Power Series Solutions of a NODE System in the Complex Domain

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Abstract

In this Capstone Project, we analyze a second order nonlinear ordinary differential equation (NODE), $y''(x) = f(y', y)$ that is impossible to solve analytically. First, using the Taylor Power Series method, we obtain a series expansion of the solution $y(x)$ about $x = 0$ for $x \in \Re$, and find that this series diverges for values of x a little above $x = 1$. This implies that the equation has a singularity in the complex domain. Therefore, we investigate this NODE by using Laurent expansions about the unknown singularity at $x = x_*$, which is called *movable* because its location depends on the initial conditions. By finding the general form of these expansions, we obtain approximate expressions for the singularity closest to $x = 0$ and thus are able to estimate the radius of convergence for different initial conditions. We also integrate numerically the solutions in the real x, y plane and demonstrate the connection of the global form of the solutions of the problem with the predictions of our laurent series expansions in the complex $x-$ plane.

1 Introduction

Nonlinear ordinary differential equations (NODEs) are of great interest to a wide variety of applications in Physics, Biology, Engineering, Economics and many other sciences. As is well–known, if the unknown quantity satisfying a given NODE is $y = y(x)$, a classical method for obtaining solutions is to expand $y(x)$ in a Taylor series for real x about a given point of interest, say $x = 0$, at which the solutions have a known regular behavior, for example, $x = 0$ may be a fixed point of the NODE. However, the NODE often possesses *movable* (i.e. initial condition dependent) singularities in the complex $x-$ domain, at which the attempted Taylor series expansion diverges and becomes useless. One of the important tasks of complex analysis, therefore, is to develop expansions *about these singularities* in the form of Laurent series and provide *alternative* solutions to the NODE, which in fact converge near the singularity, where the Taylor series solution begins to break down.

Since the location of a singularity depends on the initial conditions, we may therefore use our Laurent series expansions in an *inverse way*: We may solve for the location of the singularity as a function of the initial conditions and thus be able to predict where the singularity nearest to our point of interest is located, so as know how far our solutions expressed by Taylor series will cease to be valid. Since this task involves the inversion of power series, however, the estimates we obtain will be approximate, as their accuracy will depend on the number of terms of the series that we wish to take into account.

The main objective of this Project is to apply the above approach to a specific example of a NODE of the form

$$y''(x) = f(y', y) = -yy' + y, \tag{1}$$

which in fact may be viewed as a 2–dimensional dynamical system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -y - xy \tag{2}$$

of two variables $x(t), y(t)$ parametrized by time. This will be very useful to us later, when we view the solutions as *orbits* in the x, y plane, by finding a first integral for the system (1). However, first we will approach our equation in the form of (1) by applying in Section 2 the methodology of Taylor power series to construct an analytical solution, about $x = 0$. As is well–known, these series contain from the outset two *free* parameters which can be computed from the knowledge of the initial conditions, i.e. $y(0), y'(0)$.

The application of such Taylor series expansions in nonlinear equations such as (1) necessitates that the calculation of the coefficients $a_n, n = 0, 1, 2, ..., N$ of the series are obtained from a *nonlinear* recurrence relation. Still, this method makes it possible to obtain successive approximations of $y(x)$ containing terms of progressively higher order N. Plotting then these solutions for higher and higher values of N we can compare it with the numerically obtained solution. This comparison illustrates that the method is very effective, since by adding new terms the graph of the analytical solution approximates more and more closely the numerical solution, for *small values* of $y(0), y'(0)$ and intervals $x \in [0, X]$ in the real domain. However, as the $y(0), y'(0)$ increase and/or X increases, our series solutions *diverge* away from the true solution of the problem!

The reason for this divergence, of course, is the fact that an "invisible" *singularity* exists in the *complex x–plane*, which severely limits the convergence of our Taylor series. To illustrate this phenomenon, we have constructed graphs of $y = \lim_{N\to\infty} \sum_{n=0}^{N} a_n x^n$, for several N, and compared them with the numerically obtained solution.

To estimate the location of this "invisible" singularity, we have proceeded in Section 3 to develop the solution of (1) by Laurent series expansions in the variable $x - x_*$ about such a singularity. These expansions begin with a divergent term, and become more and more accurate exactly where the Taylor series becomes more and more *inaccurate*. These Laurent series represent the *general* solution since they contain two *free constants* one of which is the location of the singularity x_*. Thus, for any set of (real or complex) initial conditions, we can *invert* the Laurent series and obtain expressions for x_*, as well as the second free constant.

Even though the inversion of these series is a difficult task, we can still keep their first few terms and obtain *approximate* yet analytical expressions for the unknown quantities as functions of the initial conditions. These expressions provide useful estimates about x_*, which are in good agreement with what is found from the comparison with the Taylor series and thus give a satisfactory answer as to where teh Taylor series is expected to diverge.

In Section 4, we compare the results of the previous section with actual graphs of the solutions of (1) viewed as orbits of the (1) system in the x,y plane. We also show how these orbits can be analytically provided, in the case of our problem, by a first integral of the equations (1). The Project ends in Section 5, where we present our conclusions.

2 TAYLOR SERIES SOLUTIONS OF OUR EQUATION

Our goal in this Section is to solve the following second order NODE

$$y'' + yy' + y = 0 \tag{3}$$

where $y' := \mathrm{d}y/\mathrm{d}x$ and $y'' := \mathrm{d}^2y/\mathrm{d}x^2$, using Taylor series expansions about $x = 0$ and compare the results with the corresponding numerical solution. We will choose for specificity the initial conditions $y(0) = 1$ and $y'(0) = 1$, but any other choice can be similarly treated. As will become clear later, the regime around $x = 0$ is occupied by periodic solutions, however, it is not known at this point how far this regime extends. For this reason, it is a good idea to keep the values of our $y(0)$ initial condition small. The precise value of $y'(0)$ is not important so we will set it equal to zero for our convenience, as this does not limit the generality of our conclusions.

To solve the above equation (3), we will employ the Taylor series expansion about $x = 0$, namely

$$y = \sum_{n=0}^{\infty} a_n x^n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n x^n \tag{4}$$

where $N \in \mathbb{N}_0$ and a_n are the series coefficients, with

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \tag{5}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \tag{6}$$

Substituting Eqs. (4)-(6) into Eq. (3), we obtain

$$\lim_{N\to\infty} \left[\sum_{n=2}^{N} n(n-1) a_n x^{n-2} + \sum_{n=0}^{N} a_n x^n \sum_{m=1}^{N} m a_m x^{m-1} + \sum_{n=0}^{N} a_n x^n \right] = 0 \tag{7}$$

We further shift the summation index as $(n-2) \to n$ and $(m-1) \to m$ in order to have in all sums the same starting value, i.e., 0:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{n=0}^{\infty} a_n x^n = 0 \tag{8}$$

We can rewrite the above equation as

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{k+1}(k+1) a_{n-k} x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \tag{9}$$

or, equivalently,

$$\sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} a_{k+1}(k+1) a_{n-k} + a_n \right) x^n = 0\,. \tag{10}$$

Since Eq. (10) must be satisfied for all values of x, this leads to the following recurrence relation for the Taylor coefficients:

$$(n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} a_{k+1}(k+1) a_{n-k} + a_n = 0\,, \tag{11}$$

yielding the following expression for each coefficient:

$$a_{n+2} = -\frac{\sum_{k=0}^{n} a_{k+1}(k+1)a_{n-k} + a_n}{(n+2)(n+1)}. \tag{12}$$

Now, the first two terms of the solution series in Eq. (4) are determined from the initial conditions, which in our case yields $a_0 = 1, a_1 = 1$. Using then Eq. (12) we calculate below the first few of these coefficients $\{a_2, \ldots, a_7\}$ to obtain:

$$a_2 = -\frac{a_0\,1\,a_1 + a_0}{2} = -1$$
$$a_3 = -\frac{a_1\,1\,a_1 + a_2\,2\,a_0 + a_1}{6} = 0$$
$$a_4 = -\frac{a_2\,1\,a_1 + a_1\,2\,a_2 + a_0 3 a_3 + a_2}{12} = \frac{1}{3}$$
$$a_5 = -\frac{a_3 a_1 + a_2\,2\,a_2 + a_1\,3\,a_3 + a_0 4 a_4 + a_3}{20} = -\frac{1}{6}$$
$$a_6 = -\frac{a_4 a_1 + a_3 2 a_2 + a_2 3 a_3 + a_1 4 a_4 + a_0 5 a_5 + a_4}{30} = -\frac{7}{180}$$
$$a_7 = -\frac{a_5 a_1 + a_4 2 a_2 + a_3 3 a_3 + a_2 4 a_4 + a_1 5 a_5 + a_0 6 a_6 + a_5}{42} = \frac{17}{210}$$

Then, the behavior of $y(x)$ for low values of x up to the order $\mathcal{O}(x^7)$ is given as

$$y(x) = 1 + x - x^2 + \frac{1}{3}x^4 - \frac{1}{6}x^5 - \frac{7}{180}x^6 + \frac{17}{210}x^7 + \cdots \tag{13}$$

Let us now plot this solution as a function of x. As we will explain later in Section 4, we expect the solution to be periodic and therefore bounded. However, as our Taylor series is plotted for higher and higher values of x, the expansion (2) above begins to diverge at values of x close to $x = 1$. This is clearly shown by the dashed curves in Fig.1 below, where even if we include higher and higher order terms in (2) the solution may reach to a slightly greater value of x, but always blows up to infinity at some point. On the other hand, the numerically obtained solution (shown by a solid curve in Fig.1) is clearly bounded and is in fact periodic, as would be evident if we had plotted it for higher values of x.

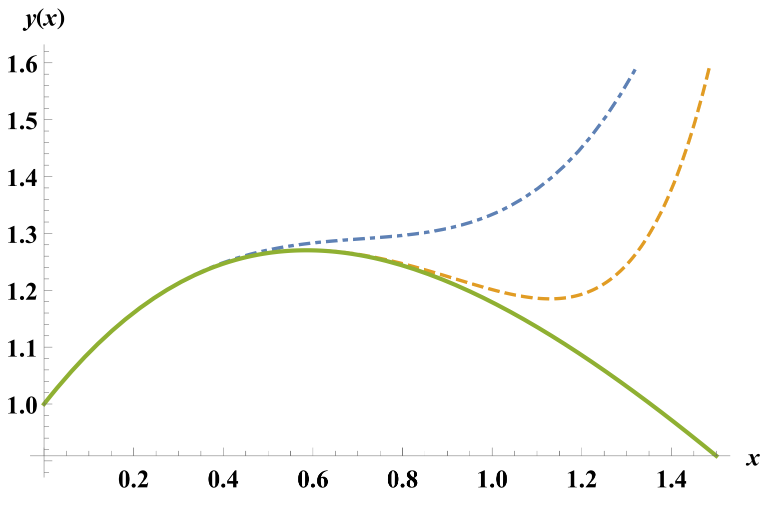

Figure 1: We plot x and y(x) of power series solutions for lower terms of approximation. The dotdashed blue line shows the first four terms and dashed orange line nine terms, while green thick line describes the general solution. The considered power series solutions are good approximated to general solution as more terms are added.

What is the reason for this phenomenon? As it is known, Taylor series around some point (here $x = 0$) converge up to the nearest singularity x_*, wherever that singularity may be located in the complex x–plane. In many problems, where the solution blows up at some real x_* this lies on the real axis and the solution of course is not periodic. However, this "invisible" singularity may lie in the complex plane, as in our case here, and hence our solution will converge up to a circle about $x = 0$ which passes by x_*! In the next Section, we will see how, by using Laurent series expansions, we may actually write our solution in a form that converges in a circle *around the singular point*, exactly where the Taylor series fails to be valid.

3 LAURENT SERIES SOLUTIONS VALID AROUND THE SINGULARITY

Let us suppose that our solution $y(x)$ diverges near this singularity as $y = \frac{c}{(x-x_*)^p}$, where $p > 0$. We call this term the most divergent of a Laurent type of expansion series, which is developed to higher orders as follows:

$$y = \frac{c}{(x-x_*)^p} + \sum_{i=0}^{\infty} c_i (x-x_*)^i \tag{14}$$

where c_i are the series coefficients and c, p are as yet undetermined constants, with

$$y' = \frac{-pc}{(x-x_*)^{p+1}} + \sum_{i=1}^{\infty} i c_i x^{i-1} \tag{15}$$

$$y'' = \frac{p(p+1)c}{(x-x_*)^{p+2}} + \sum_{i=2}^{\infty} i(i-1) c_i x^{i-2} \tag{16}$$

In order to determine the constants p and c, we need to equalize in our equation the most divergent terms in the series for y'' and yy', i.e., $y'' = -yy'$, since, as x goes to infinity, the term y in the equation is of higher order. This can be easili achieved as follows:

$$\frac{p(p+1)c}{(x-x_*)^{p+2}} = -\left(\frac{-pc^2}{(x-x_*)^{2p+1}}\right) \tag{17}$$

yielding

$$(x-x_*)^{p+2} = (x-x_*)^{2p+1} \qquad \text{and} \qquad cp(p+1-c) = 0 \tag{18}$$

It now becomes clear from Eq. (18) that $p = 1$ and $c = 2$ (for $\neq 0$), so the Laurent series solution $y(x)$ now becomes:

$$y(x) = \frac{2}{(x-x_*)} + \sum_{i=0}^{\infty} c_i (x-x_*)^i \tag{19}$$

For simplicity we introduce the variable $z = x - x_*$, so that Eq. (19) can be written as:

$$y(z) = \frac{2}{z} + \sum_{i=0}^{\infty} c_i z^i \tag{20}$$

To determine the unknown coefficients c_i, just as in the case of Taylor series, we first differentiate Eq. (20) with respect to z,

$$y' = -\frac{2}{z^2} + c_1 + 2c_2 z + 3c_3 z^2 + \cdots \tag{21a}$$

$$y'' = \frac{4}{z^3} + 2c_2 + 6c_3 z + 12c_4 z^2 + \cdots \tag{21b}$$

$$yy' = -\frac{4}{z^3} - \frac{2c_0}{z^2} + (2c_2 + c_0 c_1) + \left(c_1^2 + 4c_3 + 2c_0 c_2\right) z + (3c_1 c_2 + 6c_4 + 3c_0 c_3)\, z^2 + \cdots \tag{21c}$$

and then substitute Eq. (21) into Eq. (3) to obtain

$$-\frac{2c_0}{z^2} + \frac{2}{z} + (c_0 + 4c_2 + c_0 c_1) + \left(c_1 + c_1^2 + 10c_3 + 2c_0 c_2\right) z + (c_2 + 3c_1 c_2 + 3c_0 c_3 + 18c_4)\, z^2 + \cdots = 0 \tag{22}$$

Note that all terms of order $\frac{1}{z^3}$ have cancelled, while the term of the order $\frac{1}{z^2}$ remained as $-\frac{2c_0}{z^2}$. Since the LHS should be equal to 0, this yields $c_0 = 0$, from which it also follows that $c_4 = 0$ also. This, however, is very important as it demonstrates that the coefficient c_1 *is free* and together with x_* constitutes *the two arbitrary constants* needed to claim that our expansion (20) represents the *general solution* of the problem near this singularity!

Note, however, that throughout this analysis, there is no way to cancel the term $2/z$ present in (43a)! This is an interesting development that occurs often in the analysis of systems that are not solvable by elementary functions. It simply implies that a Laurent series expansion involving *only integer powers of* z in (20) is *not sufficient* and necessitates the inclusion of logarithmic terms in our series. This implies in our case that a term of the form $zlnz$ must be included among the first few terms of our expansion, which now reads:

$$y = \frac{2}{z} + c_1 z + dz \ln z + c_2 z^2 \cdots \tag{23}$$

and hence the first terms that need to be substituted in the original equation of our system are

$$y = \frac{2}{z} + c_1 z + dz \ln z + c_2 z^2 \cdots \tag{24}$$

$$y' = -\frac{2}{z^2} + c_1 + d \ln z + d + 2c_2 z + 3c_3 z^2 \cdots \tag{25}$$

$$y'' = \frac{4}{z^3} + \frac{d}{z} + 2c_2 + 6c_3 z + 12c_4 z^2 \cdots \tag{26}$$

$$yy' = -\frac{4}{z^3} + \frac{2d}{z} + 2c_2 + (4c_3 + c_1^2 + c_1 d + d^2 \ln z + c_1 d \ln z) z + (6c_4 + 3c_1 c_2 + dc_2 + 3dc_2 lnz)z^2 \tag{27}$$

Inserting the above equations into Eq. (3), we obtain:

$$\frac{3d}{z} + \frac{2}{z} + 4c_2 + \left(10c_3 + c_1 + c_1^2 + c_1 d + d^2 \ln^2 z + (c_1 + 1) d \ln z\right) z + (18c_4 + c_2 + 3c_1 c_2 + dc_2 + 3dc_2 \ln z)\, z^2 + \cdots = 0$$

We now use the coefficient d to cancel the terms proportional to $\frac{1}{z}$:

$$\frac{3d}{z} + \frac{2}{z} = 0 \tag{28}$$

This yields $d = -\frac{2}{3}$ and finally

$$
\begin{aligned}
y &= \frac{2}{z} + c_1 z - \frac{2}{3} z \ln z + c_3 z^3 + d_1 z^3 \ln z + d_2 z^3 \ln z^2 \cdots && \text{(29a)}\\
y' &= -\frac{2}{z^2} + c_1 - \frac{2}{3}\ln z - \frac{2}{3} + 3c_3 z^2 + 3d_1 z^2 \ln z + d_1 z^2 + 3d_2 z^2 (\ln z)^2 + 2d_2 z^2 \ln z \cdots && \text{(29b)}\\
y'' &= (6c_3 + 5d_1 + 2d_2)\, z + (6d_1 + 10d_2)\, z \ln z + 6d_2 z (\ln z)^2 \cdots && \text{(29c)}\\
yy' &= \left(4c_3 + 2d_1 + c_1^2 - \frac{2c_1}{3}\right) z + \left(4d_1 + 4d_2 - \frac{4c_1}{3} + \frac{4}{9}\right) z \ln z + \left(4d_2 + \frac{4}{9}\right) z (\ln z)^2 && \text{(29d)}
\end{aligned}
$$

where the equations for the different z- dependent terms are:

$$
\begin{aligned}
z: \quad 6c_3 + 5d_1 + 2d_2 + 4c_3 + 2d_1 + c_1^2 - 2/3c_1 + c_1 = 0 && \text{(30a)}\\
10c_3 + 7d_1 + 2d_2 + c_1/3 + c_1^2 = 0 && \text{(30b)}\\
z \ln z: \quad 6d_1 + 10d_2 + 4d_1 + 4d_2 - 4c_1/3 + 4/9 - 2/3 = 0 && \text{(30c)}\\
10d_1 + 14d_2 - 4c_1/3 - 2/9 = 0 && \text{(30d)}\\
z \ln z^2: \quad 6d_2 + 4d_2 + 4/9 = 0 && \text{(30e)}\\
d_2 = -2/45 && \text{(30f)}
\end{aligned}
$$

To determine the value of x_* for our given initial conditions, we first need to set $x = 0$ in our series

$$
y(x) = \frac{2}{x - x_*} - \frac{2(x - x_*)\ln(x - x_*)}{3} + \sum_{i=1}^{\infty} c_i (x - x_*)^i \tag{31}
$$

Keeping only the first few terms we have for the choice of initial conditions $y(0) = 1$ and $y'(0) = 1$

$$
y(0) = -\frac{2}{x_*} + c_1(-x_*) - \frac{2}{3}(-x_*)\ln(-x_*) \stackrel{!}{=} 1 \tag{32a}
$$

$$
y'(0) = -\frac{2}{x_*^2} + c_1 - \frac{2}{3}\ln(-x_*) - \frac{2}{3} \stackrel{!}{=} 1 \tag{32b}
$$

while for the particular initial conditions $y(0) = 1$ and $y'(0) = 1$ we obtain the following two equations for the unknowns c_1 and x_*:

$$
-\frac{2}{x_*^2} - c_1 + \frac{2}{3}\ln(-x_*) = \frac{1}{x_*} \tag{33}
$$

$$
-\frac{2}{x_*^2} + c_1 - \frac{2}{3}\ln(-x_*) = \frac{5}{3} \tag{34}
$$

Adding these equations, we get

$$
c_1 = -\frac{1}{2x_*} + \frac{5}{6} + \frac{2}{3}\ln(-x_*) \tag{35}
$$

and substituting in the second we find

$$-\frac{2}{x_*^2}-\left(-\frac{1}{2x_*}+\frac{5}{6}+\frac{2}{3}\ln(-x_*)\right)+\frac{2}{3}\ln(-x_*)=\frac{1}{x_*} \tag{36a}$$

$$-\frac{2}{x_*^2}-\frac{1}{2x_*}=5/6 \tag{36b}$$

$$5x_*^2+3x_*+12=0 \tag{36c}$$

$$x_*=-0.3\pm 1.52i \tag{36d}$$

$$|x_*|=\sqrt{0.09+1.52^2}=1.55 \tag{36e}$$

$$\tan\theta=-\frac{1.55}{0.3}\rightarrow\theta=1.768 \tag{36f}$$

$$x_*=1.55(-\frac{0.3}{1.55}\pm\frac{1.52\imath}{1.55})=1.55exp(i\theta) \tag{36g}$$

This is an important result. It tells us that the singularity is approximately located at a distance of nearly 1.55 from $x=0$ and explains why we found divergence of our Taylor series in Section 2.

Note that from the above, we can substitute x_* into $Eq(27)$ and find the coefficient of c_1, using the principal argument to rewrite $\ln(-x_*)$ as $ln(-x_*)=\ln(x_*)+i(\pi+\theta)$ in our equations as follows:

$$c_1=-\frac{2}{|x_*|^2}exp(-2i\theta)+\frac{2}{3}(\ln(x_*)+i(\theta+\pi))+-\frac{1}{|x_*|}exp(-i\theta) \tag{37a}$$

$$c_1=-\frac{2}{|x_*|^2}(\cos 2\theta-i\sin 2\theta)+\frac{2\ln|x_*|}{3}+\frac{2i(\theta-\pi)}{3}-\frac{\cos\theta-\imath\sin\theta}{|x_*|} \tag{37b}$$

$$c_1=0.768-0.32i+0.3-0.913i+0.126+0.632i=1.2-0.6i \tag{37c}$$

$$d_1=-\frac{14d_2}{10}+\frac{4c_1}{30}+\frac{2}{90} \tag{37d}$$

$$d_1=0.244-0.08i \tag{37e}$$

$$c_3=-0.7d_1-0.2d_2-\frac{c_1}{30}-0.1c_1^2 \tag{37f}$$

$$c_3=-0.31+0.22i \tag{37g}$$

So, from the above we can now determine all coefficients that enter in the Laurent series expansion of y, and thus have an alternative solution of our equation, which complements the results of the Taylor series solution in real x away from $x=0$. In Appendix A, at the end of the Project, we repeat the above calculation for a different initial condition and find that the solutions converge over a longer x interval. Thus, one can use these results to determine wider regions of convergence of the Taylor series solutions.

4 Solutions of the NODE as orbits in the x,y plane

As we mentioned earlier, our second order NODE can be written as a system of two first order ODEs,
$x'(t)=-yx-y,\ y'(t)=x$
where differentiation now is with respect to a real variable t that we call the time. Dividing these two equations by sides we are thus let to the ratio of two differentials that may be

interpreted as the derivative of y with respect to x:

$$\frac{dy}{dx} = \frac{x}{-y(x+1)}. \tag{38}$$

We now make the crucial observation that this equation can be integrated once directly as follows:

$$\tag{39a}$$

$$-\int y dy = \int \frac{x dx}{x+1} \tag{39b}$$

$$-\frac{y^2}{2} = \int dx - \int \frac{1 dx}{x+1} \tag{39c}$$

$$-\frac{y^2}{2} = x - \ln|x+1| + C \tag{39d}$$

$$\ln|x+1| - C = x + \frac{y^2}{2} \tag{39e}$$

where C is an arbitrary constant. We will now proceed to use this integral to plot our solutions as orbits in the x, y plane as follows: Case 1: We will assume $x+1 > 0$, whence the above expression can be exponentiated to read

$$x + 1 = ke^{x+\frac{y^2}{2}} \tag{40}$$

where the new arbitrary constant $k > 0$. Note that for $k = 0$ we find the invariant axis solution $x + 1 = 0$. Solving the above equation for y^2 now gives: Case1:

$$y^2 = 2\ln[\frac{(x+1)e^{-x}}{k}] \tag{41}$$

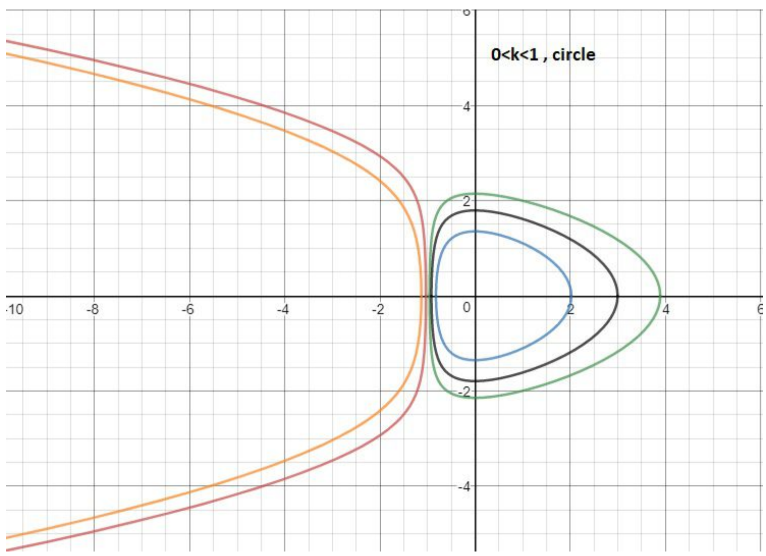

Figure 2: The case for $x > 1$

In Figure 2, we plot these solutions in the x, y plane on the right of the $x + 1 = 0$ axis. On the other hand, for Case 2: $x < -1$, we have:

$$y^2 = 2\ln[-(x+1)e^{-x}/k] \tag{42}$$

Plotting these curves on the left of the $x + 1 = 0$ axis we obtain the curves shown in Figure 3.

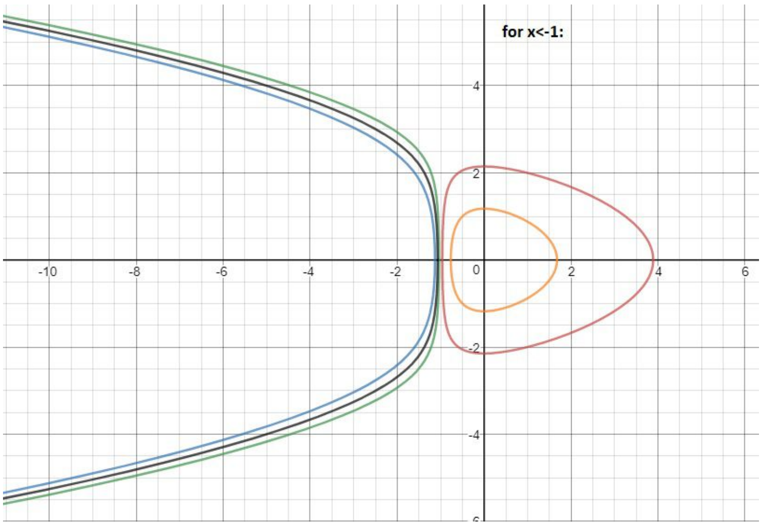

Figure 3: The case for $x < -1$

5 Conclusions

In this paper, we have shown that PSS is an applicable method to solve nonlinear ODE. In order to achieve the result, the second order nonlinear differential equation has been transformed into an equation of algebraic series. Further, it lead to a reccurence relation, which makes possible to find the series terms. This technique, in an easier way, gives us an analytic solution of our equation and can be approximated till the necessary degrees. The efficiency of the result has been illustrated by comparing an analytical and exact solutions. Thus, it is seen from the graph that accuracy of the PSSM increases by adding new terms into the series, which means that PSSM solution is an accurate and the same as the general solution. [3].

The given analytical solution does not have a full information about the behavior of the graph. Once we know it, the probability of the existence of singularity could be checked by the series expansion with a singular term, so called "Painleve test". The test is not that straightforward, therefore by adding new terms to expansion, we obtain the coefficients to find the location of singularity, which depends on initial conditions. Since, the graph of the exact solution has not got the singularity, Painleve test has been used to check by adding new terms. To sum up, this work shows that the PSSM is one of the general techniques to solve nonlinear differential equation, that is supported by graphics and also it makes possible to analyze different characteristics of ODE.

6 References

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7 Appendix A: Solutions for different initial conditions

7.1 Different initial conditions in the y, y' plane

The first case we will consider will be the initial NODE in $y = y(x)$ that we examined in Section 2 of this project. Let us recall our equations (32a) and (32b) and set the right hand sides equal to 0.1, i.e much smaller than the case $y(0) = y'(0) = 1$ we had considered in Section 2. In this case, we would expect to find different values of the radius of convergence for our Taylor series. Indeed, starting with

$$y(0) = -\frac{2}{x_*} + c_1(-x_*) - \frac{2}{3}(-x_*)\ln(-x_*) \stackrel{!}{=} 0.1 \tag{43a}$$

$$y'(0) = -\frac{2}{x_*^2} + c_1 - \frac{2}{3}\ln(-x_*) - \frac{2}{3} \stackrel{!}{=} 0.1 \tag{43b}$$

and

$$-\frac{2}{x_*^2} - c_1 + \frac{2}{3}\ln(-x_*) = \frac{0.1}{x_*} \tag{44}$$

$$-\frac{2}{x_*^2} + c_1 - \frac{2}{3}\ln(-x_*) = \frac{23}{30} \tag{45}$$

and summing the above equations, we get

$$c_1 = -\frac{1}{20x_*} + \frac{23}{60} + \frac{2}{3}\ln(-x_*) \tag{46}$$

$$\frac{23}{60}x^2 - \frac{x}{20} + 2.1 = 0 \tag{47}$$

$$x_* = 0.0652174 \pm 2.3397i \tag{48a}$$

$$|x_*| = \sqrt{0.0652174^2 + 2.33397^2} = 2.341 \tag{48b}$$

$$\tan\theta = -\frac{2.341}{0.0652} \rightarrow \theta = 1.598 \tag{48c}$$

$$x_* = 2.341(\frac{00652}{2.341} \pm \frac{2.3397i}{2.341}) = 2.341\exp(i\theta) \tag{48d}$$

This clearly demonstrates that for these initial conditions the region of convergence has grown significantly in comparison to the case we considered earlier, since the radius of the circle of convergence in the complex plane has grown to 2.341. If we wanted to complete the calculation and find also the arbitrary constant c_1 in this case we would proceed as follows:

$$c_1 = -\frac{2}{|x_*|^2}(\cos 2\theta - i\sin 2\theta) + \frac{2\ln|x_*|}{3} + \frac{2i(\theta-\pi)}{3} - \frac{\cos\theta - i\sin\theta}{|x_*|} \tag{49a}$$

$$c_1 = -0.194 - 0.9963i \tag{49b}$$

$$d_1 = -\frac{14d_2}{10} + \frac{4c_1}{30} + \frac{2}{90} \tag{49c}$$

$$d_1 = 0.0586 - 0.133i \tag{49d}$$

$$c_3 = -0.7d_1 - 0.2d_2 - \frac{c_1}{30} - 0.1c_1^2 \tag{49e}$$

$$c_3 = 0.0698 + 0.0876i \tag{49f}$$

7.2 Different initial conditions in the x,y plane

Let us suppose now that we wanted to choose our initial conditions from the x, y plane shown in Fig. 2 and Fig.3 and not from the variable $y = y(x)$, as $y(0), y'(0)$. We will thus be able to connect our results on the location of singularities with some of the geometric features present in these figures. To this end we write our variables as functions of t and obtain their Laurent series expansions as follows:

$$y(t) = \frac{2}{t-t_*} + c_1(t-t_*) - \frac{2}{3}(t-t_*)\ln(t-t_*) \tag{50}$$

and since $x = y'$:

$$x(t) = -\frac{2}{(t-t_*)^2} + c_1 - \frac{2}{3}\ln(t-t_*) - \frac{2}{3} \tag{51}$$

Hence:

$$y(0) = -\frac{2}{t_*} + c_1(-t_*) + \frac{2}{3}t_*\ln(t-t_*) \tag{52a}$$

$$x(0) = -\frac{2}{(t_*)^2} + c_1 - \frac{2}{3}\ln(-t_*) - \frac{2}{3} \tag{52b}$$

$$\frac{y(0)}{t_*} = -\frac{2}{(t_*)^2} - c_1 + \frac{2}{3}\ln(t-t_*) \tag{52c}$$

Adding them up we have:

$$x(0) + \frac{y(0)}{t_*} = -\frac{4}{(t_*)^2} - \frac{2}{3} \tag{53}$$

And for $y(0) = 0$ we get:

$$x(0) = -\frac{4}{(t_*)^2} - \frac{2}{3} \tag{54}$$

or

$$-\frac{4}{(t_*)^2} = x(0) + \frac{2}{3} \tag{55}$$

Let us suppose in the above calculations that we wanted to look for real singularities, i.e. $t_* \in \Re$, which is the case corresponding to solutions that blow up in real time. As is evident in Fig.2 and 3 these solutions occur on the left side of the $x = -1$ axis. In terms of our above calculations, $t_*^2 > 0$ in the above formula and hence we would expect real singularities, if $x(0) < -\frac{2}{3}$. This result is a good approximation of the value $x(0) < -1$ that we have obtained from our analysis of our system.