EXISTENCE AND UNIQUENESS RESULTS FOR A SYSTEM OF COUPLED KDV EQUATIONS IN WEIGHTED SOBOLEV SPACE

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Abstract. The system of Korteweg–De Vries (KdV) equations is a system of partial differential equations. It is used to describe the modeling of shallow water surface waves. In this paper, we show the well-posedness of the coupled Korteweg–De Vries system of equations with initial data under Sobolev Space and Weighted Sobolev Space for $\frac{3}{4} \leq s \leq 1$.

Contents

1. Introduction

In this paper, we are interested in the initial value problem for the Korteweg-de Vries system of equations [6] under Sobolev Space:

$$\begin{cases} u_t + a_1 u_{xxx} = b_1 uu_x + c_1 vv_x + d_1 uv_x + e_1 vu_x, \\ v_t + a_2 v_{xxx} = b_2 uu_x + c_2 vv_x + d_2 uv_x + e_2 vu_x, \qquad x \in \mathbb{R} \end{cases} . \tag{1}$$

with given initial data:

$$u_0(x,0) = u_0(x), v_0(x,0) = v_0(x). \tag{2}$$

The Korteweg-de Vries system of equations is a mathematical system of equations, that describes the behavior of waves on shallow water surfaces [2]. Firstly, it was deduced by Boussinesq [4], a French mathematician, in the 19th century [5]. Then, this system of equations was rediscovered by Korteweg and de Vries [13], Dutch mathematicians, with a simpler solution. This system of equations is important because it provides insights into the behavior of nonlinear waves, which can be applied to physical modeling and optics.

This system of equations contains also initial data, which is crucial for establishing the solution. Given initial data, we want to establish the solution. Since this system of equations is nonlinear, The KdV equation may not have explicit solutions, but it may possess weak solutions. In that sense, Sobolev spaces allow us to work with weak solutions by providing several useful properties for functions with partial derivatives.

With the use of Sobolev spaces, it is possible to establish the well-posedness of the solution for a coupled KdV system of equations. The problem (1) is said to be locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ if it has existence, uniqueness, and persistence shown in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with continuous dependence on the initial data in some [0, T] time interval [3]. This work focuses on two parts, especially the Existence and Uniqueness of the solution. Next, this paper considers a brief review of well-known well-posedness results for the system of KdV.

(1) In 1992, J. Bona, G.Ponce, J. Saut, and M. Tom [3] showed well-posedness with $a_1, b_1, a_2, c_2 = 1$ and $|a_3|\sqrt{b}_2 < 1$ for $s \geq 1$
(2) In 2008, J.Alvarez [1] have shown well-posedness for $s > 3/4$, where $a_1 = 1$, $b_1 = 6b_1$, $c_1 = 2b$, $d_2 = 3$
(3) In 2009, T. Oh [17] obtained the well-posedness for $s \geq 0$, where $a_1 = 1$, $b_1 = 0$, $c_1 = 1$, $d_1 = 0$,$e_1 = 0$ $a_2 = \alpha$, $b_2 = 0$, $c_2 = 0$, $d_2 = 1$,$e_2 = 0$ by considering bilinear estimates.
(4) In 2021, A. Muñoz and A.Pastor, [15] showed for $s > 3/4$ within $a_1 = b$, $b_1 = 6a$, $c_1 = 2r$, $d_1 = 0$,$e_1 = 0$ $a_2 = 1$, $b_2 = 0$, $c_2 = 0$, $d_2 = 3$,$e_2 = 0$ in Weighted Sobolev Space.

From the literature review, it can be seen the general trend of decreasing regularities with more general coefficients. This paper considers the exact regularity of 3/4 for Sobolev Space, which requires the introduction of Bourgain Space. This space allows working in low regularities, providing several tools such as bilinear estimates to provide well-posedness.

The second focus of this work is the establishment of well-posedness for the same regularity in the Weigthed-Sobolev Space. Weighted Sobolev space guarantees the decay of the solution at infinity with the decay of initial data, so the solution would be more consistent. The proof methodology is done similarly as in Sobolev Space, except that it requires control in the weighted norm. This control would proceed by introducing three intermediate commutators and the final nonlinear part. The control of each commutator requires the use of technical estimates and exploring the control of oscillatory integrals in Lemma 3.4.

2. Definitions and Preliminaries

This paper uses Banach's fixed point theorem 2.1 as the main tool for showing the existence of a unique solution.

Definition 2.1. *Banach's fixed point theorem* [19] *Let* $(X, \|\cdot\|\ )$ *be a complete normed vector space. If* $\Phi : X \to X$ *is a contraction mapping, that is,*

$$\|\Phi(x) - \Phi(y)\|_X \leq K\|x - y\|_X, \quad x, y \in X,$$

for some $0 < K < 1$*, then there exists a unique* $x^* \in X$ *verifying that*

$$x^* = \Phi(x^*).$$

This theorem is useful since our initial problem can be converted to the Fixed fixed-point problem as in [21] using Duhamel's principle:

$$(u, v) = (\Phi_1(u, v), \Phi_2(u, v)), \tag{3}$$

where $\Phi_1(u, v), \Phi_2(u, v)$ is defined by

$$\begin{cases} \Phi_1(u, v) = e^{a_1 t\partial_x^3} u_0 + \int_0^t e^{a_1(t-t')\partial_x^3}[b_1 u u_x + c_1 v v_x + d_1 u v_x + e_1 v u_x]\, dt' \\ \Phi_2(u, v) = e^{a_2 t\partial_x^3} v_0 + \int_0^t e^{a_2(t-t')\partial_x^3}[b_2 u u_x + c_2 v v_x + d_2 u v_x + e_2 v u_x]\, dt'. \end{cases}$$

Therefore, we focus on the well-definedness and contraction mapping as the main method in this paper. In this paper, we consider A $\lesssim$ B as $A \leq CB$, where C>0 is a constant. Furthermore, A$\sim$ B indicates $A \lesssim B$ and $B \lesssim A$ as in [14].
Next, consider the definitions of Function spaces, Bourgain spaces, Fractional Derivates, and a few lemmas which are important tools in showing the main results.

2.1. Function spaces.

Let a function $f \in L^2(\mathbb{R})$. Then its Fourier Transform $\widehat{f}$ as in [6] can be defined by:

$$\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx, \quad x \in \mathbb{R}$$

and its inverse Fourier transform by

$$f^{\wedge}(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} f(x)\, dx, \quad x \in \mathbb{R}$$

This work considers inhomogeneous Sobolev Space $H^s(\mathbb{R})$ of order s $\in \mathbb{R}$ defined through the norm:

$$||f||_{H^s(\mathbb{R})} = (\int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{f(\xi)}|^2\, d\xi)^{\frac{1}{2}}$$

This inhomogeneous Sobolev space satisfies the principle of inclusion [6] such that for any $H^{s'} \subset H^s$ such that $s \leq s'$, we have the following inequality:

$$\|f\|_{H^s} \lesssim \|f\|_{H^{s'}} \tag{4}$$

Furthermore, to measure functions regularity in the space-time domain, we futher consider mixed-norm Lebesgue spaces as in [6], $L_x^pL_T^q$ or $L_T^qL_x^p$, for $1\leq p,q\leq\infty$, given by the norms:

$$\|f\|_{L_x^pL_T^q}=\Big\{\int_{\mathbb{R}}(\int_0^T|f(x,t)|^q\,dt)^{\frac{p}{q}})\,dx\Big\}^{\frac{1}{p}}$$

, and

$$\|f\|_{L_T^qL_x^p}=\Big\{\int_0^T(\int_{\mathbb{R}}|f(x,t)|^p\,dx)^{\frac{q}{p}})\,dt\Big\}^{\frac{1}{q}}.$$

We consider some well-known results in the L_p spaces, which will be useful in this paper. The first one is the Holder's Inequality, which is defined in the following lemma considered in [18].

Lemma 2.1. *Let p, q > 1 with* $\frac{1}{p}+\frac{1}{q}=1$. *Then:*

$$\int_a^b|f(x)g(x)|\,dx\leq\Big(\int_a^b|f(x)|^p\,dx\Big)^{1/p}\Big(\int_a^b|g(x)|^q\,dx\Big)^{1/q}$$
$$=\|f(x)\|_{L^p(a,b)}\|g(x)\|_{L^p(a,b)}$$

Also, we consider Minkowski's integral inequality. It is stated in the following Lemma as in [18].

Lemma 2.2. *For* $1<p<\infty$, *we have:*

$$\Big(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}u(x,t)\,dx\Big|^p\,dt\Big)^{\frac{1}{p}}\leq\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}|u(x,t)|^p\,dt\Big)^{\frac{1}{p}}\,dx$$

Next, we consider Young's inequality [18], which is a useful property when working with convolution.

Lemma 2.3. *For* $1<p,q<\infty$ *with* $\frac{1}{p}+\frac{1}{q}\geq 1, let f\in L^p(\mathbb{R})$ *and* $g\in L^q(\mathbb{R})$. *Then* $f*g\in L^r(\mathbb{R})$ *with the following equaly* $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$. *In addition,*

$$\|f*g\||_{L^r(\mathbb{R})}\leq\|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}$$

2.2. **Bourgain space.** For $a,b,s\in\mathbb{R}$, we define the Bourgain space $X_{s,b}^a$

$$\|u\|_{X_{s,b}^a}:=\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\tau-a\xi^3|)^{2b}(1+|\xi|)^{2s}|\hat{u}(\xi,\tau)|^2d\xi d\tau\Big)^{1/2}$$

Lemma 2.4. *The following Lemma from (*[21] *states that the bilinear estimate holds for any* $s>-3/4$, $b\in(1/2,b_0(s))$ *and for all* $w_1,w_2\in R$ *with some* $b_0(s)>1/2$

i) $\|\partial_x(w_1,w_2)\|_{X_{s,b-1}^{\alpha_2}}\lesssim\|w_1\|_{X_{s,b}^{\alpha_1}}\|w_2\|_{X_{s,b}^{\alpha_1}}$,
ii) $\|\partial_x(w_1,w_2)\|_{X_{s,b-1}^{\alpha_1}}\lesssim\|w_1\|_{X_{s,b}^{\alpha_1}}\|w_2\|_{X_{s,b}^{\alpha_2}}$,
iii) $\|(\partial_xw_1)w_2\|_{X_{s,b-1}^{\alpha_1}}\lesssim\|w_1\|_{X_{s,b}^{\alpha_1}}\|w_2\|_{X_{s,b}^{\alpha_2}}$,
iv) $\|w_1(\partial_xw_2)\|_{X_{s,b-1}^{\alpha_1}}\lesssim\|w_1\|_{X_{s,b}^{\alpha_1}}\|w_2\|_{X_{s,b}^{\alpha_2}}$.

2.3. **Fractional Derivative.**

Fractional Derivative D_x^s for Fourier multiplier, where s $\in\mathbb{R}$ is defined as:

$$D_x^su(t)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}(i\xi)^s\widehat{u}(\xi)e^{it\xi}\,dt,$$

which is the same as saying that

$$\widehat{D_x^su(\xi)}=(i\xi)^s\widehat{u(\xi)}.$$

Furthermore, Plancherel's property allows rewriting $\|f\|_{H^s(\mathbb{R})}$ as

$$\|f\|_{H^s(\mathbb{R})} \lesssim \|f\|_{L^2} + \|D_x^s f\|_{L^2}.$$

Lemma 2.5. *For any $r \in \mathbb{R}$, we have that*

i) $\|uv_x\|_{L_T^2 L_x^2} \leq \|v_x\|_{L_x^\infty L_T^2} \|u\|_{L_x^2 L_T^\infty}$

ii)

$$\begin{aligned}\|D_x^r(uv_x)\|_{L_T^2 L_x^2} &\lesssim \|v_x\|_{L_T^4 L_x^\infty} \|D_x^s u\|_{L_T^\infty L_x^2} + \|u\|_{L_x^2 L_T^\infty} \|D_x^s \partial_x v\|_{L_x^\infty L_T^2} \\ &\lesssim cT^{\frac{1}{4}} \|D_x^r u\|_{L_T^\infty L_x^2} \|v_x\|_{L_T^4 L_x^\infty} + \|u\|_{L_x^2 L_T^\infty} \|D_x^r(v_x)\|_{L_x^\infty L_T^2}\end{aligned}$$

Lemma 2.5, i).

$$\begin{aligned}\|uv_x\|_{L_T^2 L_x^2} = \|uu_x\|_{L_x^2 L_T^2} &= \left(\int_{\mathbb{R}} \left(\int_0^T \|vu_x\|^2 dt\right)\right)^{\frac{1}{2}} dx \\ &\leq \int_{\mathbb{R}} \|v_x\|_{L_T^\infty}^2 \|u_x\|_{L_T^2}^2 dx \leq \|v_x\|_{L_x^\infty L_T^2} \|u\|_{L_x^2 L_T^\infty}.\end{aligned}$$

□

Lemma 2.5, ii).

$$\begin{aligned}\|D_x^r(uv_x)\|_{L_T^2 L_x^2} &\lesssim \Big\| \|v_x\|_{L_x^\infty} \|D_x^r u\|_{L_x^2} \Big\|_{L_T^2} + \|u\|_{L_x^2 L_T^\infty} \|D_x^r(v_x)\|_{L_x^\infty L_T^2} \\ &\leq cT^{\frac{1}{4}} \|D_x^r u\|_{L_T^\infty L_x^2} \|v_x\|_{L_T^4 L_x^\infty} + \|u\|_{L_x^2 L_T^\infty} \|D_x^r(v_x)\|_{L_x^\infty L_T^2}.\end{aligned}$$

□

Lemma 2.6. *The following inequalities in* [1] *hold*

i) $\|e^{at\partial_x^3} u_0\|_{H^r} \lesssim \|u_0\|_{H_r}$, $r \in \mathbb{R}$,

ii) $\|\partial_x e^{at\partial_x^3} u_0\|_{L_T^4 L_x^\infty} \lesssim \frac{1}{|a|^{\frac{1}{4}}} \|u_0\|_{H_r}$ *for* $r \geq \frac{3}{4}$,

iii) $\|D_x^r \partial_x e^{at\partial_x^3} u_0\|_{L_x^\infty L_T^2} \leq \frac{c}{|a|^{\frac{1}{2}}} ||D_x^r u_0||_{H_r}$ *for* $r \in \mathbb{R}$,

iv) $\|e^{at\partial_x^3} u_0\|_{L_x^2 L_T^\infty} \leq c_{(a,r)}(1+T)^{\frac{1}{2}} \|u_0\|_{H_r}$ *for* $r > \frac{3}{4}$.

Lemma 2.6, i). It is a group property defied in [1] □

Lemma 2.6, ii). This expression follows from Theorem 2.1 in [11].
For $\phi \in$ A define for $\gamma \geq 0$ and $(x,t) \in \mathbb{R}$

$$W_\gamma(t) u_0(x) = \int_\Omega e^{i(t\phi(\xi) + x\xi)} \|\phi''(\xi)\|^{\frac{\gamma}{2}} \widehat{u_0}(\xi) d\xi$$

Then, we have for any v $\in [0,1]$:

$$\|W_{v/2}(t) u_0\|_{L_t^q(\mathbb{R}:L^p)} \leq c\|u_0\|_2$$

Consider $\partial_x e^{at\partial_x^3} u_0$ and take $\phi(\xi)$ as $a\xi^3$ and γ as $1/2$:

$$\begin{aligned}\partial_x e^{at\partial_x^3} u_0(x) &= \int_{\mathbb{R}} e^{ix\xi} \widehat{\partial_x e^{it\partial_x^3} u_0}(\xi) d\xi = \int_{\mathbb{R}} e^{ix\xi + it\xi^3} \xi \widehat{u_0}(\xi) d\xi = \\ &= \int_{\mathbb{R}} e^{ix\xi + it\phi(\xi)} \xi^{\frac{1}{4}} \xi^{\frac{3}{4}} \widehat{u_0}(\xi) d\xi \simeq \frac{1}{|a|^{\frac{1}{4}}} \int_{\mathbb{R}} e^{ix\xi + it\phi(\xi)} (\phi'')^{\frac{1}{4}} \xi^{\frac{3}{4}} \widehat{u_0}(\xi) d\xi \\ &= \frac{1}{|a|^{\frac{1}{4}}} \int_\Omega e^{i(t\phi(\xi) + x\xi)} \|\phi''(\xi)\|^{\frac{1}{4}} \widehat{u_0}(\xi) d\xi = \frac{1}{|a|^{\frac{1}{4}}} W_{\frac{1}{2}}(\partial_\xi^{\frac{3}{4}} u_0)\end{aligned}$$

Using Thereom 2.1 in [11] and taking the norm with $v = 1$, $q = 4$, and $p > \infty$, we have the following:

$$\|\partial_x e^{at\partial_x^3} u_0(x)\|_{L^4_T L^\infty_x} = \frac{1}{|a|^{\frac{1}{4}}} \|W_{\frac{1}{2}}(\partial_\xi^{\frac{3}{4}} u_0)\|_{L^4_T L^\infty_x}$$
$$\lesssim \frac{1}{|a|^{\frac{1}{4}}} \Big(\|\partial_x^{3/4}\|_{L^2} + \|u_0\|_{L^2} \Big) \lesssim \frac{1}{|a|^{\frac{1}{4}}} \|u_0\|_{H^{3/4}}.$$

□

Lemma 2.6, iii). Observe that

$$D_x^r \partial_x \widehat{e^{at\partial_x^3}} u_0(x) = |\xi|^{r+1} e^{-iat\xi^3} \widehat{u_0(\xi)} = e^{-iat\xi^3} |\xi||\xi|^r \widehat{u_0(\xi)} = e^{-iat\xi^3} \widehat{\partial_x D_x^r u_0}(\xi).$$

Then the following equality is a case of Theorem 4.1 in [11],

$$D_x^r \partial_x e^{at\partial_x^3} u_0(x) = (e^{-iat\xi^3} \widehat{\partial_x D_x^r u_0})^{-1}(\xi) = \int_{\mathbb{R}} e^{ix\xi} e^{-iat\xi^3} \widehat{\partial_x D_x^r u_0}(\xi)\, d\xi = W(t) f(x).$$

Now applying Theorem 4.3-4.4 in [11], We can derive that:

$$\|D_x^r \partial_x e^{at\partial_x^3} u_0(x)\|^2_{L^\infty_x L^2_T} \lesssim \int_{\mathbb{R}} \frac{|\widehat{f(\xi)}|^2}{|-3a\xi^2|}\, d\xi \equiv \frac{1}{|a|} \int_{\mathbb{R}} |\widehat{f(\xi)}|^2\, d\xi.$$

□

Lemma 2.6, iv). Using proposition 2.4 in [14], we have the following property for $s > 1/2$, and for $T > 0$:

$$\|e^{at\partial_x^3} u_0(x)\|^2_{L^2_x L^\infty_T} = \Big(\int_{-\infty}^{+\infty} \sup_{t\in[-T,T]} \|e^{at\partial_x^3}(t) u_0(x)\|^2 dt \Big)^{\frac{1}{2}} \lesssim (T+1)^{\frac{1}{2}} \|u_0(x)\|_{H^s}.$$

□

2.4. **Well-posedness in Sobolev $3/4 < s \le 1$.**
This section generalizes Well-posedness results for $3/4 < s \le 1$ established by Alvarez-Samaniego and Carvajal in 2008 [1].

Proposition 2.7. *Let $3/4 < s \le 1$, $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ and $b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2 \in \mathbb{R}$. Then, for $u_0, v_0 \in H^s(\mathbb{R})$ there exist $T > 0$ and a unique solution (u, v) of Fixed Point (3) such that*

$$u(\cdot, t), v(\cdot, t) \in H^s(\mathbb{R}), \quad t \in [0, T].$$

Proof. Consider the mappings $\Phi = (\Phi_1, \Phi_2)$ defined in Theorem 2. Then, we define the norm

$$\Lambda_s^T(u) := \max_{[-T,T]} \|u(t)\|_{H_s} + \|u_x\|_{L^4_T L^\infty_x} + \|D_x^r u_x\|_{L^\infty_x L^2_T} + (1+T)^{-\frac{1}{2}} \|u\|_{L^2_x L^\infty_T} + \|u_x\|_{L^\infty_x L^2_T}$$

Define $\|(u, v)\|_{X_T} := \Lambda_s^T(u) + \Lambda_s^T(v)$.
For some M,$T > 0$ conveniently fixed later, consider the space [1]

$$X^T = \{(u, v) \in C([-T, T]; H^s(\mathbb{R})) \times C([-T, T]; H^s(\mathbb{R})); \|(u, v)\| < \infty\}.$$

and $X_M^T = \{(u, v) \in X^T; \|(u, v)\| \le M\}$
We prove that there exist $T > 0$ and $M > 0$ such that if $(u, v) \in X_T^M \times X_T^M$, then

$(u,v)=\Phi(u,v)\in X_M^T\times X_M^T$ and $\Phi: X_M^T\times X_M^T\to X_M^T\times X_M^T$ is a contraction. For that, for $(u,v)\in X_M^T\times X_T^M$ we prove that

$$\|\Phi(u,v)\|_{X^T}\le M.$$

where,

$$\begin{aligned}\|\Phi(u,v)\|_{X^T} := &\|\Phi_1(u,v)\|_{L_T^\infty H^s}+\|\Phi_2(u,v)\|_{L_T^\infty H^s}+\|\partial_x\Phi_1(u,v)\|_{L_T^4L_x^\infty}\\ &+\|\partial_x\Phi_2(u,v)\|_{L_T^4L_x^\infty}+\|D_x^s\partial_x\Phi_1(u,v)\|_{L_x^\infty L_T^2}+\|D_x^s\partial_x\Phi_2(u,v)\|_{L_x^\infty L_T^2}\\ &+\|\Phi_1(u,v)\|_{L_x^2L_T^\infty}+\|\Phi_2(u,v)\|_{L_x^2L_T^\infty}+\|\partial_x\Phi_1(u,v)\|_{L_X^\infty L_T^2}\\ &+\|\partial_x\Phi_2(u,v)\|_{L_X^\infty L_T^2}.\end{aligned}$$

Without loss of generality, we consider just the $\Phi_1(u,v)$, and the $\Phi_2(u,v)$ can proceed similarly. In the same manner, we consider just the uu_x norm, the other norms can proceed similarly. Then, consider the well-definedness of Φ:

Step 1: Φ is well-defined

We consider the well-definedness of each norm separately, then we combine the results.

Case 1: First, consider the $\|\Phi_1(u,v)\|_{L_T^\infty H^s}$ norm

$$\|\Phi_1(u,v)\|_{L_T^\infty H^s}\le\|e^{at\partial_x^3}u_0\|_{H^s}+\|\int_0^t e^{a_1(t-t')\partial_x^3}b_2uu_x(t')\,dt'\|_{L_T^\infty H^s}.$$

Linear part can be rewritten in the following way

$$\|e^{at\partial_x^3}u_0\|_{H^s}=(\int_{\mathbb{R}}(1+|\xi|^2)^s|\widehat{e^{at\partial_x^3}u_0(\xi)}|^2\,d\xi)^{\frac12}=\|u_0\|_{H^s}.$$

Next, consider the nonlinear part

$$\begin{aligned}\|\int_0^t e^{a_1(t-t')\partial_x^3}b_2uu_x(t')\,dt'\|_{L_T^\infty H^s}&=\sup_{t\in[0,T]}||\int_0^t e^{a_1(t-t')\partial_x^3}b_2uu_x(t')\,dt'||_{H^s}\\ \le\sup_{t\in[0,T]}\int_0^T\|b_2uu_x\|'_{H^s}\,dt'&\lesssim\int_0^T||uu_x||'_{H^s}\,dt'.\end{aligned}$$

Then, consider Holder's inequality to get the following results.

$$\int_0^T||uu_x||'_{H^s}\,dt'\le||1||_{L_T^2}\cdot||uu_x||_{L_T^2H^s}\le T^{1/2}\|uu_x\|_{L_T^2H^s}.$$

We apply the following equivalence, $\|f\|_{H^s}^2\sim\|f\|_{L^2}^2+\|\partial_xf\|_{L^2}^2$ and Lemma 2.5, we get the following results

$$||\Phi_1(u,v)||_{L_T^\infty H^s}\lesssim\|u_0\|_{H^s}+T^{1/2}M^2.$$

Case 2: In this subsection, we focus on $\partial_x\Phi_1(u,v)||_{L_T^4L_x^\infty}$ norm

$$\begin{aligned}||\partial_x\Phi_1(u,v)||_{L_T^4L_x^\infty}&\le||\partial_xe^{a_1t\partial_x^3}u_0||_{L_T^4L_x^\infty}+\int_0^T||\partial_xe^{a_1(t-t')\partial_x^3}b_2uu_x(t')||_{L_T^4L_x^\infty}\,dt\\ &\lesssim\frac{c}{|a|^{\frac14}}||u_0||_{H^s}+\frac{c}{|a|^{\frac14}}\int_0^T\|b_2uu_x(t')||_{L_T^4L_x^\infty}\,dt\\ &\lesssim||u_0||_{H^s}+\int_0^T\|b_2uu_x(t')||_{H^s}\,dt\\ &\lesssim||u_0||_{H^s}+\int_0^T||uu_x||'_{H^s}\,dt'\end{aligned}$$

Using previous results and Lemma 2.5, we have the following results

$$||e^{a_1 t\partial_x^3}u_0||_{L_T^4 L_x^\infty} \lesssim \|u_0\|_{H^s} + T^{\frac{1}{2}}M^2.$$

Case 3: Next, consider the $\|D_x^s\partial_x\Phi_1(u,v)||_{L_x^\infty L_T^2}$ norm

$$\begin{aligned}\|D_x^s\partial_x\Phi_1(u,v)||_{L_x^\infty L_T^2} &\le ||D_x^s\partial_x e^{a_1 t\partial_x^3}u_0||_{L_x^\infty L_T^2} \\ &\quad + \int_0^T \|D_x^s\partial_x e^{a_1(t-t')\partial_x^3}b_2 uu_x(t')\|_{L_x^\infty L_T^2}\,dt' \\ &\lesssim \frac{1}{|a|^{\frac{1}{2}}}||D_x^s u_0||_{L_x^2} + \frac{1}{|a|^{\frac{1}{2}}}\int_0^T ||D_x^s(b_2 uu_x(t')||_{L_x^2}\,dt' \\ &\lesssim ||u_0||_{H^s} + ||D_x^s uu_x||_{L_T^1 L_x^2} \lesssim ||u_0||_{H^s} + T^{\frac{1}{2}}||D_x^s uu_x||_{L_T^2 L_x^2} \\ &\lesssim \|u_0\|_{H^s} + T^{\frac{1}{2}}M^2.\end{aligned}$$

Case 4: Next, consider the $||\Phi_1(u,v)||_{L_x^2 L_T^\infty}$ norm.

$$\begin{aligned}\|\Phi_1(u,v)\|_{L_x^2 L_T^\infty} &\le (1+T)^{\frac{1}{2}}||e^{at\partial_x^3}U_0||_{L^2} + (1+T)^{\frac{1}{2}}||\int_0^t e^{a_1(t-t')\partial_x^3}b_2 uu_x(t')\,dt'||_{L^2 L_T^\infty} \\ &\le c_{(a,r)}(1+T)^{\frac{1}{2}}||u_0||_{H^s} + c_{(a,r)}(1+T)^{\frac{1}{2}}\int_0^t ||b_2 uu_x\|\,dt'_{H^s} \\ &\lesssim \|u_0\|_{H^s} + \int_0^T ||uu_x||'_{H^s}\,dt' \lesssim \|u_0\|_{H^s} + T^{\frac{1}{2}}M^2.\end{aligned}$$

Case 5 Finally, consider the $||\partial_x\Phi_1(u,v)||_{L_x^\infty L_T^2}$ norm.

$$\begin{aligned}\|\partial_x\Phi_1(u,v)\|_{L_x^\infty L_T^2} &\le ||\partial_x e^{a_1 t\partial_x^3}u_0||_{L_x^\infty L_T^2} + \int_0^T ||\partial_x e^{a_1(t-t')\partial_x^3}b_2 uu_x(t')||_{L_x^\infty L_T^2}\,dt' \\ &\lesssim \frac{1}{|a|^{\frac{1}{2}}}\|u_0\|_{L_x^2} + \frac{1}{|a|^{\frac{1}{2}}}\int_0^T \|b_2 uu_x(t')\|_{L_x^2}\,dt' \lesssim ||u_0||_{H^s} + ||uu_x||_{L_T^1 L_x^2} \\ &\lesssim ||u_0||_{H^s} + T^{\frac{1}{2}}\|uu_x\|_{L_T^2 L_x^2} \lesssim ||u_0||_{H^s} + T^{\frac{1}{2}}M^2.\end{aligned}$$

Conclusion

The substitution of all cases into $\|\Phi_1(u,v)\|_{X_T}$ and considering other norms gives

$$\|\Phi_1(u,v)\|_{X_T} \le K_1\Big[\|u_0\|_{H_x^{3/4}} + T^{1/2}M^2\Big]$$

for some $K_1, T > 0$.

Applying the same argument for the $\|\Phi_2(u,v)\|_{X_T}$, we obtain

$$\|\Phi_2(u,v)\|_{X_T} \le K_2\Big[\|v_0\|_{H_x^{3/4}} + T^{1/2}M^2\Big]$$

for some $K_2, T{>}0$.

Taking

$$M := 2(K_1+K_2)(\|u_0\|_{H_x^{3/4}} + \|v_0\|_{H_x^{3/4}})$$

and $T > 0$ small enough such that

$$M/2 + (K_1+K_2)T^{1/2}M^2 \le M$$

for some $T, M > 0$. Hence, it follows that

$$\|\Phi(u,v)\|_{X_T^M} \le M.$$

is true. □

Step: 2 Contraction in X_M^T

In this subsection, we consider the contraction mapping for the solution of fixed point problem. This mapping in X_M^T is defined by:

$$\|\Phi(u_1,v_1)(t)-\Phi(u_2,v_2)(t)\| \le k\|(u_1,v_1)-(u_2,v_2)\|$$

for $0<k<1$ and for (u_1,v_1) in X_M^T and (u_2,v_2) in X_M^T solutions with the same initial data (u_0,v_0)

Using the definition of the X_M^T space, this inequality is equivalent to the following inequality:

$$\begin{aligned}&\Lambda_s^T(\Phi_1(u_1,v_1)(t)-\Phi_1(u_2,v_2)(t))+\Lambda_s^T(\Phi_2(u_1,v_1)(t)-\Phi_2(u_2,v_2)(t))\\ &\quad\le k(\Lambda_s^T(u_1-u_2)+\Lambda_s^T(v_1-v_2))\end{aligned}$$

Without loss of generality, we will focus on the $\Phi_1(u,v)$ and $\|uu_x\|$ norm. The contraction mapping for $\Phi_2(u,v)$ and other norms can proceed similarly. The $\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)$ is defined as:

$$\begin{aligned}\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)&=e^{a_1t\partial_x^3}u_0+\int_0^t e^{a_1(t-t')\partial_x^3}[b_1u_1(u_1)_x](t')\,dt'-e^{a_1t\partial_x^3}u_0\\ &-\int_0^t e^{a_1(t-t')\partial_x^3}[b_2u_2(u_2)_x](t')\,dt'=\int_0^t e^{a_1(t-t')\partial_x^3}\Big[b_2[u_1(u_1)_x-u_2(u_2)]_x\Big](t')\,dt'\end{aligned}$$

We consider the norms of $\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)$ piece wisely

Case 1: In this subsection, we consider the $\|\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\|_{L_T^\infty H^s}$

$$\begin{aligned}\|\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\|_{L_T^\infty H^s}&=\|\int_0^t e^{a_1(t-t')\partial_x^3}b_1[u_1(u_1)_x-u_2(u_2)]_x(t')\,dt'\|_{L_T^\infty H^s}\\ &\lesssim\int_0^t\|[b_1[u_1(u_1)_x-u_2(u_2)_x]\|_{H^s}\,dt'\end{aligned}$$

Using Lemma 2.3, we will have:

$$\begin{aligned}\|\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\|_{L_T^\infty H^s}&\lesssim T^{\frac12}||(u_1)_x(u_1-u_2)||_{L_T^2H^s}+T^{\frac12}||u_2(u_1-u_2)_x||_{L_T^2H^s}\\ &\lesssim T^{\frac12}\Big[M(\Lambda_s^T(u_1-u_2))+T^{\frac14}M(\Lambda_s^T(u_1-u_2))\Big]\\ &\lesssim T^{\frac12}M\Big[\Lambda_s^T(u_1-u_2)\Big].\end{aligned}$$

Case 2 Next, consider the $\|\partial_x\Big(\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\Big)\|_{L_T^4L_x^\infty}$.

$$\begin{aligned}&\Big\|\partial_x\Big(\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\Big)\Big\|_{L_T^4L_x^\infty}\\ &\le\int_0^T\Big\|\partial_x e^{a_1(t-t')\partial_x^3}b_1[u_1(u_1)_x-u_2(u_2)]_x\Big\|_{L_T^4L_x^\infty}dt'\\ &\le\frac{c}{|a|^{\frac14}}\int_0^T\Big\|b_1[u_1(u_1)_x-u_2(u_2)]_x\Big\|_{L_T^4L_x^\infty}(t')\,dt'\\ &\lesssim\int_0^T\Big\|(u_1(u_1)_x-u_2(u_2))_x\Big\|_{H_s}(t')\,dt'.\end{aligned}$$

Using the Lemma 2.3, we got the following results:

$$\Big\|\partial_x\Big(\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\Big)\Big\|_{L_T^4L_x^\infty}\lesssim T^{\frac12}M[\Lambda_s^T(u_1-u_2)]$$

Case 3: Next, consider the $\|D_x^s\partial_x\Big(\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\Big)\|_{L_x^\infty L_T^2}$ norm

$$
\begin{aligned}
&||D_x^s\partial_x\Big(\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\Big)||_{L_x^\infty L_T^2}\\
&\leq \int_0^T \Big\|D_x^s\partial_x e^{a_1(t-t')\partial_x^3}\Big[b_1[u_1(u_1)_x-u_2(u_2)]_x\Big]\Big\|_{L_T^\infty L_x^2}(t')\,dt'\\
&\leq \frac{1}{|a|^{\frac{1}{2}}}\int_0^T \Big\|D_x^s\Big[b_1[u_1(u_1)_x-u_2(u_2)]_x\Big]\Big\|_{L_x^2}(t')\,dt'\\
&\lesssim \|D_x^s[u_1(u_1)_x-u_2(u_2)_x]\|_{L_T^1L_x^2}\\
&\lesssim T^{\frac{1}{2}}\Big[\|D_x^s[u_1(u_1)_x-u_2(u_2)_x]\|_{L_T^2L_x^2}\Big]\\
&\lesssim T^{\frac{1}{2}}\Big[\|D_x^s\Big[(u_1)_x(u_1-u_2)+u_2(u_1-u_2)_x\Big]\|_{L_T^2L_x^2}\Big]
\end{aligned}
$$

Applying, Lemman 2.5 we get the following results

$$
\begin{aligned}
&T^{\frac{1}{2}}\Big[\|D_x^s\Big[(u_1)_x(u_1-u_2)+u_2(u_1-u_2)_x\Big]\|_{L_T^2L_x^2}\Big]\\
&\lesssim T^{\frac{1}{2}}\Big[T^{\frac{1}{4}}\Lambda_s^T(u_1)\Lambda_s^T(u_1-u_2)+(1+T)^{\frac{1}{2}}\Lambda_s^T(u_1)\Lambda_s^T(u_1-u_2)\\
&\qquad +T^{\frac{1}{4}}\Lambda_s^T(u_2)\Lambda_s^T(u_1-u_2)+(1+T)^{\frac{1}{2}}\Lambda_s^T(u_2)\Lambda_s^T(u_1-u_2)\Big]\\
&\lesssim T^{\frac{3}{4}}M\Big[\Lambda_s^T(u_1-u_2)\Big]
\end{aligned}
$$

Case 4 Next, consider $\|\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\|_{L_x^2L_T^\infty}$.

$$
\begin{aligned}
||\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)||_{L_x^2L_T^\infty} &\lesssim (1+T)^{\frac{1}{2}}\int_0^T \Big\|b_1[u_1(u_1)_x-u_2(u_2)]_x\Big\|_{H_s}(t')\,dt'\\
&\lesssim (1+T)^{\frac{1}{2}}\int_0^t ||(u_1)_x(u_1-u_2)||_{H^s}\,dt'\\
&\quad +\int_0^t ||u_2(u_1-u_2)_x||_{H^s}\,dt'\\
&\lesssim T^{\frac{1}{2}}M\Big[\Lambda_s^T(u_1-u_2)\Big].
\end{aligned}
$$

Case 5 Finally, consider $\|\partial_x(\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2))\|_{L_x^\infty L_T^2}$ norm.

$$
\begin{aligned}
&\|\partial_x(\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2))\|_{L_x^\infty L_T^2}\\
&\leq \int_0^T \Big\|\partial_x e^{a_1(t-t')\partial_x^3}b_1[u_1(u_1)_x-u_2(u_2)]_x\Big\|_{L_x^\infty L_T^2}(t')\,dt'\\
&\leq \frac{1}{|a|^{\frac{1}{2}}}\int_0^T \Big\|b_1[u_1(u_1)_x-u_2(u_2)]_x\Big\|_{L_x^2}(t')\,dt'\\
&\lesssim \int_0^T \Big\|b_1\Big[(u_1)_x(u_1-u_2)+u_2(u_1-u_2)_x\Big](t')\Big\|_{L_x^2}\,dt'\\
&\lesssim T^{\frac{1}{2}}\Big[||(u_1)_x(u_1-u_2)||_{L_T^2L_x^2}+||u_2(u_1-u_2)_x||_{L_T^2L_x^2}\Big]\\
&\lesssim T^{\frac{1}{2}}M\Big[\Lambda_s^T(u_1-u_2)\Big]
\end{aligned}
$$

Conclusion The substitution of all cases into $\|\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\|_{X_T}$ and considering other norms gives

$$\|\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\|_{X_T} \le K_1 T^{\frac{1}{2}} M \Big[\Lambda_s^T(u_1-u_2)\Big] \tag{5}$$

for some $K_1, M, T > 0$.

Applying the same argument for the $\|\Phi_2(u_1,v_1)-\Phi_2(u_2,v_2)\|_{X_T}$, we obtain

$$\|\Phi_2(u_1,v_1)-\Phi_2(u_2,v_2)\|_{X_T} \le K_2 T^{\frac{1}{2}} M \Big[\Lambda_s^T(v_1-v_2)\Big] \tag{6}$$

for some $K_2, M, T > 0$.
Taking $T > 0$ small enough such that

$$T^{\frac{1}{2}} M\Big[K_1\Lambda_s^T(u_1-u_2)+K_2\Lambda_s^T(v_1-v_2)\Big] \le k\Big[\Lambda_s^T(u_1-u_2)+\Lambda_s^T(v_1-v_2)\Big]$$

for some $k \in [0,1]$. Hence, it follows that

$$\|\Phi(u_1,v_1)(t)-\Phi(u_2,v_2)(t)\|_{X_M^T} \le k\|(u_1,v_1)-(u_2,v_2)\|_{X_M^T}$$

is true.

2.5. Well-posedness in Sobolev $s = 3/4$.

This section focuses on the Well-posedness results in Sobolev $s = 3/4$ shown by Yang and Zhang in 2022 [21]. This result is considered below.

Proposition 2.8. *let $s = 3/4$, $a_1, a_2 \in \mathbb{R}\setminus\{0\}$ and $b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2 \in \mathbb{R}$. Then, for $u_0, v_0 \in H^s(\mathbb{R})$ there exist $T > 0$ and a unique solution (u, v) of Fixed Point Problem [3] such that*

$$u(\cdot,t), v(\cdot,t) \in H^{3/4}(\mathbb{R}), \quad t \in [0,T].$$

In this section, we discuss the main tools and features in dealing with lower regularity $s = 3/4$. The complete proof of this proposition is and it can found in [21].
Recall that, in the previous section Well-posedness in Sobolev Space for $s > 3/4$ was considered. The key idea was the introduction of the specific norms such that they allow to construct a contraction mapping $\|\Phi(u_1,v_1)-\Phi(u_2,v_2)\| \le k\|(u_1,v_1)-(u_2,v_2)\|$. However, this technique is not applicable under the $s = 3/4$ as since one of the essential preliminaries in Lemma 2.6 *iv*), especially$\|u\|_{L_x^2 L_T^\infty}$ does not hold since it requires condition $s > 3/4$.

Thus, in this proof, consider the Bourgain Space, which was defined in Equation 2.2. This space is a subset of $L_T^\infty H_x^s$. This allows to use of Bourgain space as a uniform tool to build the contraction mapping in the Sobolev Space through the introduction of bilinear estimates. Below consider the brief scheme of showing well-definedness for our problem. Using Lemma 2.1 in [21] and Lemma 3.2 in [21] for s ≥ 0 and considering uu_x case, we obtain the following results

$$\begin{aligned}\Phi_1(u,v)(t) &\lesssim \|e^{t\partial^3}u_0\|_{X_{s,b}^1} + \|\int_0^t e^{a_1(t-t')\partial_x^3}[b_1uu_x+c_1vv_x+d_1uv_x+e_1vu_x]\,dt'\|_{X_{s,b}^{a_1}} \\ &\lesssim \|u_0\|_{H^s} + \|\partial_x(u^2)\|_{X_{s,b-1}^{a_1}} \lesssim \|u_0\|_{H^s} + \|u\|^2_{X_{s,b}^{a_1}}.\end{aligned}$$

3. Weighted Sobolev spaces

3.1. Well-posedness in weighted Sobolev for $3/4 < s \leq 1$.

Proposition 3.1. *Let $3/4 < s \leq 1$, $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ and $b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$ such that $d_1 = e_1$ and $d_2 = e_2$. Then, for $u_0, v_0 \in H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx)$, with $0 < r \leq s/2$, there exist $T > 0$ and a unique solution (u, v) of Fixed Point Problem 3 such that*

$$u(\cdot, t), v(\cdot, t) \in H^s(\mathbb{R} \cap L^2(|x|^{2r}dx), \quad t \in [0, T].$$

Step 1: Φ is well-defined

We consider the $3/4 < s \leq 1$, so we can escape the introduction of Bourgain Space in this proof because the norm properties of the Sobolev Space for this regularity are still useful. For the Weighted case, we consider the space with conveniently chosen $M, T > 0$.

$$X^T = \{(u, v) \in C([-T, T]; H^s(\mathbb{R}) \cap L^2(|x|^{2r}) \times C([-T, T]; H^s(\mathbb{R})); \|(u, v)\| < \infty\}$$

and $X^T_M = \{(u, v) \in X^T; \|(u, v)\| \leq M\}$

By defining the norm:

$$\begin{aligned}\Lambda^T_s(u) := \max_{[-T,T]} \|u\|_{H^s} &+ \|u_x\|_{L^4_T L^\infty_x} + \|D^r_x u_x\|_{L^\infty_x L^2_T} + (1+T)^{-\frac{1}{2}} \|u\|_{L^2_x L^\infty_T} \\ &+ \|u_x\|_{L^\infty_x L^2_T} + \||x|^r u\|_{L^\infty_T L^2_x}\end{aligned}$$

Define $\|(u, v)\| := \Lambda^T_s(u) + \Lambda^T_s(v)$.

Looking for $T, M > 0$ such that Φ is well-defined in X^T_M. This section considers the analysis of the $L^\infty_T L^2_x$ of $|x|^r \Phi(u)$ since the analysis of other norms was considered in Section 2.2. Without loss of generality, we focus on uu_x norm. Other norms can proceed in similar way.

$$\begin{aligned}\||x|^r \Phi_1(u)\|_{L^\infty_T L^2_x} &\leq \||x|^r e^{a_1 t \partial^3_x} u_0\|_{L^\infty_T L^2_x} + \left\| |x|^r \int_0^t e^{a_1(t-t')\partial^3_x} b_1 u u_x(t')\, dt' \right\|_{L^\infty_T L^2_x} \\ &=: L + NL.\end{aligned}$$

The linear part can be controlled using Theorem 1 in [9]:

$$\begin{aligned}L &\leq \||x|^r e^{a_1 t \partial^3_x} u_0\|_{L^\infty_T L^2_x} + \|e^{a_1 t \partial^3_x} \Psi_{t,r}(\widehat{u_0}(\xi))\|_{L^\infty_T L^2_x} \\ &\lesssim \left\| |x|^r u_0 \right\|_{L^2_x} + (1+T)(\|u_0\|_{L^2_x} + \|D^{2r}_x u_0\|_{L^2}) \\ &\lesssim \left\| |x|^r u_0 \right\|_{L^2_x} + \|u_0\|_{H^s}.\end{aligned}$$

The nonlinear part can be controlled using

$$\begin{aligned}NL &\leq \int_0^T \|e^{a_1(t-t')\partial^3} |x|^r b_1 u u_x\|_{L^\infty_T L^2_x}\, dt' + \int_0^T \|e^{a_1(t-t')\partial^3} \Big(\Psi_{t,r} b_1 u u_x\Big)^\vee \|_{L^\infty_T L^2_x}\, dt' \\ &\leq NL_1 + NL_2.\end{aligned}$$

The control of NL_1 can be established by Holder's inequality with $q_1 = 1, p_1 = \infty, q_2 = \infty, p_2 = 2$.

$$\begin{aligned}\int_0^T \|e^{a_1(t-t')\partial^3} |x|^r b_1 u u_x\|_{L^\infty_T L^2_x}\, dt' &\lesssim \||x|^r u v_x\|_{L^1_T L^2_x} \lesssim \||x|^r u\|_{L^\infty_T L^2_x} \|v_x\|_{L^1_T L^\infty_x} \\ &\lesssim T^{1/2} \||x|^r u\|_{L^\infty_T L^2_x} \|v_x\|_{L^4_T L^\infty_x} \lesssim T^{1/2} M^2.\end{aligned}$$

The next term can be controlled through Lemma 2.5.

$$
\begin{aligned}
NL_2 &\lesssim \int_0^T \|e^{a_1(t-t')\partial^3}\Big(\Psi_{t,r} b_1 u u_x\Big)^{\vee}\|_{L^\infty_T L^2_x}\, dt' \\
&\lesssim (1+|t|)\int_0^T \|u_x u\|_{L^2_x} + D_x^{2r}\|u_x u\|_{L^2_x}\, dt' \lesssim T^{\frac{1}{2}}\Big(\|u_x u\|_{L^2_T L^2_x} + D_x^{2r}\|u_x u\|_{L^2_T L^2_x}\Big) \\
&\lesssim T^{1/2} M^2.
\end{aligned}
$$

<u>Conlusion.</u>

The substitution of all unweighted and weighted results into $\|\Phi_1(u,v)\|_{X_T}$ and considering other norms gives

$$
\|\Phi_1(u,v)\|_{X_T} \leq K_1\Big[\|u_0\|_{H_x^{3/4}} + \||x|^r u_0\|_{L^2_x} + T^{1/2}M^2\Big]
$$

for some $K_1, T > 0$.

Applying the same argument for the $\|\Phi_2(u,v)\|_{X_T}$, we obtain

$$
\|\Phi_2(u,v)\|_{X_T} \leq K_2\Big[\|v_0\|_{H_x^{3/4}} + \||x|^r v_0\|_{L^2_x} + T^{1/2}M^2\Big]
$$

for some $K_2, T{>}0$.

Taking

$$
M := 2(K_1+K_2)(\|u_0\|_{H_x^{3/4}} + \|v_0\|_{H_x^{3/4}} + \||x|^r u_0\|_{L^2_x} + \||x|^r v_0\|_{L^2_x})
$$

and $T > 0$ small enough such that

$$
M/2 + (K_1+K_2)T^{1/2}M^2 \leq M
$$

for some $T, M > 0$. Hence, it follows that

$$
\|\Phi(u,v)\|_{X_T^M} \leq M.
$$

is true.

3.2. **Well-posedness in weighted Sobolev for $s = 3/4$.**

In this section, we focus on the well-posedness in Weighted Sobolev for $s = 3/4$. We can not obtain the control for $s = 3/4$ by using Theorem 1 in [9], because we consider Bourgain setting for lower regularities. Therefore, the main focus of this section is to consider the control by setting auxiliary tools such as commutators, which would provide the following theorem.

Theorem 3.2. *Let $a_1, a_2 \in \mathbb{R}\setminus\{0\}$ and $b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$. Then, for $u_0, v_0 \in H^{3/4}(\mathbb{R}) \cap L^2(|x|^{2r}dx)$, with $0 < r \leq 3/8$, there exist $T > 0$ and a unique solution (u,v) of Fixed point Problem 3 such that*

$$
u(\cdot,t), v(\cdot,t) \in H^{3/4}(\mathbb{R}) \cap L^2(|x|^{2r}dx), \quad t \in [0,T].
$$

Proof. First, we define the norm

$$
\|(u,v)\|_{X_T} := \|u\|_{X^{a_1}_{3/4,b}} + \||x|^r u\|_{L^\infty_T L^2_x} + \|v\|_{X^{a_2}_{3/4,b}} + \||x|^r v\|_{L^\infty_T L^2_x}.
$$

For $11/12 < b < 1$ and some $M, T > 0$, that will be conveniently fixed /later, consider the space

$X^T = \{(u,v) \in C([-T,T]; H^{3/4}(\mathbb{R}) \cap L^2(|x|^{2r}) \times C([-T,T]; H^{3/4}(\mathbb{R})); \|(u,v)\|_{X_T} < \infty\}$ and $X^T_M = \{(u,v) \in X^T; \|(u,v)\|_{X_T} \leq M\}$

Recall, $\Phi = (\Phi_1, \Phi_2)$ is defined in Equation [3]. For any fixed $(u_0, v_0) \in (H^{3/4} \times H^{3/4}) \cap (L^2(|x|^{2r}dx) \times L^2(|x|^{2r}dx))$ with $s = 3/4$, $r \in (0, 3/8]$, we denote by $(u,v) =$

$\Phi(u,v)$ the solution of the system of linear inhomogeneous IVP 2.
We prove that there exist $T>0$ and $M>0$ such that if $(u,v)\in X_T^M$, then $(u,v)=\Phi(u,v)\in X_T^M$ and $\Phi: X_T^M\to X_T^M$ is a contraction.
That is, for $(u,v)\in X_T^M$ we prove that

$$\|\Phi(u,v)\|_{X_T}\le M.$$

Here

$$\begin{aligned}\|\Phi(u,v)\|_{X_T} &:= \|\Phi_1(u,v)\|_{X^{a_1}_{3/4,b}} + \|\Phi_2(u,v)\|_{X^{a_2}_{3/4,b}}\\ &\quad + \||x|^r\Phi_1(u,v)\|_{L^\infty_T L^2_x} + \||x|^r\Phi_2(u,v)\|_{L^\infty_T L^2_x}.\end{aligned}$$

Without loss of generality, this section focuses on well-defined mapping. The contraction mapping can proceed similarly. In section 2.3, the well-definedness was shown for 3/4 in the unweighted norm. This section focuses on the well-definiteness of $\||x|^r u\|_{L^\infty_T L^2_x}$ norm for $s=3/4$. We later conveniently fix $T,M>0$ such that Φ is well-defined in X^T_M.
By triangle inequality the norm $\||x|^r\Phi_1(u)\|_{L^\infty_T L^2_x}$ can be represented as follows

$$\begin{aligned}&\||x|^r\Phi_1(u)\|_{L^\infty_T L^2_x} \le \||x|^r e^{a_1 t\partial_x^3}u_0\|_{L^\infty_T L^2_x}\\ &\quad + \left\||x|^r\int_0^t e^{a_1(t-t')\partial_x^3}[b_1uu_x + c_1vv_x + d_1uv_x + e_1vu_x](t')\,dt'\right\|_{L^\infty_T L^2_x} =: L \quad + \quad NL.\end{aligned}$$

The linear part can be controlled using Theorem 1 in [9]:

$$\begin{aligned}L &\le \||x|^r e^{a_1 t\partial_x^3}u_0\|_{L^\infty_T L^2_x} + \|e^{a_1t\partial_x^3}\Psi_{t,r}(\widehat{u_0}(\xi)^\vee\|_{L^\infty_T L^2_x}\\ &\lesssim \||x|^r u_0\|_{L^2_x} + (1+T)(\|u_0\|_{L^2_x} + \|D_x^{2r}u_0\|_{L^2_x})\\ &\lesssim \||x|^r u_0\|_{L^2_x} + (1+T)\|u_0\|_{H^{3/4}_x}\end{aligned}$$

where $r\in(0,3/8]$.

Next, we consider $\psi\in C_0^\infty(\mathbb{R})$ defined in [20] to be smooth and cut off function such that assigns to 1 on $[-1,1]$ and has $supp\psi\subseteq(-2,2)$. For $T>0$, we put $\psi_T(t)=\psi(t/T)$.

We invoke the time localization estimate [8, Lemma 2.7]. Let $-1/2<\beta'\le\beta<1/2$. Then the following inequality

$$\|\psi(t/T)u\|_{X^a_{3/4,\beta'}} \lesssim T^{\beta-\beta'}\|u\|_{X^a_{3/4,\beta}} \tag{7}$$

holds for any $0<T<1$. The nonlinear part can be controlled by moving x^r inside the integral by introducing additional norms.
Without loss of generality, it is sufficient to consider uu_x case. We insert function ψ inside on the norm and then, we apply Plancherel Theorem:

$$\begin{aligned}NL &= \|\psi(T^{-1}t)|x|^r\int_0^t e^{-ia_1(t-t')\partial_x^3}(uu_x)(\xi,t')dt'\|_{L^\infty_T L^2_x}\\ &= \|\psi(T^{-1}t)D^r_\xi\Big[\int_0^t e^{ia_1(t-t')\xi^3}\mathcal{F}_x(uu_x)(\xi,t')dt'\Big]\|_{L^\infty_T L^2_\xi}\end{aligned}$$

Then, with the same approach, function ψ can be introduced inside of the integral. Therefore, the goal is to show by considering several cases the following identity

$$\psi(T^{-1}t)\mathbb{1}_{[0,t]}(t') = \psi(T^{-1}t)\mathbb{1}_{[0,t]}(t')\psi(t'(2T)^{-1}). \tag{8}$$

Case 1. First, consider $|t'|<2T$

$$\psi(T^{-1}t')=1.$$

Case 2. Next,consider $|t'| < 2T$, we show that $\psi(T^{-1}t')\mathbb{1}_{[0,t]}(t') = 0$ by considering two specific intervals. Then, we have following two equalities

$$\psi(T^{-1}t') = 0 \qquad , for \quad |t| > 2T$$
$$\mathbb{1}_{[0,t]}(t') = 0 \qquad for \quad t' \notin [0,t]$$

Applying two equations, we consider the following two subcases of the second case.
Case 2.1 $|t| \geq 2T$, then

$$\psi(T^{-1}t') = 0$$

Case 2.2 $|t| < 2T$, then
Since $|t'| > 2T$ and $|t| \leq 2T$, we have that $t' \notin [0,t]$, which implies $\psi(T^{-1}t')\mathbb{1}_{[0,t]}(t') = 0$
Therefore by equality in 8, the $\psi(T^{-1}t)$ can be inserted inside of the integral.

$$\begin{aligned}
&\|\psi(T^{-1}t)D_\xi^r\Big[\int_0^t e^{ia_1(t-t')\xi^3}\mathcal{F}_x(uu_x)(\xi,t')dt'\Big]\|_{L_T^\infty L_\xi^2}\\
&= \|D_\xi^r\Big[\int_0^t e^{ia_1(t-t')\xi^3}\psi(T^{-1}t')\mathcal{F}_x(uu_x)(\xi,t')dt'\Big]\|_{L_T^\infty L_\xi^2}.
\end{aligned}$$

Define $f(\xi,\tau) := \mathcal{F}_t\mathcal{F}_x\Big(\psi(T^{-1}t)uu_x\Big)$ and $h(\xi,\tau) := \langle\xi\rangle^{3/4}\langle\tau - a_1\xi^3\rangle^{b-1-\theta}$ for $0 < \theta < b - 11/12$. Then, we got the following results:

$$\begin{aligned}
&\Big\|D_\xi^r\int_0^t\Big(\int_{\mathbb{R}} e^{-ia_1(t-t')\xi^3}e^{it'\tau}f(\xi,\tau)\,d\tau\Big)\,dt'\Big\|_{L_T^\infty L_\xi^2}\\
&= \Big\|D_\xi^r\Big(\int_{\mathbb{R}} e^{-ia_1t\xi^3}f(\xi,\tau)\Big(\int_0^t e^{it'(\tau+a_1\xi^3)}\,dt'\Big)\,d\tau\Big)\Big\|_{L_T^\infty L_\xi^2}\\
&= \Big\|D_\xi^r\Big(\int_{\mathbb{R}} f(\xi,\tau)e^{-ia_1t\xi^3}\frac{e^{it'(\tau+a_1\xi^3)}-1}{\tau+a_1\xi^3}\,d\tau\Big)\Big\|_{L_T^\infty L_\xi^2}\\
&= \Big\|\Big(\int_{\mathbb{R}} D_\xi^r\Big(e^{-ia_1t\xi^3}\frac{e^{it'(\tau+a_1\xi^3)}-1}{(\tau+a_1\xi^3)h(\xi,\tau)}h(\xi,\tau)f(\xi,\tau)\Big)\,d\tau\Big)\Big\|_{L_T^\infty L_\xi^2}
\end{aligned}$$

Applying commutators three times gives x^r be inside of the norm.

$$\begin{aligned}
&\Big\|\Big(\int_{\mathbb{R}} D_\xi^r\Big(\frac{e^{it\tau}-e^{-ia_1t\xi^3}}{(\tau+a_1\xi^3)h(\xi,\tau)}h(\xi,\tau)f(\xi,\tau)\Big)\,d\tau\Big)\Big\|_{L_T^\infty L_\xi^2}\\
&\leq \Big\|\Big(\int_{\mathbb{R}}\Big[\frac{e^{it\tau}-e^{-ia_1t\xi^3}}{(\tau+a_1\xi^3)h(\xi,\tau)}, D_\xi^r,\Big](hf)(\xi,\tau)\Big)\,d\tau\Big)\Big\|_{L_T^\infty L_\xi^2}\\
&\quad + \Big\|\Big(\int_{\mathbb{R}}\Big(\frac{e^{it\tau}-e^{-ia_1t\xi^3}}{\tau+a_1\xi^3}\frac{1}{h(\xi,\tau)}D_\xi^r(hf)(\xi,\tau)\Big)\,d\tau\Big)\Big\|_{L_T^\infty L_\xi^2}\\
&\leq \Big\|\Big(\int_{\mathbb{R}}\Big[\frac{e^{it\tau}-e^{-ia_1t\xi^3}}{(\tau+a_1\xi^3)h(\xi,\tau)}, D_\xi^r,\Big](hf)(\xi,\tau)\Big)\,d\tau\Big)\Big\|_{L_T^\infty L_\xi^2}\\
&\quad + \Big\|\Big(\int_{\mathbb{R}}\Big(\frac{e^{it\tau}-e^{-ia_1t\xi^3}}{\tau+a_1\xi^3}\frac{1}{h(\xi,\tau)}\Big[\langle\tau-a_1\xi^3\rangle^{b-1-\theta}, D_\xi^r\Big](\langle\xi\rangle^{3/4}f)\Big)\,d\tau\Big)\Big\|_{L_T^\infty L_\xi^2}\\
&\quad + \Big\|\Big(\int_{\mathbb{R}}\Big(\frac{e^{it\tau}-e^{-ia_1t\xi^3}}{\tau+a_1\xi^3}\frac{1}{\langle\xi\rangle^{3/4}}D_\xi^r(\langle\xi\rangle^{3/4}f)\Big)\,d\tau\Big)\Big\|_{L_T^\infty L_\xi^2}
\end{aligned}$$

$$\leq \Big\|\big(\int_{\mathbb{R}} \Big[\frac{e^{it\tau}-e^{-ia_1t\xi^3}}{(\tau+a_1\xi^3)h(\xi,\tau)}, D^r_{\xi},\Big](hf)(\xi,\tau)\Big)\, d\tau\big)\Big\|_{L^\infty_T L^2_\xi}$$
$$+ \Big\|\big(\int_{\mathbb{R}} \Big(\frac{e^{it\tau}-e^{-ia_1t\xi^3}}{\tau+a_1\xi^3}\frac{1}{h(\xi,\tau)}\Big[\langle\tau-a_1\xi^3\rangle^{b-1-\theta}, D^r_\xi\Big](\langle\xi\rangle^{3/4}f)\Big)\, d\tau\big)\Big\|_{L^\infty_T L^2_\xi}$$
$$+ \Big\|\big(\int_{\mathbb{R}} \Big(\frac{e^{it\tau}-e^{-ia_1t\xi^3}}{\tau+a_1\xi^3}\Big[\langle\xi\rangle^{-3/4}, D^r_\xi\Big](\langle\xi\rangle^{3/4}f)\Big)\, d\tau\big)\Big\|_{L^\infty_T L^2_\xi}$$
$$+ \|\big(\int_{\mathbb{R}} \Big(\frac{e^{it'(\tau+a_1\xi^3)}-1}{\tau+a_1\xi^3}D^r_\xi f(\xi,\tau)\Big)\, d\tau\big)\Big\|_{L^\infty_T L^2_\xi} =: C_1+C_2+C_3+NL_1$$

Using Propositions 3.3, 3.6, 3.8 and, 3.9 we obtain the estimations for C_1,C_2,C_3, and NL_1:

$$\begin{aligned} C_1+C_2+C_3 &\lesssim \|hf\|_{L^2_\xi L^2_\tau}, \\ NL_1 &\lesssim T^{1/4}\|u\|_{X^a_{3/4,b}}\||x|^r u\|_{L^\infty_T L^2_x}. \end{aligned} \tag{9}$$

Using (7), we note that

$$\begin{aligned} \|hf\|_{L^2_\xi L^2_\tau} &= \|h\mathcal{F}_t\mathcal{F}_x\psi(T^{-1}t)uu_x\|_{L^2_\xi L^2_\tau} = \|\psi(T^{-1}t)uu_x\|_{X^{a_1}_{3/4,b-1-\theta}} \\ &\lesssim T^\theta\|uu_x\|_{X^{a_1}_{3/4,b-1}} \lesssim T^\theta\|u\|^2_{X^{a_1}_{3/4,b}}. \end{aligned} \tag{10}$$

The substitution of two results (9) and (11) into $\|\Phi_1(u,v)\|_{X_T}$ and considering other norms gives

$$\|\Phi_1(u,v)\|_{X_T} \leq K_1\Big[\|u_0\|_{H^{3/4}_x} + \||x|^r u_0\|_{L^2_x}$$
$$+ T^{\theta_1}[\|u\|^2_{X^{a_1}_{3/4,b}} + \|v\|^2_{X^{a_2}_{3/4,b}} + \|u\|_{X^{a_1}_{3/4,b}}\|v\|_{X^{a_2}_{3/4,b}} \tag{11}$$
$$+ \|u\|_{X^a_{3/4,b}}\||x|^r v\|_{L^\infty_T L^2_x} + \|u\|_{X^a_{3/4,b}}\||x|^r u\|_{L^\infty_T L^2_x} \tag{12}$$
$$+ \|v\|_{X^a_{3/4,b}}\||x|^r v\|_{L^\infty_T L^2_x} + \|v\|_{X^a_{3/4,b}}\||x|^r u\|_{L^\infty_T L^2_x}]\Big]$$
$$\leq K_1[\|u_0\|_{H^{3/4}_x} + \||x|^r u_0\|_{L^2_x} + T^{\theta_1}M^2]$$

for some $K_1, T, \theta_1 > 0$.

Applying the same argument for the $\|\Phi_2(u,v)\|_{X_T}$, we obtain

$$\|\Phi_2(u,v)\|_{X_T} \leq K_2\Big[\|v_0\|_{H^{3/4}_x} + \||x|^r v_0\|_{L^2_x}$$
$$+ T^{\theta_2}[\|u\|^2_{X^{a_1}_{3/4,b}} + \|v\|^2_{X^{a_2}_{3/4,b}} + \|u\|_{X^{a_1}_{3/4,b}}\|v\|_{X^{a_2}_{3/4,b}} \tag{13}$$
$$+ \|u\|_{X^a_{3/4,b}}\||x|^r v\|_{L^\infty_T L^2_x} + \|u\|_{X^a_{3/4,b}}\||x|^r u\|_{L^\infty_T L^2_x} \tag{14}$$
$$+ \|v\|_{X^a_{3/4,b}}\||x|^r v\|_{L^\infty_T L^2_x} + \|v\|_{X^a_{3/4,b}}\||x|^r u\|_{L^\infty_T L^2_x}]\Big]$$
$$\leq K_2[\|v_0\|_{H^{3/4}_x} + \||x|^r v_0\|_{L^2_x} + T^{\theta_2}M^2] \tag{15}$$

for some T, θ', K_2>0.
Taking

$$M := 2(K_1+K_2)(\|u_0\|_{H^{3/4}_x} + \|v_0\|_{H^{3/4}_x} + \||x|^r u_0\|_{L^2_x} + \||x|^r v_0\|_{L^2_x})$$

and $T > 0$ small enough such that

$$M/2 + (K_1+K_2)T^\theta M^2 \leq M$$

for some $T \in (0,1), \theta = \min\{\theta_1, \theta_2\}$. Hence, it follows that

$$\|\Phi(u,v)\|_{X_T^M} \leq M.$$

is true. □

3.2.1. *Analysis of C1 commutator.*

Lemma 3.3. *Let* $a_1 \in \mathbb{R}\backslash\{0\}$ *and* $11/12 < b < 1$, $0 < \theta < b - 11/12$ *and* $h(\xi,\tau) = \langle\xi\rangle^s\langle\tau - a_1\xi^3\rangle^{b-1-\theta}$. *Then for* $T \in [0,1]$ *such that*

$$\Big\| \int_{\mathbb{R}} \Big[\frac{e^{it\tau} - e^{ia_1 t\xi^3}}{(\tau - a_1\xi^3)h(\xi,\tau)}, D_\xi^r\Big](hf)(\xi,\tau)d\tau\Big\|_{L_T^\infty L_\xi^2} \lesssim \|hf\|_{L_\xi^2 L_\tau^2}.$$

The proof of the above proposition is skipped in this discussion. It can be done similarly to [16, Section 3] and [7, Section 3].

3.2.2. *Analysis of C2 commutator.* In this subsection, we provide the control of Commutator 2 defined in Section 3.2. We start by introducing two Lemmas: 3.4 and 3.5, which are essential in this proof.
First, we consider the control of the oscillated integral. We are particularly interested in $\mathbb{K}_{\alpha,\beta}(x,\tau)$ as generalization of Naha's Lemma 3.4 [16] with extra parameter τ. $\mathbb{K}_{\alpha,\beta}(x,\tau)$ is defined as

$$\mathbb{K}_{\alpha,\beta}(x,\tau) := \int_{\mathbb{R}} e^{-ix\xi}\langle\xi\rangle^{-\alpha}\langle\tau - a_1\xi^3\rangle^{-\beta}\, d\xi.$$

Without loss of generality, we consider a_1=1.

Lemma 3.4. *For* $0 \leq \alpha < 1$, $0 < \beta < 1$, $0 < \varepsilon < \beta$ *and* $x \in \mathbb{R}$ *we have that*

$$|\mathbb{K}_{\alpha,\beta}(x,\tau)| \lesssim \begin{cases} \dfrac{1}{|x|^{1-\varepsilon}(1 + |x|^{1+\varepsilon})|\tau|^{\varepsilon}}, & |\tau| < 1, \\[2ex] \dfrac{1}{|x|^{1-\beta}(1 + |x|^{1+\beta})|\tau|^{\beta+\alpha/3-1/3}}, & |\tau| \geq 1. \end{cases}$$

Proof. We prove this lemma by considering specific intervals of the x,τ values. Thus, we split our analysis into four specific considerations.

Case 1: $|x| \geq 1$ and $|\tau| < 1$
In this integral, we use the differential property of exponential term that will be useful further:

$$\frac{\partial}{\partial\xi}\Big[e^{-ix\xi}\Big] = (-ix)e^{-ix\xi} \to e^{-ix\xi} = \frac{1}{-ix}\frac{\partial}{\partial\xi}\Big[e^{-ix\xi}\Big].$$

Applying this property and also the Integration by Parts method to our inequality, we get the following equation:

$$\int_{\mathbb{R}} e^{-ix\xi}\langle\xi\rangle^{-\alpha}\langle\tau - \xi^3\rangle^{-\beta}\, d\xi = \frac{1}{-ix}\int_{\mathbb{R}} \frac{\partial}{\partial\xi}\Big[e^{-ix\xi}\Big]\langle\xi\rangle^{-\alpha}\langle\tau - \xi^3\rangle^{-\beta}\, d\xi$$
$$= \frac{1}{-ix}\int_{\mathbb{R}} e^{-ix\xi}\frac{\partial}{\partial\xi}\Big[\langle\xi\rangle^{-\alpha}\langle\tau - \xi^3\rangle^{-\beta}\Big]\, d\xi + \frac{1}{-ix}e^{-ix\xi}\langle\xi\rangle^{-\alpha}\langle\tau - \xi^3\rangle^{-\beta}|_{\xi=-\infty}^{\xi=\infty}.$$

Notice, that the boundary term is zero. Applying Integration by Parts a second time, we get the following results:

$$\int_{\mathbb{R}} e^{-ix\xi}\langle\xi\rangle^{-\alpha}\langle\tau-\xi^3\rangle^{-\beta}\,d\xi = \frac{1}{x^2}\int_{\mathbb{R}} e^{-ix\xi}\frac{\partial^2}{\partial\xi^2}\Big[\langle\xi\rangle^{-\alpha}\langle\tau-\xi^3\rangle^{-\beta}\Big]\,d\xi.$$

Next, we consider the second derivative of the last term

$$\begin{aligned}\frac{\partial^2}{\partial\xi^2}\Big[\langle\xi\rangle^{-\alpha}\langle\tau-\xi^3\rangle^{-\beta}\Big] &\sim \frac{1}{(1+\xi^2)^{\alpha/2+1}(1+(\tau-\xi^3)^2)^{\beta/2}}\\ &+\frac{\xi^2}{(1+\xi^2)^{\alpha/2+2}(1+(\tau-\xi^3)^2)^{\beta/2}}+\frac{\xi^3(\tau-\xi^3)}{(1+\xi^2)^{\alpha/2+1}(1+(\tau-\xi^3)^2)^{\beta/2+1}}\\ &+\frac{\xi(\tau-\xi^3)}{(1+\xi^2)^{\alpha/2}(1+(\tau-\xi^3)^2)^{\beta/2+1}}+\frac{\xi^4(\tau-\xi^3)^2}{(1+\xi^2)^{\alpha/2}(1+(\tau-\xi^3)^2)^{\beta/2+2}}\\ &=: A_1(\xi,\tau)+A_2(\xi,\tau)+A_3(\xi,\tau)+B_1(\xi,\tau)+B_2(\xi,\tau).\end{aligned}$$

We can show the integrals convergence of each of the derivative terms. We consider the A_1 and B_1 terms, the others can be shown in a similar manner.

$$|\int_{\mathbb{R}} e^{-ix\xi}A_1(\xi,\tau)\,d\xi| \lesssim \int_{\mathbb{R}} |\frac{1}{(1+\xi^2)^{\alpha/2+1}(1+(\tau-\xi^3)^2)^{\frac{\beta}{2}}}|\,d\xi \lesssim \int_{\mathbb{R}}\frac{d\xi}{(1+|\xi|)^{\alpha+2}} \lesssim 1$$

We proceed for B_1 term by adding and subtracting τ:

$$\begin{aligned}\int_{\mathbb{R}}\frac{|\xi|^4}{(1+|\xi|)^{\alpha}(1+|\tau-\xi^3|)^2)^{\beta+2}}\,d\xi &\lesssim \int_{\mathbb{R}}\frac{(|\tau|+|\tau-\xi^3|)^{\frac{4}{3}}}{(1+|\xi|)^{\alpha}(1+|\tau-\xi^3|)^2)^{\beta+2}}\,d\xi\\ \lesssim \int_{\mathbb{R}}\frac{1}{(1+|\xi|)^{\alpha}(1+|\tau-\xi^3|)^2)^{\beta+2/3}}\,d\xi &\lesssim \int_{\mathbb{R}}\frac{(1+|\xi^3|)^{\frac{1}{3}}}{(1+|\xi|)^{\alpha+1}(1+|\tau-\xi^3|)^2)^{\beta+2/3}}\,d\xi\\ \lesssim \int_{\mathbb{R}}\frac{(1+|\tau|+|\tau-\xi^3|)^{\frac{1}{3}}}{(1+|\xi|)^{\alpha+1}(1+|\tau-\xi^3|)^2)^{\beta+2/3}}\,d\xi &\lesssim \int_{\mathbb{R}}\frac{d\xi}{(1+|\xi|)^{\alpha+1}(1+|\tau-\xi^3|)^2)^{\beta+1/3}}\\ \lesssim \int_{\mathbb{R}}\frac{d\xi}{(1+|\xi|)^{\alpha+1}} \lesssim 1.\end{aligned}$$

Then, we can show the following results.

$$A_2, A_1 \lesssim A_1, \qquad\qquad B_2 \lesssim B_1.$$

Applying previous results and collecting all terms, we conclude the following results:

$$\begin{aligned}\int_{\mathbb{R}} e^{-ix\xi}\langle\xi\rangle^{-\alpha}\langle\tau-\xi^3\rangle^{-\beta}\,d\xi &\lesssim \frac{1}{|x|^2}\lesssim\frac{|x|^{1-\varepsilon}(1+|x|^2)|\tau|^{\varepsilon}}{|x|^{1-\varepsilon}(1+|x|^2)x^2|\tau|^{\varepsilon}}\\ &\lesssim \frac{(1+|x|)^{1-\varepsilon}}{|x|^{1-\varepsilon}(1+|x|^2)|\tau|^{\varepsilon}}\lesssim\frac{1}{|x|^{1-\varepsilon}(1+|x|^{1+\varepsilon})|\tau|^{\varepsilon}}.\end{aligned}$$

for $|\tau|<1$, $|x|>1$.

Case 2: $|x|\ge 1$ and $|\tau|\ge 1$:

In this section, we consider Case 2. The analysis of Case 2 can be referred to previous case 1, where we introduced A_1, A_2, A_3, B_1, and B_2.

Next, we consider each case separately by introducing the change of variables with

$\xi \to \tau^{\frac{1}{3}}\xi'$ and $d\xi \to \tau^{\frac{1}{3}}d\xi'$.
First, consider the integral of the A_1:

$$\begin{aligned}\frac{1}{x^2}\int_{\mathbb{R}} A_1(\xi,\tau)\,d\xi &= \frac{1}{x^2}\int_{\mathbb{R}} \frac{1}{(1+\xi^2)^{\alpha/2+1}(1+(\tau-\xi^3)^2)^{\beta/2}}\,d\xi \\ &= \frac{1}{x^2}\int_{\mathbb{R}} \frac{\tau^{1/3}}{(1+|\tau^{\frac{1}{3}}\xi|)^{\alpha+2}(1+|\tau-\tau\xi^3|)^{\beta}}\,d\xi \\ &\lesssim \frac{1}{\tau^{\alpha/3+1/3+\beta}x^2}\int_{\mathbb{R}} \frac{1}{(\frac{1}{\tau^{1/3}}+|\xi|)^{\alpha+2}(\frac{1}{\tau}+|1-\xi^3|)^{\beta}}\,d\xi \\ &\lesssim \frac{1}{\tau^{\alpha/3+1/3+\beta}x^2}\int_{\mathbb{R}} \frac{1}{(\frac{1}{\tau^{1/3}})^2|\xi|^{\alpha}(\frac{1}{\tau}+|1-\xi^3|)^{\beta}}\,d\xi.\end{aligned}$$

Then in the integral of A_1, we consider the three following intervals $|\xi| \le \frac{1}{2}, \frac{1}{2} < |\xi| \le 2$ and $2 \le |\xi|$:

$$\begin{aligned}&\frac{1}{\tau^{\frac{\alpha+1}{3}+\beta}x^2}\Big(\int_{|\xi|\le\frac{1}{2}} + \int_{\frac{1}{2}<|\xi|\le 2} + \int_{2\le|\xi|}\Big)\frac{1}{(\frac{1}{\tau^{\frac{1}{3}}})^2|\xi|^{\alpha}(\frac{1}{\tau}+|1-\xi^3|)^{\beta}}\,d\xi \\ &\lesssim \frac{1}{\tau^{\frac{\alpha-1}{3}+\beta}x^2}\int_{|\xi|\le\frac{1}{2}} \frac{1}{\frac{1}{|\xi|}^{\alpha}}\,d\xi + \frac{1}{\tau^{\frac{\alpha-1}{3}+\beta}x^2}\int_{\frac{1}{2}<|\xi|\le 2}\frac{1}{|1-\xi|^{\beta}}\,d\xi \\ &\quad + \frac{1}{\tau^{\frac{\alpha+1}{3}+\beta}x^2}\int_{2\le|\xi|}\frac{1}{(|\xi|^{\alpha}|1-\xi^3|^{\beta}}\,d\xi \lesssim \frac{1}{\tau^{\frac{\alpha-1}{3}+\beta}x^2}.\end{aligned}$$

Next, we consider the integral of A_2:

$$\begin{aligned}\frac{1}{x^2}\int_{\mathbb{R}} A_2\,d\xi &= \frac{1}{x^2}\int_{\mathbb{R}} \frac{\xi^2}{(1+\xi^2)^{\alpha/2+2}(1+(\tau-\xi^3)^2)^{\beta/2}}\,d\xi \\ &\lesssim \frac{1}{x^2}\int_{\mathbb{R}} \frac{1+|1+\xi|^2}{(1+|\xi|)^{\alpha+4}(1+(\tau-\xi^3)^2)^{\beta/2}}\,d\xi \\ &\lesssim \frac{1}{x^2}\int_{\mathbb{R}} \frac{1}{(1+|\xi|)^{\alpha+2}(1+(\tau-\xi^3)^2)^{\beta/2}}\,d\xi \\ &= \frac{1}{x^2}\int_{\mathbb{R}} A_1(\xi,\tau)\,d\xi \lesssim \frac{1}{\tau^{\frac{\alpha-1}{3}+\beta}x^2}.\end{aligned}$$

Now, consider the analysis of A_3 integral:

$$\begin{aligned}\frac{1}{x^2}\int_{\mathbb{R}} A_3\,d\xi &= \frac{1}{x^2}\int_{\mathbb{R}} \frac{\xi^3(\tau-\xi^3)}{(1+\xi^2)^{\alpha/2+1}(1+(\tau-\xi^3)^2)^{\beta/2+1}}\,d\xi \\ &\lesssim \frac{1}{x^2}\int_{\mathbb{R}} \frac{|\xi|^3(1+|\tau-\xi|^3)}{(1+|\xi|)^{\alpha+2}(1+|\tau-\xi^3|)^{\beta+2}}\,d\xi \\ &\lesssim \frac{1}{x^2}\int_{\mathbb{R}} \frac{\tau^{\frac{4}{3}}|\xi|^3}{(1+|\tau^{\frac{1}{3}}\xi|)^{\alpha+2}(1+|\tau-\tau\xi^3|)^{\beta+1}}\,d\xi \\ &\lesssim \frac{1}{\tau^{\frac{\alpha+1}{3}+\beta}x^2}\int_{\mathbb{R}} \frac{|\xi|^3}{(\frac{1}{\tau^{\frac{1}{3}}}+|\xi|)^{\alpha+2}(\frac{1}{\tau}+|1-\xi^3|)^{\beta+1}}\,d\xi.\end{aligned}$$

Recall that the last term was previously considered in A_1 analysis. Therefore, we can conclude that:

$$\frac{1}{x^2}\int_{\mathbb{R}} A_3\,d\xi \lesssim \frac{1}{\tau^{\frac{\alpha-1}{3}+\beta}x^2}.$$

We apply the same methods and techniques in integrals of B_1 and B_2 to show that:

$$\begin{aligned}\frac{1}{x^2}\int_{\mathbb{R}} B_1\,d\xi &= \frac{1}{x^2}\int_{\mathbb{R}} \frac{\xi(\tau-\xi^3)}{(1+\xi^2)^{\alpha/2}(1+(\tau-\xi^3)^2)^{\beta/2+1}}\,d\xi \\ &\lesssim \frac{1}{\tau^{\frac{\alpha+1}{3}+\beta}x^2}\int_{\mathbb{R}} \frac{1}{(\frac{1}{|\tau|^{\frac{1}{3}}}+|\xi|)^{\alpha}(\frac{1}{\tau}+|1-\xi^3|)^{\beta+\frac{2}{3}}}\,d\xi \lesssim \frac{1}{\tau^{\frac{\alpha-1}{3}+\beta}x^2},\end{aligned}$$

$$\begin{aligned}\frac{1}{x^2}\int_{\mathbb{R}} B_2\,d\xi &= \frac{1}{x^2}\int_{\mathbb{R}} \frac{\xi^4(\tau-\xi^3)^2}{(1+\xi^2)^{\alpha/2}(1+(\tau-\xi^3)^2)^{\beta/2+2}}\,d\xi \\ &\lesssim \frac{1}{\tau^{\frac{\alpha+1}{3}+\beta}x^2}\int_{\mathbb{R}} \frac{1}{(\frac{1}{|\tau|^{\frac{1}{3}}}+|\xi|)^{\alpha}(\frac{1}{\tau}+|1-\xi^3|)^{\beta+\frac{2}{3}}}\,d\xi \lesssim \frac{1}{\tau^{\frac{\alpha-1}{3}+\beta}x^2}.\end{aligned}$$

In conclusion of Case 2, if we combine all cases, we have the following results for $|x|>1$ and $|\tau|>1$:

$$\begin{aligned}\int_{\mathbb{R}} e^{-ix\xi}\langle\xi\rangle^{-\alpha}\langle\tau-\xi^3\rangle^{-\beta}\,d\xi &\lesssim \frac{1}{\tau^{\frac{\alpha-1}{3}+\beta}x^2} \lesssim \frac{|x|^{1-\beta}(1+|x|^2)}{\tau^{\frac{\alpha-1}{3}+\beta}|x|^{1-\beta}(1+|x|^2)x^2} \\ &\lesssim \frac{(1+|x|)^{1-\beta}}{\tau^{\frac{\alpha-1}{3}+\beta}|x|^{1-\beta}(1+|x|^2)} = \frac{1}{\tau^{\frac{\alpha-1}{3}+\beta}|x|^{1-\beta}(1+|x|)^{1+\beta}}.\end{aligned}$$

<u>Case 3: $|x|<1$ and $|\tau|<1$:</u>

First step, we apply the change of variable with $\xi=\tau^{\frac{1}{3}}\xi'$ and $d\xi=\tau^{\frac{1}{3}}d\xi'$. Then, we split the integral by the region $\Omega=\left[\frac{-2}{|x|},\frac{2}{|x|}\right]$ Thus, we get the following identity:

$$\begin{aligned}\int_{\mathbb{R}} e^{-ix\xi}\langle\xi\rangle^{-\alpha}\langle\tau-\xi^3\rangle^{-\beta}\,d\xi &= \int_{\mathbb{R}} e^{-ix\tau^{\frac{1}{3}}\xi}\tau^{\frac{1}{3}}(1+|\tau^{\frac{1}{3}}\xi|^2)^{-\alpha/2}(1+|\tau-\tau\xi^3|^2)^{-\beta/2}\,d\xi \\ &=: \int_{\mathbb{R}} A\,d\xi = \int_{\Omega} A\,d\xi + \int_{\mathbb{R}/\Omega} A\,d\xi =: D_1+D_2.\end{aligned}$$

Consider the analysis of the D_1 case.

$$\begin{aligned}|D_1| &= |\int_{\Omega} e^{-ix\tau^{\frac{1}{3}}\xi}\tau^{\frac{1}{3}}(1+|\tau^{\frac{1}{3}}\xi|^2)^{-\alpha/2}(1+|\tau-\tau\xi^3|^2)^{-\beta/2}\,d\xi| \\ &\lesssim \int_{|\xi|\le\frac{2}{|x|}} \frac{\tau^{\frac{1}{3}}}{(1+|\tau^{\frac{1}{3}}\xi|)^{\alpha}(1+|\tau-\tau\xi^3|)^{\beta}}\,d\xi \\ &\lesssim \int_{|\xi|\le\frac{2}{|x|}} \frac{\tau^{\frac{1}{3}-\beta-\alpha/3}}{(\frac{1}{\tau^{\frac{1}{3}}}+|\xi|)^{\alpha}(\frac{1}{\tau}+|1-\xi^3|)^{\beta}}\,d\xi \\ &\lesssim \int_{|\xi|\le\frac{2}{|x|}} \frac{\tau^{\frac{1}{3}-\beta-\alpha/3}}{\tau^{-\alpha/3}(\frac{1}{\tau}+|1-\xi^3|)^{\beta-\varepsilon+\varepsilon}}\,d\xi \\ &\lesssim \int_{|\xi|\le\frac{2}{|x|}} \frac{\tau^{\frac{1}{3}-\beta}}{\tau^{-\beta+\varepsilon}|1-\xi^3|^{\varepsilon}}\,d\xi\end{aligned}$$

$$\lesssim \tau^{\frac{1}{3}-\varepsilon}\int_{|\xi|\le\frac{2}{|x|}}\frac{1}{|1-\xi|^{\varepsilon}}\,d\xi \lesssim \frac{1}{|\tau|^{\varepsilon}}\frac{1}{|1-\xi|^{\varepsilon-1}}\Big|_{\frac{2}{|x|}} \lesssim \frac{1}{|x|^{1-3\varepsilon}|\tau|^{\varepsilon}}.$$

Next, consider the D_2 part by integrating by parts:

$$\begin{aligned}|D_2| &= |\int_{\mathbb{R}/\Omega} e^{-ix\tau^{\frac{1}{3}}\xi}\tau^{\frac{1}{3}}(1+|\tau^{\frac{1}{3}}\xi|^2)^{-\alpha/2}(1+|\tau-\tau\xi^3|^2)^{-\beta/2}\,d\xi|\\
&= |\frac{1}{x}\int_{\mathbb{R}/\Omega}\frac{d}{d\xi}\Big[e^{-ix\xi\tau^{\frac{1}{3}}}\Big](1+|\tau^{\frac{1}{3}}\xi|^2)^{-\alpha/2}(1+|\tau-\tau\xi^3|^2)^{-\beta/2}\,d\xi|\\
&\lesssim \frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\Big(\frac{1}{(\frac{1}{\tau^{\frac{1}{3}}}+|\xi|)^{\alpha}(\frac{1}{\tau}+|1-\xi^3|)^{\beta}}\Big|_{\xi=\frac{2}{|x|}}\\
&\quad+\int_{\mathbb{R}/\Omega}\Big|\frac{d}{d\xi}\Big[\frac{1}{(\frac{1}{\tau^{\frac{2}{3}}}+|\xi|^2)^{\alpha/2}(\frac{1}{\tau^2}+|1-\xi^3|^2)^{\beta/2}}\Big]\Big|\,d\xi\Big) := D_{21}+D_{22}.\end{aligned}$$

Consider the boundary term D_{21}.

$$\begin{aligned}|D_{21}| &= \frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\frac{1}{(\frac{1}{\tau^{\frac{1}{3}}}+|\xi|)^{\alpha}(\frac{1}{\tau}+|1-\xi^3|)^{\beta}}\Big|_{\xi=\frac{2}{|x|}}\\
&\lesssim \frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\frac{1}{(\frac{1}{\tau^{\frac{1}{3}}})^{\alpha}(\frac{1}{\tau})^{\beta-\varepsilon}(|1-\xi^3|)^{\varepsilon}}\Big|_{\xi=\frac{2}{|x|}}\\
&\lesssim \frac{1}{\tau^{\beta-\varepsilon+\frac{\alpha}{3}}|x|}\frac{1}{|\xi|^{\alpha}|\xi^3|^{\varepsilon}}\Big|_{\xi=\frac{2}{|x|}} \lesssim \frac{1}{\tau^{\varepsilon+\frac{\alpha}{3}}|x|^{1-3\varepsilon-\alpha}} \lesssim \frac{1}{\tau^{\varepsilon}|x|^{1-3\varepsilon}}.\end{aligned}$$

Next, consider the D_{22} term with splitting:

$$\begin{aligned}|D_{22}| &= \frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\int_{\mathbb{R}/\Omega}|\frac{d}{d\xi}\Big[\frac{1}{(\frac{1}{\tau^{\frac{2}{3}}}+|\xi|^2)^{\alpha/2}(\frac{1}{\tau^2}+|1-\xi^3|^2)^{\beta/2}}\Big]|\,d\xi\\
&\lesssim \frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\int_{\mathbb{R}/\Omega}\Big(\frac{\xi}{(\frac{1}{\tau^{\frac{2}{3}}}+|\xi|^2)^{\alpha/2+1}(\frac{1}{\tau^2}+|1-\xi^3|^2)^{\beta/2}}\,d\xi\\
&\quad+\frac{1}{(\tau^{\beta+\frac{\alpha}{3}}|x|}\int_{\mathbb{R}/\Omega}\frac{|1-\xi^3||\xi^2|}{(\frac{1}{\tau^{\frac{2}{3}}}+|\xi|^2)^{\alpha/2}(\frac{1}{\tau^2}+|1-\xi^3|^2)^{\beta/2+1}}\,d\xi.\end{aligned}$$

Notice that the second term is dominated by the first one. Therefore, we consider just the first term.

$$\frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\int_{\mathbb{R}/\Omega}\frac{\xi}{(\frac{1}{\tau^{\frac{1}{3}}})^{\alpha}|\xi|^2\frac{1}{\tau}^{\beta-\varepsilon}|1-\xi^3|^{\varepsilon}}\,d\xi \lesssim \frac{1}{\tau^{\varepsilon+\frac{\alpha}{3}}|x|}\int_{\mathbb{R}/\Omega}\frac{1}{|\xi|^{3\varepsilon+\alpha+1}}\,d\xi \lesssim \frac{1}{|x|^{1-3\varepsilon}|\tau|^{\varepsilon}}.$$

In conclusion of Case 3, we get the following results:

$$\begin{aligned}|\int_{\mathbb{R}} e^{-ix\xi}\langle\xi\rangle^{-\alpha}\langle\tau-\xi^3\rangle^{-\beta}\,d\xi| &\lesssim A_1+A_{21}+A_{22} \lesssim \frac{1}{|x|^{1-3\varepsilon}|\tau|^{\varepsilon}}+\frac{1}{|x|^{1-3\varepsilon}}\\
&\lesssim \frac{1}{|x|^{1-3\varepsilon}|\tau|^{\varepsilon}} \lesssim \frac{1}{|x|^{1-3\varepsilon}(1+|x|^{1+3\varepsilon})|\tau|^{\varepsilon}}.\end{aligned}$$

Case 4: $|x|<1$ and $|\tau|\ge 1$:

In this section, we start the analysis of Case 4 by introducing the change of variables and splitting the integral into two parts as it was shown previously in Case 3. Then, consider D_1 case.

$$
\begin{aligned}
|D_1| &= |\int_{\Omega} e^{-ix\tau^{\frac{1}{3}}\xi}\tau^{\frac{1}{3}}(1+|\tau^{\frac{1}{3}}\xi|^2)^{-\alpha/2}(1+|\tau-\tau\xi^3|^2)^{-\beta/2}\,d\xi| \\
&\lesssim \int_{|\xi|\le\frac{2}{|x|}} \frac{\tau^{\frac{1}{3}}}{(1+|\tau^{\frac{1}{3}}\xi|)^{\alpha}(1+|\tau-\tau\xi^3|)^{\beta}}\,d\xi \\
&= \frac{1}{\tau^{\beta+\frac{\alpha-1}{3}}}\int_{|\xi|\le\frac{2}{|x|}} \frac{1}{(\frac{1}{\tau^{\frac{1}{3}}}+|\xi|)^{\alpha}((\frac{1}{\tau}+|1-\xi^3|)^{\beta}}\,d\xi \\
&\lesssim \frac{1}{\tau^{\beta+\frac{\alpha-1}{3}}}\int_{|\xi|\le\frac{2}{|x|}} \frac{1}{(|\xi|)^{\alpha}((\frac{1}{\tau}+|1-\xi^3|)^{\beta}}\,d\xi.
\end{aligned}
$$

At this step, we have two singularities at $\xi=0$ and $\xi=1$. Therefore, we split the considered integrals into two parts, considering each discontinuity separately:

$$
\begin{aligned}
&\frac{1}{\tau^{\beta+\frac{\alpha-1}{3}}}\int_{|\xi|\le\frac{2}{|x|}} \frac{1}{(|\xi|)^{\alpha}((\frac{1}{\tau}+|1-\xi^3|)^{\beta}}\,d\xi \lesssim \frac{1}{\tau^{\beta+\frac{\alpha-1}{3}}}\int_{|\xi|\le\frac{2}{|x|}} \frac{1}{|\xi|^{\alpha}|1-\xi^3|^{\beta}}\,d\xi \\
&\lesssim \frac{1}{\tau^{\beta+\frac{\alpha-1}{3}}}\Big(\int_{|\xi|\le\frac{1}{2}}+\int_{\frac{1}{2}|\xi|\le\frac{2}{|x|}}\Big)\frac{1}{|\xi|^{\alpha}|1-\xi^3|^{\beta}}\,d\xi \\
&\lesssim \frac{1}{\tau^{\beta+\frac{\alpha-1}{3}}}\Big(\int_{|\xi|\le\frac{1}{2}}\frac{1}{|\xi|^{\alpha}}\,d\xi+\int_{\frac{1}{2}\le|\xi|\le\frac{2}{|x|}}\frac{1}{|1-\xi^3|^{\beta}}\,d\xi\Big) \\
&\lesssim \frac{1}{\tau^{\beta+\frac{\alpha-1}{3}}}\Big(1+\frac{1}{|x|^{1-\beta}}\Big) \lesssim \frac{1}{\tau^{\beta+\frac{\alpha-1}{3}}|x|^{1-\beta}}.
\end{aligned}
$$

Next, consider D_2 by Integration By Parts method:

$$
\begin{aligned}
|D_2| &= |\int_{\mathbb{R}/\Omega} e^{-ix\xi\tau^{\frac{1}{3}}}\tau^{\frac{1}{3}}(1+|\tau^{\frac{1}{3}}\xi|^2)^{-\alpha/2}(1+|\tau-\tau\xi^3|^2)^{-\beta/2}\,d\xi| = \\
&= |\frac{1}{x}\int_{\mathbb{R}/\Omega}\frac{d}{d\xi}\Big[e^{-ix\xi\tau^{\frac{1}{3}}}\Big](1+|\tau^{\frac{1}{3}}\xi|^2)^{-\alpha/2}(1+|\tau-\tau\xi^3|^2)^{-\beta/2}\,d\xi| \\
&= \frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\Big(\frac{1}{(\frac{1}{\tau^{\frac{1}{3}}}+|\xi|)^{\alpha}(\frac{1}{\tau}+|1-\xi^3|)^{\beta}}\Big|_{\xi=\frac{2}{|x|}} \\
&\quad+\int_{\mathbb{R}/\Omega}\frac{d}{d\xi}\Big[\frac{1}{(\frac{1}{\tau^{\frac{2}{3}}}+|\xi|^2)^{\alpha/2}(\frac{1}{\tau^2}+|1-\xi^3|^2)^{\beta/2}}\Big]\,d\xi\Big) \\
&= D_{21}+D_{22}.
\end{aligned}
$$

Consider D_{21} and D_{22} separately:

$$
\begin{aligned}
|D_{21}| &= |\frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\frac{1}{(\frac{1}{\tau^{\frac{1}{3}}}+|\xi|)^{\alpha}(\frac{1}{\tau}+|1-\xi^3|)^{\beta}}|\Big|_{\xi=\frac{2}{|x|}} \lesssim \frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\frac{1}{|\xi|^{\alpha}|1-\xi^3|^{\beta}}\Big|_{\xi=\frac{2}{|x|}} \\
&\lesssim \frac{1}{|x|^{1-\alpha-3\beta}|\tau|^{\beta+\frac{\alpha}{3}}}.
\end{aligned}
$$

Finally, we consider A_{22} case:

$$
|D_{22}| = |\frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\int_{\mathbb{R}/\Omega}\frac{d}{d\xi}\Big[\frac{1}{(\frac{1}{\tau^{\frac{2}{3}}}+|\xi|^2)^{\alpha/2}(\frac{1}{\tau^2}+|1-\xi^3|^2)^{\beta/2}}\Big]\,d\xi|
$$

$$\begin{aligned}
&\lesssim \frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\int_{\mathbb{R}/\Omega}\Big(\frac{\xi}{(\frac{1}{\tau^{\frac{2}{3}}}+|\xi|^2)^{\alpha/2+1}(\frac{1}{\tau^2}+|1-\xi^3|^2)^{\beta/2}}\,d\xi\\
&\quad+\frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\int_{\mathbb{R}/\Omega}\frac{|1-\xi^3||\xi^2|}{(\frac{1}{\tau^{\frac{2}{3}}}+|\xi|^2)^{\alpha/2}(\frac{1}{\tau^2}+|1-\xi^3|^2)^{\beta/2+1}}\,d\xi\\
&\lesssim \frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x|}\int_{\mathbb{R}/\Omega}\frac{1}{|\xi|^{\alpha+1}|\xi|^{3\beta}}\,d\xi=\frac{1}{\tau^{\beta+\frac{\alpha}{3}}|x}\frac{1}{|\xi|^{\alpha+3\beta}}\Big|_{\frac{2}{|x|}}=\frac{1}{|x|^{1-\alpha-3\beta}|\tau|^{\beta+\frac{\alpha}{3}}}.
\end{aligned}$$

To conclude in Case 4, if we combine all cases, we have the following results for $|x|<1$ and $|\tau|>1$:

$$\begin{aligned}
\int_{\mathbb{R}} e^{-ix\xi}\langle\xi\rangle^{-\alpha}\langle\tau-\xi^3\rangle^{-\beta},d\xi &\lesssim \frac{1}{\tau^{\beta+\frac{\alpha-1}{3}}|x|^{1-\beta}}+\frac{1}{|x|^{1-\alpha-3\beta}|\tau|^{\beta+\frac{\alpha}{3}}}\\
&\lesssim \frac{1}{\tau^{\beta+\frac{\alpha-1}{3}}|x|^{1-\beta}}\lesssim\frac{(1+|x|^{\beta'})}{\tau^{\beta+\frac{\alpha-1}{3}}|x|^{1-\beta}(1+|x|^{\beta'})}\\
&\lesssim \frac{1}{\tau^{\beta+\frac{\alpha-1}{3}}|x|^{1-\beta}(1+|x|^{\beta})}.
\end{aligned}$$

□

This Lemma 3.4 is a generalization of Nahas's Theorem 3.4 [16], which is essential in the control. If we apply this Lemma in our analysis, we obtain the following results:

Lemma 3.5. *Let* $0<r<3/8$ *and* $11/12<b<1$, $\theta<b-11/12$. *Then,*

$$\Big\|\mathbb{K}_{3/4,2b-1-2\theta}(\cdot,\tau)*|\cdot|^r\mathbb{K}_{0,1-b+\theta}(\cdot,\tau)\Big\|_{L^2_\tau L^1_x}<\infty.$$

Proof. We apply the Lemma 3.4 to get the following

$$\begin{aligned}
&\Big\|\mathbb{K}_{3/4,2b-1-2\theta}(\cdot,\tau)*|\cdot|^r\mathbb{K}_{0,1-b+\theta}(\cdot,\tau)\Big\|^2_{L^2_\tau L^1_x}\\
&\lesssim \Big\|\|\mathbb{K}_{3/4,2b-1-2\theta}(\cdot,\tau)\|_{L^1_x}\||\cdot|^r\mathbb{K}_{0,1-b+\theta}(\cdot,\tau)\|_{L^1_x}\Big\|^2_{L^2_\tau}\\
&\lesssim \int_{\mathbb{R}}\Big|\|\mathbb{K}_{3/4,2b-1-2\theta}(\cdot,\tau)\|_{L^1_x}\||\cdot|^r\mathbb{K}_{0,1-b+\theta}(\cdot,\tau)\|_{L^1_x}\Big|^2\,d\tau\\
&=\int_{|\tau|<1}\Big|\|\frac{1}{|x|^{1-\varepsilon}(1+|x|^{1+\varepsilon})|\tau|^{\varepsilon}}\|_{L^1_x}\||\cdot|^r\frac{1}{|x|^{1-\varepsilon}(1+|x|^{1+\varepsilon-r})|\tau|^{\varepsilon}}(\cdot,\tau)\|_{L^1_x}\Big|^2\,d\tau\\
&\quad+\int_{|\tau|\ge1}\Big|\frac{1}{|\tau|^{(2b-1-2\theta)+1/4-1/3}}\frac{1}{|\tau|^{1-b+\theta-1/3}}\Big|^2\,d\tau\\
&\lesssim \int_{|\tau<1}\frac{1}{\tau^{4\xi}}\,d\tau+\int_{|\tau|\ge1}\frac{1}{|\tau|^{2(b-\theta+1/4s-2/3)}}\,d\tau\lesssim 1
\end{aligned}$$

The last result is true provided that $\xi<\min\{1/4,2b-1-2\theta,1-b+\theta\}$ and $2(b-\theta+1/4-2/3)>1$. Hence, we have $b>\theta+11/12>11/12$ condition. □

Now, return to the main analysis of the C2 commutator by defining the following identities:

$$\begin{aligned}
\Lambda(\xi,\tau,t)&:=\langle\tau-a_1\xi^3\rangle^{1-2b+2\theta}\langle\xi\rangle^{-s},\\
g(\xi,\tau,t)&:=\frac{e^{it\tau}-e^{-ia_1t\xi^3}}{\tau+a_1\xi^3}\langle\tau-a_1\xi^3\rangle^{b-\theta},
\end{aligned}$$

Applying the following definitions,lemmas, and $|g| \lesssim \langle \tau - a_1\xi^3\rangle^{b-1-\theta}$, we introduce the following proposition.

Proposition 3.6. *Let $a_1 \in \mathbb{R}\backslash\{0\}$ and $b-\theta > 1/2$. Then there exists $T>0$ such that where $h(\xi,\tau) := \langle\xi\rangle^{3/4}\langle\tau - a_1\xi^3\rangle^{b-1-\theta}$ for $0<\theta<b-11/12$.*

$$\| \int_{\mathbb{R}} \Lambda(\xi,\tau,t) g(\xi,\tau,t)\Big[\langle \tau - a_1\xi^3\rangle^{b-1-\theta}, D_\xi^r\Big](\langle\xi\rangle^{3/4} f)(\xi,\tau)\, d\tau \|_{L^\infty_\tau L^2_\xi} \lesssim \|hf\|_{L^2_\tau L^2_\xi}.$$

Proof. Denote $F(\xi,\tau) := \langle\xi\rangle^s f(\xi,\tau)$. We can obtain the following results by considering Young's convolution inequality and Holder's Inequality. We also consider

$$\begin{aligned}
&\| \int_{\mathbb{R}} \Lambda(\xi,\tau,t) g(\xi,\tau,t)\Big[\langle \tau - a_1\xi^3\rangle^{b-1-\theta}, D_\xi^r\Big](F)(\xi,\tau)\, d\tau \|_{L^\infty_\tau L^2_\xi} \\
&\lesssim \int_{\mathbb{R}} \|\Lambda(\xi,\tau) g(\xi,\tau,t)[\langle \tau - a_1\xi^3\rangle^{b-1-\theta}, D_\xi^r](F)(\xi,\tau)\|_{L^\infty_T L^2_\xi} d\tau \\
&\lesssim \int_{\mathbb{R}} \|\Big(\mathcal{F}_\xi(\Lambda) * |x|^r \mathbb{K}_{1-b+\theta}(\cdot,\tau)\Big) * \Big(\mathcal{F}_\xi g * \mathcal{F}_\xi F\Big)\|_{L^\infty_\tau L^2_x}\, d\tau \\
&\lesssim \int_{\mathbb{R}} \Big\|\mathcal{F}_\xi(\Lambda) * \|x\|^r \mathbb{K}_{1-b+\theta}(\cdot,\tau)\Big\|_{L^1_x} \Big\|gF\Big\|_{L^\infty_T L^2_\xi} d\tau \\
&\lesssim \int_{\mathbb{R}} \Big\|\mathcal{F}_\xi(\Lambda) * \|\cdot\|^r \mathbb{K}_{1-b+\theta}(\cdot,\tau)\Big\|_{L^1_x} \Big\|\langle \tau - a_1\xi^3\rangle^{b-1-\theta}\langle\xi\rangle^{3/4} f(\xi,\tau)\Big\|_{L^\infty_T L^2_\xi} d\tau.
\end{aligned}$$

Apply Holder's inequality to have the following

$$\begin{aligned}
&\int_{\mathbb{R}} \Big\|\mathcal{F}_\xi(\Lambda) * \|\cdot\|^r \mathbb{K}_{1-b+\theta}(\cdot,\tau)\Big\|_{L^1_x} \Big\|\langle \tau - a_1\xi^3\rangle^{b-1-\theta}\langle\xi\rangle^{3/4} f(\xi,\tau)\Big\|_{L^\infty_T L^2_\xi} d\tau \\
&\lesssim \int_{\mathbb{R}} \Big\|\mathcal{F}_\xi(\Lambda) * \|\cdot\|^r \mathbb{K}_{1-b+\theta}(\cdot,\tau)\Big\|_{L^1_x} \|hf\|_{L^2_\tau L^2_\xi}\, d\tau
\end{aligned}$$

The last term can be controlled using Lemma 3.5. Therefore, we have the control of the C2 commutator:

$$\begin{aligned}
&\Big\|\mathcal{F}_\xi(\Lambda) * \|\cdot\|^r \mathbb{K}_{1-b+\theta}(\cdot,\tau)\Big\|_{L^2_\tau L^1_x} \|hf\|_{L^2_\tau L^2_\xi} \\
&= \Big\|\mathbb{K}_{s,2b-1-2\theta}(\cdot,\tau) * |\cdot|^r \mathbb{K}_{0,1-b+\theta}(\cdot,\tau)\Big\|_{L^2_\tau L^1_x} \|hf\|_{L^2_\tau L^2_\xi} \\
&\lesssim \|hf\|_{L^2_\tau L^2_\xi}.
\end{aligned}$$

□

3.2.3. *Analysis of C3 commutator.* First, we state the auxiliary lemma.

Lemma 3.7. *For $\mathcal{F} = \mathcal{F}(\xi,\tau)$ and $W = W(\xi,\tau)$, we have that*

$$\|[\langle\xi\rangle^{-s}, D_\xi^r](F)\|_{L^2_\xi(W^2)} \lesssim \|F\|_{L^2_{\tau,\xi}(W^2)}$$

for fixed $\tau \in \mathbb{R}$

Proof. We bound the commutator

$$\begin{aligned}
&\|[\langle\xi\rangle^{-s}, D_\xi^r](F)\|_{L^2_\xi(W^2)} = \|W(\xi,\tau)[\langle\xi\rangle^{-s}, D_\xi^r](F)\|_{L^2_\xi} \\
&= \|W\widehat{[\langle\xi\rangle^{-s}, D_\xi^r]}(F)(x)\|_{L^2_x} = |\widehat{W(\cdot,\tau)} * \widehat{[\langle\xi\rangle^{-s}, D_\xi^r]}(F)(x)|_{L^2_x} \\
&\leq \Big\| \int_R \Big|W\widehat{(x-z,\tau)}\Big|\Big|\widehat{[\langle\xi\rangle^{-s}, D_\xi^r]}(F)(z)\Big|\, dz\Big\|_{L^2_x}.
\end{aligned}$$

The proof of Lemma 3.4 from [6] and Fubini's Theorem allows to show that:

$$\begin{aligned}
&\Big\| \int_R \Big|\widehat{W(x-z,\tau)}\Big|\Big|\widehat{[\langle\xi\rangle^{-s}, D_\xi^r]}(F)(z)\Big|\, dz\Big\|_{L^2_x}\\
&\leq \Big\| \int_R \Big|\widehat{W(x-z,\tau)}\Big|\Big(\int_R |z-y|^r|K_s(z-y)||\widehat{F(y,z)}|\, dy\Big)\, dz\Big\|_{L^2_x}\\
&= \Big\| \int_R \Big(\int_R ||z-y|^r|K_s(z-y)||\widehat{W(x-z,\tau)}|\widehat{F(y,z)}|\, dz\Big)\, dy\Big\|_{L^2_x}.
\end{aligned}$$

Next, by applying a change of variable $z-y=m$, we get the following

$$\begin{aligned}
&\Big\| \int_R \Big(\int_R ||z-y|^r|K_s(z-y)||\widehat{W(x-z,\tau)}|\widehat{F(y,z)}|\, dz\Big)\, dy\Big\|_{L^2_x}\\
&= \Big\| \int_R \Big(\int_R ||m|^r|K_s(m)||\widehat{W((x-m)-y,\tau)}|\widehat{F(y,z)}|\, dm\Big)\, dy\Big\|_{L^2_x}\\
&= \Big\| \int_R ||m|^r|K_s(m)|\Big(\int_R |\widehat{W((x-m)-y,\tau)}|\widehat{F(y,z)}|\, dy\Big)\, dm\Big\|_{L^2_x}\\
&= \Big\| \int_R ||m|^r|K_s(m)|\Big(\widehat{W(\cdot,\tau)} * \widehat{F(\cdot,\tau)}\Big)\, dm\Big\|_{L^2_x}\\
&= \Big\|\Big(|\cdot|^r K_s\Big) * \Big(\widehat{W(\cdot,\tau)} * \widehat{F(\cdot,\tau)}\Big)(x)\Big\|_{L^2_x}.
\end{aligned}$$

Then, we apply Young's convolution inequality with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$ to have

$$\begin{aligned}
&\Big\|\Big(|\cdot|^r K_s\Big) * \Big(\widehat{W(\cdot,\tau)} * \widehat{F(\cdot,\tau)}\Big)(x)\Big\|_{L^2_x} \lesssim \Big\||x|^r K_s(x)\Big\|_{L^1_x}\Big\|\widehat{W(\cdot,\tau)} * \widehat{F(\cdot,\tau)}\Big\|_{L^2_x}\\
&= \Big\||x|^r K_s(x)\Big\|_{L^1_x}\Big\|W(\xi,\tau)F(\xi,\tau)\Big\|_{L^2_\xi} \lesssim \|F(\xi,\tau)\|_{L^2_\xi(W^2)},
\end{aligned}$$

here we recall inequality (24) from [6]

$$\Big\||x|^r K_s(x)\Big\|_{L^1_x} \leq \infty.$$

□

The next proposition is the main statement of this chapter.

Proposition 3.8. *Let $a_1 \in \mathbb{R}\backslash\{0\}$ and $b-\theta > 1/2$. Then there exists $T>0$ such that the following inequality*

$$\| \int_{\mathbb{R}} \frac{e^{it\tau} - e^{ia_1 t\xi^3}}{\tau - a_1\xi^3}[\langle\xi\rangle^{-s} D_\xi^r](\langle\xi\rangle^s f)(\xi,\tau)d\tau\|_{L^\infty_T L^2_\xi} \lesssim \|hf\|_{L^2_\tau L^2_\xi}.$$

Proof. By Lemma 3.3 [12], Lemma 2.1 [21] and Lemma 3.7, we define $h(\xi,\tau) := \langle\xi\rangle^s\langle\tau - a_1\xi^3\rangle^{b-1-\theta}$ and bound C_3

$$\begin{aligned}
&\Big\|\Big(\int_{\mathbb{R}} \Big(\frac{e^{it\tau} - e^{-ia_1 t\xi^3}}{\tau + a_1\xi^3}\Big[\langle\xi\rangle^{-s}, D_\xi^r\Big](\langle\xi\rangle^s f)\Big)\, d\tau\Big)\Big\|_{L^\infty_T L^2_\xi}\\
&\lesssim \Big\|\Big(\int_0^t e^{a_1(t-t')\partial_x^3}\mathcal{F}_x^{-1}\Big[\langle\xi\rangle^{-s}, D_\xi^r\Big]\mathcal{F}_x^{-1}\langle\xi\rangle^s\psi(T^{-1}t)uu_x\Big)\, dt'\Big)\Big\|_{L^\infty_T L^2_\xi}\\
&\lesssim \Big\|\Big(\int_0^t e^{a_1(t-t')\partial_x^3}\mathcal{F}_x^{-1}\Big[\langle\xi\rangle^{-s}, D_\xi^r\Big]\mathcal{F}_x^{-1}\langle\xi\rangle^s\psi(T^{-1}t)uu_x\Big)\, dt'\Big)\Big\|_{X^{a_1}_{s,b-1-\theta}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \Big\| \mathcal{F}_x^{-1}\Big[\langle\xi\rangle^{-s}, D_\xi^r\Big]\mathcal{F}_x^{-1}\langle\xi\rangle^s\psi(T^{-1}t)uu_x\Big)\Big\|_{X^{a_1}_{s,b-1-\theta}} \\
&= \|\langle\tau - a_1\xi^3\rangle^{b-1-\theta}[\langle\xi\rangle^{-s}, D_\xi^r](\langle\xi\rangle^s f(\xi,\tau)\|_{L^2_{\xi,\tau}} \\
&= \|[\langle\xi\rangle^{-s}, D_\xi^r](\langle\xi\rangle^s f(\xi,\tau)\|_{L^2_{\xi,\tau}\langle\tau-a_1\xi^3\rangle^{b-1-\theta}} \\
&\leq \|\langle\tau - a_1\xi^3\rangle^{b-1-\theta}\langle\xi\rangle^s f(\xi,\tau)\|_{L^2_{\xi,\tau}} \\
&= \|hf\|_{L^2_\tau L^2_\xi}.
\end{aligned}
$$

□

3.2.4. *Analysis Nonlinear Part.*

Proposition 3.9. *Let* $a_1 \in \mathbb{R}\backslash\{0\}$,$0 < r \leq 3/8$, $11/12 < b < 1$. *Then we have the following inequality holds*

$$
\| \int_0^t e^{-ia_1(t-t')\partial_x^3}|x|^r uu_x dt'\|_{L^\infty_T L^2_x} \lesssim T^{1/4}\||x|^r u\|_{L^\infty_T L^2_x}\|u\|_{X^{a_1}_{\frac{3}{4},b}}. \tag{16}
$$

Proof. We consider the following inequality using [[10], Lemma 2, i)].

$$
\begin{aligned}
\|\partial_x u\|_{L^1_T L^\infty_x} \sim \|D_x^{\frac14}(D_x^{\frac34}u)\|_{L^1_T L^\infty_x} &\lesssim T^\theta \|D_x^{\frac34}u_x\|_{L^4_T H_x^{1/4,\infty}} \\
&\lesssim T^\theta \|D_x^{\frac34}u\|_{X^{a_1}_{0,b}} \lesssim T^{1/4}\|u\|_{X^{a_1}_{3/4,b}}.
\end{aligned}
$$

Now, we focus on the main result.
Without loss of generality, we consider the uu_x term. Other cases can proceed similarly. Applying the inequality above, we have

$$
\begin{aligned}
&\Big\|\int_0^t e^{a_1(t-t')\partial_x^3}|x|^r uu_x(t')\,dt'\Big\|_{L^\infty_T L^2_x} \lesssim \Big\|(|x|^r u)(u_x)\Big\|_{L^1_T L^2_x} \\
&\quad\lesssim \Big\|\||x|^r u\|_{L^2_x}\|(u_x)\|_{L^\infty_x}\Big\|_{L^1_T} \lesssim \Big\||x|^r u\Big\|_{L^\infty_T L^2_x}\Big\|(u_x)\Big\|_{L^1_T L^\infty_x} \lesssim T^{1/4}\||x|^r u\|_{L^\infty_T L^2_x}\|u\|_{X^{a_1}_{\frac34,b}}.
\end{aligned}
$$

The proof of the proposition is complete. □

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