

## Geometric Hardy and Hardy–Sobolev inequalities on Heisenberg groups

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In this paper, we present geometric Hardy inequalities for the sub-Laplacian in half-spaces of stratified groups. As a consequence, we obtain the following geometric Hardy inequality in a half-space of the Heisenberg group with a sharp constant:

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi, \quad p > 1,$$

which solves a conjecture in the paper [S. Larson, Geometric Hardy inequalities for the sub-elliptic Laplacian on convex domain in the Heisenberg group, *Bull. Math. Sci.*

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6 (2016) 335–352]. Here,

$$\mathcal{W}(\xi) = \left( \sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2 \right)^{\frac{1}{2}}$$

is the angle function. Also, we obtain a version of the Hardy–Sobolev inequality in a half-space of the Heisenberg group:

$$\left( \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left( \frac{p-1}{p} \right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi \right)^{\frac{1}{p}} \geq C \left( \int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{\frac{1}{p^*}},$$

where  $\text{dist}(\xi, \partial\mathbb{H}^+)$  is the Euclidean distance to the boundary,  $p^* := Qp/(Q-p)$ , and  $2 \leq p < Q$ . For  $p = 2$ , this gives the Hardy–Sobolev–Maz’ya inequality on the Heisenberg group.

*Keywords:* Stratified groups; Heisenberg group; geometric Hardy inequality; half-space; sharp constant.

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### 1. Introduction

Let us recall the Hardy inequality in the half-space of  $\mathbb{R}^n$

$$\int_{\mathbb{R}_+^n} |\nabla u|^p dx \geq \left( \frac{p-1}{p} \right)^p \int_{\mathbb{R}_+^n} \frac{|u|^p}{x_n^p} dx, \quad p > 1, \tag{1.1}$$

for every function  $u \in C_0^\infty(\mathbb{R}_+^n)$ , where  $\nabla$  is the usual Euclidean gradient and  $\mathbb{R}_+^n := \{(x', x_n) | x' := (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\}$ ,  $n \in \mathbb{N}$ . There is a number of studies related to inequality (1.1), see e.g. [1, 2, 5, 12].

Filippas *et al.* in [6] established the Hardy–Sobolev inequality in the following form:

$$\left( \int_{\mathbb{R}_+^n} |\nabla u|^p dx - \left( \frac{p-1}{p} \right)^p \int_{\mathbb{R}_+^n} \frac{|u|^p}{x_n^p} dx \right)^{\frac{1}{p}} \geq C \left( \int_{\mathbb{R}_+^n} |u|^{p^*} dx \right)^{\frac{1}{p^*}}, \tag{1.2}$$

for all function  $u \in C_0^\infty(\mathbb{R}_+^n)$ , where  $p^* = \frac{np}{n-p}$  and  $2 \leq p < n$ . For a different proof of this inequality, see Frank and Loss [8] as well as Psaradakis [13].

A Hardy inequality for a half-space of the Heisenberg group was shown by Luan and Young [11] in the form

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi. \tag{1.3}$$

An alternative proof of this inequality was given by Larson in [10], where the author generalized it to any half-space of the Heisenberg group

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \frac{1}{4} \int_{\mathbb{H}^+} \frac{\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2}{\text{dist}(\xi, \partial\mathbb{H}^+)^2} |u|^2 d\xi,$$

where  $X_i$  and  $Y_i$  (for  $i = 1, \dots, n$ ) are left-invariant vector fields on the Heisenberg group,  $\nu$  is the Riemannian outer unit normal (see [9]) to the boundary. In the same

paper the following  $L^p$ -generalization of the inequality was proved

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\sum_{i=1}^n |\langle X_i(\xi), \nu \rangle|^p + |\langle Y_i(\xi), \nu \rangle|^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi.$$

Note also that the authors in [15] have extended this result to general Carnot groups, see [16–19].

The main aim of this paper is to improve the  $L^p$ -version of the geometric Hardy inequality for the sub-Laplacian in the half-spaces of stratified groups (Carnot groups), where the obtained inequality will be a natural extension of the inequality derived by the authors in [10, 15] on Heisenberg and stratified groups, respectively. In particular, we prove an inequality conjectured in [10]. Moreover, we obtain a version of the Hardy–Sobolev inequality in the setting of the Heisenberg group.

The main results of this paper are as follows:

- **Geometric  $L^p$ -Hardy inequality on  $\mathbb{G}^+$ :** Let  $\mathbb{G}^+ := \{x \in \mathbb{G} : \langle x, \nu \rangle > d\}$  be a half-space of a stratified group  $\mathbb{G}$ . Then for all  $u \in C_0^\infty(\mathbb{G}^+)$ ,  $\beta \in \mathbb{R}$  and  $p > 1$ , we have

$$\begin{aligned} \int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx &\geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{G}^+} \frac{\mathcal{W}(x)^p}{\text{dist}(x, \partial\mathbb{G}^+)^p} |u|^p dx \\ &\quad + \beta \int_{\mathbb{G}^+} \frac{\mathcal{L}_p(\text{dist}(x, \partial\mathbb{G}^+))}{\text{dist}(x, \partial\mathbb{G}^+)^{p-1}} |u|^p dx, \end{aligned}$$

where  $\mathcal{W}(x) := (\sum_{i=1}^N \langle X_i(x), \nu \rangle^2)^{1/2}$ , and  $\mathcal{L}_p$  is the  $p$ -sub-Laplacian on  $\mathbb{G}$ , see (1.6).

- **Geometric  $L^p$ -Hardy inequality on  $\mathbb{H}^+$ :** Let  $\mathbb{H}^+ := \{\xi \in \mathbb{H}^n : \langle \xi, \nu \rangle > d\}$  be a half-space of the Heisenberg group. Then for all  $u \in C_0^\infty(\mathbb{H}^+)$  and  $p > 1$  we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi,$$

where  $\mathcal{W}(\xi) := (\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2)^{1/2}$  and the constant is sharp.

- **Geometric Hardy–Sobolev inequality on  $\mathbb{H}^+$ :** Let  $\mathbb{H}^+ := \{\xi \in \mathbb{H}^n : \langle \xi, \nu \rangle > d\}$  be a half-space of the Heisenberg group. Then for all  $u \in C_0^\infty(\mathbb{H}^+)$  and  $2 \leq p < Q$  with  $Q = 2n + 1$ , there exists some  $C > 0$  such that we have

$$\left( \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi \right)^{\frac{1}{p}} \geq C \left( \int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{\frac{1}{p^*}},$$

where  $\text{dist}(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu \rangle - d$  is the distance from  $\xi$  to the boundary and  $p^* := Qp/(Q-p)$ .

### 1.1. Preliminaries on stratified groups

Let  $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$  be a stratified Lie group (or a homogeneous Carnot group), with dilation structure  $\delta_\lambda$  and Jacobian generators  $X_1, \dots, X_N$ , so that  $N$  is the

dimension of the first stratum of  $\mathbb{G}$ . Let us denote by  $Q$  the homogeneous dimension of  $\mathbb{G}$ . We refer to the recent books [7, 20] for extensive discussions of stratified Lie groups and their properties.

The sub-Laplacian on  $\mathbb{G}$  is given by

$$\mathcal{L} = \sum_{k=1}^N X_k^2. \tag{1.4}$$

We also recall that the standard Lebesgue measure  $dx$  on  $\mathbb{R}^n$  is the Haar measure for  $\mathbb{G}$  (see, e.g. [7, Proposition 1.6.6]). Each left invariant vector field  $X_k$  has an explicit form and satisfies the divergence theorem, see e.g. [7] for the derivation of the exact formula: more precisely, we can express

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^r \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x', \dots, x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}}, \tag{1.5}$$

with  $x = (x', x^{(2)}, \dots, x^{(r)})$ , where  $r$  is the step of  $\mathbb{G}$  and  $x^{(l)} = (x_1^{(l)}, \dots, x_{N_l}^{(l)})$  are the variables in the  $l$ th stratum, see also [7, Sec. 3.1.5] for a general presentation. The horizontal gradient is given by

$$\nabla_{\mathbb{G}} := (X_1, \dots, X_N),$$

and the horizontal divergence is defined by

$$\operatorname{div}_{\mathbb{G}} v := \nabla_{\mathbb{G}} \cdot v.$$

The  $p$ -sub-Laplacian has the following form:

$$\mathcal{L}_p v = \nabla_{\mathbb{G}} (|\nabla_{\mathbb{G}} v|^{p-2} \nabla_{\mathbb{G}} v). \tag{1.6}$$

Let us define the half-space on the stratified group  $\mathbb{G}$  as

$$\mathbb{G}^+ := \{x \in \mathbb{G} : \langle x, \nu \rangle > d\},$$

where  $\nu := (\nu_1, \dots, \nu_r)$  with  $\nu_j \in \mathbb{R}^{N_j}, j = 1, \dots, r$ , is the Riemannian outer unit normal to  $\partial\mathbb{G}^+$  (see [9]) and  $d \in \mathbb{R}$ . Let us define the so-called angle function

$$\mathcal{W}(x) := \sqrt{\sum_{i=1}^N \langle X_i(x), \nu \rangle^2};$$

such function was introduced by Garofalo [9] in his investigation of the horizontal Gauss map. The Euclidean distance to the boundary  $\partial\mathbb{G}^+$  is denoted by  $\operatorname{dist}(x, \partial\mathbb{G}^+)$  and defined as

$$\operatorname{dist}(x, \partial\mathbb{G}^+) = \langle x, \nu \rangle - d.$$

## 2. Geometric Hardy Inequalities

**Theorem 2.1.** *Let  $\mathbb{G}^+$  be a half-space of a stratified group  $\mathbb{G}$ . Then for all  $\beta \in \mathbb{R}$  and  $p > 1$ , we have*

$$\begin{aligned} \int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx &\geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{G}^+} \frac{\mathcal{W}(x)^p}{\text{dist}(x, \partial\mathbb{G}^+)^p} |u|^p dx \\ &\quad + \beta \int_{\mathbb{G}^+} \frac{\mathcal{L}_p(\text{dist}(x, \partial\mathbb{G}^+))}{\text{dist}(x, \partial\mathbb{G}^+)^{p-1}} |u|^p dx, \end{aligned} \tag{2.1}$$

for all  $u \in C_0^\infty(\mathbb{G}^+)$ .

**Proof of Theorem 2.1.** Let us begin with the divergence theorem, then we apply the Hölder inequality and the Young inequality, respectively. It follows for a vector field  $V \in C^\infty(\mathbb{G}^+)$  that

$$\begin{aligned} \int_{\mathbb{G}^+} \text{div}_{\mathbb{G}} V |u|^p dx &= -p \int_{\mathbb{G}^+} |u|^{p-1} \langle V, \nabla_{\mathbb{G}} |u| \rangle dx \\ &\leq p \left( \int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} |u||^p dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{G}^+} |V|^{\frac{p}{p-1}} |u|^p dx \right)^{\frac{p-1}{p}} \\ &\leq \int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} |u||^p dx + (p-1) \int_{\mathbb{G}^+} |V|^{\frac{p}{p-1}} |u|^p dx. \end{aligned}$$

By rearranging the above expression and using  $|\nabla_{\mathbb{G}} u| \geq |\nabla_{\mathbb{G}} |u||$ , we arrive at

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx \geq \int_{\mathbb{G}^+} (\text{div}_{\mathbb{G}} V - (p-1)|V|^{\frac{p}{p-1}}) |u|^p dx. \tag{2.2}$$

Now, we choose  $V$  in the following form:

$$V = \beta \frac{|\nabla_{\mathbb{G}} \text{dist}(x, \partial\mathbb{G}^+)|^{p-2}}{\text{dist}(x, \partial\mathbb{G}^+)^{p-1}} \nabla_{\mathbb{G}} \text{dist}(x, \partial\mathbb{G}^+), \tag{2.3}$$

that is

$$|V|^{\frac{p}{p-1}} = |\beta|^{\frac{p}{p-1}} \frac{|\nabla_{\mathbb{G}} \text{dist}(x, \partial\mathbb{G}^+)|^p}{\text{dist}(x, \partial\mathbb{G}^+)^p}.$$

Also, we have

$$\begin{aligned} |\nabla_{\mathbb{G}} \text{dist}(x, \partial\mathbb{G}^+)|^p &= |(X_1 \text{dist}(x, \partial\mathbb{G}^+), \dots, X_N \text{dist}(x, \partial\mathbb{G}^+))|^p \\ &= |(\langle X_1(x), \nu \rangle, \dots, \langle X_N(x), \nu \rangle)|^p \\ &= \left( \sum_{i=1}^N \langle X_i(x), \nu \rangle^2 \right)^{\frac{p}{2}} = \mathcal{W}(x)^p. \end{aligned}$$

Indeed, let us show that  $\langle X_i(x), \nu \rangle = X_i \langle x, \nu \rangle$ :

$$\begin{aligned}
 X_i(x) &= \underbrace{(0, \dots, 1, \dots, 0)}_i, \underbrace{a_{i,1}^{(2)}(x'), \dots, a_{i,N_2}^{(2)}(x'), \dots}_{N_2}, \\
 &\quad \underbrace{a_{i,1}^{(r)}(x', x^{(2)}, \dots, x^{(r-1)}), \dots, a_{i,N_r}^{(r)}(x', x^{(2)}, \dots, x^{(r-1)})}_{N_r}, \\
 \langle X_i(x), \nu \rangle &= \nu'_i + \sum_{l=2}^r \sum_{m=1}^{N_l} a_{i,m}^{(l)}(x', x^{(2)}, \dots, x^{(r-1)}) \nu_m^{(l)},
 \end{aligned}$$

and

$$\begin{aligned}
 \langle x, \nu \rangle &= \sum_{k=1}^N x'_k \nu'_k + \sum_{l=2}^r \sum_{m=1}^{N_l} x_m^{(l)} \nu_m^{(l)}, \\
 X_i \langle x, \nu \rangle &= \nu'_i + \sum_{l=2}^r \sum_{m=1}^{N_l} a_{i,m}^{(l)}(x', x^{(2)}, \dots, x^{(r-1)}) \nu_m^{(l)}.
 \end{aligned}$$

A direct calculation shows that

$$\begin{aligned}
 \operatorname{div}_{\mathbb{G}} V &= \beta \frac{\nabla_{\mathbb{G}} (|\nabla_{\mathbb{G}} \operatorname{dist}(x, \partial \mathbb{G}^+)|^{p-2} \nabla_{\mathbb{G}} \operatorname{dist}(x, \partial \mathbb{G}^+))}{\operatorname{dist}(x, \partial \mathbb{G}^+)^{p-1}} \\
 &\quad |\nabla_{\mathbb{G}} \operatorname{dist}(x, \partial \mathbb{G}^+)|^{p-2} \nabla_{\mathbb{G}} \operatorname{dist}(x, \partial \mathbb{G}^+) \\
 &\quad - \beta(p-1) \frac{\operatorname{dist}(x, \partial \mathbb{G}^+)^{p-2} \nabla_{\mathbb{G}} \operatorname{dist}(x, \partial \mathbb{G}^+)}{\operatorname{dist}(x, \partial \mathbb{G}^+)^{2(p-1)}} \\
 &= \beta \frac{\mathcal{L}_p(\operatorname{dist}(x, \partial \mathbb{G}^+))}{\operatorname{dist}(x, \partial \mathbb{G}^+)^{p-1}} - \beta(p-1) \frac{|\nabla_{\mathbb{G}} \operatorname{dist}(x, \partial \mathbb{G}^+)|^p}{\operatorname{dist}(x, \partial \mathbb{G}^+)^p}.
 \end{aligned}$$

So, we get

$$\begin{aligned}
 \operatorname{div}_{\mathbb{G}} V - (p-1)|V|^{\frac{p}{p-1}} &= -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \frac{|\nabla_{\mathbb{G}} \operatorname{dist}(x, \partial \mathbb{G}^+)|^p}{\operatorname{dist}(x, \partial \mathbb{G}^+)^p} \\
 &\quad + \beta \frac{\mathcal{L}_p(\operatorname{dist}(x, \partial \mathbb{G}^+))}{\operatorname{dist}(x, \partial \mathbb{G}^+)^{p-1}}.
 \end{aligned}$$

Putting the above expression into inequality (2.2), we arrive at

$$\begin{aligned}
 \int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx &\geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{G}^+} \frac{\left(\sum_{i=1}^N \langle X_i(x), \nu \rangle\right)^{\frac{p}{2}}}{\operatorname{dist}(x, \partial \mathbb{G}^+)^p} |u|^p dx \\
 &\quad + \beta \int_{\mathbb{G}^+} \frac{\mathcal{L}_p(\operatorname{dist}(x, \partial \mathbb{G}^+))}{\operatorname{dist}(x, \partial \mathbb{G}^+)^{p-1}} |u|^p dx,
 \end{aligned}$$

completing the proof. □

### 2.1. Preliminaries on the Heisenberg group

Let us give a brief introduction of the Heisenberg group. Let  $\mathbb{H}^n$  be the Heisenberg group, that is, the set  $\mathbb{R}^{2n+1}$  equipped with the group law

$$\xi \circ \tilde{\xi} := (z + \tilde{z}, t + \tilde{t} + 2 \operatorname{Im}\langle z, \tilde{z} \rangle),$$

where  $\xi := (x, y, t) \in \mathbb{H}^n$ ,  $x := (x_1, \dots, x_n)$ ,  $y := (y_1, \dots, y_n)$ ,  $z := (x, y) \in \mathbb{R}^{2n}$  is identified by  $\mathbb{C}^n$ , and  $\xi^{-1} = -\xi$  is the inverse element of  $\xi$  with respect to the group law. The dilation operation of the Heisenberg group with respect to the group law has the following form:

$$\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t) \quad \text{for } \lambda > 0.$$

The Lie algebra  $\mathfrak{h}$  of the left-invariant vector fields on the Heisenberg group  $\mathbb{H}^n$  is spanned by

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \quad \text{for } 1 \leq i \leq n,$$

$$Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t} \quad \text{for } 1 \leq i \leq n,$$

and with their (nonzero) commutator

$$[X_i, Y_i] = -4 \frac{\partial}{\partial t}.$$

The horizontal gradient of  $\mathbb{H}^n$  is given by

$$\nabla_H := (X_1, \dots, X_n, Y_1, \dots, Y_n),$$

so the sub-Laplacian on  $\mathbb{H}^n$  is given by

$$\mathcal{L} := \sum_{i=1}^n (X_i^2 + Y_i^2).$$

Let us define the half-space of the Heisenberg group by

$$\mathbb{H}^+ := \{\xi \in \mathbb{H}^n : \langle \xi, \nu \rangle > d\},$$

where  $\nu := (\nu_x, \nu_y, \nu_t)$  with  $\nu_x, \nu_y \in \mathbb{R}^n$  and  $\nu_t \in \mathbb{R}$  is the Riemannian outer unit normal to  $\partial\mathbb{H}^+$  (see [9]) and  $d \in \mathbb{R}$ . Let us define the so-called angle function

$$\mathcal{W}(\xi) := \sqrt{\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2}.$$

The Euclidean distance to the boundary  $\partial\mathbb{H}^+$  is denoted by  $\operatorname{dist}(\xi, \partial\mathbb{H}^+)$  and defined by

$$\operatorname{dist}(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu \rangle - d.$$

### 2.2. Consequences on the Heisenberg group

As a consequence of Theorem 2.1, we have the following inequality.

**Corollary 2.2.** *Let  $\mathbb{H}^+$  be a half-space of the Heisenberg group  $\mathbb{H}^n$ . Then for all  $u \in C_0^\infty(\mathbb{H}^+)$  and  $p > 1$ , we have*

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi, \tag{2.4}$$

where the constant is sharp.

**Remark 2.3.** Note that inequality (2.4) was conjectured in [10] which is a natural extension of inequality (1.3) in [11]. Also, the sharpness of inequality (2.4) was proved by choosing  $\nu := (1, 0, \dots, 0)$  and  $d = 0$  in the paper [10].

**Proof of Corollary 2.2.** Let us rewrite the inequality in Theorem 2.1 in terms of the Heisenberg group as follows:

$$\begin{aligned} \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi &\geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi \\ &\quad + \beta \int_{\mathbb{H}^+} \frac{\mathcal{L}_p(\text{dist}(\xi, \partial\mathbb{H}^+))}{\text{dist}(\xi, \partial\mathbb{H}^+)^{p-1}} |u|^p d\xi. \end{aligned}$$

In the case of the Heisenberg group, we need to show that the last term vanishes to prove Corollary 2.2. Indeed, we have

$$\begin{aligned} \mathcal{L}_p(\text{dist}(\xi, \partial\mathbb{H}^+)) &= \nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}}\langle \xi, \nu \rangle - d|^{p-2} \nabla_{\mathbb{G}}\langle \xi, \nu \rangle - d) \\ &= \nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}}\langle \xi, \nu \rangle|^{p-2} \nabla_{\mathbb{G}}\langle \xi, \nu \rangle) \\ &= \sum_{i=1}^n (X_i(|\nabla_{\mathbb{G}}\langle \xi, \nu \rangle|^{p-2} \langle X_i(\xi), \nu \rangle) + Y_i(|\nabla_{\mathbb{G}}\langle \xi, \nu \rangle|^{p-2} \langle Y_i(\xi), \nu \rangle)) \\ &= \sum_{i=1}^n X_i \left( \left( \sum_{j=1}^n ((\langle X_j(\xi), \nu \rangle)^2 + (\langle Y_j(\xi), \nu \rangle)^2) \right)^{\frac{p-2}{2}} \langle X_i(\xi), \nu \rangle \right) \\ &\quad + \sum_{i=1}^n Y_i \left( \left( \sum_{j=1}^n ((\langle X_j(\xi), \nu \rangle)^2 + (\langle Y_j(\xi), \nu \rangle)^2) \right)^{\frac{p-2}{2}} \langle Y_i(\xi), \nu \rangle \right) \\ &= (p-2) \sum_{i=1}^n \left( \left( \sum_{j=1}^n ((\langle X_j(\xi), \nu \rangle)^2 + (\langle Y_j(\xi), \nu \rangle)^2) \right)^{\frac{p-2}{2}-1} \right. \\ &\quad \left. \times \langle Y_i(\xi), \nu \rangle \langle X_i \langle Y_i(\xi), \nu \rangle \rangle \langle X_i(\xi), \nu \rangle \right) \end{aligned}$$

$$\begin{aligned}
 & + (p-2) \sum_{i=1}^n \left( \left( \sum_{j=1}^n (\langle X_j(\xi), \nu \rangle)^2 + (\langle Y_j(\xi), \nu \rangle)^2 \right)^{\frac{p-2}{2}-1} \right. \\
 & \left. \times \langle X_i(\xi), \nu \rangle \langle Y_i(\xi), \nu \rangle \right) = 0,
 \end{aligned}$$

since

$$\begin{aligned}
 \langle X_i(\xi), \nu \rangle &= \nu_{x,i} + 2y_i \nu_t, & \langle Y_i(\xi), \nu \rangle &= \nu_{y,i} - 2x_i \nu_t, \\
 X_i \langle X_i(\xi), \nu \rangle &= 0, & Y_i \langle Y_i(\xi), \nu \rangle &= 0, \\
 Y_i \langle X_i(\xi), \nu \rangle &= 2\nu_t, & X_i \langle Y_i(\xi), \nu \rangle &= -2\nu_t,
 \end{aligned}$$

where  $\xi := (x, y, t)$  with  $x, y \in \mathbb{R}^n$  and  $t \in \mathbb{R}$ ,  $\nu := (\nu_x, \nu_y, \nu_t)$  with  $\nu_x := (\nu_{x,1}, \dots, \nu_{x,n})$  and  $\nu_y := (\nu_{y,1}, \dots, \nu_{y,n})$ . Then, we have

$$\begin{aligned}
 X_i(\xi) &= (\underbrace{0, \dots, 1, \dots, 0}_n, \underbrace{0, \dots, 0}_n, 2y_i), \\
 Y_i(\xi) &= (\underbrace{0, \dots, 0}_n, \underbrace{0, \dots, 1, \dots, 0}_n, -2x_i).
 \end{aligned}$$

So, we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi.$$

Now, we optimize by differentiating the above inequality with respect to  $\beta$ , so that we have

$$\frac{p}{p-1} |\beta|^{\frac{1}{p-1}} + 1 = 0,$$

which leads to

$$\beta = -\left(\frac{p-1}{p}\right)^{p-1}.$$

Using this value of  $\beta$ , we arrive at

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi.$$

We have finished the proof of Corollary 2.2. □

### 3. Geometric Hardy–Sobolev Inequalities

In this section, we present the geometric Hardy–Sobolev inequality in the half space on the Heisenberg group.

### 3.1. A lower estimate for the geometric Hardy type inequalities

We start with an estimate for the remainder.

**Lemma 3.1.** *Let  $\mathbb{H}^+$  be a half-space of the Heisenberg group  $\mathbb{H}^n$ . Then for  $p \geq 2$ , there exists a constant  $C_p > 0$  such that*

$$\begin{aligned}
 E_p[u] &= \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{|\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi \\
 &\geq C_p \int_{\mathbb{H}^+} |\text{dist}(\xi, \partial\mathbb{H}^+)|^{p-1} |\nabla_H v|^p d\xi,
 \end{aligned} \tag{3.1}$$

for all  $u \in C_0^\infty(\mathbb{H}^+)$ , where  $\text{dist}(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu \rangle - d$  is the distance from  $\xi$  to the boundary,  $C_p = (2^{p-1} - 1)^{-1}$ , and  $u(\xi) = \text{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} v(\xi)$ .

The Euclidean version of such a lower estimate to the Hardy inequality was established by Barbaris *et al.* [4].

**Proof of Lemma 3.1.** Let us begin by recalling once again the angle function, denoted by  $\mathcal{W}$ ,

$$\begin{aligned}
 |\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|^p &= |(X_1 \langle \xi, \nu \rangle, \dots, X_n \langle \xi, \nu \rangle, Y_1 \langle \xi, \nu \rangle, \dots, Y_n \langle \xi, \nu \rangle)|^p \\
 &= \left( \sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + \langle Y_i(\xi), \nu \rangle^2 \right)^{\frac{p}{2}} = \mathcal{W}(\xi)^p.
 \end{aligned} \tag{3.2}$$

Note that  $X_i \langle \xi, \nu \rangle$  is equal to  $\langle X_i(\xi), \nu \rangle$ , see the proof of Theorem 2.1. This expression  $|\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|^p = \mathcal{W}(\xi)^p$  will be used later. For now, we will estimate the following form:

$$E_p[u] := \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi. \tag{3.3}$$

To estimate this, we introduce the following transformation:

$$u(\xi) = \text{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} v(\xi). \tag{3.4}$$

By inserting it into (3.3) and using (3.2), we have

$$\begin{aligned}
 E_p[u] &= \int_{\mathbb{H}^+} \left| \frac{p-1}{p} \text{dist}(\xi, \partial\mathbb{H}^+)^{-\frac{1}{p}} \nabla_H \text{dist}(\xi, \partial\mathbb{H}^+) v + \text{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} \nabla_H v \right|^p d\xi \\
 &\quad - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |\text{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} v|^p d\xi \\
 &\geq \int_{\mathbb{H}^+} \left| \frac{p-1}{p} \text{dist}(\xi, \partial\mathbb{H}^+)^{-\frac{1}{p}} v + \text{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} \frac{\nabla_H v}{\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)} \right|^p |\mathcal{W}(\xi)|^p \\
 &\quad - \left| \frac{p-1}{p} \text{dist}(\xi, \partial\mathbb{H}^+)^{-\frac{1}{p}} v \right|^p |\mathcal{W}(\xi)|^p d\xi.
 \end{aligned}$$

Then for  $p \geq 2$  and  $A, B \in \mathbb{R}^n$ , we have that

$$|A + B|^p - |A|^p \geq C_p |B|^p + p|A|^{p-2} A \cdot B,$$

where  $C_p = (2^{p-1} - 1)^{-1}$  (see [4, Lemma 3.3]). By taking

$$A := \frac{p-1}{p} \text{dist}(\xi, \partial\mathbb{H}^+)^{-\frac{1}{p}} v \quad \text{and} \quad B := \text{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} \frac{\nabla_H v}{\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)},$$

then we have the following lower estimate:

$$\begin{aligned} E_p[u] &\geq \int_{\mathbb{H}^+} |\mathcal{W}(\xi)|^p (|A + B|^p - |A|^p) d\xi \\ &\geq C_p \int_{\mathbb{H}^+} \text{dist}(\xi, \partial\mathbb{H}^+)^{p-1} |\nabla_H v|^p \frac{\mathcal{W}(\xi)^p}{|\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|^p} d\xi + \left(\frac{p-1}{p}\right)^{p-1} \\ &\quad \times \int_{\mathbb{H}^+} |\mathcal{W}(\xi)|^p |\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|^{p-2} (\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+) \cdot \nabla_H |v|^p) d\xi \\ &\geq C_p \int_{\mathbb{H}^+} \text{dist}(\xi, \partial\mathbb{H}^+)^{p-1} |\nabla_H v|^p d\xi. \end{aligned}$$

In the last line, we have used (3.2) and we dropped the last term on the right-hand side. This completes the proof of Lemma 3.1.  $\square$

### 3.2. Geometric Hardy–Sobolev inequality in the half-space on $\mathbb{H}^n$

Now, we are ready to obtain the geometric Hardy–Sobolev inequality in the half-space on the Heisenberg group  $\mathbb{H}^n$ .

**Theorem 3.2.** *Let  $\mathbb{H}^+$  be a half-space of the Heisenberg group  $\mathbb{H}^n$ . For  $2 \leq p < Q$  with  $Q = 2n + 1$  there exists some  $C > 0$  such that for every function  $u \in C_0^\infty(\mathbb{H}^+)$  we have*

$$\left( \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi \right)^{\frac{1}{p}} \geq C \left( \int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{\frac{1}{p^*}}, \tag{3.5}$$

where  $p^* := Qp/(Q-p)$  and  $\text{dist}(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu \rangle - d$  is the distance from  $\xi$  to the boundary.

**Remark 3.3.** Note that for  $p = 2$  we have the Hardy–Sobolev–Maz’ya inequality in the following form:

$$\left( \int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \frac{1}{4} \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^2}{\text{dist}(\xi, \partial\mathbb{H}^+)^2} |u|^2 d\xi \right)^{\frac{1}{2}} \geq C \left( \int_{\mathbb{H}^+} |u|^{2^*} d\xi \right)^{\frac{1}{2^*}}, \tag{3.6}$$

where  $2^* := 2Q/(Q-2)$ .

**Proof of Theorem 3.2.** Our key ingredient of proving the Hardy–Sobolev inequality in the half-space of  $\mathbb{H}^n$  is the  $L^1$ -Sobolev inequality, or the Gagliardo–Nirenberg inequality. It has been established on the Heisenberg group by Baldi *et al.* in [3].

The  $L^1$ -Sobolev inequality on the Heisenberg group follows in the form:

$$c \left( \int_{\mathbb{H}^n} |g|^{\frac{Q}{Q-1}} d\xi \right)^{\frac{Q-1}{Q}} \leq \int_{\mathbb{H}^n} |\nabla_H g| d\xi,$$

for some  $c > 0$ , for every function  $g \in W^{1,1}(\mathbb{H}^n)$ . Now, let us set  $g = |u|^{p^*(1-1/Q)}$ , then we obtain

$$\begin{aligned} c \left( \int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{\frac{Q-1}{Q}} &\leq \left| \frac{p(Q-1)}{Q-p} \right| \int_{\mathbb{H}^+} |u|^{\frac{Q(p-1)}{Q-p}} |\nabla_H |u|| d\xi, \\ &\leq \left| \frac{p(Q-1)}{Q-p} \right| \int_{\mathbb{H}^+} |u|^{\frac{Qp}{Q-p} \frac{(p-1)}{p}} |\nabla_H u| d\xi, \\ &= \left| \frac{p(Q-1)}{Q-p} \right| \int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi. \end{aligned}$$

We have used  $|\nabla_H |u|| \leq |\nabla_H u|$  (see [14, Proof of Theorem 2.1]). Then, we arrive at

$$C_1 \left( \int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{\frac{Q-1}{Q}} \leq \int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi, \tag{3.7}$$

where  $C_1 := c \left| \frac{Q-p}{p(Q-1)} \right| > 0$ . Let us estimate the right-hand side of inequality (3.7).

Again, we use a ground transform  $u(\xi) = \text{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} v(\xi)$  which leads to

$$\begin{aligned} &\int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi \\ &= \int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} \left| \text{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} \nabla_H v + \frac{p-1}{p} \text{dist}(\xi, \partial\mathbb{H}^+)^{-1/p} \right. \\ &\quad \left. \times \nabla_H \text{dist}(\xi, \partial\mathbb{H}^+) v \right| d\xi \\ &\leq \int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} \text{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} |\nabla_H v| d\xi \\ &\quad + \frac{p-1}{p} \int_{\mathbb{H}^+} \text{dist}(\xi, \partial\mathbb{H}^+)^{p^*(1-1/p)^2 - 1/p} \\ &\quad \times |\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)| |v|^{p^*(1-1/p)+1} d\xi \\ &= I_1 + \frac{p-1}{p} I_2. \end{aligned}$$

In the last line we have denoted two integrals by  $I_1$  and  $I_2$ , respectively. Also, for simplification we denote  $\alpha := p^*(1-1/p)^2 + 1 - 1/p$ . First, we estimate  $I_2$  using

integrations by parts

$$\begin{aligned}
 I_2 &= \int_{\mathbb{H}^+} \text{dist}(\xi, \partial\mathbb{H}^+)^{\alpha-1} |\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)| |v|^{\alpha p/(p-1)} d\xi \\
 &= \frac{1}{\alpha} \int_{\mathbb{H}^+} \langle \nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)^{\alpha}, \nabla_H \text{dist}(\xi, \partial\mathbb{H}^+) \rangle \frac{|v|^{\alpha p/(p-1)}}{|\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|} d\xi \\
 &= -\frac{1}{\alpha} \int_{\mathbb{H}^+} \text{dist}(\xi, \partial\mathbb{H}^+)^{\alpha} \nabla_H \left( \frac{\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+) |v|^{\alpha p/(p-1)}}{|\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|} \right) d\xi \\
 &= -\frac{1}{\alpha} \int_{\mathbb{H}^+} \text{dist}(\xi, \partial\mathbb{H}^+)^{\alpha} \\
 &\quad \times \left( \frac{\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+) \nabla_H |v|^{\alpha p/(p-1)}}{|\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|} \right. \\
 &\quad \left. - \frac{\langle \nabla_H \text{dist}(\xi, \partial\mathbb{H}^+), \nabla_H |\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)| \rangle |v|^{\alpha p/(p-1)}}{|\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|^2} \right) d\xi \\
 &= -\frac{1}{\alpha} \int_{\mathbb{H}^+} \text{dist}(\xi, \partial\mathbb{H}^+)^{\alpha} \frac{\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+) \nabla_H |v|^{\alpha p/(p-1)}}{|\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|} d\xi \\
 &\leq -\frac{p}{p-1} \int_{\mathbb{H}^+} \text{dist}(\xi, \partial\mathbb{H}^+)^{\alpha} |v|^{\alpha p/(p-1)-1} |\nabla_H v| d\xi \\
 &= \frac{p}{p-1} \int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} \text{dist}(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} |\nabla_H v| d\xi \leq \frac{p}{p-1} I_1.
 \end{aligned}$$

We have used  $|\nabla_H |u|| \leq |\nabla_H u|$ , and

$$\mathcal{L}(\text{dist}(\xi, \partial\mathbb{H}^+)) = \sum_{i=1}^n (X_i \langle X_i(\xi), \nu \rangle + Y_i \langle Y_i(\xi), \nu \rangle) = 0,$$

since

$$\begin{aligned}
 \langle X_i(\xi), \nu \rangle &= \nu_{x,i} + 2y_i \nu_t, & \langle Y_i(\xi), \nu \rangle &= \nu_{y,i} - 2x_i \nu_t, \\
 X_i \langle X_i(\xi), \nu \rangle &= 0, & Y_i \langle Y_i(\xi), \nu \rangle &= 0,
 \end{aligned}$$

where  $\xi := (x, y, t)$  with  $x, y \in \mathbb{R}^n$  and  $t \in \mathbb{R}$ ,  $\nu := (\nu_x, \nu_y, \nu_t)$  with  $\nu_x := (\nu_{x,1}, \dots, \nu_{x,n})$  and  $\nu_y := (\nu_{y,1}, \dots, \nu_{y,n})$ . Also, we have

$$\begin{aligned}
 &\langle \nabla_H \text{dist}(\xi, \partial\mathbb{H}^+), \nabla_H |\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)| \rangle \\
 &= \frac{2\nu_t}{|\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|} \\
 &\quad \times \underbrace{((2x_1 \nu_t - \nu_{y,1})(\nu_{x,1} + 2y_1 \nu_t) + \dots + (2x_n \nu_t - \nu_{y,n})(\nu_{x,n} + 2y_n \nu_t))}_n \\
 &\quad + \underbrace{(\nu_{y,1} - 2x_1 \nu_t)(\nu_{x,1} + 2y_1 \nu_t) + \dots + (\nu_{y,n} - 2x_n \nu_t)(\nu_{x,n} + 2y_n \nu_t)}_n = 0,
 \end{aligned}$$

since

$$\begin{aligned} |\nabla_H |\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|} &= (X_1 |\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|, \dots, X_n |\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|, \\ &\quad \times Y_1 |\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|, \dots, Y_n |\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|) \\ &= \frac{2\nu_t}{|\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+)|} \\ &\quad \times \underbrace{(2x_1\nu_t - \nu_{y,1}, \dots, 2x_n\nu_t - \nu_{y,n})}_n \underbrace{(\nu_{x,1} + 2y_1\nu_t, \dots, \nu_{x,n} + 2y_n\nu_t)}_n, \end{aligned}$$

and

$$\nabla_H \text{dist}(\xi, \partial\mathbb{H}^+) = \underbrace{(\nu_{x,1} + 2y_1\nu_t, \dots, \nu_{x,n} + 2y_n\nu_t)}_n, \underbrace{(\nu_{y,1} - 2x_1\nu_t, \dots, \nu_{y,n} - 2x_n\nu_t)}_n.$$

As we see that integral  $I_2$  can be estimated by integral  $I_1$ . From this estimation, we know that

$$\int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi \leq 2I_1. \tag{3.8}$$

Now, it comes to estimate  $I_1$  by using the Hölder inequality

$$\begin{aligned} I_1 &= \int_{\mathbb{H}^+} \{|u|^{p^*(1-1/p)}\} \{\text{dist}(\xi, \partial\mathbb{H}^+)^{(p-1)/p} |\nabla_H v|\} d\xi \\ &\leq \left( \int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{1-1/p} \left( \int_{\mathbb{H}^+} \text{dist}(\xi, \partial\mathbb{H}^+)^{p-1} |\nabla_H v|^p d\xi \right)^{1/p} \\ &\leq C_p^{-1/p} \left( \int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{1-1/p} \\ &\quad \times \left( \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left( \frac{p-1}{p} \right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi \right)^{1/p}. \end{aligned}$$

In the last line we have used Lemma 3.1. Inserting the estimate of  $I_1$  in (3.8), we arrive at

$$\begin{aligned} &\int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi \\ &\leq 2C_p^{-1/p} \left( \int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{1-1/p} \\ &\quad \times \left( \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left( \frac{p-1}{p} \right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi \right)^{\frac{1}{p}}. \end{aligned}$$

Plugging the above estimate in (3.7), we have

$$C_1 \left( \int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{\frac{Q-1}{Q}} \leq 2C_p^{-1/p} \left( \int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{\frac{p-1}{p}} \\ \times \left( \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left( \frac{p-1}{p} \right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi \right)^{\frac{1}{p}}.$$

By collecting terms, we finish the proof of Theorem 3.2.  $\square$

### 3.3. Consequence of Theorem 3.2

Let us demonstrate our result in a particular case when  $p = 2$ .

**Corollary 3.4.** *Let  $\mathbb{H}^+ := \{\xi = (x, y, t) \in \mathbb{H}^n \mid t > 0\}$  be a half-space of the Heisenberg group  $\mathbb{H}^n$ . Then for every function  $u \in C_0^\infty(\mathbb{H}^+)$  taking  $d = 0$ , we have*

$$\left( \int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi \right)^{\frac{1}{2}} \geq C \left( \int_{\mathbb{H}^+} |u|^{2^*} d\xi \right)^{\frac{1}{2^*}}, \quad (3.9)$$

where  $2^* := 2Q/(Q - 2)$ ,  $Q = 2n + 2$ , with  $C > 0$  independent of  $u$ .

**Proof of Corollary 3.4.** We have the following left-invariant vector fields:

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \quad \text{and} \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t},$$

with the commutator

$$[X_i, Y_i] = -4 \frac{\partial}{\partial t}.$$

Then for  $\xi = (x_1, \dots, x_n, y_1, \dots, y_n, t)$  and  $\nu = (\overbrace{0, \dots, 0}^n, \overbrace{0, \dots, 0}^n, 1)$ , we get

$$\langle X_i(\xi), \nu \rangle = 2y_i \quad \text{and} \quad \langle Y_i(\xi), \nu \rangle = -2x_i,$$

where

$$X_i(\xi) = (\overbrace{0, \dots, 0}^i, 1, \dots, 0, \overbrace{0, \dots, 0}^n, 2y_i),$$

$$Y_i(\xi) = (\overbrace{0, \dots, 0}^n, \overbrace{0, \dots, 0}^i, 1, \dots, 0, -2x_i).$$

Thus, we arrive at

$$\frac{\mathcal{W}(\xi)^2}{\text{dist}(\xi, \partial\mathbb{H}^+)^2} = 4 \frac{|x|^2 + |y|^2}{t^2}. \quad (3.10)$$

Plugging the above expression into inequality (3.6), we obtain

$$\left( \int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi \right)^{\frac{1}{2}} \geq C \left( \int_{\mathbb{H}^+} |u|^{2^*} d\xi \right)^{\frac{1}{2^*}},$$

showing (3.9).  $\square$

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## References

- [1] A. Ancona, On strong barriers and an inequality of Hardy for domains  $\mathbb{R}^n$ , *J. London Math. Soc.* **43** (1986) 274–290.
- [2] F. G. Avkhadiiev and A. Laptev, On a sharp Hardy inequality for convex domains, *Around the Research of Vladimir Mazya I*, International Mathematical Series, Vol. 11 (Springer, New York, 2010).
- [3] A. Baldi, B. Franchi and P. Pansu, Gagliardo–Nirenberg inequalities for differential forms in Heisenberg groups, *Math. Ann.* **365** (2016) 1633–1667.
- [4] G. Barbatis, S. Filippas and A. Tertikas, A unified approach to improved  $L^p$  Hardy inequalities with best constants, *Trans. Am. Math. Soc.* **356**(6) (2004) 2169–2196.
- [5] E. B. Davies, A review of Hardy inequalities. The Maz’ya anniversary Collection, *Oper. Theory Adv. Appl.* **2** (1999) 55–67.
- [6] S. Filippas, V. Maz’ya and A. Tertikas, Critical Hardy–Sobolev inequalities, *J. Math. Pures Appl.* **87** (2007) 37–56.
- [7] V. Fischer and M. Ruzhansky, *Quantization on Nilpotent Lie Groups*, Progress in Mathematics, Vol. 314 (Birkhäuser, 2016).
- [8] R. L. Frank and M. Loss, The Hardy–Sobolev–Maz’ya inequalities for arbitrary domains, *J. Math. Pures Appl.* **97**(1) (2012) 39–54.
- [9] N. Garofalo, Geometric second derivative estimates in Carnot groups and convexity, *Manuscripta Math.* **126** (2008) 353–373.
- [10] S. Larson, Geometric Hardy inequalities for the sub-elliptic Laplacian on convex domain in the Heisenberg group, *Bull. Math. Sci.* **6** (2016) 335–352.
- [11] J. Luan and Q. Yang, A Hardy type inequality in the half-space on  $\mathbb{R}^n$  and Heisenberg group, *J. Math. Anal. Appl.* **347**(2) (2008) 645–651.
- [12] B. Opic and A. Kufner, Hardy-type inequalities, *Pitman Res. Notes Math. Ser.* **219** (1990) 333 pp.
- [13] G. Psaradakis, Hardy-Sobolev-Maz’ya and related inequalities in the half space, *J. Elliptic Parabol. Equ.* **3** (2017) 127–134.
- [14] M. Ruzhansky, B. Sabitbek and D. Suragan, Weighted  $L^p$ -Hardy and  $L^p$ -Rellich inequalities with boundary terms on stratified Lie groups, *Rev. Math. Complutense* **32** (2019) 19–35.
- [15] M. Ruzhansky, B. Sabitbek and D. Suragan, Subelliptic geometric Hardy type inequalities on half-spaces and convex domains, *Ann. Funct. Anal.* (2020), <https://doi.org/10.1007/s43034-020-00067-9>.
- [16] M. Ruzhansky, B. Sabitbek and D. Suragan, Hardy and Rellich inequalities for anisotropic  $p$ -sub-Laplacians, *Banach J. Math. Anal.* **14** (2020) 380–398.
- [17] M. Ruzhansky and D. Suragan, On horizontal Hardy, Rellich, Caffarelli-Kohn-Nirenberg and  $p$ -sub-Laplacian inequalities on stratified groups, *J. Differential Equations* **262** (2017) 1799–1821.

- [18] M. Ruzhansky and D. Suragan, Layer potentials, Kac’s problem, and refined Hardy inequality on homogeneous Carnot groups, *Adv. Math.* **308** (2017) 483–528.
- [19] M. Ruzhansky and D. Suragan, Local Hardy and Rellich inequalities for sums of squares, *Adv. Diff. Equations* **22** (2017) 505–540.
- [20] M. Ruzhansky and D. Suragan, *Hardy Inequalities on Homogeneous Groups*, Progress in Mathematics, Vol. 327 (Birkhäuser, 2019), 588 pp.