

Project Title: Finite element solutions of the nonlinear RAMP Black-Scholes model

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Abstract

The main purpose of our Capstone project is to study the Risk-Adjusted Pricing Methodology (RAMP) Black-Scholes model and to find the finite element solutions of the nonlinear Black-Scholes equation. The RAMP is one of the many nonlinear models in option pricing considering factors which affect the volatility in the original Black-Scholes equation. This model can be simplified to a nonlinear parabolic equation in a new variable which equals the product of Gamma and the price of the underlying asset. Galerkin finite element method is applied to the parabolic equation. Two types of solutions will be presented: one using the linear elements and the other using quadratic elements. Local finite element equations for the linear and quadratic elements are derived with some specific interpolations of the nonlinear terms. Numerical solutions are obtained and compared to the results in literature. The explanation of the discrepancies will be given together with the future goals of this study.

1 Introduction

An option is a concept related to the finance and economics. An option is a right, but not an obligation, that allows to buy or sell some asset-usually stocks or bonds, which is subjected to definite conditions [1]. There exist several types of options. The most often used ones are an “American option”, and a “European option”. The main difference between these options is in the time of exercise. So, an “American option” can be exercised at any moment up to the time of expiration. While a “European option” can be used only on a specified date [1]. Other important concepts are the “exercise price” or “striking price” - the amount of money that is paid for the underlying stock when the option is exercised, and “expiration date” or the “maturity date” - the last day when

the option can be exercised [1]. As it can be seen, the concept of “option” is important because it gives an opportunity to secure the right of buying or selling a certain asset for predetermined price. So, it can save its holder from losing money by overpaying for a good, and it can also allow to buy a stock for a lower price. However, what is the right way of evaluating the price of an option itself? The answer to this question was found by Fischer Black and Myron Scholes in 1973-the creators of the following partial differential equation.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad (1)$$

where V is the price of the option, S the price of the underlying asset, r the interest rate, t the time, σ the volatility of the stock price. This is a linear parabolic partial differential equation which has been studied extensively [3].

1.1 Boundary and terminal-time conditions of the linear Black-Scholes equation

The classical Black-Scholes equation (1) can be classified as a linear parabolic equation by the standard theory of partial differential equations, see, e.g., the lecture notes by Chan [3]. A parabolic equations also can be classified as a *forward* or a *backward* parabolic equation. If in an parabolic equation the signs of the terms involving derivatives are the same, when they are located on the same side of an equation, then it is *backward parabolic*, otherwise it is *forward parabolic* [3]. Thus, the Black-Scholes equation is *backward parabolic*. Generally, a partial differential equation has infinitely many solutions. Thus, in order to obtain the unique option price some boundary conditions are to be imposed Now we state the standard the boundary conditions, see, e.g., citechan. Given the fact that we have $\partial^2 V / \partial S^2$ term, we should impose two conditions on V relative to S , and one condition on V relative to t , since we have $\partial V / \partial t$. The Black-Scholes equation is backward, so we also need a *final* condition on V [3].

For simplicity, we consider the European call option. Let the option price be denoted by $V(S, t)$ with the exercise price E , stock price S and expiration date T , so we have the following conditions:

- *Final* condition - payoff at the maturity date T :

$$V(S, T) = \max(S - E, 0), S \geq 0. \quad (2)$$

- Boundary conditions

1. When the price of the stock is zero, the payoff will be equal to zero. Therefore, the option is worthless:

$$V(0, t) = 0, t \geq 0. \quad (3)$$

2. When the price of the stock tends to infinity, $S \rightarrow \infty$, then the chances that the option will be exercised increase and the value of the exercise price loses its importance. So, as $S \rightarrow \infty$, the price of the option equals to the price of the stock minus the exercise price that we have to pay to exchange for the stock. So, for all $t > 0$,

$$V(S, T) \sim S - Ee^{-r(T-t)}, S \rightarrow \infty. \quad (4)$$

1.2 Assumptions of the linear Black-Scholes equation

In the derivation of equation (1), Black and Scholes assumed the "ideal conditions" in the market for the stock and for the option [1]:

- The short-term interest rate r is known and is constant through time.
- The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is "lognormal". The variance rate of the return on the stock is constant.
- The stock pays no dividends or other distributions.
- The option is European, that is, it can only be exercised at the time of maturity.
- There are no transaction costs in buying or selling the stock or the option.
- It is possible to borrow any fraction of the price of a security to buy it or to hold it at the short-term interest rate.
- There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date [1].

However, such "ideal conditions" never take place in reality. Due to transaction costs, large investor preferences, and incomplete markets, the option price is "unrealistic" for the classical equation (1) to model in such conditions. Research based on more realistic assumptions results in many strongly or fully nonlinear, possibly degenerate, parabolic diffusion-convection equations in option price, where both the volatility σ and the drift μ can depend on the time t , the stock price S or the derivatives of the option price V itself [4]. That is why the nonlinear models of the Black-Scholes equation become increasingly popular among financial analysts and academicians for option price modeling. Since the nonlinear Black-Scholes equations are based on more realistic assumptions, such as transaction costs, risks from an unprotected portfolio, large investor's preferences or illiquid markets, the volatility, the drift and the option price [4], they can provide more accurate results.

2 Nonlinear Black-Scholes models

There exist numerous nonlinear Black-Scholes models. In this work, we would like to pay a particular attention to the model concerned with the volatility which is affected by the transaction costs of the underlying asset.

Since the main goal of the Black-Scholes model is to “hedge the position without any risk”, continuous portfolio adjustments are necessary [4]. However, when the transaction costs are introduced, such adjustments of the portfolio become expensive, given the fact that infinite number of transactions is required. So, the aim is obtain the “balance between the transaction costs, which are required to rebalance the portfolio and the implied costs of hedging errors” [4]. Thus, the value of the option can be over- or under-estimated. In order to find the remedy for this case, alternative relaxation strategies were explored and several models were derived.

2.1 Leland’s model

Leland in his model proposes to relax the hedging conditions by trading at discrete times in order to reduce the cost of the portfolio adjustment [4]. In the Leland’s model the transaction cost, $\frac{\kappa}{2}|\Delta|S$, is taken to be proportional to the “monetary value of the assets bought or sold” [4]. Here κ stands for the round trip transaction cost per unit dollar of the transaction and Δ denotes the number of assets bought ($\Delta > 0$) or sold ($\Delta < 0$) at price S . So, the modified volatility will look like

$$\tilde{\sigma}^2 = \sigma^2(1 + Le \operatorname{sgn}(V_{SS}))$$

, where σ denotes the original volatility and Le is the *Leland number*:

$$Le = \sqrt{\frac{2}{\pi}} \frac{\kappa}{\sigma \sqrt{\delta t}}$$

, where δt represents the transaction frequency.

2.2 Barles’ and Soner’s model

The model proposed by Barles and Soner was derived on the basis of the “utility function approach of Hodges and Neuberger, that was further developed by Davis et al” [4]. The volatility imposed in their model is the following:

$$\tilde{\sigma}^2 = \sigma^2(1 + \Psi(e^{r(T-t)} a^2 S^2 V_{SS}))$$

, where $a = \kappa/\sqrt{\epsilon}$ and $\Psi(x)$ represents the solution of the following equation:

$$\Psi'(x) = \frac{\Psi(x) + 1}{2\sqrt{x\Psi(x)} - x}, x \neq 0$$

. After some manipulations, the final form of the volatility is:

$$\tilde{\sigma}^2 = \sigma^2(1 + e^{r(T-t)} a^2 S^2 V_{SS}).$$

2.3 The risk adjusted pricing methodology

This model was derived by M. Kratka and further modified by Jandačka and Ševčovič [4]. Here the minimization of the rate of transaction costs is made by obtaining the optimal time-lag δt between the transactions. The volatility then becomes:

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + 3 \left(\frac{C^2 M}{2\pi} S V_{SS} \right)^{\frac{1}{3}} \right),$$

where $M \geq 0$ is the transaction cost measure and $C \geq 0$ is the risk premium measure.

For completeness, we would like to introduce the *Risk adjusted pricing methodology* (RAPM) developed, see [2], in more detail.

3 The RAPM model

3.1 Introduction

As it was mentioned earlier, the solution of the classical Black-Scholes equation (1) gives the price of the option in the situations where the transaction costs and the risk from a volatile portfolio are negligible. Thus, once these concepts are introduced, the solution becomes worthless. In order to evaluate the option price correctly, some modifications of the classical Black-Scholes equation have to be made. So as to preserve the delta hedge, “one has to make frequent portfolio adjustments yielding thus a substantial increase in transaction costs. On the other hand, rare portfolio adjustments leads to the increase of the risk from a volatile (unprotected) portfolio” [2].

During the evaluation of the option price, one important question arises how to incorporate both transaction costs and the risk from a volatile portfolio into the option price equation. An answer to this question was found by Jandačka and Ševčovič. They constructed the RAPM model on the basis of the Black-Scholes equation, where the transaction costs are derived using the approach of T. Hoggard, A. E. Whalley, and P. Wilmott [2]. As far as the risk from a volatile portfolio is concerned, it was defined by the variance of the synthesized portfolio. Both of these concepts (transaction cost, risk of the volatile portfolio) depend on the time lag between two consecutive transactions. In order to obtain the optimal length of the hedge time interval they propose to minimize their sum [2].

3.2 Derivation of the RAPM model

In the paper “On the risk-adjusted pricing-methodology-based valuation of vanilla options and explanation of the volatility smile” Jandačka and Ševčovič make the following assumptions:

- The price of the asset $S = S(t)$, $t \geq 0$, follows a geometric Brownian motion with a drift ρ and standard deviation $\sigma > 0$, paying no dividends, that is:

$$dS = \rho S dt + \sigma S dW,$$

where dW defines the differential of the standard Wiener process. The main drawback of this assumption is that the volatility is treated as a constant.

- Construct a synthesized portfolio Π that contains one option with a price V and δ assets with a price S per one asset:

$$\Pi = V + \delta S. \tag{5}$$

According to the classical theory of the Black-Scholes model, the next step is to examine the differential of (8). While the right-hand side of

the derivative of (8) can be differentiated using Itô's formula, while the expression $\Delta\Pi(t) = \Pi(t + \Delta t) - \Pi(t)$ on the left-hand side is expressible as:

$$\Delta\Pi(t) = r\Pi\Delta t,$$

where $r > 0$ is a risk-free interest rate of a zero-coupon bond. However, in reality such assumption is not acceptable, so a new term measuring the total risk is required. Thus, the change of a portfolio Π consists of two parts:

1. The contribution from the risk-free interest rate: $r\Pi\Delta t$
2. The contribution from the total risk premium: $r_R S \Delta t$. Here r_R is a risk premium per unit asset price. Note that r_R is composed of transaction risk premium r_{TC} and the portfolio volatility risk premium r_{VP} : $r_R = r_{TC} + r_{VP}$.

Therefore, the change of portfolio can be expressed as:

$$\Delta\Pi(t) = r\Pi\Delta t + (r_{TC} + r_{VP})S\Delta t. \quad (6)$$

Let us now observe how $r_{TC} + r_{VP}$ are related to other quantities, such as σ , S , V , and the derivatives of V .

3.2.1 Modeling transaction costs

In order to obtain the value of the transaction costs, Jandačka and Ševčovič derive the coefficient of transaction costs r_{TC} from (9).

Let C represent the round trip transaction cost per unit dollar of transaction. Thus,

$$C = \frac{S_{ask} - S_{bid}}{S},$$

where S_{ask} and S_{bid} are the Ask and Bid prices of the stock. So, "the market price offers for selling and buying assets." Let $S = (S_{ask} + S_{bid})/2$ be the mid-value of the stock price.

The transaction cost is equal to $C|k|S/2$, where k denotes the number of the sold assets ($k < 0$) or bought assets ($k > 0$). Therefore, after one-time step Δt , the value of the portfolio $\Pi = V + \delta S$ changes to $\Delta\Pi = \Delta V + \delta\Delta S - C|k|S/2$. It can be seen that the value of k (number of sold or bought assets) depends on the one-time step change of δ , thus, $k = \Delta\delta$. So, the portfolio is $\Delta\Pi = \Delta V + \delta\Delta S - C|\Delta\delta|S/2$. By assuming that the portfolio adjustments happen according to the δ -hedging strategy, $\delta = -\partial_S V$, we arrive to the following value of the portfolio:

$$\Delta\Pi = \Delta V + \delta\Delta S - r_{TC}S\Delta t,$$

where the r_{TC} is equal to:

$$r_{TC} = \frac{C\sigma S}{\sqrt{2\pi}} |\partial_S^2 V| \frac{1}{\sqrt{\Delta t}} \quad (7)$$

With the increase in the time-lag Δt between portfolio adjustments, the transaction costs decrease. Thus, in order to minimize transaction costs, we have to take a larger time lag Δt . However, by choosing a larger time lag, one may end up with a higher risk from an unprotected portfolio [2].

3.2.2 Modeling risk from a volatile portfolio

Volatility of a fluctuating portfolio can be measured by the variance of relative increments of the replicating portfolio $\bar{\Pi} = V + \delta S$, that is, by the term $\text{var}((\Delta\bar{\Pi})/S)$ [2]. So,

$$r_{VP} = R \frac{\text{var}(\Delta\bar{\Pi}/S)}{\Delta t}$$

, where R is *risk premium coefficient* that denotes the marginal value of investor's exposure to a risk.

By Itô's formula to the differential, we have $\Delta\bar{\Pi} = \Delta V + \delta\Delta S$. From this we attain the following equation:

$$\Delta\bar{\Pi} = (\partial_S v + \delta)\sigma S\Delta W + \frac{1}{2}\sigma^2 S^2 \Gamma(\Delta W)^2 + \mathcal{G},$$

where $\Gamma = \partial_S^2 V$ and $\mathcal{G} = (\partial_S V + \delta)\rho S\Delta t + \partial_t V\Delta t$ is a deterministic term, that is, $(\mathcal{G}) = \mathcal{G}$ in the lowest order Δt -term approximation [2]. Thus,

$$\Delta\bar{\Pi} - E(\Delta\bar{\Pi}) = (\partial_S V + \delta)\sigma S\phi\sqrt{\Delta t} + \frac{1}{2}\sigma^2 S^2(\phi^2 - 1)\Gamma\Delta t,$$

where ϕ is a random variable with the standard normal distribution such that $\Delta W = \phi\sqrt{\Delta t}$ [2]. Therefore, the variance of $\Delta\bar{\Pi}$ can be calculated through the formula:

$$\text{var}(\Delta\bar{\Pi}) = E [(\Delta\bar{\Pi} - E(\Delta\bar{\Pi}))^2] = E \left[\left((\partial_S V + \delta)\sigma S\phi\sqrt{\Delta t} + \frac{1}{2}\sigma^2 S^2 \Gamma(\phi^2 - 1)\Delta t \right)^2 \right].$$

Following the δ -hedging of portfolio adjustments assumption, we take $\delta = -\partial_S V$. Since $E((\phi^2 - 1)^2) = 2$, the formula for the risk premium r_{VP} is:

$$r_{VP} = \frac{1}{2}R\sigma^4 S^2 \Gamma^2 \Delta t. \quad (8)$$

Since the value of r_{VP} is proportional to the time lag Δt , the larger time interval leads to the higher risk exposure for an investor [2].

3.2.3 Gamma hedging strategy based on the RAPM model

As it was mentioned before, the total risk premium $r_R = r_{TC} + r_{VP}$ is composed of two parts: transaction costs premium, r_{TC} , and the risk from a volatile portfolio, r_{VP} (formulas (10) and (11), respectively). Assuming that the investor is risk averse, the next step will be to minimize the value of the risk premium r_R . In order to do this, the optimal time-lag Δt between two subsequent portfolio adjustments has to be found. So, the following function should be minimized:

$$\Delta t \rightarrow r_R = r_{TC} + r_{VP} = \frac{C|\Gamma|\sigma S}{\sqrt{2\pi}} + \frac{1}{2}R\sigma^4 S^2 \Gamma^2 \Delta t.$$

According to Jandačka and Ševčovič, the minimum is obtained at

$$\Delta t_{opt} = \frac{K^2}{\sigma^2 |S\Gamma|^{2/3}}, \text{ where } K = \left(\frac{C}{R} \frac{1}{\sqrt{2\pi}} \right)^{1/3}. \quad (9)$$

Thus,

$$r_R(\Delta t_{opt}) = \frac{3}{2} \left(\frac{C^2 R}{2\pi} \right)^{1/3} \sigma^2 |S\Gamma|^{4/3},$$

and we obtain the transaction cost risk adjusted volatility which gives us a nonlinear Black-Scholes equation.

3.2.4 The risk-adjusted Black-Scholes equation

Because of the risk adjusted volatility discussed in the previous sections, we now have the following nonlinear Black-Scholes equation

$$\partial_t V + \frac{\sigma^2}{2} S^2 (1 - \mu(S\Gamma)^{1/3}) \Gamma = r(V - S\partial_S V), \quad (10)$$

where

$$\Gamma = \partial_S^2 V, \mu = 3 \left(\frac{C^2 R}{2\pi} \right)^{1/3}$$

It should be noted that (10) is a backward parabolic equation if and only if

$$\beta(H) = \frac{\sigma^2}{2} (1 - \mu H^{1/3}) H, \quad (11)$$

is an increasing function in the variable $H := S\Gamma = S\partial_S^2 V$ (by maximum principle argument). Hence, in order to satisfy the parabolicity condition (11), we assume the following condition for our subsequent analysis and numerical simulations:

$$S\Gamma < \left(\frac{3}{4\mu} \right)^3.$$

3.3 Behavior near the exercise time

Let us observe the change in the price of the option $V = V(S, t)$ near the exercise date T , that is, when $T - t$ is small. Jandačka and Ševčovič assume that the “time-lag Δt between consecutive portfolio adjustments is small compared to $T - t$ ”. They propose the condition on $\Delta t_{opt} \ll T - t$ to prohibit any portfolio modifications when the time t is near the exercise time T . So, the time interval $(0, T)$ now gets divided into two parts:

- $(0, t_*)$ when the option price is determined by the risk-adjusted Black-Scholes equation
- (t_*, T) when portfolio adjustments are disallowed and the option price is determined by the linear Black-Scholes equation.

where t_* is the switching time. Now the next question is: “How the switching time is determined?” In order to obtain the value of t_* , the last portfolio adjustment moment $0 < t_* < T$ before the maturity date T . This task is accomplished by assuming that the hedging process acts in accordance with the optimal time lag Δt_{opt} that was acquired from (12). “More precisely, the switching time t_* can be determined from the implicit equation:

$$T - t_* = \min \Delta t_{opt}(S, t_*), S > 0$$

which, after some modifications become:

$$T - t_* = \frac{C}{R\sigma^2}$$

Since t_* is the variable of time, it should be positive. In order to validate the sign of t_* we need to impose additional condition [2]:

$$C < \sigma^2 RT.$$

3.4 Solution of the risk-adjusted Black-Scholes equation

The solution of the risk-adjusted Black-Scholes equation is a continuous function $V = V(S, T), S \in (0, \infty), t \in [0, T]$, that fulfills the boundary and final conditions such that [2]:

- $V(S, t)$ is a classical solution to the Black-Scholes equation

$$\partial_t V + \frac{\sigma^2}{2} S^2 \Gamma = r(V - S \partial_S V), S > 0$$

on the time interval (t_*, T) , that in its turn satisfies the final condition at $t = T$:

$$V(0, t) = 0, \frac{V(S, t)}{S} \rightarrow 1,$$

as

$$S \rightarrow \infty, V(S, T) = \max(S - E, 0).$$

- $V(S, t)$ is a smooth solution to the equation

$$\partial_t V + \frac{\sigma^2}{2} S^2 (1 - \mu(S\Gamma)^{1/3} \Gamma) = r(V - S\partial_S V),$$

on the time interval $t \in (0, t_*)$ that satisfies $V(S, t_*) = V_*(S)$, where $t_* = T - \frac{C}{R\sigma^2}$ is the switching time and $V_*(S) = \lim_{t \rightarrow t_*^+} V(S, t)$.

$$V(0, t) = 0, \frac{V(S, t)}{S} \rightarrow 1 \text{ as } S \rightarrow \infty, V_*(S) = \lim_{t \rightarrow t_*^+} V(S, t).$$

From (13) following quasilinear parabolic equation can be obtained:

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + r\partial_x H, \quad (12)$$

where $H = H(x, \tau), \tau \in (\tau_*, T), x \in \mathbb{R}$

- Initial condition:

$$H(x, \tau_*) = \frac{N'(d)}{\sigma\sqrt{T_*}}, \tau_* = T - t_*.$$

- Boundary conditions:

$$H(-\infty, \tau) = H(\tau, \infty) = 0, \tau \in (0, T).$$

Out of computational considerations, the ∞ condition was changed to $x = \pm L$. So we have the update boundary condition:

$$H(-1.5, \tau) = H(1.5, \tau) = 0, \tau \in (\tau_*, T).$$

4 Finite element solutions

We want to solve the (15) using the basic procedures of Galerkin Finite Elements Method (FEM) [5]. Initially, two linear elements is used, then one quadratic, and finally, three linear elements are used to examine numerical convergence.

4.1 Weak formulation

Using the equation (15) we can implement the numerical solution technique. So, we multiply the (15) by the shape function $\phi(x)$ and take the integral from $-L$ to L .

$$\int_{-L}^L H_\tau \phi(x) dx = - \int_{-L}^L \beta_x(H) dx + \beta_x(H) \phi(x) \Big|_{-L}^{+L} + \int_{-L}^L \beta_x(H) \phi(x) dx + r \int_{-L}^L H_x \phi(x) dx$$

$$H(x, \tau) = \sum_{i=1}^{NG} h_i(\tau) \phi_i(x),$$

where NG is the number of Global nodes.

4.2 The two linear element solution

Local finite linear element system is:

$$\begin{aligned}
& \frac{l^{(e)}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{H}_1^{(e)} \\ \dot{H}_2^{(e)} \end{bmatrix} = \\
& -\frac{\sigma^2}{2l^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} H_1^{(e)} \\ H_2^{(e)} \end{bmatrix} + \frac{\sigma^2 \mu}{2l^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (H_1^{(e)})^{\frac{4}{3}} \\ (H_2^{(e)})^{\frac{4}{3}} \end{bmatrix} + \frac{1}{2} \left(\frac{\sigma^2}{2} + r \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} H_1^{(e)} \\ H_2^{(e)} \end{bmatrix} \\
& -\frac{3\mu\sigma^2}{4} \begin{bmatrix} -\frac{H_2^{(e)}}{H_1^{(e)}-H_2^{(e)}} & \frac{H_2^{(e)}}{H_1^{(e)}-H_2^{(e)}} \\ \frac{H_1^{(e)}}{H_1^{(e)}-H_2^{(e)}} & -\frac{H_1^{(e)}}{H_1^{(e)}-H_2^{(e)}} \end{bmatrix} \begin{bmatrix} (H_1^{(e)})^{\frac{4}{3}} \\ (H_2^{(e)})^{\frac{4}{3}} \end{bmatrix} - \frac{3\mu\sigma^2\mu}{14} \begin{bmatrix} \frac{1}{H_1^{(e)}-H_2^{(e)}} & \frac{1}{H_1^{(e)}-H_2^{(e)}} \\ \frac{1}{H_1^{(e)}-H_2^{(e)}} & -\frac{1}{H_1^{(e)}-H_2^{(e)}} \end{bmatrix} \begin{bmatrix} (H_1^{(e)})^{\frac{7}{3}} \\ (H_2^{(e)})^{\frac{7}{3}} \end{bmatrix} \\
& + \begin{bmatrix} -\beta_x(H) \text{ at } x_1^{(e)} \\ \beta_x(H) \text{ at } x_2^{(e)} \end{bmatrix}
\end{aligned}$$

By assembling two local systems of linear elements we obtain the global system:

$$\begin{aligned}
& \frac{l^{(e)}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{H}_1^{(e)} \\ \dot{H}_2^{(e)} \\ \dot{H}_3^{(e)} \end{bmatrix} = \\
& -\frac{\sigma^2}{2l^{(e)}} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} H_1^{(e)} \\ H_2^{(e)} \\ H_3^{(e)} \end{bmatrix} + \frac{\sigma^2 \mu}{3l^{(e)}} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} (H_1^{(e)})^{\frac{4}{3}} \\ (H_2^{(e)})^{\frac{4}{3}} \\ (H_3^{(e)})^{\frac{4}{3}} \end{bmatrix} \\
& + \frac{1}{2} \left(\frac{\sigma^2}{2} + r \right) \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} H_1^{(e)} \\ H_2^{(e)} \\ H_3^{(e)} \end{bmatrix} \\
& + \frac{3\sigma^2 \mu}{8} \begin{bmatrix} -\frac{H_2^{(e)}}{H_1^{(e)}-H_2^{(e)}} & \frac{H_2^{(e)}}{H_1^{(e)}-H_2^{(e)}} & 0 \\ \frac{H_1^{(e)}}{H_1^{(e)}-H_2^{(e)}} & -\frac{H_1^{(e)}}{H_1^{(e)}-H_2^{(e)}} - \frac{H_3^{(e)}}{H_2^{(e)}-H_3^{(e)}} & \frac{H_3^{(e)}}{H_2^{(e)}-H_3^{(e)}} \\ 0 & \frac{H_2^{(e)}}{H_2^{(e)}-H_3^{(e)}} & -\frac{H_2^{(e)}}{H_2^{(e)}-H_3^{(e)}} \end{bmatrix} \begin{bmatrix} (H_1^{(e)})^{\frac{4}{3}} \\ (H_2^{(e)})^{\frac{4}{3}} \\ (H_3^{(e)})^{\frac{4}{3}} \end{bmatrix} \\
& -\frac{3\sigma^2 \mu}{14} \begin{bmatrix} \frac{1}{H_1^{(e)}-H_2^{(e)}} & \frac{1}{H_1^{(e)}-H_2^{(e)}} & 0 \\ \frac{1}{H_1^{(e)}-H_2^{(e)}} & -\frac{1}{H_1^{(e)}-H_2^{(e)}} - \frac{1}{H_2^{(e)}-H_3^{(e)}} & \frac{1}{H_2^{(e)}-H_3^{(e)}} \\ 0 & \frac{1}{H_2^{(e)}-H_3^{(e)}} & -\frac{1}{H_2^{(e)}-H_3^{(e)}} \end{bmatrix} \begin{bmatrix} (H_1^{(e)})^{\frac{7}{3}} \\ (H_2^{(e)})^{\frac{7}{3}} \\ (H_3^{(e)})^{\frac{7}{3}} \end{bmatrix} \begin{bmatrix} -\beta_x(H)|_{x=-L} \\ 0 \\ \beta_x(H)|_{x=L} \end{bmatrix}
\end{aligned}$$

Applying the boundary conditions we obtain the following Global system:

$$\frac{l^{(e)}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{H}_2^{(e)} \\ 0 \end{bmatrix} =$$

$$\begin{aligned}
& -\frac{\sigma^2}{2l^{(e)}} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ H_2^{(e)} \\ 0 \end{bmatrix} + \frac{\sigma^2 \mu}{3l^{(e)}} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ (H_2^{(e)})^{\frac{4}{3}} \\ 0 \end{bmatrix} \\
& + \frac{1}{2} \left(\frac{\sigma^2}{2} + r \right) \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ H_2^{(e)} \\ 0 \end{bmatrix} \\
& + \frac{3\sigma^2 \mu}{8} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ (H_2^{(e)})^{\frac{4}{3}} \\ 0 \end{bmatrix} \\
& - \frac{3\sigma^2 \mu}{14} \begin{bmatrix} \frac{1}{H_2^{(e)}} & -\frac{1}{H_2^{(e)}} & 0 \\ -\frac{1}{H_2^{(e)}} & 0 & \frac{1}{H_2^{(e)}} \\ 0 & \frac{1}{H_2^{(e)}} & -\frac{1}{H_2^{(e)}} \end{bmatrix} \begin{bmatrix} 0 \\ (H_2^{(e)})^{\frac{7}{3}} \\ 0 \end{bmatrix} + \begin{bmatrix} -\beta_x(H)|_{x=-L} \\ 0 \\ \beta_x(H)|_{x=L} \end{bmatrix}
\end{aligned}$$

From the second row we get the equation:

$$\frac{2}{3}l^{(e)}\dot{H}_2 = -\frac{\sigma^2}{2l^{(e)}}H_2 + \frac{2\sigma^2\mu}{3l^{(e)}}H_2^{\frac{4}{3}}$$

where $H_2 = H_2(\tau)$

Using $l^{(1)} = l^{(1)} = 1.5$, our H function at the fixed time τ_1 will look like:

$$\dot{H}_2 = -\frac{2\sigma^2}{3}H_2 + \frac{4\sigma^2\mu}{9}H_2^{\frac{4}{3}} - \frac{6\sigma^2\mu}{7}H_2^{\frac{1}{3}}$$

With the help of the MATLAB ode45, we solve the above nonlinear ODE for the nodal solution $H_2(t)$ and obtain the finite element solution:

$$H(x, t) = \chi^{(1)} \frac{(2x+3)}{3} H_2(t) + \chi^{(2)} \frac{(3-2x)}{3} H_2(t)$$

4.3 The quadratic element solution

Local system for the quadratic element is :

$$\begin{aligned}
& \frac{l^{(e)}}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} \dot{H}_1^{(e)} \\ \dot{H}_2^{(e)} \\ \dot{H}_3^{(e)} \end{bmatrix} = \\
& -\frac{\sigma^2}{6l^{(e)}} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 8 & -8 \\ -1 & -8 & 7 \end{bmatrix} \begin{bmatrix} H_1^{(e)} \\ H_2^{(e)} \\ H_3^{(e)} \end{bmatrix} + \frac{2\sigma^2\mu}{9l^{(e)}} \left(|H_1|^{\frac{1}{3}} + |H_2|^{\frac{1}{3}} + |H_3|^{\frac{1}{3}} \right) \begin{bmatrix} 7 & -8 & 1 \\ -8 & 8 & -8 \\ -1 & -8 & 7 \end{bmatrix} \begin{bmatrix} H_1^{(e)} \\ H_2^{(e)} \\ H_3^{(e)} \end{bmatrix} \\
& + \frac{\sigma^2}{6l^{(e)}} \begin{bmatrix} 3 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} H_1^{(e)} \\ H_2^{(e)} \\ H_3^{(e)} \end{bmatrix} - \frac{2\sigma^2\mu}{9l^{(e)}} \left(|H_1|^{\frac{1}{3}} + |H_2|^{\frac{1}{3}} + |H_3|^{\frac{1}{3}} \right) \begin{bmatrix} 3 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} H_1^{(e)} \\ H_2^{(e)} \\ H_3^{(e)} \end{bmatrix}
\end{aligned}$$

$$+\frac{r}{3l^{(e)}} \begin{bmatrix} 3 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 3 \end{bmatrix} + \begin{bmatrix} -\beta_x(H)|_{x=-L} \\ 0 \\ \beta_x(H)|_{x=L} \end{bmatrix}$$

Applying the boundary conditions, we obtain the following Global system:

$$\begin{aligned} & \frac{l^{(e)}}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{H}_2^{(e)} \\ 0 \end{bmatrix} = \\ & -\frac{\sigma^2}{6l^{(e)}} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 8 & -8 \\ -1 & -8 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ H_2^{(e)} \\ 0 \end{bmatrix} + \frac{2\sigma^2\mu}{9l^{(e)}} |H_2|^{\frac{1}{3}} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 8 & -8 \\ -1 & -8 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ H_2^{(e)} \\ 0 \end{bmatrix} \\ & +\frac{\sigma^2}{6l^{(e)}} \begin{bmatrix} 3 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ H_2^{(e)} \\ 0 \end{bmatrix} - \frac{2\sigma^2\mu}{9l^{(e)}} |H_2|^{\frac{1}{3}} \begin{bmatrix} 3 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ H_2^{(e)} \\ 0 \end{bmatrix} \\ & +\frac{r}{3l^{(e)}} \begin{bmatrix} 3 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 3 \end{bmatrix} + \begin{bmatrix} -\beta_x(H)|_{x=-L} \\ 0 \\ \beta_x(H)|_{x=L} \end{bmatrix} \end{aligned}$$

From second row we get the following ODE in H_2 :

$$\dot{H}_2 = -\frac{5}{18}\sigma^2 H_2 + \frac{10}{27}\sigma^2\mu |H_2|^{\frac{1}{3}} H_2$$

and the finite element solution:

$$H(x, t) = \frac{2.25 - x^2}{2.25} H_2(t), -L < x < L$$

4.4 Three linear element solution

Following the same procedures conducted before, we obtain the interpolation function in terms of H function:

$$\chi^{(1)}(x + 1.5)H_2 + \chi^{(2)}(H_2 + H_3) + \chi^{(3)}(1.5 - x)H_3$$

4.5 Evaluation of the option price

After obtaining the solution in terms of $H(t)$ at the fixed time τ_1 , we can find the option price, V , using the formula below [2], as

$$\begin{aligned} V(S, T - \tau) &= e^{-r(\tau - \tau_*)} V(S e^{r(\tau - \tau_*)}, T - \tau_*) \\ &+ S \int_{\tau_*}^{\tau} \beta \left(H \left(\ln \left(\frac{S}{E} \right) + r(\tau - \theta), \theta \right) \right) d\theta \end{aligned} \quad (13)$$

for any $S > 0$ and $\tau \in (\tau_*, T)$.

In order to find $V(Se^{r(\tau-\tau_*)}, T-\tau_*)$ we obtain the value of $\lim_{\tau \rightarrow \tau_*^+} V(Se^{r(\tau-\tau_*)}, T-\tau)$, where $V(Se^{r(\tau-\tau_*)}, T-\tau)$ is calculated from an explicit classical Black-Scholes formula for Call options [1].

$$V_{ec}(S, T-\tau) = SN(d_1) - Ee^{-r\tau}N(d_2),$$

where

$$d_1 = \frac{\left(\ln\left(\frac{S}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right)}{\sigma\sqrt{\tau}}, d_2 = d_1 - \sigma\sqrt{\tau}$$

Integral in (16) can be calculated using trapezoidal rule for numerical integration:

$$V(S, T-\tau) = e^{-r(\tau-\tau_*)}V(Se^{r(\tau-\tau_*)}, T-\tau_*) + Sk \sum_{i=p+1}^j \beta\left(H\left(\ln\left(\frac{S}{E}\right) + r\tau - rik, ik\right)\right)$$

for $j = p+1, \dots, m$, $\tau_* \approx pk$, where k is the time step.

4.6 Comparison of the results

Results		
Finite element method	Two linear element	0.8169
	One quadratic element	0.8225
	Three linear elements	1.2605
Finite difference method	from the results of M. Jandačka and D. Ševčovič	1.5000
	Solution of the linear problem	0.7580

5 Conclusion

We can see that with just three linear finite elements, the numerical solution obtained is already comparable with results were obtained by Jandačka and Ševčovič in 2005 who used finite difference in combinations with the Newton's method. Thus we can conjecture that by proceeding implementation of FEM with more elements we can get results of more precision. Our next natural steps are to develop a general finite element code for the problem and to develop a theory of existence and uniqueness of solutions to the problem, to the semi- and fully discrete problems with convergence and stability and error analysis.

6 Graphs

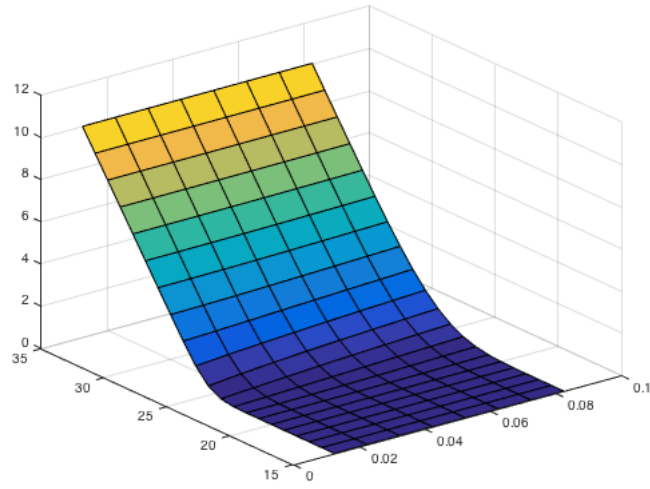


Figure 1: Graph of the classical solution.

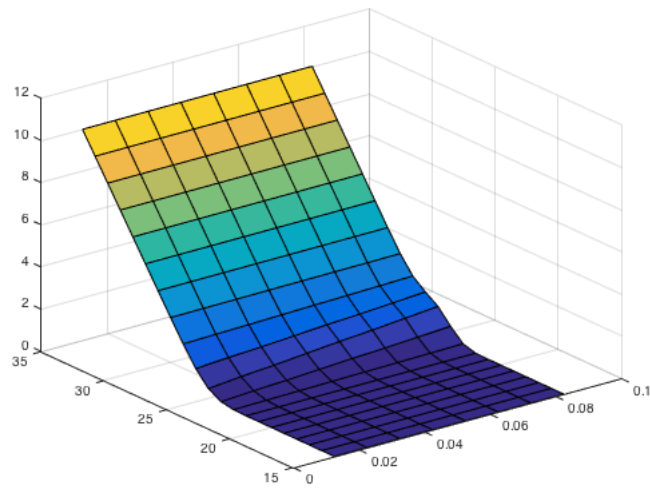


Figure 2: Graph of the two linear elements solution.

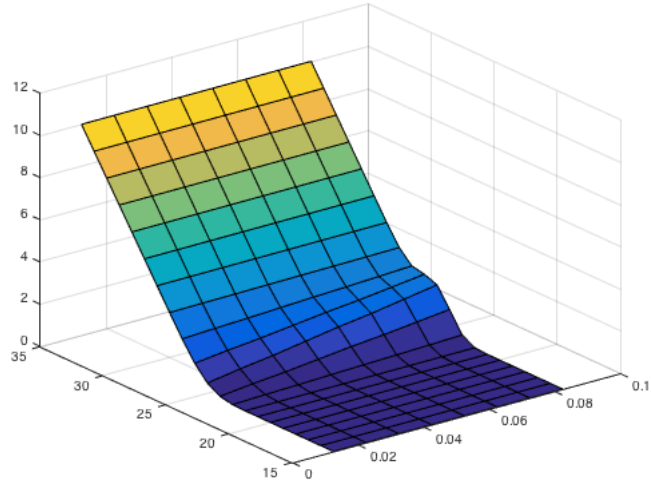


Figure 3: Graph of the three linear elements solution.

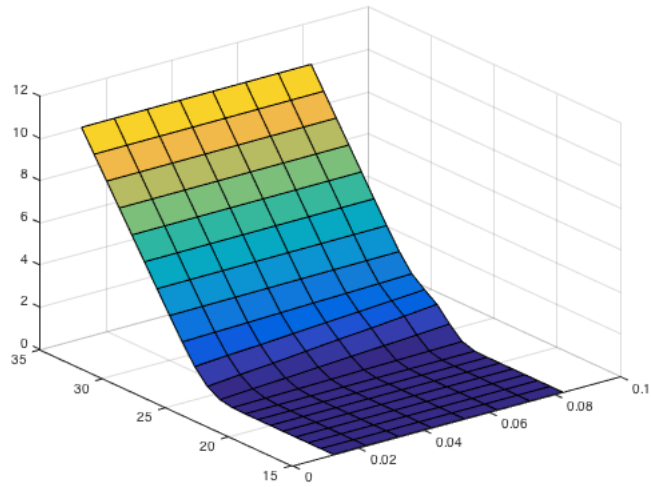


Figure 4: Graph of the one quadratic element solution.

7 MATLAB code

```

clc;
r = 0.02; %risk free interest rate;
sig = 0.4; %sigma=volatility
E = 25; %Strike price
T = 21/252 %Maturity
S=24.6; %Price of asset

C=0.01; %Transaction costs

R =5; %Risk premium coefficient
tau=0.0556;
if ((C*R)<(pi/8)) && ((C/R)<(sig^2*T)) %condition for solution existence
    disp('satisfied existence sondition');
end
m = 3*(C^2*R/(2*pi))^(1/3); %mu
t = C/(R*sig^2) %switching time
d = (log(S/E)+(r + (sig^2)/2)*t)/(sig*t^(1/2));
h = 1/((2*pi*t)^(1/2)*sig*exp((d^2)/2)); %initial condition for Gamma-equation
d1= (log(S*exp(r*(tau-t))/E)+(r + (sig^2)/2)*t)/(sig*t^(1/2)); %d1=argument for Normal distribution
d2=d1-sig*sqrt(t);

D1=(log(S/E)+(r + (sig^2)/2)*tau)/(sig*tau^(1/2));
D2=D1-sig*sqrt(tau);
V=S*normcdf(D1)-E*exp(-r*tau)*normcdf(D2) %linear solution

f1 = @(e,u)[-2/3*sig^2*u(1)+4/9*m*sig^2*u(1)^(4/3)]; %equation for 2 linear elements
[Time,H] = ode45(f1,[t T], h);
[p p]=min(abs(Time-tau));
timestep=Time(2)-Time(1);
x=log(S/E)+r*tau;
%plot(Time, H);
sum=0;
for i=1:p
    if (x-r*Time(i))<0
        beta=1/3*(2*(x-r*Time(i))+3)*H(i);
    end
    if (x-r*Time(i))>0
        beta=1/3*(3-2*(x-r*Time(i)))*H(i);
    end
    sum=sum+sig^2/2*(1-m*(abs(beta))^(1/3))*beta;
end
%option price, 2 linear elements
sum;
V1=exp(-r*(tau-t))*(S*exp(r*(tau-t))*normcdf(d1)-E*exp(-r*t)*normcdf(d2))+sum*S*timestep
sum =0;

%solve the general equation for quadratic element

f2 = @(w,u)[-5/18*sig^2*u(1)+10/27*m*sig^2*(abs(u(1)))^(1/3)*u(1)];
[Time,A] = ode45(f2,[t T], h);
for i=1:p
    beta=(2.25-(x-r*Time(i))^2)/2.25*A(i);

```

```

sum=sum+sig^2/2*(1-m*(abs(beta))^(1/3))*beta;
end
sum;
%option price, quadratic element
V2=exp(-r*(tau-t))*(S*exp(r*(tau-t))*normcdf(d1)-E*exp(-r*t)*normcdf(d2))+sum*S*timestep

sum=0;
%system of equations, 3 linear elements
f3 = @(w,u)[-3*sig^2*(3/5*u(1)-2/5*u(2))+2*sig^2*m*(u(1)*(3/5*u(1)^(1/3)+1/3*u(2)^(1/3))+u(2)*(-1/3*
[Time,C] = ode45(f3,[t T],[h h]);
for i=1:p
    if (x-r*Time(i))<-0.5
        beta=(x-r*Time(i)+1.5)*C(i,1);
    end
    if ((x-r*Time(i))>-0.5) && ((x-r*Time(i))<0.5)
        beta=C(i,1)+C(i,2);
    end
    if (x-r*Time(i))>0.5
        beta=(1.5-(x-r*Time(i)))*C(i,2);
    end
    sum=sum+sig^2/2*(1-m*(abs(beta))^(1/3))*beta;
end
sum;
%option price, 3 linear elements
V3=exp(-r*(tau-t))*(S*exp(r*(tau-t))*normcdf(d1)-E*exp(-r*t)*normcdf(d2))+sum*S*timestep

```

8 References

References

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