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Remainders of the higher-order Poincaré inequalities for Baouendi-Grushin vector fields

by

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Abstract

The main result of this thesis work is a proof of the higher-order Poincaré inequalities for Baouendi-Grushin vector fields on non-smooth domains. The finding is successfully applied to the study of the higher-order Fisher-KPP equation. And some classical vector calculus' theorems for Baouendi-Grushin vector fields are discussed.

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Chapter 1

Introduction

1.1 Notations

Hereinafter, despite sections Historical review and Celebrated inequalities, we consider x, y, z, and m, k, n as follows. $z = (x, y) = (x_1, ..., x_m, y_1, ..., y_k) \in \mathbb{R}^m \times \mathbb{R}^k = \mathbb{R}^n$ with $m, k \geqslant 1$ and $m + k = n$.

Throughout this work we assume that $\Omega \subset \mathbb{R}^n$ is a finite, multiply connected, open domain with a piecewise smooth boundary $\partial\Omega$.

λ_1 is the first eigenvalue of the negative Dirichlet Baouendi-Grushin operator and e_1 is a corresponding eigenfunction in $\overline{\Omega}$ [MP09, Theorem 6.4]:

$$\begin{cases} -\Delta_\gamma e_1 = \lambda_1 e_1 & \text{in } \ \Omega, \\ e_1 = 0 & \text{on } \ \partial\Omega. \end{cases}$$

l is always regarded as an arbitrarily chosen natural number. Whenever it is used, we imply the statement holds for all natural numbers $\{1, 2, ...\}$.

1.2 Preliminaries

Let us consider the vector fields

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, ..., m, \qquad Y_j = |x|^\gamma \frac{\partial}{\partial y_j}, \quad j = 1, ..., k, \quad \gamma \geqslant 0.$$

The corresponding gradient is defined by

$$\nabla_\gamma := (X_1, ..., X_m, Y_1, ..., Y_k) = (\nabla_x, |x|^\gamma \nabla_y).$$

The Baouendi-Grushin operator on $\mathbb{R}^{m+k}$ is defined by

$$\Delta_\gamma = \sum_{i=1}^{m} X_i^2 + \sum_{j=1}^{k} Y_j^2 = \Delta_x + |x|^{2\gamma}\Delta_y = \nabla_\gamma \cdot \nabla_\gamma,$$

where Δ_x and Δ_y stand for the standard Laplacians in the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$ respectively.

The anisotropic dilation attached to Δ_γ on $\mathbb{R}^n$ is defined by

$$\delta_\lambda(z) = \left(\lambda x, \lambda^{1+\gamma} y\right)$$

for $\lambda > 0$, and the homogeneous dimension with respect to this dilation is

$$Q = m + (1 + \gamma)k. \tag{1.1}$$

The corresponding distance function from the origin on $\mathbb{R}^n$ is defined by

$$\rho(z) = \rho(x,y) = \left(|x|^{2(1+\gamma)} + (1+\gamma)^2|y|^2 \right)^{\frac{1}{2(1+\gamma)}} . \tag{1.2}$$

A fundamental result of the classical vector fields' theory is Gauss-Ostrogradsky divergence theorem. Although in this treatise we deal with a non-classical case, the statement plays a crucial role here. For the sake of mathematical precision, we claim the theorem as follows [IP82, ch. 7, sec. 3, Th. 7.5, p. 188].

Theorem (***Gauss-Ostrogradsky divergence theorem***).
Let $\Omega \subset \mathbb{R}^n$ be a finite, multiply connected, open domain with piecewise smooth boundary $\partial\Omega$ and let $f \in C^1(\overline{\Omega})$. Then

$$\int\limits_{\Omega} \mathbf{div} f \, dz = \oint\limits_{\partial\Omega} f \cdot \mathbf{n} \, d\nu,$$

or, in a more convenient notation,

$$\int\limits_{\Omega} \nabla \cdot f \, dz = \oint\limits_{\partial\Omega} f \cdot d\boldsymbol{\nu}.$$

Note that the set Ω complies exactly the same conditions as in **??**. In fact, we simply follow the restrictions on the set imposed by Ilyin and Poznyak.

Another classical vector fields' result, which will play an essential role in this thesis, is Green's first identity. Thanks to ***Gauss-Ostrogradsky divergence theorem*** and [Wei], we formulate the theorem as follows.

Theorem (***Green's first identity***).
Let $\Omega \subset \mathbb{R}^n$ be a finite, multiply connected, open domain with piecewise smooth boundary $\partial\Omega$ and let $f \in C^2(\overline{\Omega})$, $g \in C^1(\overline{\Omega})$. Then

$$\int\limits_{\Omega} \Delta f \, g \, dz = \oint\limits_{\partial\Omega} f \, \nabla g \cdot d\boldsymbol{\nu} - \int\limits_{\Omega} \nabla f \cdot \nabla g \, dz.$$

1.3 Historical review

This historical review is inspired by Lorenzo D'Ambrosio and Sandra Lucente' introductory review in the article [DL03]. They have given a clue to delineate a connection between the Tricomi and the Baouendi-Grushin operators: the Baouendi-Grushin operator is a specific generalization of the Tricomi operator. Therefore it is interesting to trace development of the Tricomi type operators' theory. The aims of this review is to show evolution of study of the second order mixed-type partial differential equations on the example of the Tricomi and the Baouendi-Grushin operators and, notably, honour those who have made significant contribution to the theory of the above-mentioned operators. The primary question this review endeavors to answer is why the theory of the Tricomi equation have not developed to contain the Baouendi-Grushin operator as a particular case.

- 1923, Francesco Giacomo Tricomi pioneeres in the study of mixed-type equations, particularly,

  $$y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

  He proved existence and uniqueness of a boundary-value problem for this equation with a very specific boundary enclosing the degenaration point $y = 0$. Later, this problem for mixed type equations would be named after him [Tri23].

- 1926, Erik Albert Holmgren determines Green's function for a boundary-value problem with the equation
$$y^m z_{xx} + z_{yy} = 0, \quad m \in \mathbb{N},$$
and boundary C to be some analytic curve ("the normal curve") [Hol26].

- 1935, Sven Gellerstedt solves boundary-value problem for the equation
$$y^{2m-1} z_{xx} + z_{yy} - cz = F(x, y), \quad m \in \mathbb{N}, \ c > 0$$
with "assumptions, which are essentially the same as those made by Tricomi in [Tri23]" (translated from French from the thesis) [Gel35].

- 1938, Sven Gellerstedt investigates boundary problems for the equation
$$y^{2m-1} z_{xx} + z_{yy} = 0, \quad m \in \mathbb{N}$$
with another boundary profiles [Gel38].

- 1945, Feliks Isidorovich Frankl' commences a new stage in the study of mixed-type equations. He studied the Tricomi problem for the Chaplygin equation
$$K(y)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (K(0) = 0, \ K'(y) > 0).$$
He proved uniqueness of this problem with the condition
$$F(y) = 2\left(\frac{K}{K'}\right)' + 1 \tag{1.3}$$
to be non-negative when $y < 0$. Particularly, these conditions are satisfied by $K(y) = y^{2l-1}$ [Fra45].

- 1953, 1955, Murray Harold Protter generalizes Frankl's uniqeness theorem: $F(y)$, as in (1.3), is permitted to take negative values for $y < 0$ [Pro53], [Pro55].

- 1954, Murray Harold Protter proves uniqueness theorem in the three-dimensional case. The equation is
$$K(z)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2 u}{\partial z^2} = 0,$$
and the function K is continuous and
$$K(0) = 0, \quad K'(z) \neq 0 \ \text{ for } \ z < 0,$$
while F from (1.3) is subject to $F(z) > 0$ for $z \leqslant 0$. As the author assures, the results can be translated to the multidimensional case [Pro55]
$$K(t)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \ldots + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\partial^2 u}{\partial t^2} = 0.$$

- 1967, Mohammed Salah Baouendi establishes hypoellipticity [THÉORÈME (III. 2).] of a class of operators
$$A(D) = \Lambda^*\Lambda + \varphi^\rho B(D)\varphi^\rho, \tag{1.4}$$
where Λ is an operator of order 1 defined in $\overline{V}$ and transversal to ∂V, $B(D)$ is an elliptic operator of order 2 in $\overline{V}$, φ is a function C^∞ such that $\varphi = 0$ on ∂V. Finally ρ is a positive integer.

It is essential to note that the author does not mention the Baouendi-Grushin operator itself. Indeed, the operator

$$\frac{\partial^2 u}{\partial x^2} + |x|^{2r} \frac{\partial^2 u}{\partial y^2}, \quad r \in \mathbb{N}$$

complies perfectly with *Corollaire du THÉORÈME (III. 2)* (Chapitre III, section 1), which deals with elliptic operators of type (1.4), degenerate inside the domain.

The hypoellipticity result is obtained by extending the author's discoveries for the case when the operator (1.4) is degenerate on the boundary of the domain [Bao67].

- 1970, Viktor Vasil'evich Grushin establishes hypoellitcity of the operator

$$\Delta_x + |x|^{2r} \Delta_y, \quad x \in \mathbb{R}^m,\ y \in \mathbb{R}^k,\ r \in \mathbb{N}.$$

Despite chronological pioneership of Baouendi in this discovery, Grushin rightfully deserves his merit in the study of the operator. Not only he mentioned operators of the form

$$\Delta_x^l + |x|^{2r} \Delta_y^l, \quad r, l \in \mathbb{N},$$

but also used a new-at-the-time theory of differential operators which he refers to works of Lars Valter Hörmander, "the foremost contributor to the modern theory of linear partial differential equations" (Wolf Foundation. The 1988 Wolf Foundation Prize In Mathematics) [Gru70].

- 2009, Dario Daniele Monticelli and Kevin Ray Payne acquire the most seminal result on the theory of the Baouendi-Grushin operator to the moment: existence and positivity of the principal (first) eigenvalue and a.e. strict positivity (essential infimum is positive) of an associated eigenfunction of the negative Dirichlet Baouendi-Grushin operator [MP09].

As we can see, study of the Tricomi type equations is a relatively old topic. First, there was only a two-dimensional equation

$$y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (1923)$$

with no modulus and power of the coefficient. The next step was considering the natural power of the coefficient

$$y^m z_{xx} + z_{yy} = 0, \quad m \in \mathbb{N}. \ (1926) \tag{1.5}$$

An inhomogeneous variant of the equation with a lower-order term came in 1935

$$y^{2m-1} z_{xx} + z_{yy} - cz = F(x, y), \qquad m \in \mathbb{N},\ c > 0.$$

Back to (1.5), only 12 years later it would be investigated under different boundary conditions.

A new stage of the study started in 1945 with the equation

$$K(y) \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (K(0) = 0,\ K'(y) > 0).$$

While the boundary was still leaving of Tricomi's type, the coefficient was being further generalized. The struggle for a wider class of coefficient functions continued during 1953–1955.

The next generalization came in 1954 when a three-dimensional case was firstly considered

$$K(z) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial^2 u}{\partial z^2} = 0$$

with a potential extension to a multidimensional setting.

Protter's 1954 work is the only found work on the dimension $\geqslant 3$. Wo, we conclude that the study of the Tricomi equation stopped dimensional extension. A good insight on why this happened is provided in [PHNS17]: "...However, the multidimensional case is rather different, and there is no general understanding of the situation. Even the question of well posedness is not completely resolved".

1.4 Celebrated inequalities

In this section, Q stands for the homogeneous dimension from (1.1), ρ is the corresponding distance function (1.2).

As discovered by Nicola Garofalo [Gar93], the classical Hardy inequality [RS10] can be extended to the Baouendi-Grushin Laplacian as follows:

$$\int_{\mathbb{R}^n} |\nabla_\gamma u|^2 \, dz \geqslant \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|x|^{2\gamma}}{\rho^{2(1+\gamma)}} u^2 \, dz$$

where $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$.

Further, Lorenzo D'Ambrosio found the following weighted L^p Hardy type inequality [D'A04]:

$$\int_V |\nabla_\gamma u|^p \, |x|^{\beta-\gamma p} \rho^{(1+\gamma)p-\alpha} \, dz \geqslant \left(\frac{Q+\beta-\alpha}{p}\right)^p \int_V |u|^p \, |x|^\beta \rho^{-\alpha} \, dz,$$

where $u \in C_0^\infty(V)$, $V \subset \mathbb{R}^n$ is open, ρ is the distance function (1.2), $p > 1$, $\alpha, \beta \in \mathbb{R}$ such that $Q > \alpha - \beta - p$ and $m > \gamma p - \beta$. Moreover, the constant $[\,(Q+\beta-\alpha)/p\,]^p$ is sharp whenever $0 \in V$.

After, Ismail Kombe and Abdullah Yener in [KY18] derived a general weighted Hardy type inequality with a remainder term: if $0 \leqslant v \in C^1(\mathbb{R}^n)$ and $0 \leqslant w \in L^1_{loc}(\mathbb{R}^n)$ and $0 < \varphi \in C^\infty(\mathbb{R}^n)$ satisfy

$$-\nabla_\gamma \cdot \left(v \, |\nabla_\gamma \varphi|^{p-2} \, \nabla_\gamma \varphi\right) \geqslant w \varphi^{p-1}$$

almost everywhere in $\mathbb{R}^n$, then there exists a positive constant $c_p = c(p)$ such that for $p \geqslant 2$

$$\int_{\mathbb{R}^n} v|\nabla_\gamma u|^p \, dz \geqslant \int_{\mathbb{R}^n} w|u|^p \, dz + c_p \int_{\mathbb{R}^n} v \left|\nabla_\gamma \frac{u}{\varphi}\right|^p \varphi^p \, dz$$

and for $1 < p < 2$

$$\int_{\mathbb{R}^n} v|\nabla_\gamma u|^p \, dz \geqslant \int_{\mathbb{R}^n} w|u|^p \, dz + c_p \int_{\mathbb{R}^n} \frac{v \left|\nabla_\gamma \frac{u}{\varphi}\right|^2 \varphi^2}{\left(\left|\frac{u}{\varphi} \nabla_\gamma u\right| + \left|\nabla_\gamma \frac{u}{\varphi}\right| u\right)^{2-p}} \, dz,$$

where $u \in C_0^\infty(\mathbb{R}^n)$ is arbitrarily chosen.

Another type of famous connecting norm of a function and of its derivative is Rellich inequality. The pioneer of this property for Baouendi-Grushin vector fields might be Ismail Kombe who have proven two Rellich type inequalities in [Kom07]. The first one:

$$\int_{\mathbb{R}^n} \frac{\rho^\alpha}{|\nabla_\gamma \rho|^2} |\Delta_\gamma u|^2 \, dz \geqslant \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \int_{\mathbb{R}^n} \rho^\alpha \frac{|\nabla_\gamma \rho|^2}{\rho^4} |u|^2 \, dz, \tag{1.6}$$

where $u \in C_0^\infty(\,\mathbb{R}^n \setminus \{0\}\,)$ and $\alpha > 2$. Moreover, the constant $(Q+\alpha-4)^2\,(Q-\alpha)^2/16$ is sharp.

The second Rellich type inequality:

$$\int_{\mathbb{R}^n} \rho^\alpha \frac{|\Delta_\gamma u|^2}{|\nabla_\gamma \rho|^2} \, dz \geqslant \frac{(Q-\alpha)^2}{4} \int_{\mathbb{R}^n} \rho^\alpha \frac{|\nabla_\gamma u|^2}{\rho^2} \, dz$$

where $u \in C_0^\infty(\,\mathbb{R}^n \setminus \{0\}\,)$ and $2 < \alpha < Q$. Furthermore, the constant $(Q-\alpha)^2/4$ is sharp.

Later, in 2012, Shou-feng Shen and Yong-yang Jin prove L^p type Rellich inequlity (1.6)

$$\int\limits_{\mathbb{R}^n} |\Delta_\gamma u|^2 \rho^{\alpha+2p} |\nabla_\gamma \rho|^{2-2p} dz \geqslant \left(\frac{(Q+\alpha)\left((p-1)(Q-2)-\alpha-2 \right)}{p^2} \right)^p \int\limits_{\mathbb{R}^n} |u|^2 \rho^\alpha |\nabla_\gamma \rho|^2 dz$$

where $p>1$, $-Q<\alpha<(p-1)(Q-2)-2$, $u\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$. Moreover, the constant

$$\left(\frac{(Q+\alpha)\left((p-1)(Q-2)-\alpha-2 \right)}{p^2} \right)^p$$

is sharp [SJ12].

The most enormous success in studying Rellich type inequalities for Baouendi-Grushin vector fields is L^p Rellich inequality by Kombe and Yener [KY17]. The result is as follows: for $p>1$ if $0\leqslant a\in C^2(\mathbb{R}^n)$ and $0\leqslant b\in L^1_{loc}(\mathbb{R}^n)$ and $0<\varphi\in C^\infty(\mathbb{R}^n)$ satisfy

$$\Delta_\gamma\big(a\,|\Delta_\gamma v|^{p-2}\,\Delta_\gamma v\big)\geqslant bv^{p-1} \quad \text{and} \quad -\Delta_\gamma v>0$$

almost everywhere in $\mathbb{R}^n$, then

$$\int\limits_{\mathbb{R}^n} a|\Delta_\gamma u|^p\,dz \geqslant \int\limits_{\mathbb{R}^n} b|u|^p\,dz$$

for any $u\in C_0^\infty(\mathbb{R}^n)$.

The Caffarelli-Kohn-Nirenberg inequality has been successfully translated to Baouendi-Grushin vector fields by Manli Song and Wenjuan Li. The inequality

$$\left(\int\limits_{\mathbb{R}^n} \left(\frac{|x|^\gamma}{\rho^\gamma}\right)^{(\alpha-\gamma)r} \rho^{\gamma r}|u|^r dz \right)^{\frac{1}{r}} \leqslant C \left(\int\limits_{\mathbb{R}^n} \rho^{\alpha p}|\nabla_\gamma u|^p dz \right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^n} \left(\frac{|x|^\gamma}{\rho^\gamma}\right)^{(\alpha-\beta)q} \rho^{\beta q}|u|^q dz \right)^{\frac{1-a}{q}}$$

with conditions for the parameters to be found in [SL16].

While the previous setting is the most general at the moment, the constant C is still unknown. To the contrary, in a narrower context the Caffarelli-Kohn-Nirenberg inequality for Baouendi-Grushin vector fields have been established with a sharp constant. With $k=1$, i.e. $\mathbb{R}^n=\mathbb{R}^{m+1}=\{\,(x,y)\ :\ x\in\mathbb{R}^m, y\in\mathbb{R}\,\}$, and $\gamma=1$, it was studied by Joshua Flynn [Fly20]: for all $(a,b)\in\mathbb{R}^2$

$$\widehat{C}\int\limits_{\mathbb{R}^{m+1}} |u|^2\,\frac{\psi}{\rho^{a+b+1}}\,dz \leqslant \left(\int\limits_{\mathbb{R}^{m+1}} \frac{|\nabla_\gamma u|^2}{\rho^{2b}}\,dz \right)^{\frac{1}{2}} \left(\int\limits_{\mathbb{R}^{m+1}} |u|^2\,\frac{\psi}{\rho^{2a}}\,dz \right)^{\frac{1}{2}}$$

where $u\in C_0^\infty(\,\mathbb{R}^n\setminus\{0\}\,)$,

$$\widehat{C}=\frac{|Q-(a+b+1)|}{2} \quad \text{and} \quad \psi=\frac{|x|^2}{\rho^2}.$$

The constant $\widehat{C}$ is sharp for appropriate choices of the pair (a,b).

It is also worth to note the works [MPP19], [AL11], [LRY19] as a contribution to the study of the famous inequalities involving Baouendi-Grushin vector fields. In [MPP19] the authors obtained the L^2-Poincare inequality for the Baouendi-Grushin vector fields with a non-explicit constant, while in [AL11] and [LRY19] the authors showed refined versions of the Hardy inequality for the Baouendi-Grushin vector fields.

1.5 Pillar works

The book which have enabled ***Gauss-Ostrogradsky divergence theorem for Baouendi-Grushin vector fields*** and ***Green's first identity for Baouendi-Grushin vector fields*** is "Fundamentals of Mathematical Analysis" by Vladimir Aleksandrovich Ilyin and Eduard Grigorievich Poznyak [IP82]. They prove the Green's formula for a special but sufficiently wide class of domains [IP82, ch. 7, sec. 1]. Then they establish a number of auxiliary statements needed to prove the theorem in its original setting in terms of domain. With robust confidence that the technique works in our case, we provide the proof only for Ilyin and Poznyak's special class of domains.

The first article to note is [SY21] by Durvudkhan Suragan and Nurgissa Yessirkegenov. The authors propose a simple and yet creative approach to prove the Poincaré inequality for Baouendi-Grushin vector fields. Let us restate their main result:

Theorem (**Theorem 1.1**). *Let $V \subset \mathbb{R}^{m+k}$ be a set supporting the divergence formula. Then we have*

$$0 \leqslant \int\limits_V \left| \nabla_\gamma u - \frac{\nabla_\gamma \varphi}{\varphi} u \right|^2 dz = \int\limits_V \left(|\nabla_\gamma u|^2 + \frac{\Delta_\gamma \varphi}{\varphi} |u|^2 \right) dz$$

for all $u \in \dot{H}_0^{1,\gamma}(V)$ and any twice differentiable φ. The equality case holds if and only if u is proportional to φ.

Here $\dot{H}_0^{1,\gamma}(V)$ is a Sobolev space is obtained as completion of $C_0^\infty(V)$ with respect to the norm

$$\|f\|_{\dot{H}_0^{1,\gamma}(V)} = \left(\int\limits_V |\nabla_\gamma f|^2 \, dz \right)^{\frac{1}{2}}.$$

The proof of **Theorem 1.1** is implemented with applying elementary calculus' tricks and, as we supplement in this thesis, ***Gauss-Ostrogradsky divergence theorem for Baouendi-Grushin vector fields***. Then the authors use effectively the fact that an eigenfunction associated with the first eigenvalue of $-\Delta_\gamma$ is a.e. strictly positive by taking φ equal to this eigenfunction. This promptly brings us the following result:

Theorem (**Corollary 1.2**). *Suppose that the negative Dirichlet Baouendi-Grushin operator on V has a positive eigenvalue λ and a corresponding positive eigenfunction φ. Then we have*

$$\frac{1}{\lambda} \int\limits_V \left| \nabla_\gamma u - \frac{\nabla_\gamma \varphi}{\varphi} \right|^2 dz = \frac{1}{\lambda} \int\limits_V |\nabla_\gamma u|^2 \, dz - \int\limits_V |u|^2 \, dz$$

for all $u \in \dot{H}_0^{1,\gamma}(V)$.

The approach is a methodologically invaluable example of using elementary calculus as well as a powerful application of the spectral property of the Baouendi-Grushin operator.

Further, the authors accomplish a powerful application of **Theorem 1.1** to prove existence of blow-up solutions to a Dirichlet initial-boundary value problem for the Baouendi-Grushin operator. The problem is as follows.

$$\begin{cases} u_t(t,x,y) = \Delta_\gamma u + f\big(u(t,x,y)\big), & (t,x,y) \in (0,+\infty) \times V, \\ u(t,x,y) = 0, & (t,x,y) \in [0,+\infty) \times \partial V, \\ u(0,x,y) = u_0(x,y), & x, y \in \overline{V}, \end{cases} \tag{1.7}$$

where $V \subset \mathbb{R}^{m+k}$ is an open, bounded, and connected set such that $V \setminus \Sigma$ consists of countably many connected components [MP09, Remark 6.5]; f is locally Lipschitz continuous on $\mathbb{R}$ with the property that $f(0) = 0$ and $f(u) > 0$ for $u > 0$. It is also assumed that the initial data $u_0 \in C^1(\overline{V})$ is a non-negative and non-trivial function with $u_0 = 0$ on ∂V.

For the initial-boundary value problem (1.7) the authors prove the following result:

Theorem (**Theorem 1.5**). *Assume that there exist constants $\alpha > 2$ and θ such that f satisfies*

$$\alpha F(u) \leqslant uf(u) + \beta u^2 + \alpha\theta$$

for all $u > 0$, where $F(u) = \int_0^u f(s)ds$, $0 < \beta \leqslant \lambda_1(\alpha - 2)/2$. If the initial data $u_0 \in C^1(\overline{V})$ with $u_0 = 0$ on ∂V satisfies

$$-\frac{1}{2}\int\limits_V |\nabla_\gamma u_0|^2\, dz + \int\limits_V (F(u_0) - \theta)\, dz > 0,$$

then the nonnegative solution to the problem (1.7) blows up at a finite time T^*, that is,

$$\lim_{t \to T^*} \int\limits_0^t \int\limits_V u^2(\tau, z) dz d\tau = +\infty.$$

Moreover, the blow-up time T^* satisfies

$$0 < T^* \leqslant \frac{M}{\sigma \int_V u_0^2\, dz},$$

where $\sigma = \sqrt{\alpha/2} - 1$,

$$M := \frac{\left(1 + \sqrt{\alpha/2}\right)\left(\int_V u_0^2\, dz\right)^2}{2(\alpha - 2)\left(-1/2\int_V |\nabla_\gamma u_0|^2\, dz + \int_V \left(F(u_0) - \theta\right) dz\right)}.$$

The next article is [OS20] by Tohru Ozawa and Durvudkhan Suragan. This is the most important article for the current work since the main result of the whole thesis is fully inspired by their findings. The authors' primary discovery is as follows:

Theorem (**Theorem 2.1**). *Let $V \subset M$. Let $\varphi > 0$ be a strictly positive eigenfunction of $-\mathcal{L}$ with an eigenvalue λ, that is, $-\mathcal{L}\varphi = \lambda\varphi$ on V. Then for any $u \in C_0^\infty(V)$ we have*

$$\begin{aligned} &\left|X^{2r}u\right|^2 - \lambda^{2r}|u|^2 = \\ &\sum_{j=0}^{r-1} \lambda^{2(r-1-j)}\left(\left|\mathcal{L}^{j+1}u + \lambda\mathcal{L}^j u\right|^2 + 2\lambda\left|X\mathcal{L}^j u - \frac{X\varphi}{\varphi}\mathcal{L}^j u\right|^2\right) + \\ &2\sum_{j=0}^{r-1}\lambda^{2(r-1-j)+1} X \cdot \left(\frac{X\varphi}{\varphi}|\mathcal{L}^j u|^2 - \mathcal{L}^j u X\mathcal{L}^j u\right), \end{aligned}$$

where $r = 1, 2, ...,$ and

$$\begin{aligned} &\left|X^{2r+1}u\right|^2 - \lambda^{2r+1}|u|^2 = \left|X\mathcal{L}^r u - \frac{X\varphi}{\varphi}\mathcal{L}^r u\right|^2 + \\ &\sum_{j=0}^{r-1}\lambda^{2(r-j)-1}\left(|\mathcal{L}^{j+1}u + \lambda\mathcal{L}^j u|^2 + 2\lambda\left|X\mathcal{L}^j u - \frac{X\varphi}{\varphi}\mathcal{L}^j u\right|^2\right) + \\ &2\sum_{j=0}^{r-1}\lambda^{2(r-j)} X \cdot \left(\frac{X\varphi}{\varphi}|\mathcal{L}^j u|^2 - \mathcal{L}^j u X\mathcal{L}^j u\right) + X \cdot \left(\frac{X\varphi}{\varphi}|\mathcal{L}^r u|^2\right), \end{aligned}$$

where $r = 0, 1, ...$

Clarification: M is "a smooth n-dimension manifold of a volume form $d\nu$", i.e. an n-dimensional manifold smooth w.r.t. the Baouendi-Grushin vector fields. The authors augment the result of the previous work [SY21]: one can consider **Theorem 2.1** as an inductive extension of **Theorem 1.1**. Once the theorem is proved, an immediate corollary is provided:

Theorem (**Theorem 2.2**). *Let $V \subset \mathbb{R}^n$ be a connected domain, for which the divergence theorem is true. We have the following remainder of the higher order Steklov inequality*

$$\int_V \left|\nabla^{2r} u\right|^2 ds - \widetilde{\lambda}_1^{2r} \int_V |u|^2 ds =$$

$$\sum_{j=0}^{r-1} \widetilde{\lambda}_1^{2(r-1-j)} \left(\int_V \left|\Delta^{j+1} u + \widetilde{\lambda}_1 \Delta^j u\right|^2 ds + 2\widetilde{\lambda}_1 \int_V \left|\nabla \Delta^j u - \frac{\nabla \widetilde{e}_1}{\widetilde{e}_1} \Delta^j u\right|^2 ds \right) \geqslant 0,$$

where $r = 1, 2, ...,$ and

$$\int_V \left|\nabla^{2r+1} u\right|^2 ds - \widetilde{\lambda}_1^{2r+1} \int_V |u|^2 ds = \int_V \left|\nabla \Delta^r u - \frac{\nabla \widetilde{e}_1}{\widetilde{e}_1} \Delta^r u\right|^2 ds +$$

$$\sum_{j=0}^{r-1} \widetilde{\lambda}_1^{2(r-j)-1} \left(\int_V \left|\Delta^{j+1} u + \widetilde{\lambda}_1 \Delta^j u\right|^2 ds + 2\widetilde{\lambda}_1 \int_V \left|\nabla \Delta^j u - \frac{\nabla \widetilde{e}_1}{\widetilde{e}_1} \Delta^j u\right|^2 ds \right) \geqslant 0,$$

where $r = 0, 1, ...,$ for all $u \in C_0^\infty(V)$. Here $\widetilde{e}_1$ is the ground state of the (minus) Dirichlet Laplacian in V and $\widetilde{\lambda}_1$ is the corresponding eigenvalue. The equality cases hold if and only if u is proportional to $\widetilde{e}_1$.

In analogy with **Corollary 1.2**, T. Ozawa and D. Suragan fruitfully use classical ***Gauss-Ostrogradsky divergence theorem*** in the proof of **Theorem 2.2**.

And the final article is [KST21] by Ardak Kashkynbayev, Durvudkhan Suragan, and Berikbol Torebek which inspired Applications chapter of this thesis. The results of the chapter are the higher-order Baouendi-Grushin analogues of theorems from the article.

1.6 Structure

Chapter Main results, apart from the higher-order Poincaré inequalities for Baouendi-Grushin vector fields themeselves, consists of assertions necessary to prove the Poincaré inequalities and crucial in the proofs of chapter Applications. Chapter Proofs of Main results contains a rigorous proof of each mathematical statements from Main results. Chapter Applications consists of applications of the Poincaré inequalities to the higher-order Fisher-KPP equation on Baouendi-Grushin vector fields.

Chapter 2

Main results

Following V. A. Ilyin and E. G. Poznyak, we introduce a special class of domains. Let V be a singly connected finite domain with piecewise smooth boundary ∂V. We assume that every straight line parallel to any coordinate axis intersects ∂V at most in two points. Such domains will be called ***type-K domains*** [IP82, ch. 7, sec. 1, subsec. 2, p. 171–172]. We will consider ***type-K domains with respect to*** y_j ***axes***, which means that the previous statement holds only for axes $y_1, ..., y_k$.

We also introduce the following notation for vector-valued functions $f : \Omega \to \mathbb{R}^n$:

$$f = (f_1, ..., f_n) = (f_1, ..., f_m, f_{m+1}, ..., f_{m+k}) = (\,\boldsymbol{f_1}, \boldsymbol{f_2}\,),$$

where

$$\boldsymbol{f_1} = (f_1, ..., f_m), \quad \boldsymbol{f_2} = (f_{m+1}, ..., f_{m+k}).$$

2.1 Classical vector calculus' theorems for Baouendi-Grushin vector fields

Theorem (***Gauss-Ostrogradsky divergence theorem for Baouendi-Grushin vector fields on type-K domains with respect to*** y_j ***axes***)**.** *Let $V \subset \mathbb{R}^n$ be a type-K domain with respect to y_j axes, $f : \overline{V} \to \mathbb{R}^n$ — a continuously differentiable function. Then*

$$\int\limits_V \nabla_\gamma \cdot f \, dz = \oint\limits_{\partial V} (\,\boldsymbol{f_1}, |x|^\gamma \boldsymbol{f_2}\,) \cdot d\boldsymbol{\nu}.$$

Corollary (***Gauss-Ostrogradsky divergence theorem for Baouendi-Grushin vector fields***)**.** *Let $f : \overline{\Omega} \to \mathbb{R}^n$ be a continuously differentiable function. Then*

$$\int\limits_\Omega \nabla_\gamma \cdot f \, dz = \oint\limits_{\partial \Omega} (\,\boldsymbol{f_1}, |x|^\gamma \boldsymbol{f_2}\,) \cdot d\boldsymbol{\nu}.$$

Theorem (***Green's first identity for Baouendi-Grushin vector fields on type-K domains with respect to*** y_j ***axes***)**.** *Let $V \subset \mathbb{R}^n$ be a type-K domain with respect to y_j axes, $f \in C^2(\overline{V})$, $g \in C^1(\overline{V})$. Then*

$$\int\limits_V \Delta_\gamma f \, g \, dz = \oint\limits_{\partial V} g \, \nabla_{2\gamma} f \cdot d\boldsymbol{\nu} - \int\limits_V \nabla_\gamma f \cdot \nabla_\gamma g \, dz.$$

Corollary (***Green's first identity for Baouendi-Grushin vector fields***)**.** *Let $f \in C^2(\overline{\Omega})$, $g \in C^1(\overline{\Omega})$. Then*

$$\int\limits_\Omega \Delta_\gamma f \, g \, dz = \oint\limits_{\partial \Omega} g \, \nabla_{2\gamma} f \cdot d\boldsymbol{\nu} - \int\limits_\Omega \nabla_\gamma f \cdot \nabla_\gamma g \, dz.$$

This result gives birth to a lemma which will be crucial in Applications chapter.

Lemma (***The reduction formula***)**.** *Let* $u \in C^{2l}(\overline{\Omega})$ *and*

$$\nabla_{2\gamma}\big(\nabla_\gamma^{2(r-1)}u\big)\,\nabla_\gamma^{2(l-r)}u = 0 \quad on \;\; \partial\Omega \qquad for \;\; r = 1, ..., l.$$

Then

$$\int\limits_\Omega u\,\Delta_\gamma^l u\,dz = (-1)^l \int\limits_\Omega \big(\nabla_\gamma^l u\big)^2\,dz.$$

2.2 Integrability criterion for degenerate domains

Lemma (***The integrability criterion for degenerate domains***)**.**
If $\{x = 0\} \cap \Omega \neq \varnothing$ *and* $\gamma \in \mathbb{R}^+ \setminus \mathbb{N}$*, then the integral*

$$\int\limits_\Omega \big|\nabla_\gamma^r u\big|^2\,dz$$

converges for all $u \in C_0^\infty(\Omega)$ *if and only if* $r < m/2 + 2(\gamma + 1)$, $r \in \mathbb{N}$.

2.3 Representation formulae

For convenience of the reader, we provide the definition of the iterated gradient as follows:

$$\nabla_\gamma^{2r}\,u = \Delta_\gamma^r\,u, \quad \nabla_\gamma^{2r+1}\,u = \nabla_\gamma\,\Delta_\gamma^r\,u, \qquad r = 0, 1, 2...$$

The theorem is a particular case of **Theorem 2.1** from [OS20].

Theorem (***The representation formulae***)**.** *Let* $\{x = 0\} \cap \Omega = \varnothing$*. Then for any* $u \in C_0^\infty(\Omega)$

$$\begin{aligned}&\big|\nabla_\gamma^{2r}\,u\big|^2 - \lambda_1^{2r}|u|^2 = \\ &\quad \sum_{j=0}^{r-1} \lambda_1^{2(r-1-j)} \left(|\Delta_\gamma^{j+1}\,u + \lambda_1\,\Delta_\gamma^j\,u|^2 + 2\lambda_1 \left|\nabla_\gamma\,\Delta_\gamma^j\,u - \frac{\nabla_\gamma\,e_1}{e_1}\Delta_\gamma^j\,u\right|^2 \right) + \\ &\quad 2\sum_{j=0}^{r-1} \lambda_1^{2(r-1-j)+1}\,\nabla_\gamma \cdot \left(\frac{\nabla_\gamma\,e_1}{e_1}|\Delta_\gamma^j\,u|^2 - \Delta_\gamma^j\,u\,\nabla_\gamma\,\Delta_\gamma^j\,u\right),\end{aligned} \tag{2.1}$$

where $r = 1, 2, ...,$ *and*

$$\begin{aligned}&\big|\nabla_\gamma^{2r+1}\,u\big|^2 - \lambda_1^{2r+1}|u|^2 = \left|\nabla_\gamma\,\Delta_\gamma^r\,u - \frac{\nabla_\gamma\,e_1}{e_1}\Delta_\gamma^r\,u\right|^2 + \\ &\quad \sum_{j=0}^{r-1} \lambda_1^{2(r-j)-1} \left(\big|\Delta_\gamma^{j+1}\,u + \lambda_1\,\Delta_\gamma^j\,u\big|^2 + 2\lambda_1 \left|\nabla_\gamma\,\Delta_\gamma^j\,u - \frac{\nabla_\gamma\,e_1}{e_1}\Delta_\gamma^j\,u\right|^2 \right) + \\ &\quad 2\sum_{j=0}^{r-1} \lambda_1^{2(r-j)}\,\nabla_\gamma \cdot \left(\frac{\nabla_\gamma\,e_1}{e_1}\,\big|\Delta_\gamma^j\,u\big|^2 - \Delta_\gamma^j\,u\,\nabla_\gamma\,\Delta_\gamma^j\,u\right) + \nabla_\gamma \cdot \left(\frac{\nabla_\gamma\,e_1}{e_1}\,\big|\Delta_\gamma^r\,u\big|^2\right),\end{aligned} \tag{2.2}$$

where $r = 0, 1, ...$

2.4 The higher-order Poincaré inequalities

We recall that the definition of λ_1 and e_1 and the conditions on Ω are given in Notations.

Theorem (***Remainders of the higher-order Poincaré inequalities for Baouendi–Grushin vector fields***). *Assume that $\lambda_1 > 0$ and $e_1 > 0$ exist in $\overline{\Omega}$ [MP09]. Then, for all $u \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \left|\nabla_\gamma^{2r} u\right|^2 dz - \lambda_1^{2r} \int_\Omega |u|^2 dz =$$

$$\sum_{j=0}^{r-1} \lambda_1^{2(r-1-j)} \left(\int_\Omega \left|\Delta_\gamma^{j+1} u + \lambda_1 \Delta_\gamma^j u\right|^2 dz + 2\lambda_1 \int_\Omega \left|\nabla_\gamma \Delta_\gamma^j u - \frac{\nabla_\gamma e_1}{e_1} \Delta_\gamma^j u\right|^2 dz \right) \geqslant 0,$$

where $r = 1, 2, \ldots$ $(1 \leqslant r < m/4 + \gamma + 1)$ and

$$\int_\Omega \left|\nabla_\gamma^{2r+1} u\right|^2 dz - \lambda_1^{2r+1} \int_\Omega |u|^2 dz = \int_\Omega \left|\nabla_\gamma \Delta_\gamma^r u - \frac{\nabla_\gamma e_1}{e_1} \Delta_\gamma^r u\right|^2 dz$$

$$\sum_{j=0}^{r-1} \lambda_1^{2(r-j)-1} \left(\int_\Omega \left|\Delta_\gamma^{j+1} u + \lambda_1 \Delta_\gamma^j u\right|^2 dz + 2\lambda_1 \int_\Omega \left|\nabla_\gamma \Delta_\gamma^j u - \frac{\nabla_\gamma e_1}{e_1} \Delta_\gamma^j u\right|^2 dz \right) \geqslant 0,$$

where $r = 0, 1, \ldots$ $(0 \leqslant r < m/4 + \gamma + 1/2)$, when $\{x = 0\} \cap \Omega = \varnothing$ $(\{x = 0\} \cap \Omega \neq \varnothing$ and $\gamma \in \mathbb{R}^+ \setminus \mathbb{N})$. The equality cases hold if and only if $u = \alpha e_1$, $\alpha \in \mathbb{R}$.

Because the right-hand sides are non-negative, we have the Poincaré inequalities. And as we have found the case when the right-hand sides are zero, we immediately obtain sharpness of the constants λ_1^{2r} and λ_1^{2r+1} respectively.

Corollary (***The higher-order Poincaré inequalities for Baouendi-Grushin vector fields***). *For all $u \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega |\nabla_\gamma^r u|^2 dz \geqslant \lambda_1^r \int_\Omega |u|^2 dz \tag{2.3}$$

with $r = 1, 2, \ldots$ when $\{x = 0\} \cap \Omega = \varnothing$ and $r < m/2 + 2(\gamma + 1)$, $r \in \mathbb{N}$, when $\{x = 0\} \cap \Omega \neq \varnothing$ and $\gamma \in \mathbb{R}^+ \setminus \mathbb{N}$.

Constant λ_1^r is sharp: it is achieved an eigenfunction corresponding to λ_1.

Remark. The inequalities are an immediate conclusion of the theorem ***Remainders of the higher-order Poincaré inequalities for Baouendi-Grushin vector fields***, since the remainder part is always non-negative — we unite both sequences of the formulae into a single one. Due to ***The integrability criterion for degenerate domains*** we differentiate between the cases when the hyperplane $\{x = 0\}$ does and does not intersect the domain, which bears the upper bound of the index r in the intersection case.

Chapter 3

Proofs of Main results

3.1 Proofs of classical vector calculus' theorems for Baouendi-Grushin vector fields

Proof of **Gauss-Ostrogradsky divergence theorem for Baouendi-Grushin vector fields on type-$\boldsymbol{K}$ domains with respect to y_j axes**.

We have

$$\int\limits_V \nabla_\gamma \cdot f\, dz = \int\limits_V \left(\sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} + |x|^\gamma \sum_{j=1}^{k} \frac{\partial f_{m+j}}{\partial y_j} \right) dz =$$

$$\int\limits_V \nabla \cdot f\, dz + \int\limits_V (\,|x|^\gamma - 1)\sum_{j=1}^{k} \frac{\partial f_{m+j}}{\partial y_j}\, dz.$$

Since for ∇ classical ***Gauss-Ostrogradsky divergence theorem*** holds, we focus on the last integral.

Denote the orthogonal projection on the hyperplane without y_j axis as

$$V_{y_j} = \mathop{\mathbf{Proj}}_{\mathbb{R}^n \setminus \{y_j\}} V.$$

Due to the projectability property of V, for each $z^j \in V_{y_j}$ there is a single segment — the intersection of V and the line, crossing z^j and parallel to y_j axis. Denote the lower and the upper ends of this segment as y_j^- and y_j^+, $y_j^- \leqslant y_j^+$. Also denote

$$z^j := (x_1, ..., x_m, y_1, ..., y_{j-1}, y_{j+1}, ..., y_k),$$
$$(\, z^j, \widetilde{y}_j(z^j)\,) := (x_1, ..., x_m, y_1, ..., y_{j-1}, \widetilde{y}_j(z^j), y_{j+1}, ..., y_k).$$

Here $\widetilde{y}_j$ is some real-valued function of z^j.

Because set V is intersected at most twice by lines parallel to axes $y_1, ..., y_k$, we proceed

$$\int\limits_V (\,|x|^\gamma - 1)\sum_{j=1}^{k} \frac{\partial f_{m+j}}{\partial y_j}\, dz = \sum_{j=1}^{k} \int\limits_V (\,|x|^\gamma - 1)\frac{\partial f_{m+j}}{\partial y_j}\, dz = \sum_{j=1}^{k} \int\limits_{V_{y_j}} \int\limits_{y_j^-}^{y_j^+} (\,|x|^\gamma - 1)\frac{\partial f_{m+j}}{\partial y_j}\, dy_j\, dz^j$$

$$= \sum_{j=1}^{k} \int\limits_{V_{y_j}} (\,|x|^\gamma - 1)\, f_{m+j} \Bigg|_{y_j^-}^{y_j^+} dz^j.$$

The integral of the Newton-Leibniz differences are

$$\int\limits_{V_{y_j}} (\,|x|^\gamma - 1)\, f_{m+j} \Big|_{y_j^-}^{y_j^+} dz^j =$$

$$\int\limits_{V_{y_j}} (\,|x|^\gamma - 1)\, f_{m+j}(\,z^j, y_j^+(z^j)\,)\, dz^j - \int\limits_{V_{y_j}} (\,|x|^\gamma - 1)\, f_{m+j}(\,z^j, y_j^-(z^j)\,)\, dz^j =$$

$$\oint\limits_{\partial V} (\,|x|^\gamma - 1)\, f_{m+j}\, d\nu,$$

since the intermediate integrals correspond themselves to the upper and lower parts of boundary ∂V with the direction of circulation about the boundary positive w.r.t. y_j axis. The minus before the integral over the lower part of ∂V changes the direction of the circulation outwards. Thereby, we obtain

$$\int\limits_V (\,|x|^\gamma - 1) \sum_{j=1}^k \frac{\partial f_{m+j}}{\partial y_j}\, dz =$$

$$\sum_{j=1}^k \oint\limits_{\partial V} (\,|x|^\gamma - 1)\, f_{m+j}\, d\nu = \oint\limits_{\partial V} (\,|x|^\gamma - 1) \sum_{j=1}^k f_{m+j}\, d\nu.$$

Thus,

$$\int\limits_V \nabla_{\boldsymbol{\gamma}} \cdot f\, dz = \oint\limits_{\partial V} \sum_{i=1}^m f_i + \sum_{j=1}^k f_{m+j}\, d\nu + \oint\limits_{\partial V} (\,|x|^\gamma - 1) \sum_{j=1}^k f_{m+j}\, d\nu =$$

$$\oint\limits_{\partial V} \sum_{i=1}^m f_i + |x|^\gamma \sum_{j=1}^k f_{m+j}\, d\nu = \oint\limits_{\partial V} (\,\boldsymbol{f_1}, |x|^\gamma \boldsymbol{f_2}\,) \cdot d\boldsymbol{\nu}.$$

□

Proof of **Green's first identity for Baouendi-Grushin vector fields on type-*K* domains with respect to y_j axes**

We have

$$\int\limits_V \Delta_{\boldsymbol{\gamma}} f\, g\, dz = \int\limits_V \left(\sum_{i=1}^m \frac{\partial^2 f_i}{\partial x_i^2} + |x|^{2\gamma} \sum_{j=1}^k \frac{\partial^2 f_{m+j}}{\partial y_j^2} \right) g\, dz =$$

$$\int\limits_V \Delta f g\, dz + \int\limits_V (\,|x|^{2\gamma} - 1) \sum_{j=1}^k \frac{\partial^2 f_{m+j}}{\partial y_j^2}\, g\, dz.$$

For the ordinary ∇ operator we know that classical ***Green's first identity*** works. Therefore we focus on the last integral.

Again, we denote the orthogonal projection on the hyperplane without y_j axis as

$$V_{y_j} = \operatorname*{\mathbf{Proj}}_{\mathbb{R}^n \setminus \{y_j\}} V.$$

As before, we denote

$$z^j := (x_1, ..., x_m, y_1, ..., y_{j-1}, y_{j+1}, ..., y_k),$$
$$\big(\, z^j, \widetilde{y}_j(z^j)\,\big) := \big(x_1, ..., x_m, y_1, ..., y_{j-1}, \widetilde{y}_j(z^j), y_{j+1}, ..., y_k\big).$$

For a fixed $z^j \in V_{y_j}$ we consider the line passing through z^j and parallel to y_j axis. Then y_j^- and y_j^+ are regarded as the lowest and the greatest values of y_j along this line such that $\big(\, z^j, y_j^-(z^j)\,\big), \big(\, z^j, y_j^+(z^j)\,\big) \in \partial V$.

Because V is intersected at most twice by lines parallel to axes $y_1, ..., y_k$, we have

$$\int\limits_V \big(\, |x|^{2\gamma} - 1\big) \sum_{j=1}^{k} \frac{\partial^2 f_{m+j}}{\partial y_j^2}\, g\, dz =$$

$$\sum_{j=1}^{k} \int\limits_V \big(\, |x|^{2\gamma} - 1\big) \frac{\partial^2 f_{m+j}}{\partial y_j^2}\, g\, dz = \sum_{j=1}^{k} \int\limits_{V_{y_j}} \int\limits_{y_j^-}^{y_j^+} \big(\, |x|^{2\gamma} - 1\big) \frac{\partial^2 f_{m+j}}{\partial y_j^2}\, g\, dy_j dz^j =$$

$$\sum_{j=1}^{k} \int\limits_{V_{y_j}} \left(\big(\, |x|^{2\gamma} - 1\big) \frac{\partial f_{m+j}}{\partial y_j}\, g \Bigg|_{y_j^-}^{y_j^+} - \int\limits_{y_j^-}^{y_j^+} \big(\, |x|^{2\gamma} - 1\big) \frac{\partial f_{m+j}}{\partial y_j} \frac{\partial g_{m+j}}{\partial y_j}\, dy_j \right) dz^j.$$

The integrals of the Newton-Leibniz difference terms are

$$\int\limits_{V_{y_j}} \big(\, |x|^{2\gamma} - 1\big) \frac{\partial f_{m+j}}{\partial y_j}\, g \Bigg|_{y_j^-}^{y_j^+} dz^j =$$

$$\int\limits_{V_{y_j}} \big(\, |x|^{2\gamma} - 1\big) \frac{\partial f_{m+j}}{\partial y_j}\big(\, z^j, y_j^+(z^j)\,\big)\; g\big(\, z^j, y_j^+(z^j)\,\big)\, dz^j -$$

$$\int\limits_{V_{y_j}} \big(\, |x|^{2\gamma} - 1\big) \frac{\partial f_{m+j}}{\partial y_j}\big(\, z^j, y_j^-(z^j)\,\big)\; g\big(\, z^j, y_j^-(z^j)\,\big)\, dz^j =$$

$$\oint\limits_{\partial V} \big(\, |x|^{2\gamma} - 1\big) \frac{\partial f_{m+j}}{\partial y_j}\, g\; d\nu.$$

Thereby

$$\int\limits_V \big(\, |x|^{2\gamma} - 1\big) \sum_{j=1}^{k} \frac{\partial^2 f_{m+j}}{\partial y_j^2}\, g\, dz =$$

$$\sum_{j=1}^{k} \left(\oint\limits_{\partial V} \big(\, |x|^{2\gamma} - 1\big) \frac{\partial f_{m+j}}{\partial y_j}\, g\; d\nu - \int\limits_V \big(\, |x|^{2\gamma} - 1\big) \frac{\partial f_{m+j}}{\partial y_j} \frac{\partial g_{m+j}}{\partial y_j}\, dz \right) =$$

$$\oint\limits_{\partial V} \big(\, |x|^{2\gamma} - 1\big) \sum_{j=1}^{k} \frac{\partial f_{m+j}}{\partial y_j}\, g\; d\nu - \int\limits_V \big(\, |x|^{2\gamma} - 1\big) \sum_{j=1}^{k} \frac{\partial f_{m+j}}{\partial y_j} \frac{\partial g_{m+j}}{\partial y_j}\, dz.$$

Back to the initial equality, we acquire

$$\int\limits_V \Delta_\gamma f\, g\, dz =$$

$$\oint\limits_{\partial V} \left(\sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} + \sum_{j=1}^{k} \frac{\partial f_{m+j}}{\partial y_j} \right) g\, d\nu - \int\limits_V \left(\sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} \frac{\partial g_i}{\partial x_i} + \sum_{j=1}^{k} \frac{\partial f_{m+j}}{\partial y_j} \frac{\partial g_{m+j}}{\partial y_j} \right) dz +$$

$$\oint\limits_{\partial V} \left(|x|^{2\gamma} - 1 \right) \sum_{j=1}^{k} \frac{\partial f_{m+j}}{\partial y_j}\, g\ d\nu - \int\limits_V \left(|x|^{2\gamma} - 1 \right) \sum_{j=1}^{k} \frac{\partial f_{m+j}}{\partial y_j} \frac{\partial g_{m+j}}{\partial y_j}\, dz =$$

$$\oint\limits_{\partial V} g \left(\sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} + |x|^{2\gamma} \sum_{j=1}^{k} \frac{\partial f_{m+j}}{\partial y_j} \right) d\nu - \int\limits_V \left(\sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} \frac{\partial g_i}{\partial x_i} + |x|^{2\gamma} \sum_{j=1}^{k} \frac{\partial f_{m+j}}{\partial y_j} \frac{\partial g_{m+j}}{\partial y_j} \right) dz =$$

$$\oint\limits_{\partial V} g\, \nabla_{2\gamma} f \cdot d\boldsymbol{\nu} - \int\limits_V \nabla_\gamma f \cdot \nabla_\gamma g\, dz.$$

□

Proof of **The reduction formula**. We possess

$$\int\limits_\Omega u\, \Delta_\gamma^l u\, dz = \int\limits_\Omega u\, \nabla_\gamma^{2l} u\, dz$$

with boundary conditions

$$\nabla_{2\gamma} \left(\nabla_\gamma^{2(r-1)} u \right) \nabla_\gamma^{2(l-r)} u \Big|_{\partial\Omega} = 0, \quad r = 1, ..., l.$$

Applying ***Green's first identity for Baouendi-Grushin vector fields*** ($f = \nabla_\gamma^{2(l-1)} u$, $g = u$), we get

$$\int\limits_\Omega u\, \nabla_\gamma^{2l} u\, dz = \oint\limits_{\partial\Omega} u\, \nabla_{2\gamma} \left(\nabla_\gamma^{2(l-1)} u \right) \cdot d\boldsymbol{\nu} - \int\limits_\Omega \nabla_\gamma u \cdot \nabla_\gamma^{2l-1} u\, dz.$$

At the same time ($f = u$, $g = \nabla_\gamma^{2(l-1)} u$) we have

$$\int\limits_\Omega \nabla_\gamma^2 u\, \nabla_\gamma^{2(l-1)} u\, dz = \oint\limits_{\partial\Omega} \nabla_\gamma^{2(l-1)} u\ \nabla_{2\gamma} u \cdot d\boldsymbol{\nu} - \int\limits_\Omega \nabla_\gamma u \cdot \nabla_\gamma^{2l-1} u\, dz.$$

Recalling that both integrands of the surface integrals are zero, we acquire

$$\int\limits_\Omega u\, \nabla_\gamma^{2l} u\, dz = \int\limits_\Omega \nabla_\gamma^2 u\, \nabla_\gamma^{2(l-1)} u\, dz.$$

Thereby we have the base of induction for

$$\int\limits_\Omega \nabla_\gamma^{2j} u\, \nabla_\gamma^{2(l-j)} u\, dz = \int\limits_\Omega \nabla_\gamma^{2(j+1)} u\, \nabla_\gamma^{2(l-j-1)} u\, dz.$$

Assume that the assertion is true for some $j = r < \lfloor l/2 \rfloor$. Then for $j = r + 1$ we have

$$\int\limits_{\Omega} \nabla_{\gamma}^{2(r+1)} u \, \nabla_{\gamma}^{2(l-r-1)} u \, dz =$$

$$\oint\limits_{\partial\Omega} \nabla_{\gamma}^{2(r+1)} u \nabla_{2\gamma} \big(\nabla_{\gamma}^{2(l-r-1)-2} u \big) \cdot d\boldsymbol{\nu} - \int\limits_{\Omega} \nabla_{\gamma}^{2(r+1)+1} u \cdot \nabla_{\gamma}^{2(l-r-1)-1} u \, dz.$$

and

$$\int\limits_{\Omega} \nabla_{\gamma}^{2(r+2)} u \, \nabla_{\gamma}^{2(l-r-2)} u \, dz =$$

$$\oint\limits_{\partial\Omega} \nabla_{\gamma}^{2(l-r-2)} u \, \nabla_{2\gamma} \big(\nabla_{\gamma}^{2(r+2)-2} u \big) \cdot d\boldsymbol{\nu} - \int\limits_{\Omega} \nabla_{\gamma}^{2(r+2)-1} u \cdot \nabla_{\gamma}^{2(l-r-2)+1} u \, dz.$$

We apply the boundary conditions and obtain

$$\int\limits_{\Omega} \nabla_{\gamma}^{2(r+1)} u \, \nabla_{\gamma}^{2(l-r-1)} u \, dz = \int\limits_{\Omega} \nabla_{\gamma}^{2(r+2)} u \, \nabla_{\gamma}^{2(l-r-2)} u \, dz.$$

Thus, due to the induction, we have

$$\int\limits_{\Omega} \nabla_{\gamma}^{2j} u \, \nabla_{\gamma}^{2(l-j)} u \, dz = \int\limits_{\Omega} \nabla_{\gamma}^{2(j+1)} u \, \nabla_{\gamma}^{2(l-j-1)} u \, dz, \quad j = 0, 1, ..., \lfloor l/2 \rfloor,$$

which implies

$$\int\limits_{\Omega} u \, \nabla_{\gamma}^{2l} u \, dz = \int\limits_{\Omega} \nabla_{\gamma}^{2\lfloor l/2 \rfloor} u \, \nabla_{\gamma}^{2(\, l - \lfloor l/2 \rfloor \,)} u \, dz$$

When l is even, we have

$$\int\limits_{\Omega} u \, \nabla_{\gamma}^{2l} u \, dz = \int\limits_{\Omega} \nabla_{\gamma}^{l} u \, \nabla_{\gamma}^{l} u \, dz = \int\limits_{\Omega} \left| \nabla_{\gamma}^{l} u \right|^2 dz.$$

When l is odd, meaning $\lfloor l/2 \rfloor = (l-1)/2$, it yields

$$\int\limits_{\Omega} u \, \nabla_{\gamma}^{2l} u \, dz = \int\limits_{\Omega} \nabla_{\gamma}^{l-1} u \, \nabla_{\gamma}^{l+1} u \, dz.$$

Now we use ***Green's first identity for Baouendi-Grushin vector fields*** one more time:

$$\int\limits_{\Omega} \nabla_{\gamma}^{l-1} u \, \nabla_{\gamma}^{l+1} u \, dz = \oint\limits_{\partial\Omega} \nabla_{\gamma}^{l-1} u \, \nabla_{2\gamma} \big(\nabla_{\gamma}^{l-1} u \big) \cdot d\boldsymbol{\nu} - \int\limits_{\Omega} \nabla_{\gamma}^{l} u \cdot \nabla_{\gamma}^{l} u \, dz.$$

With the boundary condition

$$\nabla_{2\gamma} \big(\nabla_{\gamma}^{2(r-1)} u \big) \, \nabla_{\gamma}^{2(l-r)} u \Big|_{\partial\Omega} = 0$$

for $r = (l+1)/2$, it follows that

$$\int\limits_{\Omega} u \, \nabla_{\gamma}^{2l} u \, dz = - \int\limits_{\Omega} \left| \nabla_{\gamma}^{l} u \right|^2 dz$$

and this completes the proof. $\square$

3.2 Proof of Integrability criterion for degenerate domains

Proof. Let r be even. Then we have

$$\nabla_\gamma^r = \Delta_\gamma^{r/2} = \left(\sum_{i=1}^m \frac{\partial^2}{\partial x_i^2} + |x|^{2\gamma} \sum_{j=1}^k \frac{\partial^2}{\partial y_j^2} \right)^{r/2} = \tag{3.1}$$

$$\sum_{\substack{r_1+...+r_n=r/2 \\ 0\leqslant r_i \leqslant r/2}} \binom{r/2}{r_1, ..., r_n} \prod_{i=1}^m \left(\frac{\partial^2}{\partial x_i^2} \right)^{r_i} \prod_{j=1}^k \left(|x|^{2\gamma} \frac{\partial^2}{\partial y_j^2} \right)^{r_{m+j}} \tag{3.2}$$

Potential singularities would come from iterated partial second derivatives of the function $|x|^{2\alpha}$. Once applied a partial derivative yields

$$\frac{\partial^2}{\partial x_i^2}|x|^{2\alpha} = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}(x_1^2 + x_2^2 + ... + x_m^2)^\alpha \right) = \frac{\partial}{\partial x_i}\Big(2\alpha x_i (x_1^2 + x_2^2 + ... + x_m^2)^{\alpha-1} \Big)$$
$$= 2\alpha(x_1^2 + x_2^2 + ... + x_m^2)^{\alpha-1} + 4\alpha(\alpha-1)x_i^2(x_1^2 + x_2^2 + ... + x_m^2)^{\alpha-2}$$

Thus

$$\frac{\partial^2}{\partial x_i^2}|x|^{2\alpha} = 2\alpha|x|^{2(\alpha-1)} + 4\alpha(\alpha-1)x_i^2\,|x|^{2(\alpha-2)} \tag{3.3}$$

Because further we will switch to the polar coordinates w.r.t. x_1,..., x_m variables, we take into account that $x_i = O(|x|)$. Therefore

$$\frac{\partial^2}{\partial x_i^2}|x|^{2\alpha} = O\big(\,|x|^{2(\alpha-1)}\,\big)$$

Thereby, the lowest power of the distance function in (3.1) is produced by the iterated derivatives of the type

$$\left(\frac{\partial^2}{\partial x_{i_1}^2} \right)^{r_1} \left(\frac{\partial^2}{\partial x_{i_2}^2} \right)^{r_2} \cdots \left(\frac{\partial^2}{\partial x_{i_m}^2} \right)^{r_m} \left(|x|^{2\gamma} \frac{\partial^2}{\partial y_j^2} \right)$$

with $r_1 + r_2 + ... + r_m = r/2 - 1$. Hence, for $u \in C_0^\infty(\Omega)$ we have

$$\Delta_\gamma^{r/2} u = O\big(\,|x|^{2(\gamma - r/2+1)}\,\big)$$

and the integral of $\left|\Delta_\gamma^{r/2}\, u\right|^2$ converges if the integral

$$\int_\Omega |x|^{4(\gamma-r/2+1)}\,dz$$

is finite. Convergence of this intgeral is equivalent to intgerability of the power function in a small region containing the singularity $\{x = 0\}$. Therefore, it suffices to prove that the following integral converges:

$$\int_{B\times\Pi} |x|^{4(\gamma-r/2+1)}\,dz$$

where $B \times \Pi \subset \Omega$,

$$B = \{\, x \in \mathbb{R}^m \,:\, |x| \leqslant \delta \,\},\ \delta > 0, \quad \Pi = [\,y_1', y_1''\,] \times [\,y_2', y_2''\,] \times ... \times [\,y_k', y_k''\,].$$

Due to absence of y-dependence and radial symmetry of the integrand, we have

$$\int\limits_{B\times\Pi} |x|^{4(\gamma-r/2+1)}\,dz = |\Pi| \int\limits_{B} |x|^{4(\gamma-r/2+1)}\,dx = |\Pi|\,\omega_{m-1}\int\limits_{0}^{\delta} \rho^{4(\gamma-r/2+1)} r^{m-1}\,d\rho$$

with $|\Pi|$ — the volume of Π and ω_{m-1} — the volume of the unit ball in $\mathbb{R}^{m-1}$. The last integral converges if and only if

$$4(\gamma-r/2+1)+m-1>-1 \iff 2(\gamma-r/2+1)>m/2 \iff r<m/2+2(\gamma+1)$$

This bound cannot be augmented for $C_0^\infty(\Omega)$ functions.

Let Π be some non-empty parallelepiped as before and $\Omega=\{|x|\leqslant\varepsilon\}\times\Pi$, $\varepsilon>1$. Take some $j\in\{1,...,m\}$. As a counterexample, we consider a function

$$u(x,y)=\frac{1}{2}\,y_j^2\,h(x)$$

where

$$h(x)=\begin{cases}1, & |x|\leqslant 1,\\ h_{1,\delta}(x), & 1\leqslant|x|\leqslant\delta,\\ 0, & \delta\leqslant|x|\leqslant\varepsilon.\end{cases}$$

Here $1<\delta<\varepsilon$ and

$$h_{1,\delta}(x)=\exp\left(e^{-\dfrac{1}{|x|^2-\delta}}\left(-\frac{1}{\varepsilon-|x|^2}\right)\right).$$

One can easily check that $u\in C_0^\infty(\Omega)$. As the integral over annulus $\{1\leqslant|x|\leqslant\delta\}$ is finite, we consider only the integral over $\{|x|\leqslant 1\}$. The Baouendi-Grushin operator in this region yields

$$\Delta_\gamma u=|x|^{2\gamma}.$$

It follows from (3.3) that $\Delta_\gamma^{r/2}u$ produces a term

$$2^{r/2-1}|x|^{2(\gamma-r/2+1)}.$$

Due to the inverse triangle inequality

$$\left|\int\limits_{\Omega} f+g\,dz\right| \geqslant \left|\,\left|\int_\Omega f\,dz\right|-\left|\int_\Omega g\,dz\right|\,\right|,$$

with $f=2^{2(r/2-1)}|x|^{4(\gamma-r/2+1)}$ and g equal to other terms of $\left|\Delta_\gamma^{r/2}\,u\right|^2$, the integral

$$\int\limits_{\Omega} |\Delta_\gamma^{r/2}\,u|^2\,dz \tag{3.4}$$

diverges if the integral

$$\int\limits_{\Omega} |x|^{4(\gamma-r/2+1)}\,dz$$

is infinite. As we found out before, this happens when

$$r\geqslant m/2+2(\gamma+1),$$

which means the upper bound $m/2+2(\boldsymbol{\gamma}+1)$ is necessary and sufficient in $C_0^\infty(\Omega)$.

Now, we consider the case when r is odd. We have

$$\nabla_{\boldsymbol{\gamma}}^r=\nabla_{\boldsymbol{\gamma}}\,\Delta_{\boldsymbol{\gamma}}^{(r-1)/2}=$$

$$\left(\frac{\partial}{\partial x_1}\,\Delta_{\boldsymbol{\gamma}}^{(r-1)/2},\;\ldots,\;\frac{\partial}{\partial x_m}\,\Delta_{\boldsymbol{\gamma}}^{(r-1)/2},\;|x|^{\gamma}\,\frac{\partial}{\partial y_1}\,\Delta_{\boldsymbol{\gamma}}^{(r-1)/2},\;\ldots,\;|x|^{\gamma}\,\frac{\partial}{\partial y_k}\,\Delta_{\boldsymbol{\gamma}}^{(r-1)/2}\right)$$

and

$$\Delta_{\boldsymbol{\gamma}}^{(r-1)/2}=\sum_{\substack{r_1+\ldots+r_n=(r-1)/2\\ 0\leqslant r_i\leqslant(r-1)/2}}\binom{(r-1)/2}{r_1,\ldots,r_n}\prod_{i=1}^{m}\left(\frac{\partial^2}{\partial x_i^2}\right)^{r_i}\prod_{j=1}^{k}\left(|x|^{2\gamma}\frac{\partial^2}{\partial y_j^2}\right)^{r_{m+j}}.$$

In analogy with the previous reasoning, we consider iterated partial derivatives. The resulting lowest-power singular term is produced by

$$\frac{\partial}{\partial x_i}\left(\frac{\partial^2}{\partial x_{i_1}^2}\right)^{r_1}\cdots\left(\frac{\partial^2}{\partial x_{i_m}^2}\right)^{r_m}\left(|x|^{2\gamma}\,\frac{\partial^2}{\partial y_j^2}\right).$$

with $r_1+\ldots+r_m=(r-1)/2-1$. We have

$$\frac{\partial}{\partial x_i}|x|^{2\alpha}=2\alpha x_i|x|^{2(\alpha-1)}$$

and

$$\frac{\partial^2}{\partial x_i^2}|x|^{2\alpha}=2\alpha|x|^{2(\alpha-1)}+4\alpha(\alpha-1)x_i^2\,|x|^{2(\alpha-2)}.$$

As before, taking into account the polar change of variables w.r.t. x_1, ..., x_m, we have

$$\frac{\partial}{\partial x_i}|x|^{2\alpha}=O\big(\,|x|^{2\alpha-1}\,\big)$$

and

$$\frac{\partial^2}{\partial x_i^2}|x|^{2\alpha}=O\big(\,|x|^{2(\alpha-1)}\,\big)\,.$$

For $u\in C_0^\infty(\Omega)$, this yields

$$\nabla_{\boldsymbol{\gamma}}\,\Delta_{\boldsymbol{\gamma}}^{(r-1)/2}u=O\big(\,|x|^{2(\,\boldsymbol{\gamma}-(r-1)/2+1\,)-1}\,\big)=O\big(\,|x|^{2(\boldsymbol{\gamma}-r/2+1)}\,\big)\,.$$

Repeating the previous reasoning, we acquire

$$r<m/2+2(\boldsymbol{\gamma}+1)$$

and that the bound cannot be augmented with a counterexample as before. □

3.3 Proof of Remainders of the higher-order Poincaré inequalities for Baouendi-Grushin vector fields

Proof. Case 1. $\{x=0\}\cap\Omega=\varnothing$

Because the course of the argument is the same for both sequences of the formulae (2.1) and (2.2), we perform the proof only for the first sequence.

As Ω is smooth w.r.t. Baouendi-Grushin vector fields, the formulae (2.1) work for all points of the set. Integration of (2.1) over Ω yields

$$\int\limits_\Omega \left|\nabla_\gamma^{2r} u\right|^2 dz - \int\limits_\Omega \lambda_1^{2r}|u|^2\,dz =$$

$$\int\limits_\Omega \sum_{j=0}^{r-1}\lambda_1^{2(r-1-j)}\left(|\Delta_\gamma^{j+1} u+\lambda_1\,\Delta_\gamma^j u|^2+2\lambda_1\left|\nabla_\gamma\,\Delta_\gamma^j u-\frac{\nabla_\gamma\, e_1}{e_1}\Delta_\gamma^j u\right|^2\right)dz\,+$$

$$\int\limits_\Omega 2\sum_{j=0}^{r-1}\lambda_1^{2(r-1-j)+1}\,\nabla_\gamma\cdot\left(\frac{\nabla_\gamma\, e_1}{e_1}|\Delta_\gamma^j u|^2-\Delta_\gamma^j u\,\nabla_\gamma\,\Delta_\gamma^j u\right)dz=$$

$$\sum_{j=0}^{r-1}\lambda_1^{2(r-1-j)}\int\limits_\Omega\left(|\Delta_\gamma^{j+1} u+\lambda_1\,\Delta_\gamma^j u|^2+2\lambda_1\left|\nabla_\gamma\,\Delta_\gamma^j u-\frac{\nabla_\gamma\, e_1}{e_1}\Delta_\gamma^j u\right|^2\right)dz\,+$$

$$2\sum_{j=0}^{r-1}\lambda_1^{2(r-1-j)+1}\int\limits_\Omega\nabla_\gamma\cdot\left(\frac{\nabla_\gamma\, e_1}{e_1}|\Delta_\gamma^j u|^2-\Delta_\gamma^j u\,\nabla_\gamma\,\Delta_\gamma^j u\right)dz.$$

The last sum of the integrals vanishes due to ***Gauss-Ostrogradsky divergence theorem for Baouendi-Grushin vector fields***, since

$$u\in C_0^\infty(\Omega)\qquad\Longrightarrow\qquad \frac{\partial^s u}{\partial x_i^s},\ \frac{\partial^s u}{\partial y_j^s}\in C_0^\infty(\Omega)\quad \forall i,\forall j,\ \forall s\in\{1,...,2r\},$$

implying

$$\frac{\nabla_\gamma\, e_1}{e_1}|\Delta_\gamma^j u|^2-\Delta_\gamma^j u\,\nabla_\gamma\,\Delta_\gamma^j u=0\quad\text{on }\ \partial\Omega,\qquad \forall j\in\{0,1,...,r-1\}.$$

Thus

$$\int\limits_\Omega \left|\nabla_\gamma^{2r} u\right|^2 dz - \int\limits_\Omega \lambda_1^{2r}|u|^2\,dz =$$

$$\sum_{j=0}^{r-1}\lambda_1^{2(r-1-j)}\int\limits_\Omega\left(|\Delta_\gamma^{j+1} u+\lambda_1\,\Delta_\gamma^j u|^2+2\lambda_1\left|\nabla_\gamma\,\Delta_\gamma^j u-\frac{\nabla_\gamma\, e_1}{e_1}\Delta_\gamma^j u\right|^2\right)dz\geqslant 0.$$

Case 2. $\{x = 0\} \cap \Omega \neq \varnothing$

Due to ***The integrability criterion for degenerate domains*** the integral

$$\int\limits_{\Omega} |\nabla_\gamma^{2r} u|^2 \, dz$$

converges for $2 \leqslant 2r < m/2 + 2(\gamma + 1)$, which implies the upper bound for the upper index.

We take $\varepsilon > 0$ small enough to keep the ball $\{ |x| < \varepsilon\}$ inside Ω. As $\Omega \setminus \{|x| < \varepsilon\}$ is a smooth domain w.r.t. Baouendi-Grushin vector fields, ***The representation formulae*** work. We integrate (2.1) over this deformed set:

$$\int\limits_{\Omega\setminus\{|x|<\varepsilon\}} \left|\nabla_\gamma^{2r} u\right|^2 \, dz - \int\limits_{\Omega\setminus\{|x|<\varepsilon\}} \lambda_1^{2r} |u|^2 \, dz =$$

$$\sum_{j=0}^{r-1} \lambda_1^{2(r-1-j)} \int\limits_{\Omega\setminus\{|x|<\varepsilon\}} \left(|\Delta_\gamma^{j+1} u + \lambda_1 \Delta_\gamma^j u|^2 + 2\lambda_1 \left| \nabla_\gamma \Delta_\gamma^j u - \frac{\nabla_\gamma e_1}{e_1} \Delta_\gamma^j u \right|^2 \right) dz$$

$$+ \sum_{j=0}^{r-1} 2\lambda_1^{2(r-1-j)+1} \int\limits_{\Omega\setminus\{|x|<\varepsilon\}} \nabla_\gamma \cdot \left(\frac{\nabla_\gamma e_1}{e_1} |\Delta_\gamma^j u|^2 - \Delta_\gamma^j u \, \nabla_\gamma \Delta_\gamma^j u \right) dz,$$

Now we take limit $\varepsilon \to 0+$. Because all the functions are integrable according to the reasoning of ***The integrability criterion for degenerate domains*** (we can regard the integrands as $O(|\nabla_\gamma^{2r} u|^2)$ functions), we have

$$\int\limits_{\Omega} \left|\nabla_\gamma^{2r} u\right|^2 \, dz - \int\limits_{\Omega} \lambda_1^{2r} |u|^2 \, dz =$$

$$\sum_{j=0}^{r-1} \lambda_1^{2(r-1-j)} \int\limits_{\Omega} \left(|\Delta_\gamma^{j+1} u + \lambda_1 \Delta_\gamma^j u|^2 + 2\lambda_1 \left| \nabla_\gamma \Delta_\gamma^j u - \frac{\nabla_\gamma e_1}{e_1} \Delta_\gamma^j u \right|^2 \right) dz$$

$$+ \sum_{j=0}^{r-1} 2\lambda_1^{2(r-1-j)+1} \int\limits_{\Omega} \nabla_\gamma \cdot \left(\frac{\nabla_\gamma e_1}{e_1} |\Delta_\gamma^j u|^2 - \Delta_\gamma^j u \, \nabla_\gamma \Delta_\gamma^j u \right) dz,$$

Then we apply the same reasoning as in *case 1.*

Equality cases.

Necessity.

We put $u = \alpha e_1$ into (2.1). The right-hand side becomes

$$\sum_{j=0}^{r-1} \lambda_1^{2(r-1-j)} \int\limits_{\Omega} \left(\left|\Delta_\gamma^{j+1} (\alpha e_1) + \lambda_1 \Delta_\gamma^j (\alpha e_1)\right|^2 + 2\lambda_1 \left| \nabla_\gamma \Delta_\gamma^j (\alpha e_1) - \frac{\nabla_\gamma (\alpha e_1)}{\alpha e_1} \Delta_\gamma^j (\alpha e_1) \right|^2 \right) dz =$$

$$\sum_{j=0}^{r-1} \lambda_1^{2(r-1-j)} \int\limits_{\Omega} \left(\alpha^2 \left|\Delta_\gamma^{j+1} e_1 + \lambda_1 \Delta_\gamma^j e_1\right|^2 + 2\lambda_1 \alpha^2 \left| \nabla_\gamma \Delta_\gamma^j e_1 - \frac{\nabla_\gamma e_1}{e_1} \Delta_\gamma^j e_1 \right|^2 \right) dz =$$

$$\alpha^2 \sum_{j=0}^{r-1} \lambda_1^{2(r-1-j)} \int\limits_{\Omega} \left(\left|(-\lambda_1)^{j+1} e_1 + \lambda_1 (-\lambda_1)^j e_1\right|^2 + 2\lambda_1 \left| \nabla_\gamma (-\lambda_1)^j e_1 - \frac{\nabla_\gamma e_1}{e_1} (-\lambda_1)^j e_1 \right|^2 \right) dz =$$

$$\alpha^2 \sum_{j=0}^{r-1} \lambda_1^{2(r-1-j)} \int\limits_{\Omega} \left(\lambda_1^{2j} \left|-\lambda_1 e_1 + \lambda_1 e_1\right|^2 + 2\lambda_1 \lambda_1^{2j} \left|\nabla_\gamma e_1 - \nabla_\gamma e_1\right|^2 \right) dz = 0$$

Sufficiency.

We have

$$\sum_{j=0}^{r-1} \lambda_1^{2(r-1-j)} \int\limits_{\Omega} \left(|\Delta_{\gamma}^{j+1} u + \lambda_1 \Delta_{\gamma}^{j} u|^2 + 2\lambda_1 \left| \nabla_{\gamma} \Delta_{\gamma}^{j} u - \frac{\nabla_{\gamma} e_1}{e_1} \Delta_{\gamma}^{j} u \right|^2 \right) dz = 0,$$

which, particularly ($j = 0$), implies

$$\int\limits_{\Omega} |\Delta_{\gamma} u + \lambda_1 u|^2 \, dz = 0.$$

Thereby, we have

$$\Delta_{\gamma} u + \lambda_1 u = 0 \quad \text{or} \quad \Delta_{\gamma} u = -\lambda_1 u,$$

with the boundary condition

$$u = 0 \quad \text{on} \ \partial\Omega,$$

as we consider functions with a compact support. A solution to this problem is the first eigenfunction e_1, up to the multiplicative constant. □

Chapter 4

Applications

4.1 Introduction

As an application, we consider the higher-order Fisher-KPP equation on Baouendi-Grushin vector fields. We discuss boundedness of global solutions, asymptotic behavior of global solutions and blow-up of solutions. Moreover, we extend the obtained results to the time-fractional higher-order Fisher-KPP equation on Baouendi-Grushin vector fields.

The celebrated Fisher-KPP (Fisher-Kolmogorov-Petrovsky-Piskunov) equation is

$$v_t - \Delta v = mv(1-v), \quad s \in \mathbb{R}^n,\ t > 0. \tag{4.1}$$

Replacing v by $1-w$ (with $m=1$) in the Fisher-KPP equation one can easily derive

$$w_t - \Delta w = w(w-1), \quad s \in \mathbb{R}^n,\ t > 0. \tag{4.2}$$

Here, we consider a subelliptic version of the Fisher-KPP equation with the Baouendi-Grushin sub-Laplacian Δ_γ of an odd power in the space variables:

$$u_t - \Delta_\gamma^{2l-1} u = u(u-1) \ \text{ in } \Omega,\ t > 0, \tag{4.3}$$

with an initial condition

$$u(\cdot, 0) = u_0(\cdot) \ \text{ in } \Omega, \tag{4.4}$$

and with a boundary condition

$$\nabla_{2\gamma}\big(\nabla_\gamma^{2(r-1)} u\big)\, \nabla_\gamma^{2(l-r)} u = 0 \ \text{ on } \ \partial\Omega,\ t > 0 \quad \text{for } \ r = 1, ..., l. \tag{4.5}$$

Despite existence being the first priority in studying mathematical problems, we leave the issue out of the scope. A possible resolvation could be [Paz83, ch. 6, sec. 1, th. 1.4, p. 185]. The core of the problem is checking that Δ_γ^{2l-1} generates C_0-semigroup.

Our primary goal in this chapter is to analyse solutions of (4.3)–(4.5). We prove boundedness of global solutions when the initial data is continuous and bounded between 0 and 1. We also discuss large-time behavior of global solutions. Furthermore, under a certain condition, we show that a solution of (4.3)–(4.5) blows up in a finite time. To the best of our knowledge, these results are new since we have not succeeded to find those in the existing literature. We believe that it is of essential interest for the theory of higher-order subelliptic operators since the sub-Laplacian serves as a model operator.

In the second part of this chapter we deal with the following time-fractional higher-order Fisher-KPP equation:

$$\partial_t^\alpha u - \Delta_\gamma^{2l-1} u = u(u-1), \quad \text{in } \Omega,\ t > 0,$$

where ∂_t^α is the Caputo fractional derivative. We extend our arguments from the first part of this chapter to an initial-boundary value problem for the time-fractional higher-order Fisher-KPP equation.

4.2 Main results

The following theorem was encouraged by work [DFR14].

4.2.1 Boundedness of global solutions

Theorem (**Boundedness of global solutions**). *Suppose that* $u_0 \in C(\overline{\Omega})$ *satisfies the estimate* $0 \leqslant u_0 \leqslant 1$. *Then a global solution to the problem* (4.3) *with the initial-boundary conditions* (4.4)–(4.5) $u \in C^{2(2l-1),1}(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$ *possesses the property*

$$0 \leqslant u(z,t) \leqslant 1$$

for $(z,t) \in \Omega \times (0,\infty)$.

Proof. Firstly, we show that $u \geqslant 0$. Multiplying both sides of equation (4.3) by $\tilde{u} := \min\{u, 0\}$ and integrating over Ω, we obtain

$$\int\limits_{\Omega} \tilde{u}_t \cdot \tilde{u}\, dz - \int\limits_{\Omega} \Delta_{\gamma}^{2l-1}\, \tilde{u} \cdot \tilde{u}\, dz = \int\limits_{\Omega} \tilde{u}^2(\tilde{u}-1)\, dz.$$

Putting the second term on the right hand side and using monotonicity w.r.t. the integrand of Lebesgue integral, we get

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega} \tilde{u}^2\, dz \leqslant \int\limits_{\Omega} \Delta_{\gamma}^{2l-1}\, \tilde{u} \cdot \tilde{u}\, dz + \max_{\Omega}\{\tilde{u}-1\} \int\limits_{\Omega} \tilde{u}^2\, dz.$$

Now we apply ***The reduction formula*** which yields

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega} \tilde{u}^2\, dz \leqslant -\int\limits_{\Omega} \left(\nabla_{\gamma}^{2l-1}\tilde{u}\right)^2\, dz + \max_{\Omega}\{\tilde{u}-1\} \int\limits_{\Omega} \tilde{u}^2\, dz.$$

By using ***The higher-order Poincaré inequalities for Baouendi-Grushin vector fields*** we acquire

$$\frac{d}{dt}\int\limits_{\Omega} \tilde{u}^2(z,t)\, dz \lesssim \int\limits_{\Omega} \tilde{u}^2(z,t)\, dz, \tag{4.6}$$

which, due to Grönwall's lemma, implies $\int_{\Omega} \tilde{u}^2\,(z,t)\; dz \equiv 0$ as $\tilde{u}\,(z,0) \equiv 0$, hence $\tilde{u} = 0$. Therefore, $u(z,t) \geqslant 0$ for $(z,t) \in \Omega \times (0,\infty)$.

Now we show that $u \leqslant 1$. Multiplying both sides of equation (4.3) by $\hat{u} := \min\{1-u, 0\}$ and integrating over Ω, we get

$$\int\limits_{\Omega} \hat{u}_t \cdot \hat{u}\, dz - \int\limits_{\Omega} \Delta_{\gamma}^{2l-1}\, \hat{u} \cdot \hat{u}\, dz = \int\limits_{\Omega} \hat{u}^2\,(\hat{u}-1)\; dz.$$

Repeating the above calculations for $\hat{u} := \min\{1-u, 0\}$ we get

$$\frac{d}{dt}\int\limits_{\Omega} \hat{u}^2\,(z,t)\; dz \lesssim \int\limits_{\Omega} \hat{u}^2\,(z,t)\; dz.$$

This implies $\int_{\Omega} \hat{u}^2\,(z,t)\; dz \equiv 0$ as $\hat{u}\,(z,0) \equiv 0$, hence $\hat{u} = 0$. Therefore, we have $u(z,t) \leqslant 1$ for $(z,t) \in \Omega \times (0,\infty)$. □

4.2.2 Large-time behavior of global solutions

Theorem (**Large-time behavior of global solutions**). *If* $0 \leqslant u_0 \leqslant 1$ *and* $u_0 \in C(\overline{\Omega})$*, then a solution to the problem* (4.3)–(4.5) *satisfies the estimate*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leqslant \exp\big(-\lambda_1^{2l-1}\,t\big)\,\|u_0\|_{L^2(\Omega)}, \;\; t > 0. \tag{4.7}$$

Proof. Due to **Boundedness of global solutions** we have $0 \leqslant u \leqslant 1$, implying $u(u-1) \leqslant 0$. Then u satisfies the inequality

$$u_t - \Delta_\gamma^{2l-1}\,u \leqslant 0, \quad z \in \Omega,\; t > 0, \tag{4.8}$$

with the initial-boundary conditions (4.4)–(4.5).

Multiplying both sides of the inequality (4.8) by u and integrating over Ω, we establish

$$\frac{1}{2}\frac{d}{dt}\int\limits_\Omega u^2\,dz - \int\limits_\Omega \Delta_\gamma^{2l-1}\,u\cdot u\,dz \leqslant 0.$$

Due to ***The reduction formula*** we acquire

$$\frac{d}{dt}\int\limits_\Omega u^2\,dz + 2\int\limits_\Omega (\nabla_\gamma^{2l-1}\,u)^2\,dz \leqslant 0.$$

By using ***The higher-order Poincaré inequalities for Baouendi-Grushin vector fields*** we obtain

$$E'(t) + 2\lambda_1^{2l-1}E(t) \leqslant 0, \quad t > 0,$$

where $E(t) = \int_\Omega u^2\,dz$.

Due to Grönwall's lemma

$$E(t) \leqslant E(0)\exp\big(-2\lambda_1^{2l-1}\,t\big)$$

which is the estimate (4.7) squared. □

4.2.3 Blow-up of solutions

Theorem (**Blow-up solutions**). *If* $1 + \lambda_1^{2l-1} < \int\limits_\Omega u_0 e_1\,dz = H_0$*, then a solution to the problem* (4.3)–(4.5)*, satisfying*

$$\oint\limits_{\partial\Omega} \Delta_\gamma^{2(l-r)}u\;\nabla_{2\gamma}e_1\cdot d\boldsymbol{\nu} = 0, \quad r = 1, ..., l, \tag{4.9}$$

blows up in the finite time $T^* = \log|H_0/(H_0 - 1 - \lambda_1^{2l-1})|$.

Proof. Multiplying the equation (4.3) by e_1 and integrating over Ω leads to

$$\int\limits_\Omega u_t(z,t)e_1(z)dz - \int\limits_\Omega \Delta_\gamma^{2l-1}u(z,t)e_1(z)dz = \int\limits_\Omega u(z,t)\,(u(z,t)-1)\,e_1(z)dz. \tag{4.10}$$

Let us set $H(t) = \int_\Omega u(z,t)e_1(z)\,dz$.

Due to ***Green's first identity for Baouendi-Grushin vector fields***

$$-\int\limits_\Omega \Delta_\gamma^{2l-1}u\,e_1\,dz = -\oint\limits_{\partial\Omega} e_1\nabla_{2\gamma}\big(\Delta_\gamma^{2l-2}u\big)\cdot d\boldsymbol{\nu} + \int\limits_\Omega \nabla_\gamma\big(\Delta_\gamma^{2l-2}u\big)\cdot\nabla_\gamma e_1\,dz.$$

Simultaneously,

$$\lambda_1 \int\limits_\Omega \Delta_\gamma^{2l-2} u\, e_1\, dz = - \int\limits_\Omega \Delta_\gamma^{2l-2} u\, \Delta_\gamma e_1\, dz =$$

$$- \oint\limits_{\partial\Omega} \Delta_\gamma^{2l-2} u\, \nabla_{2\gamma} e_1 \cdot d\boldsymbol{\nu} + \int\limits_\Omega \nabla_\gamma \big(\Delta_\gamma^{2l-2} u\big) \cdot \nabla_\gamma e_1\, dz.$$

Recalling that

$$e_1 = 0 \ \text{ on } \ \partial\Omega \quad \text{ and } \quad \oint\limits_{\partial\Omega} \Delta_\gamma^{2(l-1)} u \ \nabla_{2\gamma} e_1 \cdot d\boldsymbol{\nu} = 0$$

we obtain

$$- \int\limits_\Omega \Delta_\gamma^{2l-1} u\, e_1\, dz = \lambda_1 \int\limits_\Omega \Delta_\gamma^{2l-2} u\, e_1\, dz.$$

As we have (4.9), repeating the procedure another $2l-2$ times, we acquire

$$- \int\limits_\Omega \Delta_\gamma^{2l-1} u\, e_1\, dz = \lambda_1^{2l-1} \int\limits_\Omega u\, e_1\, dz.$$

Using the obtained equality and, thanks to Jensen's inequality,

$$H^2(t) \leqslant \int\limits_\Omega u^2(z,t) e_1(z) dz,$$

from (4.10) we have

$$H'(t) + (1 + \lambda_1^{2l-1}) H(t) \geqslant H^2(t), \quad H_0 := H(0) = \int\limits_\Omega u_0 e_1 dz. \tag{4.11}$$

Due to the comparison principle [LL69, ch. 1, sec. 4, th. 1.4.1, p. 15] we have

$$H(t) \geqslant \frac{1 + \lambda_1^{2l-1}}{1 - \exp\big((1 + \lambda_1^{2l-1})(t - T^*) \big)},$$

where $T^* = \log |H_0/(H_0 - 1 - \lambda_1^{2l-1})|$. □

4.3 Time-fractional extension

In this section, we consider the time-fractional higher-order version of the Fisher-KPP equation

$$\partial_t^\alpha u - \Delta_\gamma^{2l-1} u = u(u-1), \quad z \in \Omega, \ t > 0, \tag{4.12}$$

supplemented with the initial and boundary conditions (4.4)–(4.5). Here ∂_t^α is the Caputo fractional derivative

$$\partial_t^\alpha u(z,t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t (t-\tau)^{-\alpha} u_\tau(z,\tau)\, d\tau$$

of order $\alpha \in (0,1]$, defined for a differentiable w.r.t. τ function u.

4.3.1 Boundedness of global solutions

Theorem (**Time-fractional boundedness of global solutions**). *Let $u_0 \in C(\overline{\Omega})$ be such that $0 \leqslant u_0 \leqslant 1$. Then a global solution to the problem (4.12) supplemented with the initial-boundary conditions (4.4) and (4.5) satisfies $0 \leqslant u(z,t) \leqslant 1$ for all $(z,t) \in \Omega \times (0,\infty)$.*

Proof. First, we show that $u \geqslant 0$. Multiplying both sides of the Fisher-KPP equation (4.12) by $\tilde{u} := \min(u, 0)$ and integrating over Ω, we obtain

$$\int_{\Omega} \partial_t^{\alpha} \tilde{u} \cdot \tilde{u}\, dz - \int_{\Omega} \Delta_{\gamma}^{2l-1}\, \tilde{u} \cdot \tilde{u}\, dz = \int_{\Omega} \tilde{u}^2 (\tilde{u} - 1)\, dz.$$

Then, repeating the same procedure as in the proof of the theorem **Boundedness of global solutions** and using an estimate [AAK17]

$$2v(t)\partial_t^{\alpha} v(t) \geqslant \partial_t^{\alpha} v^2(t),$$

we obtain

$$\partial_t^{\alpha} \int_{\Omega} \tilde{u}^2(z,t)\, dz \lesssim \int_{\Omega} \tilde{u}^2(z,t)\, dz. \tag{4.13}$$

By denoting $\int_{\Omega} \tilde{u}^2(z,t)\, dz = \varphi(t)$ in (4.13) we get

$$\begin{cases} \partial_t^{\alpha} \varphi(t) \lesssim \varphi(t), \\ \varphi(0) = 0, \end{cases}$$

which implies $\tilde{u} = 0$ [Wu20]. Therefore $u \geqslant 0$.

The case $u \leqslant 1$ is proved similarly as in the proof of the theorem **Boundedness of global solutions**. □

4.3.2 Large-time behavior of global solutions

Theorem (**Time-fractional large-time behavior of global solutions**). *Let us assume that $0 \leqslant u_0 \leqslant 1$ and $u_0 \in C(\overline{\Omega})$. Then a solution to the problem (4.12) supplemented with the initial-boundary conditions (4.4) and (4.5) satisfies the estimate*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \lesssim \frac{1}{1 + (\lambda_1^{2l-1}/2)\, t^{\alpha}} \|u_0\|_{L^2(\Omega)} \lesssim t^{-\alpha}, \quad t > 0. \tag{4.14}$$

Proof. Thanks to **Time-fractional boundedness of global solutions** we have $u(u-1) \leqslant 0$. Thereby, u satisfies

$$\partial_t^{\alpha} u - \Delta_{\gamma}^{2l-1}\, u \leqslant 0, \quad z \in \Omega,\ t > 0, \tag{4.15}$$

with the initial-boundary conditions (4.4)–(4.5).

Multiplying both sides of the inequality (4.15) by u and using the estimate (see [AAK17])

$$2v(t)\partial_t^{\alpha} v(t) \geqslant \partial_t^{\alpha} v^2(t),$$

The reduction formula and ***The higher-order Poincaré inequalities for Baouendi-Grushin vector fields***, we acquire

$$\partial_t^{\alpha} E(t) + \frac{\lambda_1^{2l-1}}{2} E(t) \leqslant 0, \quad t > 0,$$

where $E(t) = \int_{\Omega} u^2\, dz$.

According to [Wu20] we have $E(t) \leqslant \widetilde{E}(t)$, where $\widetilde{E}(t)$ is the solution to the problem

$$\begin{cases} \partial_t^\alpha \widetilde{E}(t) + \dfrac{\lambda_1^{2l-1}}{2}\widetilde{E}(t) = 0, \quad t > 0, \\ \widetilde{E}(0) = \widetilde{E}_0 = \int_\Omega u_0^2(z)\, dz, \end{cases} \tag{4.16}$$

The unique solution to the problem (4.16) has a form [KST06, ch. 2, sec. 4, Lemma 2.23, p.98]

$$\widetilde{E}(t) = \widetilde{E}_0 E_\alpha\left(-\frac{\lambda_1^{2l-1}}{2}\, t^\alpha\right),$$

where $E_\alpha(s)$ is the Mittag-Leffler function [KST06, ch. 1, sec. 8, p. 40]:

$$E_\alpha(s) = \sum_{r=0}^{\infty} \frac{s^r}{\Gamma(\alpha r + 1)}.$$

By means of the inequality

$$E_\alpha(-s) \lesssim \sum_{r=0}^{\infty} (-s)^r = \frac{1}{1+s}, \quad s > 0,$$

we get the desired estimate (4.14). □

4.3.3 Blow-up of solutions

Theorem (**Time-fractional blow-up solutions**). *If* $1+\lambda_1^{2l-1} < \int\limits_\Omega u_0 e_1\, dz = H_0$, *then a solution to the problem* (4.12) *with the initial-boundary conditions* (4.4) *and* (4.5), *satisfying*

$$\oint\limits_{\partial\Omega} \Delta_\gamma^{2r} u\, \nabla_{2\gamma} e_1 \cdot d\boldsymbol{\nu} = 0, \quad r = 1, ..., l, \tag{4.17}$$

blows up in a finite time.

Proof. Multiplying the equation (4.3) by e_1 and integrating over Ω lead to

$$\partial_t^\alpha \int\limits_\Omega u(z,t)e_1(z)\, dz - \int\limits_\Omega \Delta_\gamma^{2l-1} u(z,t)e_1(z)\, dz = \int\limits_\Omega u(z,t)(\, u(z,t) - 1\,)e_1(z)\, dz. \tag{4.18}$$

Let us set $H(t) = \int_\Omega u(z,t)e_1(z)\, dz$.

Repeating the course of the argument from the theorem **Blow-up solutions** we acquire

$$-\int\limits_\Omega \Delta_\gamma^{2l-1} u\, e_1\, dz = \lambda_1^{2l-1} \int\limits_\Omega u\, e_1\, dz.$$

Thanks to Jensen's inequality we have

$$H^2(t) \leqslant \int\limits_\Omega u^2(z,t)e_1(z)\, dz.$$

Combining the obtained equality and inequality we get

$$\partial_t^\alpha H(t) + (1 + \lambda_1^{2l-1})H(t) \geqslant H^2(t).$$

Let $\widetilde{H}(t) = H(t) - (1 + \lambda_1^{2l-1})$. Then we get

$$\partial_t^\alpha \widetilde{H}(t) \geqslant \widetilde{H}(t)\left(\widetilde{H}(t) + 1 + \lambda_1^{2l-1}\right) \geqslant \widetilde{H}(t)\left(\widetilde{H}(t) + 1\right). \tag{4.19}$$

As $0 \leqslant \widetilde{H}_0 = \widetilde{H}(0)$, then from the results in [DFR14] the solution to the inequality (4.19) blows up in a finite time. □

Remark. The theorem **Time-fractional blow-up solutions** works, particularly, when

$$\Delta_\gamma^{2r} u = 0 \ \text{ on } \ \partial\Omega, \quad r = 1, ..., l.$$

Chapter 5

Conclusion

In this work, we have proven the higher-order Poincaré inequalities for Baouendi-Grushin vector fields. The basic contribution to the proof does undoubtedly belong to Tohru Ozawa and Durvudkhan Suragan [OS20]. Our contribution is the thorough proof of the inequalities for domains non-smooth w.r.t. the Baouendi-Grushin vector fields. To be precise, we have proceeded on the case when the domain possesses the degeneracy $\{x = 0\}$: one should treat this delicately as an intersection with the hyperlane opens the way to potential division by zero.

Another result to be appreciated is classical vector calculus' theorems for Baouendi-Grushin vector fields. With immense gratitude I credit developing of these theorems to Vladimir Aleksandrovich Ilyin and Eduard Grigorievich Poznyak who have provided thorough proofs of the analogous theorems in the ordinary case for a special class of domains and showed a way to extend the result for the common type of domains [IP82, ch. 7, sec. 1 and 3]. As far as we have searched, classical vector fields' theorems for Baouendi-Grushin vector fields have not been proven and even properly formulated. Thus the conducted proofs may be regarded as an educational contribution to the theory of Baouendi-Grushin vector fields.

And "Applications" chapter should be fully credited to Ardak Kashkynbayev, Durvudkhan Suragan, and Berikbol T. Torebek [KST21]: all the results they have obtained for the Fisher-KPP equation on the Heisenberg group are entirely translated to the case of Baouendi-Grushin vector fields. We have successfully augmented their findings by considering the higher-order Baouendi-Grushin operator of an odd power.

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