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A proof of Anđelić-Fonseca conjectures on the determinant of some Toeplitz matrices and their generalization

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ABSTRACT

In this work, we present and prove the explicit formula for the determinant of a class of $n \times n$ nonsymmetric Toeplitz matrices. Setting up one of the nonzero subdiagonals to zero results in special pentadiagonal Toeplitz matrices, whose determinant formulas are conjectured by Anđelić and Fonseca in Anđelić [Some determinantal considerations for pentadiagonal matrices. Linear Multilinear Algebra. 2020. DOI:10.1080/03081087.2019.1708845]. By using the explicit formula with this setting, the conjectures are therefore proved.

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1. Introduction

In this note, we derive an explicit formula for the determinant of an $n \times n$ six-diagonal Toeplitz matrix

$$B_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & a & b \\ 1 & 1 & 1 & \ddots & \ddots & 0 & a \\ 1 & 1 & 1 & 1 & \ddots & \ddots & 0 \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}_{n \times n}, \quad (1)$$

where $a, b \in \mathbb{R}$. Setting (i) $a = 0$ and $b = 1$ and (ii) $a = 1$ and $b = 0$ lead to pentadiagonal Toeplitz matrices, where the nonzero subdiagonals are not immediately above and below the diagonal. For B_n with these specific settings, the determinant formulas are given by Anđelić and Fonseca as Conjectures 6.1 and 6.2, respectively, in [1]. The two conjectures are stated below, now as theorems, to make our presentation self-contained.

Theorem 1.1 (Conjecture 6.1 of [1]; $a = 0, b = 1$): Let

$$A_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 1 & 1 & \ddots & \ddots & \ddots & 0 \\ 1 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}_{n \times n}.$$

Then

$$|A_n| = \begin{cases} 1, & \text{if } n \equiv 0 \pmod{4}, \\ 2, & \text{if } n \equiv 1 \pmod{4}, \\ -1, & \text{if } n \equiv 2 \pmod{4}, \\ 0, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Theorem 1.2 (Conjecture 6.2 of [1]; $a = 1, b = 0$): Let

$$M_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 1 & 0 \\ 1 & 1 & 1 & \ddots & \ddots & \ddots & 1 \\ 1 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}_{n \times n}.$$

Then

$$|M_n| = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{4}, \\ 2, & \text{if } n \equiv 1 \pmod{4}, \\ 3, & \text{if } n \equiv 2 \pmod{4}, \\ 1, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Since the matrix A_n and M_n are just two special cases of (1), a proof of the Anđelić-Fonseca conjectures can be done using the explicit determinant formula for the more general matrix (1). This formula is stated as Theorem 2.1, whose proof is discussed in Section 2.

Notation. The notation $|\cdot|$ denotes the determinant of the matrix in the argument, whose size is known and indicated by the subscript, like B_n . In addition, $|\cdot|_n$ denotes the determinant of an $n \times n$ matrix, whose entries are shown in the argument.

2. Proof of conjectures

This section is devoted to the derivation of the determinant formula for the matrix B_n defined in (1), which is used to prove Theorems 1.1 and 1.2.

We begin by stating our general result.

Theorem 2.1: *Let B_n be the $n \times n$ matrix given by (1), with $n \geq 5$. Then*

$$|B_n| = \begin{cases} (a-1)^2, & \text{if } n \equiv 0 \pmod{4}, \\ a^2 + b + 1, & \text{if } n \equiv 1 \pmod{4}, \\ a^2 + 2a - b, & \text{if } n \equiv 2 \pmod{4}, \\ a^2, & \text{if } n \equiv 3 \pmod{4}. \end{cases} \quad (2)$$

Notice that the result in the above theorem leads to Theorems 1.1 and 1.2 if a and b are set accordingly.

To prove Theorem 2.1, we need two lemmas. We shall start with the following important lemma.

Lemma 2.2: *Let*

$$|C_n| = \begin{vmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & \cdots & \vdots \\ 1 & 1 & 1 & 1 & 0 & \cdots & \vdots \\ 0 & 1 & 1 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{vmatrix}_n.$$

Then $|C_n| = |C_{n-4}|$ for $n \geq 5$.

Proof: The proof is based on a sequence of cofactor expansions, starting with cofactor expansion along the first row of C_n . In this case,

$$|C_n| = (-1)^{1+1} \begin{vmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & \cdots & \vdots \\ 1 & 1 & 1 & 1 & 0 & \cdots & \vdots \\ 0 & 1 & 1 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{vmatrix}_{n-1}$$

$$\begin{aligned}
& + (-1)^{1+2} \begin{vmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & \cdots & \vdots \\ 0 & 1 & 1 & 1 & 0 & \cdots & \vdots \\ 0 & 1 & 1 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{vmatrix}_{n-1} \\
& =: |C_{n-1}| - |T_{n-1}|.
\end{aligned}$$

The cofactor expansion on T_{n-1} along the first row results in $|T_{n-1}| = |C_{n-2}| - |C_{n-3}|$.

Thus, $|C_n| = |C_{n-1}| - |C_{n-2}| + |C_{n-3}|$, and hence $|C_{n-1}| = |C_{n-2}| - |C_{n-3}| + |C_{n-4}|$, for $n \geq 5$. Finally, substitution of $|C_{n-1}|$ into $|C_n|$ leads to $|C_n| = |C_{n-2}| - |C_{n-3}| + |C_{n-4}| - |C_{n-2}| + |C_{n-3}| = |C_{n-4}|$. ■

We remark here that one can directly prove Theorem 1.1 using the cofactor expansion of $|A_n|$ along the first row and applying the recurrence relation in Lemma 2.2. For the latter, $|C_1|$, $|C_2|$, $|C_3|$, and $|C_4|$ need to be computed.

Lemma 2.3: *Let*

$$|T_n| = \begin{vmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & a \\ 1 & 1 & 1 & \ddots & \ddots & \ddots & 0 \\ 1 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 1 \end{vmatrix}_n$$

Then

$$|T_n| = |C_{n-2}|(1 + a(-1)^{n+1}) - 2|C_{n-3}| + 2|C_{n-4}| - |C_{n-5}|,$$

for $n \geq 6$.

Proof: Consider the cofactor expansion of $|T_n|$ along the first row as follows:

$$|T_n| = \begin{vmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 1 & \ddots & \ddots & \ddots & 0 \\ 1 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \cdots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 1 \end{vmatrix}_{n-1} - \begin{vmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 1 & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \cdots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 1 \end{vmatrix}_{n-1}$$

$$\begin{aligned}
 & + a(-1)^{n+1} \begin{vmatrix} 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \ddots & \ddots & 0 \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 & 1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{vmatrix}_{n-1} \\
 & =: |M_{n-1}| - |N_{n-1}| + a(-1)^{n+1} |V_{n-1}|. \tag{3}
 \end{aligned}$$

The first determinant on the right-hand side, $|M_{n-1}|$, can be expressed in terms of $|C|$'s via cofactor expansion along the last column of M_{n-1} : $|M_{n-1}| = |C_{n-2}| - |C_{n-3}|$. For the middle term, applying cofactor expansion along the first column yields the relation

$$\begin{aligned}
 |N_{n-1}| &= \begin{vmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 1 & \ddots & \ddots & \ddots & 0 \\ 1 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 1 \end{vmatrix}_{n-2} - \begin{vmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 1 & \ddots & \ddots & \ddots & 0 \\ 1 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 1 \end{vmatrix}_{n-2} \\
 & =: |M_{n-2}| - |\tilde{N}_{n-2}| \\
 & = |M_{n-2}| - |M_{n-3}|,
 \end{aligned}$$

To get the last equation, we have used the relation $|\tilde{N}_{n-2}| = |M_{n-3}|$, which is obtained from the cofactor expansion along the first row of $\tilde{N}_{n-2}$. Finally, for the last term on the right-hand side of (3), we expanded along the last row of V_{n-1} and used the fact that $|\cdot^T| = |\cdot|$ to get $|V_{n-1}| = |C_{n-2}|$.

Combining all results, we get

$$\begin{aligned}
 |T_n| &= (|C_{n-2}| - |C_{n-3}|) - (|C_{n-3}| - |C_{n-4}|) + (|C_{n-4}| - |C_{n-5}|) + a(-1)^{n+1} |C_{n-2}| \\
 &= |C_{n-2}|(1 + a(-1)^{n+1}) - 2|C_{n-3}| + 2|C_{n-4}| - |C_{n-5}|.
 \end{aligned}$$

■

We now prove Theorem 2.1.

Proof of Theorem 2.1: Consider the cofactor expansion of $|B_n|$ along the last column as follows:

$$\begin{aligned}
 |B_n| &= b(-1)^{n+1} \begin{vmatrix} 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \ddots & \ddots & 0 \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 1 \end{vmatrix}_{n-1} \\
 &+ a(-1)^{n+2} \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & a \\ 1 & 1 & 1 & 1 & \ddots & \ddots & 0 \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 1 \end{vmatrix}_{n-1} \\
 &- \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & a \\ 1 & 1 & 1 & \ddots & \ddots & \ddots & 0 \\ 1 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 1 \end{vmatrix}_{n-1} + \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & a \\ 1 & 1 & 1 & \ddots & \ddots & \ddots & 0 \\ 1 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{vmatrix}_{n-1} \\
 &=: b(-1)^{n+1}|C_{n-1}| + a(-1)^{n+2}|G_{n-1}| - |T_{n-1}| + |P_{n-1}|. \tag{4}
 \end{aligned}$$

The cofactor expansion of $|G_{n-1}|$ along the first row results in

$$|G_{n-1}| = \begin{vmatrix} 1 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 & 1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{vmatrix}_{n-2} - \begin{vmatrix} 1 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 & 1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{vmatrix}_{n-2}$$

$$\begin{aligned}
 & + a(-1)^n \begin{vmatrix} 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 1 \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{vmatrix}_{n-2} \\
 & = |C_{n-2}| - |C_{n-3}| + (-1)^n a,
 \end{aligned}$$

after expanding the middle determinant along the the first column, followed by the use of the rule $|\cdot^T| = |\cdot|$.

We now expand the last determinant on the right-hand side of (4) along the first row to get the relation $|P_{n-1}| = |C_{n-2}| - |H_{n-2}| + a(-1)^n |C_{n-2}|$, where

$$|H_{n-2}| = \begin{vmatrix} 1 & 1 & 0 & \dots & \dots & 0 & 0 \\ 1 & 1 & 1 & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 1 & 1 & 1 \\ 0 & \dots & \dots & 0 & 1 & 1 & 1 \end{vmatrix}_{n-2}.$$

The cofactor expansion along the first column yields $|H_{n-2}| = |C_{n-3}| - |C_{n-4}|$. Hence, $|P_{n-1}| = |C_{n-2}|(1 + a(-1)^n) - |C_{n-3}| + |C_{n-4}|$.

Substitution of $|G_{n-1}|$ and $|P_{n-1}|$ to (4) and applying Lemma 2.3 lead to, for $n \geq 7$,

$$|B_n| = 3|C_n| + |C_{n-1}|(b(-1)^{n+1} - 2) + 2(|C_{n-2}| - |C_{n-3}|)(1 + a(-1)^n) + a^2.$$

Note that

$$|C_1| = 1, \quad |C_2| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}_2 = 0, \quad |C_3| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}_3 = 0, \quad |C_4| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix}_4 = 1.$$

Combining this with Lemma 2.2, we have $|C_{4k+1}| = 1$, $|C_{4k+2}| = 0$, $|C_{4k+3}| = 0$ and $|C_{4k+4}| = 1$ for $k = 0, 1, 2, \dots$. Using this relation in the equation for $|B_n|$ above leads to the theorem result for $n \geq 7$. For $n = 5, 6$, the determinant of B_n can be computed using the cofactor expansion along the first row and the fact that $|C_5| = 1$ and $|C_6| = 0$. ■

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Reference

- [1] Anđelić M, da Fonseca C. M. Some determinantal considerations for pentadiagonal matrices. *Linear Multilinear Algebra*. 2020. DOI:[10.1080/03081087.2019.1708845](https://doi.org/10.1080/03081087.2019.1708845)