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Orbital dynamics around black holes and massive strings

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Abstract

For this thesis, we studied equatorial orbits around Schwarzschild black holes and around axially symmetric vacuum solutions known as Levi-Civita metric. We then compared trajectories of test particles falling into both types of objects. One of the observations is that graph of effective potential for Levi-Civita metric changes depending on the value of σ, linear energy density of the source. The metric has a stable timelike circular geodesics only at values $\sigma < 1/4$. Another observation is that metric does not show asymptotical flatness as $r \to \infty$ like Schwarzschild does.

1 Introduction

Our understanding of gravity started with Newton's theory that described it as a force between two masses that is inversely proportional to the distance between them:

$$\vec{F} = G\frac{m_1 m_2}{|\vec{r_2} - \vec{r_1}|^2}\hat{r} \tag{1}$$

where G is the gravitational constant, m_1 and m_2 are the two masses, $\vec{r_1}$ and $\vec{r_2}$ are their position vectors, and $\hat{r}$ is the unit vector pointing from one mass to the other.

This law explained many phenomena including orbital movement of planets, physics of tides, and projectile motion on the surface of the Earth. Newton's law of gravity was so profound that it only met criticism after three centuries. It was inconsistent with some experimental observations in the early 20th century, particularly those related to the precession of the perihelion of Mercury. This gave a hint that we needed a new theory which emerged after Albert Einstein's theory of Special Relativity (SR).

In the late 19th century, physicists wanted to measure the motion of the Earth relative to the ether - an assumed medium of propagation of light at the time. Two physicists, Michelson and Morley, came up with an experimental design where they measure the difference in phase of light rays that travel the same distance but in perpendicular direction to each other. One light ray was supposed to travel faster due to traveling against the direction of Earth's rotation but results showed almost no difference in time of propagation. Speed of light was constant in all directions, so this led to heated discussions on the nature of light's medium which ended with Einstein's SR theory.

Newtonian gravity assumes that time is absolute and that gravitational effects propagate instantaneously across space [1]. In contrast, SR introduced relative time and postulated that the speed of light is the maximum speed limit for any information transfer, including gravity. These two premises fundamentally contradicted the assumptions of Newton's theory, creating the need for a new theory of gravitation with the principles of SR and that would capture many phenomena that were explained by Newton's theory.

One of the key insights of SR is the existence of an invariant quantity that all inertial observers can agree upon – the spacetime interval between two events. Time and space are not separate anymore but combined into four-dimensional continuum called spacetime. Consequently we can no longer use familiar distance formula such as Euclidean one from classical physics and have to replace them with alternatives like the Minkowski metric that describes flat spacetime. It is expressed as [1]:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(cdt)^2 + dx^2 + dy^2 + dz^2 \tag{2}$$

where c is the speed of light, $g_{\mu\nu}$ is metric tensor which defines the geometry of spacetime. This equation describes the geometry of flat spacetime and forms the mathematical foundation upon which SR is built.

While developing SR, Einstein also noticed an interesting similarity between the effects of acceleration and gravity. For instance, an observer in a space shuttle cannot distinguish whether the force they feel is due to gravity or constant acceleration. Einstein coined this observation as the equivalence principle – the idea that locally, the effects of gravity are indistinguishable from those of acceleration. In a combination with SR, this principle guided Einstein toward a more complete theory, which became General Relativity (GR).

GR revolutionized our understanding of gravity. Instead of treating gravity as a force acting at a distance, Einstein reinterpreted it as a manifestation of the curvature of spacetime caused by the presence of mass and energy. As Hartle precisely said it, "The central idea of general relativity is that gravity arises from the curvature of spacetime – the four-dimensional union of space and time. Gravity is geometry" [1].

In this geometric view, objects in free fall are not subjected to any force but move along geodesics – paths that represent the "straightest possible" trajectories in curved spacetime [1]. In empty space, such as Minkowski described in equation 2, geodesics are straight lines, while in the presence of any mass or energy, the spacetime gets curved and geodesics follow this curvature despite not appearing to be the shortest path to us. Simplest illustration of this is comparing shortest path between any two cities on a globe and on a flat map used in schools. Curved geometry can also be described using equation 2 but with different $g_{\mu\nu}$, so the last part of the equation will be different. The geodesic equation is given as:

$$\ddot{x}^{\sigma} + \Gamma^{\sigma}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0 \tag{3}$$

Here, $\ddot{x}^{\mu}$ and $\dot{x}^{\mu}$ represent the second and first derivatives of the coordinate x^{μ} with respect to proper time, and the Christoffel symbols $\Gamma^{\sigma}_{\mu\nu}$ account for the properties of the geometry and use of curved or non-inertial coordinate systems.

The Christoffel symbols are calculated from the metric tensor $g_{\mu\nu}$. Specifically, they are given by:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\mu\rho} - \partial_{\rho}g_{\mu\nu}) \tag{4}$$

In this equation, $g^{\sigma\rho}$ is the inverse of the metric tensor $g_{\sigma\rho}$, and the partial derivatives describe how the metric changes with position. These terms allow to compare vectors at different points on a curved manifold.

The link between spacetime curvature and its physical sources – mass, energy, momentum, and pressure – is described by the Einstein Field Equations (EFE):

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{5}$$

On the left-hand side, $G_{\mu\nu}$ is the Einstein tensor, which depends on the Ricci curvature tensor $R_{\mu\nu}$ and the Ricci scalar R. These geometric quantities describe the intrinsic curvature of spacetime and are obtained from $g_{\mu\nu}$ and its derivatives. On the right-hand side, $T_{\mu\nu}$ is the stress-energy tensor that has information on the energy density, momentum, and stress (pressure and shear) of the matter and fields present in spacetime. Constants G and c represent Newton's gravitational constant and the speed of light, respectively. They will be set as $G = c = 1$ for simplicity in further equations.

The Einstein Field Equations are a system of ten coupled nonlinear second-order partial differential equations. They are to GR what Maxwell's equations are to electromagnetism. Just as Maxwell's equations relate electric sources to electromagnetic fields, the EFEs relate the distribution of mass-energy, gravitational sources, to the curvature of spacetime, gravitational fields. Usually we must make assumptions on one side of the equation to obtain results on the other side because otherwise there are many mathematical solutions that bear no physical meaning.

A particularly interesting assumption is saying that stress-energy tensor is zero:

$$T_{\mu\nu} = 0 \tag{6}$$

Substituting this into the EFEs simplifies them to:

$$R_{\mu\nu} = 0 \tag{7}$$

It results in important class of solutions to the EFEs – the vacuum solutions, which describe spacetime in regions devoid of any matter or energy. Even though they may seem too simple, these vacuum equations still allow meaningful models, such as the Schwarzschild solution, which describes the spacetime geometry surrounding a spherical non-rotating mass, or Weyl solutions, which are more general case of Schwarzschild with axial symmetry instead of spherical symmetry. These solutions form the basis for our modern understanding of black holes and gravitational waves.

2 Introduction to the Schwarzschild Solution

The vacuum solutions are considered class of simple solutions and among them the simplest non-flat solution to Einstein's equations is the Schwarzschild solution [2], discovered by Karl Schwarzschild in 1916. This solution describes the gravitational field outside a non-rotating, spherical mass M. While the solution applies to the vacuum region outside a sphere of radius r_b, we can extend the solution to the case where the mass is concentrated at a central point, resulting in the description of a static black hole.

2.1 Derivation of the Schwarzschild Metric

The Schwarzschild solution is derived under several key assumptions: staticity, spherical symmetry and vacuum. The first means that the metric components do not depend on the time coordinate x^0. The second describes a solution of which the metric can be partially described using the metric on the unit sphere, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The third implies that the energy-momentum tensor vanishes. Given these assumptions, the line element can be expressed as:

$$ds^2 = g_{AB}dx^A dx^B + Cd\Omega^2 \tag{8}$$

where A, B = 0, 1, and g_{AB} are functions of x^1 only. We choose the function $C(x^1)$ such that slices of constant x^0 and x^1 have a surface area $A = 4\pi r^2$. This leads to $C = r^2$, allowing us to use r as the radial coordinate x^1. By setting the time coordinate as t, we can diagonalize the 2×2 part of the metric $g_{AB}dx^A dx^B$. Solving Einstein's equations with these conditions yields the Schwarzschild metric [2]:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{9}$$

The coordinates t, r, θ, ϕ are known as Schwarzschild coordinates. From equation 9, we see that r has several values which yield interesting results.

2.2 Asymptotic Flatness

As r approaches infinity, the Schwarzschild metric reduces to the Minkowski metric in spherical coordinates:

$$ds^2 \approx -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{10}$$

This property, called asymptotic flatness, indicates that spacetime becomes flat at spatial infinity.

2.3 Singularities

At values $r = 0$ and $r = 2M$ the Schwarzschild metric exhibits two notable singularities. In former case, the metric coefficient $g_{00} = -(1 - \frac{2M}{r})$ diverges. This is a true singularity where spacetime curvature becomes infinite. In latter case, the metric coefficient $g_{11} = \frac{1}{1-\frac{2M}{r}}$ diverges. This is a coordinate singularity that can be removed by changing coordinates, and $r_s = 2M$ is known as the Schwarzschild radius, defining the event horizon.

2.4 Geodesics for Massive Test Particles

We consider a test particle moving on a geodesic $x^\mu(\lambda)$ with 4-velocity $u^\mu = dx^\mu/d\lambda$. The Lagrangian per unit mass is:

$$2\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2M}{r}\right)} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 \tag{11}$$

Since $\mathcal{L}$ does not depend on t and ϕ, we have two conserved quantities: energy E and angular momentum L:

$$-E = \frac{\partial\mathcal{L}}{\partial\dot{t}} = -\left(1 - \frac{2M}{r}\right)\dot{t} \qquad\qquad L = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2\sin^2\theta\dot{\phi}$$

These conserved quantities and normalization of four-velocity $u^\mu u_\mu = -1$ can be used to simplify the equations of motion as well as setting $\theta = \pi/2$ which will not make equations less general because the metric has spherical symmetry and motion occurs on a plane:

$$2\mathcal{L} = -1 = -\frac{E^2}{\left(1 - \frac{2M}{r}\right)} + \frac{\dot{r}^2}{\left(1 - \frac{2M}{r}\right)} + \frac{L^2}{r^2} \tag{12}$$

If we multiply both sides by $\frac{1-\frac{2M}{r}}{2}$ and rearrange terms, we will obtain classical expression for total energy:

$$\frac{E^2 - 1}{2} = E_{new} = \frac{\dot{r}^2}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} \tag{13}$$

Now equation depends on r and $\dot{r}$ only while other terms are constants. The left-hand side is total energy, the first term on right-hand side is kinetic energy and the rest is an effective potential U_{eff}.

2.5 Circular Orbits

The conditions for circular orbits are $\dot{r} = 0$ and $\ddot{r} = 0$. From the equation 13 we may see the conditions on U_{eff} from imposing $r\ddot{}=\dot{r} = 0$:

$$\begin{cases} E_{new}^2 = U_{eff}(r) = -\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} \\ \frac{dU_{eff}}{dr} = \frac{M}{r^2} - \frac{L^2}{r^3} + \frac{3ML^2}{r^4} = 0 \end{cases} \tag{14}$$

From evaluating the second expression of equation 14 and finding the roots, we can obtain the extrema of the first expression which are the radii of possible circular orbits:

$$r_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12M^2L^2}}{2M} \tag{15}$$

For $L^2 < 12M^2$, there are no extrema, and particles spiral into the central singularity. For $L^2 > 12M^2$, there are two extrema as seen in figure 1: a maximum and a minimum, corresponding to unstable and stable circular orbits, respectively. At these values particles can be at the same distance from the center of the black hole and not fall into singularity. The one closest to the center of the black hole is called innermost stable circular orbit (ISCO) and it occurs at $L^2 = 12M^2$, with a radius:

$$r_{ISCO} = 6M \tag{16}$$

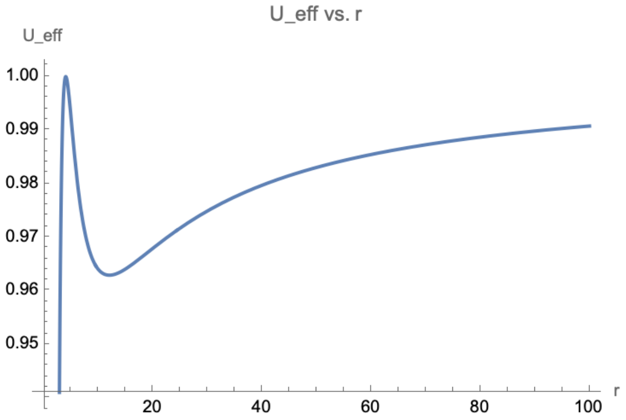

Figure 1: Effective potential for the motion of a massive test particle with $L^2 > 12M^2$ in the Schwarzschild geometry.

2.6 Radial Fall and Escape Velocity

For a particle falling radially ($\dot{\theta} = \dot{\phi} = 0$) from infinity with zero initial velocity, the equation of motion is:

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}} \tag{17}$$

The proper time to reach the singularity at r = 0 is finite:

$$\tau(0) = \frac{2r_0^{3/2}}{3\sqrt{2M}} \tag{18}$$

where r_0 is the initial position of the particle. On the other hand the coordinate time t for a radially falling particle as measured by an observer at infinity diverges as $r \to 2M$, meaning the observer at infinity never sees the particle cross the event horizon. The escape velocity for a particle moving radially outward from a radius $r > 2M$ is:

$$v_{esc} = \sqrt{\frac{2M}{r}} \tag{19}$$

At the event horizon ($r = 2M$), the escape velocity equals the speed of light.

2.7 Relativistic Free Fall and Newtonian Force

By reintroducing the mass m, the speed of light c, and the gravitational constant G, we can rewrite the equation of motion for a radially falling particle and derive an expression that resembles the Newtonian gravitational force:

$$m\frac{d^2r}{d\tau^2} = -\frac{GmM}{r^2} \tag{20}$$

In General Relativity, this equation describes a particle in free fall in curved spacetime.

3 Axially Symmetric Solutions and Weyl Spacetime

In general relativity, we assume symmetric shape of many astrophysical objects to simplify EFEs and make it possible to find exact solutions. Spherical symmetry is often unrealistic so we assume symmetry along one axis of rotation – axial symmetry. It includes many shapes like spheres, cylinders, elongated spheres and so on. Despite using word "rotation" in a definition of axial symmetry, we will be considering non-rotating bodies that have deformed from spherically symmetric shape. Weyl spacetime is an important class of solutions that describes static, axially symmetric vacuum spacetimes [3]. These solutions are particularly relevant for understanding the gravitational field outside non-rotating, deformed bodies.

3.1 Weyl Metric

The Weyl metric is typically expressed as [4]:

$$ds^2 = -e^{2\Psi}dt^2 + e^{-2\Psi}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right] \tag{21}$$

where t is the time coordinate, ρ and z are cylindrical coordinates, ϕ is the azimuthal angle, Ψ and γ are functions of ρ and z only [3].

3.2 Einstein's Equations for Weyl Spacetime

Weyl spacetime describes vacuum, so as explained in equation 7, the EFEs reduce to the Ricci tensor being zero. It simplifies further due to the symmetries. The key equation is the Laplace equation in cylindrical coordinates for the metric function Ψ:

$$\frac{\partial^2 \Psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Psi}{\partial \rho} + \frac{\partial^2 \Psi}{\partial z^2} = 0 \tag{22}$$

The other metric function, γ, is determined by Ψ through the following relations [4]:

$$\begin{cases} \frac{\partial \gamma}{\partial \rho} = \rho \left[\left(\frac{\partial \Psi}{\partial \rho}\right)^2 - \left(\frac{\partial \Psi}{\partial z}\right)^2 \right] \\ \frac{\partial \gamma}{\partial z} = 2\rho \frac{\partial \Psi}{\partial \rho}\frac{\partial \Psi}{\partial z} \end{cases} \tag{23}$$

Thus, solving the Laplace equation for Ψ provides the fundamental solution, and γ can be obtained through line integration.

3.3 Example: Schwarzschild in Weyl Coordinates

The Schwarzschild solution, which is has a special case of cylindrical symmetry – spherical symmetry, can be expressed in Weyl coordinates [4]:

$$\Psi_{Sch} = \frac{1}{2} \ln \frac{L - M}{L + M} \tag{24}$$

where $L = \frac{1}{2}(l_+ + l_-)$, $l_+ = \sqrt{\rho^2 + (z+M)^2}$, $l_- = \sqrt{\rho^2 + (z-M)^2}$, M is the mass of the black hole.

Interestingly, Schwarzschild which is considered gravitational pure monopole is not Weyl monopole. It is obvious if we express Ψ as infinite sum:

$$\Psi = \sum \frac{a_n}{r^{n+1}} P_n(\cos\theta) \tag{25}$$

Weyl monopole is defined with only non-zero a_0 coefficient and is called Curzon solution [4]. However, for Schwarzschild, coefficient a_n is given as:

$$\begin{cases} a_{2n} = -\frac{M^{2n+1}}{2n+1} \\ a_{2n+1} = 0 \end{cases} \tag{26}$$

4 The Chazy-Curzon metric

The Chazy-Curzon metric can be obtained at such values of Ψ and γ that will be plugged into equation 21 [5] [6]:

$$\begin{cases} \psi = -\frac{m}{\sqrt{\rho^2+z^2}} \\ \gamma = -\frac{m^2\rho^2}{2(\rho^2+z^2)^2} \end{cases} \tag{27}$$

The metric is static, axially symmetric and vacuum solution. The metric is not asymptotically flat. It is derived as point-like mass but due to unusual behavior of singularity, it is sometimes interpreted as an extended source – line mass.

The singularity is at $\rho = 0$, $z = 0$ and it is physical singularity because Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ which is considered curvature scalar that does not depend on coordinates also blows up. Interestingly, it is anisotropic singularity, meaning that it blows up differently as one approaches as $z = 0$, $\rho \to 0$ than $\rho = 0$, $z \to 0$. This property makes us doubt whether it is truly point-like mass or extended one. Another interesting property of Curzon metric's singularity is its nakedness. It has no event horizon that covers it which violates cosmic censorship. Thus, Chazy-Curzon metric is often used as toy model rather than real one. However, it is great model to study behavior of test particles in axially symmetric spacetime [5] [6].

5 Levi-Civita metric

If Chazy-Curzon can be interpreted as line mass, its extension to infinity is Levi-Civita metric. It also was not observed but it is intriguing model to study. Sometimes to observe interesting astrophysical phenomena in the field of GR, scientists study trajectories of test particles around massive objects with different metrics and then search for effects of behavior of test particle in studied metric. In this thesis, we consider behavior of test particles in another metric apart from Schwarzschild – Levi-Civita.

The Levi-Civita metric is one of the most fundamental exact solutions to Einstein's field equations, describing a static, cylindrically symmetric vacuum spacetime. Unlike the Schwarzschild solution, which models a point mass in an asymptotically flat universe, the Levi-Civita metric represents the gravitational field of an infinite line mass [3]. Some scientists interpret Levi-Civita as the limiting case of γ metric also known as Zipoy–Voorhees metric [7][8], which "in Newtonian image source is given by a finite rod of matter" [9]. γ metric itself is often linked to Schwarzschild metric through one of the parameters. The presence of an infinitely long source leads to significant differences in the global structure of spacetime, making the Levi-Civita metric an important tool in both theoretical and astrophysical applications.

In cylindrical coordinates the Levi-Civita metric takes the form [10]:

$$ds^2 = -f(\rho)dt^2 + e^{\mu(\rho)}(d\rho^2 + dz^2) + l(\rho)d\phi^2 \tag{28}$$

where, t, ρ, z, ϕ are cylindrical coordinates with t - time, ρ - radius in xy plane, z - axis from xyz coordinates, ϕ - angle in xy plane,

$$f(\rho) = a\rho^{4\sigma}, e^{\mu(\rho)} = \frac{1}{a}\rho^{4\sigma(2\sigma-1)}, l(\rho) = \frac{1}{a}\rho^{2(1-2\sigma)} \tag{29}$$

Here, a and σ are constants that are associated with angular defect and linear energy density, respectively. The metric represents infinite line mass and has timelike circular geodesics only if $\sigma < \frac{1}{4}$ [10].

For Levi-Civita $g_{\mu\nu}$ looks as follows:

$$g_{\mu\nu} = \begin{pmatrix} -a\rho^{4\sigma} & 0 & 0 & 0 \\ 0 & \frac{\rho^{4\sigma(2\sigma-1)}}{a} & 0 & 0 \\ 0 & 0 & \frac{\rho^{4\sigma(2\sigma-1)}}{a} & 0 \\ 0 & 0 & 0 & \frac{\rho^{2(1-2\sigma)}}{a} \end{pmatrix} \tag{30}$$

Its non-zero Christoffel symbols are calculated as follows:

$$\begin{cases} \Gamma^t_{t\rho} = \Gamma^t_{\rho t} = \frac{2\sigma}{\rho} \\ \Gamma^\rho_{tt} = 2a^2\sigma\rho^{-8\sigma^2+8\sigma-1} \\ \Gamma^\rho_{\rho\rho} = \frac{2\sigma(2\sigma-1)}{\rho} \\ \Gamma^\rho_{zz} = -\frac{2\sigma(2\sigma-1)}{\rho} \\ \Gamma^\rho_{\phi\phi} = (2\sigma-1)\rho^{1-8\sigma^2} \\ \Gamma^z_{\rho z} = \Gamma^z_{z\rho} = \frac{2\sigma(2\sigma-1)}{\rho} \\ \Gamma^z_{\rho\phi} = \Gamma^z_{\phi\rho} = \frac{1-2\sigma}{\rho} \end{cases} \tag{31}$$

The Levi-Civita metric satisfies the Einstein field equations in vacuum, reducing the Ricci tensor to zero. To further study trajectories of massive particles, we will use following form of Lagrangian equation for massive particles [11]:

$$\mathcal{L} = \kappa = g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} \tag{32}$$

where κ comes from the normalization of 4-velocity $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \dot{x}^\mu\dot{x}_\mu = \kappa$. $\kappa = -1$ for massive particles and $\kappa = 0$ for photons.[1].

6 Results

For every symmetry, there must be a conserved quantity. Schwarzschild metric is static and spherically symmetric, meaning that $\mathcal{L}$ does not depend on t and ϕ and therefore we have conservation of energy E and angular momentum L of the test particle. For Levi-Civita we additionally have translational invariance along the line, so $\mathcal{L}$ does not depend on z and therefore there is extra conserved quantity Z:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \dot{t}} = -E = 4f\dot{t} \\ \frac{\partial \mathcal{L}}{\partial \dot{z}} = Z = 4e^{\mu}\dot{z} \\ \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = L = 4l\dot{\phi} \end{cases} \tag{33}$$

We can use the conserved quantities to reduce the system to a one dimensional equation of motion:

$$\mathcal{L} = -\frac{E^2}{4f} + e^{\mu}\dot{\rho}^2 + \frac{Z^2}{4e^{\mu}} + \frac{L^2}{4l} = \kappa \tag{34}$$

The scope of this thesis will cover motion on a plane perpendicular to the infinite line, thus we can set Z=0. Plugging in the values of functions $f(\rho), e^{\mu(\rho)}, l(\rho)$ from equation 29:

$$-\frac{E^2}{4a\rho^{4\sigma}} + \frac{1}{a}\rho^{4\sigma(2\sigma-1)}\dot{\rho}^2 + \frac{L^2}{4\frac{1}{a}\rho^{2(1-2\sigma)}} = \kappa \tag{35}$$

Expressing equation for E^2:

$$E^2 = 4\rho^{8\sigma^2}\dot{\rho}^2 + \frac{a^2L^2}{\rho^{2(1-4\sigma)}} - \kappa \cdot 4a\rho^{4\sigma} \tag{36}$$

We want to have a simple $E^2 = \dot{r}^2 + U_{eff}$ form. Currently kinetic energy is non-linear, so we change coordinates:

$$\dot{r}^2 = 4\rho^{8\sigma^2}\dot{\rho}^2 \tag{37}$$

This simplifies to:

$$r = \frac{2}{4\sigma^2+1}\rho^{4\sigma^2+1} \tag{38}$$

$$\rho = \left(\frac{4\sigma^2+1}{2}r\right)^{\frac{1}{4\sigma^2+1}} = \left(\frac{\alpha}{2}r\right)^{\frac{1}{\alpha}} \tag{39}$$

where $\alpha = 4\sigma^2 + 1$.

Consequently equation 36 is reexpressed as $E^2 = \dot{r}^2 + U_{eff}$ where the effective potential for Levi-Civita metric is as follows:

$$U_{eff} = a^2L^2\left(\frac{\alpha}{2}r\right)^{\frac{2(4\sigma-1)}{\alpha}} - \kappa \cdot 4a\left(\frac{\alpha}{2}r\right)^{\frac{4\sigma}{\alpha}} \tag{40}$$

where r, defined in equation 38, rescales equation of total energy obtained and match classical form of Kinetic energy $\frac{1}{2}m\dot{r}^2$.

To get circular orbits, we repeat the process described in equation 14:

$$\begin{cases} U_{eff} = E^2 = a^2L^2\left(\frac{\alpha}{2}r\right)^{\frac{2(4\sigma-1)}{\alpha}} - \kappa\cdot 4a\left(\frac{\alpha}{2}r\right)^{\frac{4\sigma}{\alpha}} \\ \frac{\partial U_{eff}}{\partial r} = a^2L^2(4\sigma-1)\left(\frac{\alpha}{2}r\right)^{\frac{2(4\sigma-1)}{\alpha}-1} - \kappa\cdot 8a\sigma(\frac{\alpha}{2}r)^{\frac{4\sigma}{\alpha}-1} = 0 \end{cases} \tag{41}$$

From this set of equations we get expressions for E and L to get circular orbits given some radius:

$$\begin{cases} E^2 = a^2L^2\left(\frac{\alpha}{2}r\right)^{\frac{2(4\sigma-1)}{\alpha}} - \kappa\cdot 4a\left(\frac{\alpha}{2}r\right)^{\frac{4\sigma}{\alpha}} \\ L^2 = \kappa\cdot\frac{8\sigma}{a(4\sigma-1)}(\frac{\alpha}{2}r)^{\frac{2-4\sigma}{\alpha}} \end{cases} \tag{42}$$

Combining these equations, we get expression for effective potential of circular orbits:

$$U_{eff} = E^2 = 4a\kappa\cdot\frac{2\sigma-1}{4\sigma-1}\left(\frac{\alpha}{2}r\right)^{\frac{4\sigma}{\alpha}} \tag{43}$$

Here, we see condition $\sigma \neq \frac{1}{4}$ is necesary for U_{eff} not to blow up. Furthermore, if $\sigma > 1/4$ there is no potential well, and consequently no circular orbits. Interestingly, for photons ($\kappa = 0$), the effective potential is zero.

Setting $\kappa = -1$ for massive particles, a = 1 and plotting $\sigma = 0.1$, $\sigma = 0.4$, $\sigma = 0.9$ we get the figure 2 shown below:

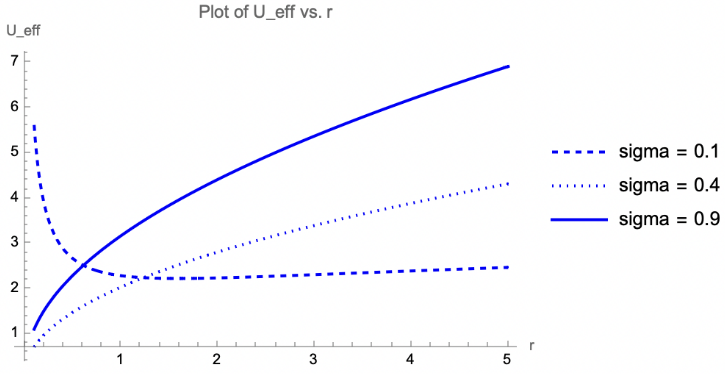

Figure 2: Effective Potential vs radius for Levi-Civita at different σ, a=1, $\kappa = -1$

It is clear that only at $\sigma = 0.1$ the function has minima – stable circular orbit.

Now we are interested in trajectory of the particle in such potential and slightly perturbed energy. To obtain it, we consider expressions for $\dot{\phi}$ and $\dot{r}$ from equations 33 and 40:

$$\begin{cases} \dot{\phi} = \frac{L}{4l} = \frac{aL}{4}\left(\frac{\alpha}{2}r\right)^{2(2\sigma-1)} \\ \dot{r}^2 = E^2 - U_{eff} = \sqrt{E^2 - a^2L^2\left(\frac{\alpha}{2}r\right)^{\frac{2(4\sigma-1)}{\alpha}} - \kappa\cdot 4a\left(\frac{\alpha}{2}r\right)^{\frac{4\sigma}{\alpha}}} \end{cases} \tag{44}$$

Combining two equations, we get differential equation that can be solved via separation of variables and numerical integration in Mathematica:

$$\frac{\dot{r}}{\dot{\phi}} = \frac{\sqrt{E^2 - a^2L^2\left(\frac{\alpha}{2}r\right)^{\frac{2(4\sigma-1)}{\alpha}} - \kappa\cdot 4a\left(\frac{\alpha}{2}r\right)^{\frac{4\sigma}{\alpha}}}}{\frac{aL}{4}\left(\frac{\alpha}{2}r\right)^{2(2\sigma-1)}} \tag{45}$$

We have axial symmetry, so initial value ϕ can be anything and we can choose zero for convenience:

$$\phi - \phi_0 = \int_{r_0}^{r} \frac{\frac{aL}{4}\left(\frac{\alpha}{2}s\right)^{2(2\sigma-1)}}{\sqrt{E^2 - a^2L^2\left(\frac{\alpha}{2}s\right)^{\frac{2(4\sigma-1)}{\alpha}} - \kappa\cdot 4a\left(\frac{\alpha}{2}s\right)^{\frac{4\sigma}{\alpha}}}} ds \tag{46}$$

where s is dummy variable for radius coordinate r.

If we plug in values for E and L slightly larger than those in equation 42, and calculate integral for massive particles in range of r from 0 to 1.4, we notice that the particle is striving to single value of radius as energy is less perturbed. This indicates that we indeed have circular orbits at given E and L in equation 42:

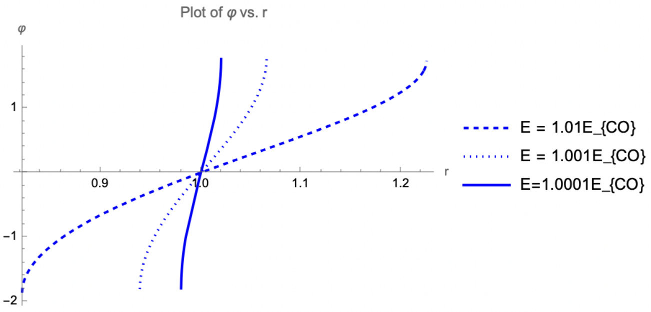

Figure 3: Trajectory of massive particle given as ϕ vs. r in Levi-Civita spacetime with $E = 1.01E_{CO}, 1.001E_{CO}, 1.0001E_{CO}$, a = 1, $\sigma = \frac{1}{8}$, $\phi_0 = 0$, $r_0 = 1$, $L = L_{CO} = \frac{1}{2}$, $E_{CO} = \frac{3}{2}$

7 Discussion and Conclusion

This work compared two vacuum, static geometries: Schwarzschild and Levi-Civita. The former is spherically symmetric and the simplest possible solution of EFEs apart from Minkowski, while the latter belongs to Weyl class of solutions that describe cylindrically symmetric spacetime. The former can be interpreted as pure gravitational monopole, while the latter as infinite line mass. We used analytical derivations and numerical simulations to get the expressions for effective potential, plots of trajectories of massive particles under energy perturbations, and values of energy and angular momentum through which we obtained circular orbits.

From the analysis of Schwarzschild metric, we reaffirmed known facts about bound and unbound orbits such as value of ISCO $r_{ISCO} = 6M$ or radii of possible circular orbits as mentioned in equations 16 and 15 respectively. Particularly interesting was the figure 1 showing effective potential for the motion of a massive particle in Schwarzschild geometry. It shows two extrema, asymptotical flatness as $r \to \infty$ and will be compared to similar graph for Levi-Civita. Another noteworthy moment is that Schwarzschild has two conserved quantities – energy due to staticity and total angular momentum due to spherical symmetry as mentioned in section 2.4.

In analysis of Levi-Civita metric, we had more nuanced dynamics. It has a parameter σ which must be less than $1/4$ to obtain any circular orbits as explained after equation 43. Regarding symmetries, Levi-Civita has conserved energy due to independence from time, angular momentum around z-axis due to cylindrical symmetry, and translational constant Z due to translational symmetry along z-axis. The metric has only a minima as illustrated in figure 2. Another difference from effective potential of Schwarzschild is that Levi-Civita does not show asymptotical flatness at any value of σ except for $\sigma = 0$ at which the whole metric is flat.

The research of Levi-Civita metric is interesting and may be useful in exploring mechanisms of black hole formation in non-spherical scenarios. Most of the massive objects in the Universe are not perfectly spherical. Some models suggest that the collapse of non-spherical objects are more likely to form string-like geometries such as Levi-Civita rather than spherical ones such as Schwarzschild. It is due to the fact that oblate spheroids have boundaries from perihelion to aphelion. Mass at perihelion gets pulled more as the relation between gravitational force and distance is inverse. Consequently, massive object should squeeze into cylindrical shape rather than sphere. This kind of objects may be described by axially symmetric solutions such as the γ metric. In the limit of extremely large mass and infinite length of cylinder, we obtain massive string with surrounding spacetime described by Levi-Civita metric [9]. Interestingly the γ metric is "continuously linked with Schwarzschild space-time" as well as to the Levi-Civita spacetime [9]. Thus, Levi-Civita metric is worth investigating.

Further research can be conducted on motion of photons, studying trajectory on a plane not perpendicular to the axis of symmetry ($Z \neq 0$) and comparing Levi-Civita with Curzon or γ metrics.

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