Finite-Time Output-Feedback Passification of Uncertain Fractional-Order Neural Networks having Time-Varying Delays

Advicer: Ardak Kashkynbayev Student: Tomiris Serikova

April 2024

1 Abstract

The purpose of the study was to investigate the finite-time stability and output feedback finite-time passification of fractional order uncertain neural networks with time-varying delay. We derived conditions for finite-time boundedness and finite-time passivity using linear matrix inequality, Lyapunov functional, and Schur complement Lemma we reached finite-time stability of the system. Certainly derivation and integration for fractional order calculus, that we used for neural network system, differs from conventional integer order calculus, by its hereditary characteristics. The findings have substantial implications for the design and control of complex neural network systems, paving the way for improved robustness and reliability in real-world applications.

2 Introduction

Fractional order calculus extends conventional integer-order calculus. However, compared with integer order derivatives, the generalization of dynamical equations by employing fractional derivatives gave more accurate mathematical modelling of dynamics of real-life physical phenomena, such as fluid mechanics [7], biological models [8], etc. In this way, implementing fractional order calculus into neural network system formulation has attracted much attention from the research community. The application of neural networks has shown great promise in many different areas, such as natural language processing, image and speech recognition, signal processing, and many other spheres. As a result, fractional-order neural networks inherited infinite memory properties and hereditary characteristics of fractional derivatives. FONN can better manage processes requiring the history of the function because the current state depends on a series of past states. Several research papers forecast the successful implementation of fractional-order neural networks in various spheres, especially biological models and viscoelastic systems.

In biological and artificial neural networks, there is asynchronous interaction between neurons, which leads to time delays between the processing of input data and the corresponding output of the system. In addition, time-varying delays in neural networks used in electronics are inevitable because of the switching speed of amplifiers. That is why it is vital to consider time-varying delays during system formulation. Also, the system can have parametric uncertainties arising from modelling inaccuracies and/or changes in the model's environment. For instance, data on neuron firing rate and weight coefficient are gathered and analyzed using statistical methods, which leads to some discrepancies from actual data and destabilization of the system. Fortunately, the range of input data is known, and we can include some parametric uncertainties. In real-life applications, both time-varying delays and parameter uncertainties are unavoidable, so that we will consider uncertain neural network systems with time-varying delays.

Various problems on dynamical behaviour of FONN have been investigated, [3], [4] reached global asymptotic stability for fractional order network with and without delay, [5] investigated the Mittag–Leffler stability of fractional-order nonlinear dynamic systems using Lyapunov direct method, in all cases the system approaches an equilibrium point in infinite time. In various practical scenarios, achieving finite-time convergence of the system is a desirable goal. The finite-time stability technique offers a distinct advantage by effectively managing specified time intervals and providing an estimate of the system's bound within a finite duration.

On the other hand, ensuring the system's stability is paramount, but maintaining passivity is equally vital for bolstering the resilience of the neural network system. The passive systems account for the energy transfer between the system and the outside environment, enabling it to preserve internal stability. Passification analysis involves developing a passive system using an appropriate control approach to minimize the components that are either excessively passive or not passive in the original system. In [6], Thuan et al. investigated output feedback passification of fractional-order static neural networks. Motivated by the above discussion, this paper focuses on the finite-time passivity analysis problem for fractional-order neural networks with uncertainties and a bounded time-varying delay via an output feedback controller.

3 Problem Statement and Preliminaries

Before outlining the neural network model, it is essential first to discuss key definitions and lemmas related to the fractional integral in the sense of Riemann-Liouville as ${}_0I_t^\alpha$ and the fractional derivative in the sense of Caputo as ${}_0^CD_t^\alpha$, both of which are of order $\alpha > 0$.

Definition 1 *[1] The fractional integral operator in the sense of Riemann-Liouville for a function $g(\cdot)$, with order $\beta > 0$, is defined as:*

$$ {}_0I_t^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s)\, ds, $$

such that $\Gamma(\cdot)$ noticed the standard gamma function, given by $\Gamma(u) = \int_0^\infty e^{-x} x^{u-1}\, dx$, for $u > 0$.

Definition 2 *[1] The fractional-order derivative operator for a function $g(t)$ with order $\beta > 0$ is formulated in the sense of Caputo as follows:*

$$ {}_0^C D_t^\beta g(t) = \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{g^{(m)}(s)}{(t-s)^{\beta+1-m}} \, ds, \quad t \geq 0, \, m-1 < \beta \leq m, $$

where m is a positive integer. Specifically, for values of β where $0 < \beta < 1$, the derivative takes the form:

$$ {}_0^C D_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\dot{g}(s)}{(t-s)^\beta} \, ds, \quad t \geq 0. $$

Lemma 1 *[1] If $y(t)$ belongs to $C^m([0,+\infty),\mathbb{R})$ and $m-1 < \beta < m$ is satisfied ($m \geq 1$, m being a positive integer), then the following expression holds:*

$$ {}_0I_t^\beta \left({}_0^C D_t^\beta y(t) \right) = y(t) - \sum_{j=0}^{m-1} \frac{t^j}{j!} y^{(j)}(0). $$

Particularly, when $0 < \beta < 1$, the relationship simplifies to:

$$ {}_0I_t^\beta ({}_0^C D_t^\beta y(t)) = y(t) - y(0). $$

Let's explore the following system of fractional-order nonlinear equations, which includes uncertainties and time-varying delays:

$$ \begin{cases} {}_0^C D_t^\alpha x(t) = -[A + \Delta A(t)]x(t) + [C + \Delta C(t)]f(x(t)) \\ \qquad\qquad +[D + \Delta D(t)]f(x(t-\sigma(t))) + W\omega(t) + Eu(t), \\ y(t) = Mf(x(t)) + Vf(x(t-\sigma(t))) + N\omega(t), \\ z(t) = Qx(t), \\ u(t) = Bz(t), \ t \geq 0. \end{cases} \tag{1} $$

In this context, $0 < \alpha < 1$ signifies the fractional-order derivative, while $x(t) = (x_1(t), \ldots, x_n(t))^T$ represents the state vector in $\mathbb{R}^n$. $y(t)$ and $z(t)$ are the output and output measurement vectors, respectively, each in $\mathbb{R}^p$, with n indicating the number of neurons. The disturbance input is represented by $\omega(t) \in \mathbb{R}^m$; the activation function is given by $f(x(t)) = (f_1(x_1(t)), \ldots, f_n(x_n(t)))^T$ in $\mathbb{R}^n$, and $f(x(t-\sigma(t))) = (f_1(x_1(t-\sigma(t))), \ldots, f_n(x_n(t-\sigma(t)))^T$ includes a time-varying delay, both in $\mathbb{R}^n$. The system's output-feedback controller is $u(t) \in \mathbb{R}^q$; $A = \text{diag}\{a_1, a_2, \ldots, a_n\}$ is a positive diagonal matrix in $\mathbb{R}^{n\times n}$. The interconnection weight matrices for $f(x(t))$ and $f(x(t-\sigma(t)))$ are C and D, respectively, both in $\mathbb{R}^{n\times n}$. W in $\mathbb{R}^{n\times m}$, B in $\mathbb{R}^{q\times p}$, and other matrices such as M, N, V, E, and Q are specified real matrices in various dimensions. Perturbation terms $\Delta A(t) = G_a F_a(t) H_a$, $\Delta C(t) = G_c F_c(t) H_c$, and $\Delta D(t) = G_d F_d(t) H_d$ involve known real constant matrices G_a, G_c, G_d, H_a, H_c, and H_d, with appropriate dimensions. The matrices $F_a(t)$, $F_c(t)$, and $F_d(t)$ are unknown real time-varying matrices fulfilling $F_a^T(t)F_a(t) \leq I$, $F_c^T(t)F_c(t) \leq I$, $F_d^T(t)F_d(t) \leq I$ for all $t \geq 0$.

Assumption 1 *[10] For the continuously bounded differentiable non-linear activation functions $f(\cdot)$ indexed by $i \in \mathcal{J}_n$, with $f_i(0) = 0$ for $i \in \mathcal{J}_n$, positive scalars k_i^- and k_i^+ exist for $i \in \mathcal{J}_n$. This ensures that for any x_1 and x_2 in $\mathbb{R}$ with $x_1 \neq x_2$, the following inequality holds:*

$$k_i^- \leq \frac{f_i(x_1) - f_i(x_2)}{x_1 - x_2} \leq k_i^+.$$

Utilizing these constants k_i^- and k_i^+ for $i \in \mathcal{J}_n$, the matrices K_1 and K_2 are postulated as:

$$K_1 = diag\{k_1^- k_1^+, k_2^- k_2^+, \ldots, k_n^- k_n^+\}$$

and

$$K_2 = diag\left\{\frac{k_1^- + k_1^+}{2}, \frac{k_2^- + k_2^+}{2}, \ldots, \frac{k_n^- + k_n^+}{2}\right\}.$$

Assumption 2 *[9] The time-varying disturbance input $\omega(t) \in \mathbb{R}^m$ satisfies the condition:*

$$\exists d > 0 : \omega^T(t)\omega(t) < d, \quad \textit{for all } t \in [0, T_f].$$

Assumption 3 *[11] The continuous time-varying delay function $\sigma(t)$ adheres to the condition $0 \leq \sigma(t) \leq \sigma$, such that σ is a known constant.*

Definition 3 *[9] Let T_f, c_1, c_2 (where $c_1 < c_2$), and d be positive values, and $R \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix. The uncertain neural network system (1) with $y(t) = 0$, is named robustly finite-time stable concerning T_f, c_1, c_2, and d if $x_0^T(t)Rx_0(t) \leq c_1$ and $x^T(t)Rx(t) < c_2$ $\forall \omega(t) \in \mathbb{R}^m$ fulfilling Assumption 2.*

Definition 4 *[9] The uncertain neural network system (1) is considered finite-time passive regarding T_f, c_1, c_2, d, and R if it fulfills the following conditions:*

(i) Under the condition where $y(t) \equiv 0$, the uncertain neural network system (1) demonstrates robust finite-time stability regarding (c_1, c_2, T_f, R, d).

(ii) For all $t \in [0, T_f]$, in the absence of initial conditions, there exists a positive scalar γ such that the inequality below holds:

$$2\, {}_0I_t^\alpha(y^T(t)\omega(t)) \geq -\gamma\, {}_0I_t^\alpha(\omega^T(t)\omega(t)).$$

The auxiliary lemmas below serve as essential tools for deriving the main results presented in this study.

Lemma 2 *[11] For any differentiable vector-valued function $\varrho(t)$ in $\mathbb{R}^n$ and any time instant $t \geq t_0$, and for any quadratic function $\mathrm{V}(\varrho(t)) = \varrho^\top(t)C\varrho(t)$, the following inequality holds:*

$${}_0^C D_t^\alpha(\mathrm{V}(\varrho(t))) \leq 2\varrho^\top(t)C\, {}_0^C D_t^\alpha \varrho(t), \textit{ for all } \alpha \in (0,1), \textit{ and for all } t \geq t_0 \geq 0,$$

where $C > 0$ in $\mathbb{R}^{n \times n}$.

Lemma 3 *[9] For constant matrices C_1, C_2, and C_3, with suitable dimensions such that $C_2 = C_2^T > 0$ and $C_1 = C_1^T$, the inequality $C_1 + C_3^T C_2^{-1} C_3 < 0$ holds iff*

$$\begin{pmatrix} C_1 & C_3^T \\ C_3 & -C_2 \end{pmatrix} < 0.$$

Lemma 4 *[12] Given real matrices C_1, C_2, and $C(t)$ of appropriate dimensions, where $C(t)$ satisfies $C^T(t)C(t) \leq I$, and for any constant $\delta > 0$, the inequality below holds:*

$$C_1 C(t) C_2 + (C_1 C(t) C_2)^T \leq \delta^{-1} C_1 C_1^T + \delta C_2^T C_1.$$

4 Main Results

In this section, we first outline the derivation of the condition that ensures the finite-time stability of a fractional-order neural network system (1), considering both parameter uncertainties and time-varying delay.

Theorem 1: Given that Assumptions 1-3 are met, suppose T_f, c_1, c_2, d, and R are positive constants. The system (1), with $y(t) \equiv 0$ as its output, achieves finite-time stability concerning (T_f, c_1, c_2, d, R) if there exists a positive-definite symmetric matrix $P \in \mathbb{R}^{n \times n}$, diagonal matrices $L_1 > 0$ and $L_2 > 0$, and constants $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon_4 > 0$. This holds under the condition that the following LMI is satisfied:

$$\begin{bmatrix} \Theta_{11} & 0 & PC + K_2 L_1 & PD & PG_a & PG_c & PG_d & PW \\ * & \Theta_{22} & 0 & K_2 L_2 & 0 & 0 & 0 & 0 \\ * & * & \Theta_{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Theta_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & * & -\epsilon_2 I & 0 & 0 \\ * & * & * & * & * & * & -\epsilon_3 I & 0 \\ * & * & * & * & * & * & * & -\epsilon_4 I \end{bmatrix} < 0, \tag{2a}$$

$$\lambda_2 c_1 + \frac{d_{e3}}{\Gamma(\alpha+1)} T_f^{\alpha} \leq \lambda_1 c_2, \tag{2b}$$

where $\overline{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}}$, $\lambda_1 = \lambda_{min}(\overline{P})$, $\lambda_2 = \lambda_{max}(\overline{P})$,
$\Theta_{11} = -PA - A^T P + PEBQ + Q^T B^T E^T P + \epsilon_1 H_a^T H_a - K_1 L_1$,
$\Theta_{22} = -K_1 L_2$,
$\Theta_{33} = \epsilon_2 H_c^T H_c - L_1$,
$\Theta_{44} = \epsilon_3 H_d^T H_d - L_2$.

Proof. Let us consider the basic quadratic Lyapunov-Kraskovskii functional

$$V(x(t)) = x^T(t) P x(t).$$

Lemma 2 suggests that for $0 < \alpha < 1$, the fractional derivative of the Lyapunov functional $V(x(t))$ along the trajectory is applicable.

$$
\begin{aligned}
{}_0^C D_t^\alpha V(x(t)) \leq & 2x^T(t)P\, {}_0^C D_t^\alpha x(t) \\
= & x^T(t)(-PA - A^T P + PEBQ + Q^T B^T E^T P)x(t) \\
& - 2x^T(t)PG_aF_a(t)H_a x(t) + 2x^T(t)PCf(x(t)) \\
& + 2x^T(t)PG_cF_c(t)H_c f(x(t)) + 2x^T(t)PDf(x(t-\sigma(t))) \\
& + 2x^T(t)PG_dF_d(t)H_d f(x(t-\sigma(t))) + 2x^T(t)PW\omega(t). \quad (3)
\end{aligned}
$$

Using Lemma 4, we have the following inequalities for the uncertainties and disturbance input containing terms

$$
-2x^T(t)PG_aF_a(t)H_a x(t) \leq \epsilon_1^{-1} x^T(t)PG_aG_a^T Px(t) + \epsilon_1 x^T(t)H_a^T H_a x(t), \quad (4)
$$

$$
\begin{aligned}
& 2x^T(t)PG_cF_c(t)H_c f(x(t)) \\
& \qquad \leq \epsilon_2^{-1} x^T(t)PG_cG_c^T Px(t) + \epsilon_2 f^T(x(t))H_c^T H_c f(x(t)), \quad (5)
\end{aligned}
$$

$$
\begin{aligned}
& 2x^T(t)PG_dF_d(t)H_d f(x(t-\sigma(t))) \\
& \qquad \leq \epsilon_3^{-1} x^T(t)PG_dG_d^T Px(t) + \epsilon_3 f^T(x(t-\sigma(t)))H_d^T H_d f(x(t-\sigma(t))), \quad (6)
\end{aligned}
$$

$$
2x^T(t)PW\omega(t) \leq \epsilon_4^{-1} x^T(t)PWW^T Px(t) + \epsilon_4 \omega^T(t)\omega(t). \quad (7)
$$

As stated by Assumption 1 about neuron activation function, the following inequalities are applicable for any diagonal matrices $L_1, L_2 > 0$

$$
\begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \begin{bmatrix} K_1L_1 & -K_2L_1 \\ -K_2L_1 & L_1 \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \leq 0, \quad (8)
$$

$$
\begin{bmatrix} x(t-\sigma(t)) \\ f(x(t-\sigma(t))) \end{bmatrix}^T \begin{bmatrix} K_1L_2 & -K_2L_2 \\ -K_2L_2 & L_2 \end{bmatrix} \begin{bmatrix} x(t-\sigma(t)) \\ f(x(t-\sigma(t))) \end{bmatrix} \leq 0. \quad (9)
$$

By replacing parameters with uncertainties (4) to (7) and adding inequalities that were derived from Lipschitz conditions (8) and (9), we get

$$
\begin{aligned}
{}_0^C D_t^\alpha V(x(t)) \leq & x^T(t)(-PA - A^T P + PEBQ + Q^T B^T E^T P)x(t) \\
& + \epsilon_1^{-1} x^T(t)PG_aG_a^T Px(t) + \epsilon_1 x^T(t)H_a^T H_a x(t) + 2x^T(t)PCf(x(t)) \\
& + \epsilon_2^{-1} x^T(t)PG_cG_c^T Px(t) + \epsilon_2 f^T(x(t))H_c^T H_c f(x(t)) \\
& + 2x^T(t)PDf(x(t-\sigma(t))) + \epsilon_3^{-1} x^T(t)PG_dG_d^T Px(t) \\
& + \epsilon_3 f^T(x(t-\sigma(t)))H_d^T H_d f(x(t-\sigma(t))) + \epsilon_4 \omega^T(t)\omega(t) \\
& + \epsilon_4^{-1} x^T(t)PWW^T Px(t) - x^T(t)K_1L_1x(t) + f^T(x(t))K_2L_1x(t) \\
& + x^T(t)K_2L_1 f(x(t)) - f^T(x(t))L_1 f(x(t)) \\
& - x^T(t-\sigma(t))K_1L_2x(t-\sigma(t)) + f^T(x(t-\sigma(t)))K_2L_2x(t-\sigma(t)) \\
& + x^T(t-\sigma(t))K_2L_2 f(x(t-\sigma(t))) \\
& - f^T(x(t-\sigma(t)))L_2 f(x(t-\sigma(t))). \quad (10)
\end{aligned}
$$

The above inequality can be equivalently reformulated into matrix form as

$$ {}_{0}^{C}D_{t}^{\alpha}V(x(t)) \leq \eta^{T}(t)\Omega(t)\eta(t) + \epsilon_{3}\omega^{T}(t)\omega(t), \tag{11} $$

where $\eta(t) = [x(t),\ x(t-\sigma(t)),\ f(x(t)),\ f(x(t-\sigma(t)))]^{T}$.
From LMI (2a), we can reach the following constraint.

$$ {}_{0}^{C}D_{t}^{\alpha}V(x(t)) \leq \epsilon_{3}\omega^{T}(t)\omega(t). \tag{12} $$

Supposed that Assumption 2 is satisfied, so by integrating both sides of (12) with order $0 < \alpha < 1$ from 0 to t $(0 < t < T_f)$ and implementing Lemma 1, we get

$$ \begin{aligned} x^{T}(t)Px(t) &\leq x^{T}(0)Px(0) +_{0} I_{t}^{\alpha}(\epsilon_{3}\omega^{T}(t)\omega(t)) \\ &= x^{T}(0)Px(0) + \frac{\epsilon_{3}}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}\omega^{T}(t)\omega(t)ds \\ &\leq x^{T}(0)Px(0) + \frac{d\epsilon_{3}}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}ds \\ &\leq x^{T}(0)Px(0) + \frac{d\epsilon_{3}}{\Gamma(\alpha+1)}T_{f}. \end{aligned} \tag{13} $$

From the property of the Lyapunov functional, it is obvious that they were derived in [9]

$$ x^{T}(t)Px(t) = x^{T}(t)R^{\frac{1}{2}}\overline{P}R^{\frac{1}{2}}x(t) \geq \lambda_{min}(\overline{P})x^{T}(t)Rx(t) = \lambda_{1}x^{T}(t)Rx(t), \tag{14} $$

where $\lambda_{min}(\overline{P})x^{T}(t)Rx(t) \leq \lambda_{1}c_{2}$.
Furthermore,

$$ \begin{aligned} x^{T}(0)Px(0) =& x^{T}(0)R^{\frac{1}{2}}\overline{P}R^{\frac{1}{2}}x(0) \\ &\leq \lambda_{\max}(P)x^{T}(0)Rx(0) \\ &= \lambda_{2}x^{T}(0)Rx(0) \\ &\leq \lambda_{2}c_{1}. \end{aligned} \tag{15} $$

From (14) and (15), one has

$$ \lambda_{1}x^{T}(t)Rx(t) = \lambda_{1}c_{1} \leq \lambda_{2}c_{1} + \frac{d\epsilon_{3}}{\Gamma(\alpha+1)}T_{f}. \tag{16} $$

Upon integrating equations (13), (14), and (15), we arrive at a condition equivalent to (2b). This indicates that the system (1), with the output $y(t) = 0$, is both finite-time bounded and finite-time stable regarding (T_f, c_1, c_2, d, R).

Theorem 2: Assuming that Assumptions 1-3 are upheld, and given positive constants T_f, c_1, c_2, d, and a symmetric matrix R, the system (1) achieves finite-time passivity concerning (T_f, c_1, c_2, d, R) if there exists a positive-definite symmetric matrix $P \in \mathbb{R}^{n\times n}$, diagonal matrices $L_1 > 0$ and $L_2 > 0$, and constants $\epsilon_1 > 0$, $\epsilon_2 > 0$,

$\epsilon_3 > 0$, $\epsilon_4 > 0$. This holds under the condition that the following LMI is satisfied:

$$\begin{bmatrix} \Theta_{11} & 0 & PC+K_2L_1 & PD & 0 & PG_a & PG_c & PG_d & PW \\ * & \Theta_{22} & 0 & K_2L_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Theta_{33} & 0 & -M^T & 0 & 0 & 0 & 0 \\ * & * & * & \Theta_{44} & -V^T & 0 & 0 & 0 & 0 \\ * & * & * & * & \Theta_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\epsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\epsilon_2 I & 0 & 0 \\ * & * & * & * & * & * & * & -\epsilon_3 I & 0 \\ * & * & * & * & * & * & * & * & -\epsilon_4 I \end{bmatrix} < 0, \tag{17a}$$

$$\lambda_2 c_1 + \frac{d_{e3}}{\Gamma(\alpha+1)} T_f^\alpha \leq \lambda_1 c_2, \tag{17b}$$

where $\overline{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}}$, $\lambda_1 = \lambda_{min}(\overline{P})$, $\lambda_2 = \lambda_{max}(\overline{P})$,
$\Theta_{11} = -PA - A^T P + PEBQ + Q^T B^T E^T P + \epsilon_1 H_a^T H_a - K_1 L_1$,
$\Theta_{22} = -K_1 L_2$,
$\Theta_{33} = \epsilon_2 H_c^T H_c - L_1$, $\Theta_{44} = \epsilon_3 H_d^T H_d - L_2$,
$\Theta_{55} = \epsilon_4 I - (N + N^T + \gamma I)$.

Proof. When $y(t) = 0$, both (a) and (b) reduce to (5a) and (5b), suggesting that the system (1) achieves finite-time stability. To establish the finite-time passivity of the system (1) for non-zero output measurement, we will employ the positive definite quadratic function utilized in the proof of Theorem 1:

$${}_0^C D_t^\alpha V(x(t)) - 2y^T(t)\omega(t) - \gamma\omega^T(t)\omega(t) \leq \kappa^T(t)\Omega'\kappa(t), \tag{18}$$

where the coefficient vector is
$\kappa(t) = [x(t),\ x(t-\sigma(t)),\ f(x(t)),\ f(x(t-\sigma(t))),\ \omega(t)]^T$ and

$$\Omega' = \begin{bmatrix} \Omega_{11} & 0 & PC+K_2L_1 & PD & 0 \\ 0 & \Omega_{22} & 0 & K_2L_2 & 0 \\ C^TP+K_2L_1 & 0 & \Omega_{33} & 0 & -M^T \\ D^TP & K_2L_2 & 0 & \Omega_{44} & -V^T \\ 0 & 0 & -M^T & -V^T & \Omega_{55} \end{bmatrix}$$

such that
$\Omega_{11} = -PA - A^TP - K_1L_1 + \epsilon_1 H_a^T H_a + \epsilon_1^{-1} PG_aG_a^TP + \epsilon_2 PG_cG_c^TP + \epsilon_3 PG_dG_d^TP + \epsilon_4 PWW^TP$,
$\Omega_{22} = -K_1L_2$,
$\Omega_{33} = \epsilon_2 H_c^T H_c - L_1$,
$\Omega_{44} = \epsilon_3 H_d^T H_d - L_2$,
$\Omega_{55} = \epsilon_4 I - (N + N^T + \gamma I)$.

By Schur Complement Lemma (Lemma 3) in LMI (17a), this implies that $\Omega' < 0$ and it directed the inequality (18) should be negative semi-definite, thus,

$${}_0^C D_t^\alpha V(x(t)) - 2y^T(t)\omega(t) - \gamma\omega^T(t)\omega(t) \leq 0. \tag{19}$$

Set a variable $J(t)$ as passivity performance measure, then

$$J(t) = {}_0I_t^\alpha(-2y^T(t)\omega(t) - \gamma\omega^T(t)\omega(t))$$
$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(2y^T(t)\omega(t) - \gamma\omega^T(t)\omega(t)),\ t \in [0, T_f]. \tag{20}$$

But the inequality (20) does not give any useful information; thus, using Lemma 1 and noting the zero initial condition of Lyapunov functional $V(x(0)) = 0$, we get the following result of $J(t)$:

$$\begin{aligned} J(t) =& {}_0I_t^\alpha\bigg({}_0^C D_t^\alpha V(x(t)) - 2y^T(t)\omega(t) - \gamma\omega^T(t)\omega(t)\bigg) - [V(x(t)) - V(x(0))] \\ \leq& {}_0I_t^\alpha\bigg({}_0^C D_t^\alpha V(x(t)) - 2y^T(t)\omega(t) - \gamma\omega^T(t)\omega(t)\bigg). \end{aligned} \tag{21}$$

Due to $V(x(t)) \geq 0$, and from (19) and (21), we obtain that $J(t) \leq 0$. Therefore, we can come to the following output

$$2{}_0I_t^\alpha(y^T(t)\omega(t)) \geq -\gamma{}_0I_t^\alpha(\omega(t)\omega^T(t)). \tag{22}$$

From these results, the conclusion arises that system (1) is a finite-time passive.

5 Conclusion

This study explores the finite-time stability and finite-time passification of uncertain fractional-order neural networks subject to time-varying delays. We utilize an output measurement-based feedback controller and capitalize on the properties of the Caputo fractional order derivative applied to quadratic functions, alongside employing a range of matrix inequalities. As a result, we obtained adequate conditions using linear matrix inequalities to confirm the finite-time stability and passification of uncertain fractional-order neural networks subject to time-varying delays.

References

[1] He, W., Qian, F., & Cao, J. (2017). Pinning-controlled synchronization of delayed neural networks with distributed-delay coupling via impulsive control. *Neural Networks*, 85, 1–9. https://doi.org/10.1016/j.neunet.2016.09.002

[2] Marat Akhmet, Mehmet Onur Fen, & Mokhtar Kirane. (2015). Almost periodic solutions of retarded SICNNs with a functional response on the piecewise constant argument. Neural Computing and Applications, 27(8), 2483–2495. https://doi.org/10.1007/s00521-015-2019-4

[3] Wang, F., Yang, Y., & Hu, M. (2015). Asymptotic stability of delayed fractional-order neural networks with impulsive effects. Neurocomputing, 154, 239-244. https://doi.org/10.1016/j.neucom.2014.11.068

[4] Liang, X.-B., & Si, J. (2001). Global exponential stability of neural networks with globally Lipschitz continuous activations and its application to linear variational inequality problem. IEEE Transactions on Neural Networks, 12(2), 349-359. https://doi.org/10.1109/72.914529

[5] Li, Y., Chen, Y., & Podlubny, I. (2010). Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability. Computers & Mathematics with Applications, 59(5), 1810-1821. https://doi.org/10.1016/j.camwa.2009.08.019

[6] Thuan, M. V., Thanh, N. T., Huyen, N. T. T., & Hong, D. T. (2023). Output feedback passification of a class of fractional-order static neural networks. Transactions of the Institute of Measurement and Control, 45(16), 3147-3158. https://doi.org/10.1177/01423312231163915

[7] Tripathi, D., Pandey, S. K., & Das, S. (2010). Peristaltic flow of viscoelastic fluid with fractional Maxwell model through a channel. Applied Mathematics and Computation, 215(10), 3645-3654. https://doi.org/10.1016/j.amc.2009.10.027

[8] Magin, R. L., & Ovadia, M. (2008). Modeling the cardiac tissue electrode interface using fractional calculus. Journal of Vibration and Control, 14(9-10), 1431-1442. https://doi.org/10.1177/1077546307087439

[9] Thuan, M. V., Huong, D. C., & Hong, D. T. (2019). New results on robust finite-time passivity for fractional-order neural networks with uncertainties. Neural Processing Letters, 50, 1065–1078. https://doi.org/10.1007/s11063-018-9902-9

[10] Shafiya, M., & Nagamani, G. (2022). New finite-time passivity criteria for delayed fractional-order neural networks based on Lyapunov function approach. Chaos, Solitons & Fractals, 158, Article 112005. https://doi.org/10.1016/j.chaos.2022.112005

[11] Sau, N. H., Thuan, M. V., & Huyen, N. T. T. (2020). Passivity analysis of fractional-order neural networks with time-varying delay based on LMI approach. Circuits, Systems, and Signal Processing, 39, 5906–5925. https://doi.org/10.1007/s00034-020-01450-6

[12] Zeng, H.-B., He, Y., Wu, M., & Xiao, H.-Q. (2014). Improved conditions for passivity of neural networks with a time-varying delay. IEEE Transactions on Cybernetics, 44(6), 785-792. https://doi.org/10.1109/TCYB.2013.2272399

[13] Shafiya, M., & Nagamani, G. (2022). New finite-time passivity criteria for delayed fractional-order neural networks based on Lyapunov function approach. Chaos, Solitons & Fractals, 158, 112005. https://doi.org/10.1016/j.chaos.2022.112005