(u_{k,1})$. We show that $\beta_\epsilon \in (0,1)$ is bounded from zero and 1 as $\epsilon \to 0$. First, if $\beta_\epsilon \to 0$, then

$$\liminf_{\epsilon} \liminf_{k} \|u_{k,2}\|^2_{D^{1,2}} \geq I,$$

but we see from (2.10) and (2.11) that

$$\liminf_{\epsilon} \left(\liminf_{k} \|u_{k,1}\|^2_{D^{1,2}}\right) + \liminf_{\epsilon} \left(\liminf_{k} \|u_{k,2}\|^2_{D^{1,2}}\right) \leq I,$$

which is a contradiction with $\liminf_\epsilon \liminf_k \|u_k\|_{D^{1,2}} > 0$. The case $\lim_\epsilon \beta_\epsilon \geq 1$ is excluded, by an argument similar to above by replacing $u_{k,2}$ by $u_{k,1}$. So we can assume that $\lim_\epsilon \beta_\epsilon \in (0,1)$, and then we get by (2.10) and (2.11) for small ϵ that

$$I \geq \liminf_{k} \|u_{k,1}\|^2_{D^{1,2}(\mathbb{R}^4)} + \liminf_{k} \|u_{k,2}\|^2_{D^{1,2}(\mathbb{R}^4)} - C\sqrt{\epsilon} \geq I_{\lim_\epsilon \beta_\epsilon} + I_{1-\lim_\epsilon \beta_\epsilon} - o(\epsilon).$$

This contradicts (2.9), and the dichotomy case is ruled out.

Compactness. The concentration-compactness principle shows now that there exists $\{y_k\} \subset \mathbb{R}^4$ such that for any $\epsilon > 0$, there is $R > 0$ such that

$$\int_{B^c_R(y_k)} \rho_k \, \mathrm{d}x \leq \epsilon \tag{2.12}$$

for all k. For simplicity, we denote the sequence $u_k(\cdot - y_k)$ by the same u_k. Then, there is $u \in D^{1,2}(\mathbb{R}^4)$ such that $u_k \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^4)$, a.e. in $\mathbb{R}^4$ and $\mathfrak{K}(u) < \infty$ by Fatou's lemma.

If $u \equiv 0$, then we show that $\mathfrak{K}(u_k) \to 0$ as $k \to \infty$. From (2.12), we have

$$\int_{B^c_R(0)} \bar{u}_k^2 K_{c,\kappa} * u_k^2 \, \mathrm{d}x \leq \epsilon$$

for sufficiently large R and for all k. Since $u_k \to 0$ in $L^1(B_R(0))$, we get the claim by an argument similar to the evanescence case.

To complete the proof, we prove that $\mathfrak{K}(u) = 1$. It is trivial that $\mathfrak{K}(u) \leq 1$ because of $\|\nabla u\|^2_{L^2(\mathbb{R}^4)} \leq I$. We prove that $\mathfrak{K}(u) \geq 1$. By Lemma 2.4, there are $\{x_j\}_J \subset \mathbb{R}^4$ and $\{\nu_j\}_J \subset (0,\infty)$ such that

$$|u_k|^4 \to |u|^4 + \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \|u_k\|^4_{L^4(\mathbb{R}^4)} \to \|u\|^4_{L^4(\mathbb{R}^4)} + \sum_{j\in J} \nu_j \qquad \text{as } k \to \infty. \tag{2.13}$$

We prove that

$$1 = \lim_k \mathfrak{K}(u_k) \leq \mathfrak{K}(u) + \gamma \sum \nu_j. \tag{2.14}$$

Referring to (2.12), we get for all k that $\sum_{j\in J, x_j \notin B_R(0)} \nu_j \leq \epsilon$ and

$$\int_{B^c_R(0)} \left|\bar{u}_k^2 K_{c,\kappa} * u_k^2\right| + |u_k|^4 \, \mathrm{d}x \leq \epsilon, \qquad \int_{B^c_R(0)} \left|\bar{u}^2 K_{c,\kappa} * u^2\right| + |u|^4 \, \mathrm{d}x \leq \epsilon.$$

Lemma 2.5 shows that

$$\int_{\mathbb{R}^4\setminus B_R(0)} \left|\bar{u}_k^2 K_{c,\kappa} * u^2 - \bar{u}^2 K_{c,\kappa} * u^2 - (\overline{u_k - u})^2 K_{c,\kappa} * (u_k - u)^2\right| \, \mathrm{d}x \to 0 \qquad \text{as } k \to \infty.$$

To get (2.14), it is enough to show that

$$\liminf_k \int_{B_R(0)} (\overline{u_k - u})^2 K_{c,\kappa} * (u_k - u)^2 \, \mathrm{d}x \le \gamma \sum_{j \in J} \nu_j.$$

But it is known that $|u_k - u|^4 \to \sum_j \nu_j \delta_{x_j}$, and by an argument similar to the dichotomy case, we obtain that

$$\liminf_k \int_{B_R} \left|\overline{(u_k - u)}^2 K_{c,\kappa} * (u_k - u)^2\right| \, \mathrm{d}x \le \gamma \liminf_k \|u_k - u\|_{L^4}^4 = \gamma \sum_j \nu_j.$$

Lemma 2.4 and (2.14) give

$$1 \le \mathfrak{K}(u) + \gamma \sum_j \nu_j, \qquad 1 \ge \|\nabla u\|^2_{L^2(\mathbb{R}^4)} + I^0 \sqrt{\sum_j \nu_j}.$$

If $\mathfrak{K}(u) \le 0$, then $\gamma \sum_j \nu_j \ge 1$ and

$$I > I^0 \sqrt{\sum_j \nu_j} \ge I^0 \gamma^{-\frac{1}{2}} = I^\infty.$$

This contradiction completes the proof of existence of a minimizer for (2.3).

Corollary 2.6. *There exists a ground state solution of (2.1).*

Proof. If u is a minimizer of (2.3), then there is $\theta \in \mathbb{C}$ such that

$$\mathfrak{R}\left\langle \nabla u, \nabla v \right\rangle = 2\theta \mathfrak{R}\left\langle uv, K_{c,\kappa} * u^2 \right\rangle$$

for all $v \in D^{1,2}(\mathbb{R}^4)$. By taking $v = u$ and $v = -\mathrm{i}u$, we get $\theta \in \mathbb{R}$ is nonzero, and then u by a scaling satisfies

$$\left\langle \nabla u, \nabla v \right\rangle = 2\left\langle uv, K_{c,\kappa} * u^2 \right\rangle.$$

Remark 2.3 shows that u is a minimizer of (2.5). Consequently, it is a ground state. ☐

Define

$$\iota_c = \inf\left\{S(u, v),\ (u, v) \in \mathfrak{G}_c\right\}$$

and

$$\mathfrak{M}_c = \left\{(u, v) \in \mathfrak{G}_c,\ S(u, v) = \iota_c\right\},$$

where $\mathfrak{G}_c = \left\{(u, v) \in H\setminus\{(0, 0)\},\ \mathfrak{X}(u, v) = \langle S'(u, v), (u, v)\rangle = 0\right\}$. Analogous to Fukaya et al. [4, Proposition 2.1], we can get the following result.

Corollary 2.7. *There holds $\mathfrak{M}_c = G_c$.*

Proposition 2.1. *Let $u \in D^{1,2}(\mathbb{R}^4)$ be a nontrivial solution of (2.1). Then $u \in L^\infty(\mathbb{R}^4)$. Moreover, it is a classical solution of (2.1).*

Proof. To get the regularity of solutions (2.1), we note that for any $q \in [4/3, 4)$ from (2.1), the Hölder inequality, and the embedding $D^{1,2}(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$ that

$$\|-\Delta u\|_{L^q(\mathbb{R}^4)} \lesssim \|u\|_{L^4(\mathbb{R}^4)} \left\|K_{c,\kappa} * u^2\right\|_{L^{\frac{4-q}{4q}}(\mathbb{R}^4)} \lesssim \|u\|^2_{L^4(\mathbb{R}^4)}.$$

This means that $u \in \dot{H}^{2,q}(\mathbb{R}^4)$ for any $q \in [4/3, 4)$. Now, if we use the Hardy–Littlewood–Sobolev inequality for the Riesz potential of order 2 (see, e.g., Linares & Ponce [10]), we get for any $r > 4$ and $q \in [4/3, 2)$ that

$$\|u\|_{L^r(\mathbb{R}^4)} \lesssim \left\|\bar{u}K_{c,\kappa} * u^2\right\|_{L^q(\mathbb{R}^4)} \lesssim \|u\|_{L^4(\mathbb{R}^4)} \left\|K_{c,\kappa} * u^2\right\|_{L^{\frac{4-q}{4q}}(\mathbb{R}^4)} \lesssim \|u\|^2_{L^4(\mathbb{R}^4)}$$

provided $\frac{1}{r} + \frac{1}{2} = \frac{1}{q}$. Hence, $u \in L^r(\mathbb{R}^4)$ for any $4 \le r < \infty$. The Gagliardo–Nirenberg inequality implies that $u \in H^{2,q}(\mathbb{R}^4)$ for any $q \in [4/3, 4]$. By repeating the above argument, we obtain that $u \in H^{2,q}(\mathbb{R}^4)$ for all $2 \le q < \infty$. The Sobolev embedding shows that $u \in L^\infty(\mathbb{R}^4)$ and u is a classical solution of (2.1). ☐

2.1 | Global existence

Having proved the existence of the ground stats, we are able to find some conditions, by the potential well theory, under which the existence of global solutions of (1.1) is guaranteed. To do so, we define the submanifolds

$$\mathscr{A}_c^{\pm} = \{(u,v) \in H,\ S(u,v) < \iota_c, \pm\mathfrak{X}(u,v) \ge 0\}.$$

These sets are invariants under the flow of (1.1).

Lemma 2.8. *If* $(u_0, v_0) \in \mathscr{A}_c^{\pm}$, *then the solution* $(u(t), v(t)) \in C^1([0,T); H)$ *of (1.1) belongs to* $\mathscr{A}_c^{\pm}$ *for all* $0 \le t < T$.

Proof. We consider $\mathscr{A}_c^{+}$, and $\mathscr{A}_c^{-}$ is treated in the same fashion.

Assume by contradiction that there is $t_0 \in (0,T)$ such that $(u(t_0), v(t_0)) \neq \mathscr{A}_c^{+}$. It is clear that $S(u(t), v(t)) < \iota_c$ for all $t \in [0,T)$, so that $\mathfrak{X}(u(t_0), v(t_0)) < 0$. This implies by the continuity that there is $t_1 \in (0, t_0)$ such that $\mathfrak{X}(u(t_1), v(t_1)) = 0$. This means by the uniqueness of $(u(t), v(t))$ of (1.1) that $(u(t_1), v(t_1)) \in \mathfrak{G}_c$; whence, $(u(t_1), v(t_1)) \in G_c$. This leads us to the contradiction

$$\iota_c \le S(u(t_1), v(t_1)) < \iota_c.$$

☐

Proposition 2.2. *If* $(u_0, v_0) \in \mathscr{A}_c^{+}$, *then the unique solution* $(u(t), v(t)) \in C^1([0,T); H)$ *of (1.1) exists globally in time.*

Proof. First, we note that $\mathscr{A}_c^{+} \neq \emptyset$. Suppose by contradiction that $T < +\infty$, so that

$$\lim_{t \to T^-} \|(u(t), v(t))\|_H = +\infty.$$

Then we have from Lemma 2.8 and the conservation law $\mathbb{F}$ that

$$\begin{aligned} \iota_c \ge S(u(t), v(t)) &= S(u(t), v(t)) - \frac{1}{3}\mathfrak{X}S(u(t), v(t)) \\ &= \frac{1}{6}\left(\|\nabla u\|^2_{L^2(\mathbb{R}^4)} + \|\nabla v\|^2_{L^2(\mathbb{R}^4)} + 2P(u(t), v(t))\right) \approxeq \|(u(t), v(t))\|^2_H; \end{aligned}$$

which completes the proof. ☐

AUTHOR CONTRIBUTIONS

Amin Esfahani: Conceptualization; methodology; investigation; writing—original draft; writing—review and editing.

ACKNOWLEDGEMENTS

The author would like to thank the referees for their insightful comments, which greatly enriched the quality and the structure of this work. A. E. is supported by Nazarbayev University under the Faculty Development Competitive Research Grants Program for 2023–2025 (grant number 20122022FD4121).

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How to cite this article: A. Esfahani, *Traveling waves for a nonlinear Schrödinger system with quadratic interaction in* $\mathbb{R}^4$, Math. Meth. Appl. Sci. **47**(2024), 12121–12131, DOI 10.1002/mma.10158.