Application of the Lambert Functions in Solving Transcendental Equations

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Abstract

The Lambert W function: $w = W(z)$ is defined as the solution of the equation $we^w = z$ where e is the base of the natural logarithm. There are several equations of the form $e^x\left(\frac{a_1x+a_2}{a_3x+a_4}\right) = a_5$, $e^x(a_1 + a_2\sqrt{x}) = a_3$, $e^x\left(\frac{a_1+a_2\sqrt{x}}{a_3+a_4\sqrt{x}}\right) = a_5$, e.t.c, which can be solved using corresponding generalized Lambert W functions. These equations occur in different fields in math and engineering, such as Magnetic Micro-Electro-Mechanical-Structures, Chemical Engineering, and other fields. This project shows solutions to these equations, the branch structure of the Lambert functions, expanding solutions into a series, and analyzing the radius of convergence.

Keywords: Transcendental equation; Lambert W function; r-Lambert function; Quadratic Lambert function; Principal branch; Dead-Core phenomena; Mass Peclet number; Magnetic Micro-Electro-Mechanical Structures; Dynamic pull-in; Generalized Lambert W function; Series expansion; Radius of convergence; Stirling approximation; Newton's Method.

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1 Introduction

1.1 History of Lambert W function

The history of the Lambert function began in 1758 when Johann Heinrich Lambert studied the equation: $x^m + px = q$. Leonhard Euler learned of

Lambert's results in 1764 and studied the more general equation: $x^{\alpha} - x^{\beta} = (\alpha - \beta)\, \upsilon x^{\alpha+\beta}$. Euler looked at the special case when $\alpha = \beta$:

$$\lim_{\alpha \to \beta} \frac{x^{\alpha} - x^{\beta}}{\alpha - \beta} = \upsilon x^{\alpha+\beta} \Rightarrow x^{\alpha} lnx = \upsilon x^{2\alpha} \Rightarrow lnx = \upsilon x^{\alpha} \Rightarrow lnx^{\alpha} = \alpha \upsilon x^{\alpha}$$

Substituting $y = x^{a}$ and $u = av$, we get:

$$lny = uy \Rightarrow y = e^{uy} \Rightarrow ye^{-uy} = 1\,;$$

Substituting $y = -\frac{z}{u}$, we get:

$$-\frac{z}{u}e^{z} = 1 \Rightarrow ze^{z} = -u\,;$$

Substituting $-u = a$ we get the equation $ze^{z} = a$ which has a solution $z = W(a)$.

Although Leonhard Euler had worked on the function in addition to Joann Lambert, Euler did not claim to name the function after himself but chose to credit Lambert for the earliest work on the subject. Thus, the solution to the equation $ze^{z} = a$ is called the Lambert function.

1.2 Properties of Lambert W function

The Lambert W function $w = W\,(z)$ is the inverse function of $z = we^{w}$. It is the multiple-valued function, i.e., there exists w, which may have more than one z. In this regard, the Lambert function can be divided into several branches - parts of the function in which one value of w corresponds to at most one value of z.

In real line, Lambert W function $y = W\,(x)$ has two branches: the principal branch $W_0(x)$ and the branch $W_{-1}(x)$. The domain of the principal branch $W_0(x)$ is: $x \in [-\frac{1}{e}, \infty)$, the range of $W_0(x)$ is: $y \in [-1, \infty)$. The domain of the another branch $W_{-1}(x)$ is: $x \in [-\frac{1}{e}, 0)$, the range of $W_{-1}(x)$ is: $y \in (-\infty, -1]$. In the complex plane, there are infinitely many branches $W_k(z)$, where $k \in \mathbb{Z}$.

2 Literature Review

Before proceeding to the analysis of the generalizations of Lambert functions, the Lambert function itself was studied. This function and its properties are described in detail in the book "The Lambert W Function: Its Generalizations and Applications" (Mező, 2022). Then, studying the application of the Lambert W function, sources from different areas of mathematics where transcendental equations are used were considered. (Amirali, 2022, He et al., 2021, Larsen and Marx, 2012) Then, while working on our own paper: Detection of Pull-in and Periodic Solutions of magMEMS Model with Current-Carrying Filament (Abdildayev et al., 2025), we tried to solve the equation $-s^2 - 2Kln|1-s| = 0$, but this equation could not be solved through the Lambert function, so we began to study generalizations of the Lambert function. Thanks to sources (Mező et al., 2020) and (Mező, 2022), the branch structure and the Taylor series for r and the quadratic Lambert functions were used. However, most of the results in the study were derived by themselves, since the generalization data are not yet well understood, and therefore, more detailed information about these generalized functions (e.g., equation transformation and Taylor series convergence radius analysis) is lacking.

3 Generalized Lambert functions

3.1 r-Lambert function

r-Lambert function $W_r(u)$ at the point u is a real or complex number that satisfies the equation

$$W_r(u)\, e^{W_r(u)} + rW_r(u) = u\,; \tag{1}$$

3.1.1 Transformation of a transcendental equation: $e^x\left(\frac{a_1x+a_2}{a_3x+a_4}\right) = a_5$

Our transcendental equation is:

$$e^x\left(\frac{a_1x+a_2}{a_3x+a_4}\right) = a_5\,; \tag{2}$$

The goal is to transform it to equation (2), so we could find one of its solutions.

$$e^x\left(\frac{a_1x+a_2}{a_3x+a_4}\right) = a_5 \Rightarrow e^x\left(\frac{x+\frac{a_2}{a_1}}{a_3x+a_4}\right) = \frac{a_5}{a_1} \Rightarrow e^x\left(\frac{x+\frac{a_2}{a_1}}{x+\frac{a_4}{a_3}}\right) = \frac{a_3\cdot a_5}{a_1}\,;$$

Substituting $y = x + \frac{a_2}{a_1}$ we obtain:

$$
\begin{aligned}
&e^{y-\frac{a_2}{a_1}}\left(\frac{y}{y+\frac{a_4}{a_3}-\frac{a_2}{a_1}}\right)=\frac{a_3\cdot a_5}{a_1}\Rightarrow\\
&\Rightarrow e^{y}\left(\frac{y}{y+\frac{a_4}{a_3}-\frac{a_2}{a_1}}\right)=e^{\frac{a_2}{a_1}}\frac{a_3\cdot a_5}{a_1}\Rightarrow\\
&\Rightarrow ye^{y}=e^{\frac{a_2}{a_1}}\frac{a_3\cdot a_5}{a_1}\left(y+\frac{a_4}{a_3}-\frac{a_2}{a_1}\right)\Rightarrow\\
&\Rightarrow ye^{y}-e^{\frac{a_2}{a_1}}\frac{a_3\cdot a_5}{a_1}y=e^{\frac{a_2}{a_1}}\frac{a_3\cdot a_5}{a_1}\left(\frac{a_4}{a_3}-\frac{a_2}{a_1}\right)\ ;
\end{aligned}
$$

Substituting $r=-e^{\frac{a_2}{a_1}}\frac{a_3\cdot a_5}{a_1}$ we obtain:

$$
ye^{y}+ry=r\left(\frac{a_2}{a_1}-\frac{a_4}{a_3}\right)\ ;\ w=r\left(\frac{a_2}{a_1}-\frac{a_4}{a_3}\right)\Rightarrow ye^{y}+ry=w\ ; \tag{3}
$$

where $y=W_r(w)$;

3.1.2 Taylor Series approximation for r-Lambert function

In order to find the value of r-Lambert function at some point, we will use Taylor Series approximation for r-Lambert function $W_r\left(x\right)$ around $r=0$, from Mező, 2022, which is:

$$
x=\left(W_0\left(w\right)-\frac{W_0\left(w\right)^2}{w\left(1+W_0\left(w\right)\right)}r-\frac{1}{2}\frac{W_0\left(w\right)^3\left(W_0\left(w\right)^2-2\right)}{w^2\left(W_0\left(w\right)+1\right)^3}r^2-\cdots\right)-\frac{a_2}{a_1}\ ; \tag{4}
$$

where $w=r\left(\frac{a_2}{a_1}-\frac{a_4}{a_3}\right)$ and $r=-e^{\frac{a_2}{a_1}}\frac{a_3\cdot a_5}{a_1}$.

3.1.3 Newton's Method in r-Lambert function

Depending on our parameters, the r-Lambert function can have up to 3 real solutions, each of which is found in a different branches. However, with the Taylor series we could find only one solution. In order to find others, we will use Newton's Method to numerically approximate these solutions:

$$
x_{n+1}=x_n-\frac{\left(x_n+\frac{a_2}{a_1}\right)e^{\left(x_n+\frac{a_2}{a_1}\right)}+r\left(x_n+\frac{a_2}{a_1}\right)-w}{\left(x_n+\frac{a_2}{a_1}\right)e^{\left(x_n+\frac{a_2}{a_1}\right)}+e^{\left(x_n+\frac{a_2}{a_1}\right)}+r}\ ; \tag{5}
$$

where $w=r\left(\frac{a_2}{a_1}-\frac{a_4}{a_3}\right)$ and $r=-e^{\frac{a_2}{a_1}}\frac{a_3\cdot a_5}{a_1}$.

3.1.4 The branch structure the$e^x\left(\frac{a_1x+a_2}{a_3x+a_4}\right)=a_5$

Depending on the parameter $r \neq 0$, The r-Lambert function has one, two, or three real branches. Since we want to properly use Newton's method to find other real solutions, we will look at the branch structure of the r-Lambert function that was studied by Mező, 2022. Let $\alpha = W_{-1}(re) - 1 - \frac{a_2}{a_1}$ and $\beta = W_0(-re) - 1 - \frac{a_2}{a_1}$. where W_{-1} and W_0 are the two branches of the classical Lambert function. Then:

1. If $r \geq \frac{1}{e^2}$ then we have only one real branch:

   $W_{r,0}(x) : (-\infty, +\infty)$ is a strictly increasing function.

2. If $0 < r < \frac{1}{e^2}$ then we have only three real branches:

   $W_{r,0}(x) : (-\infty, \alpha]$ is a strictly increasing function.

   $W_{r,1}(x) : [\alpha, \beta]$ is a strictly decreasing function.

   $W_{r,2}(x) : [\beta, +\infty)$ is a strictly increasing function.

3. If $r < 0$ then we have only two real branches:

   $W_{r,0}(x) : (-\infty, \beta)$ is a strictly decreasing function.

   $W_{r,1}(x) : [\beta, +\infty)$ is also a strictly increasing function.

3.1.5 Number of solutions of the equation: $e^x\left(\frac{a_1x+a_2}{a_3x+a_4}\right)=a_5$

Depending on the parameters, the number of real branches varies from 0 to 3. If $a_5 = 0$, then this equation has 0 or 1 real root. These graphs show examples of equations that have all possible numbers of solutions:

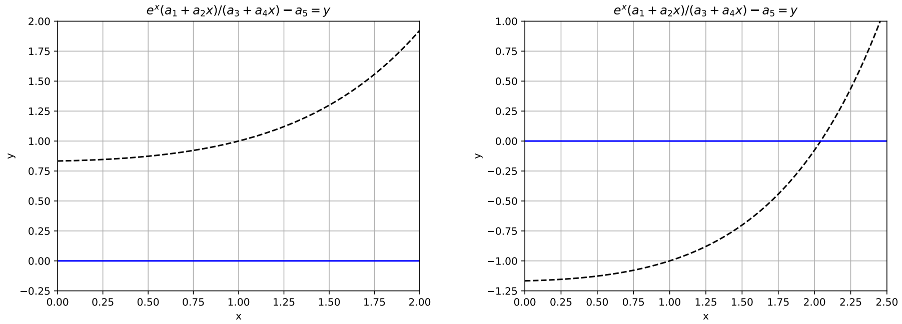

Figure 1: Equation with parameters, which have 0 and 1 roots

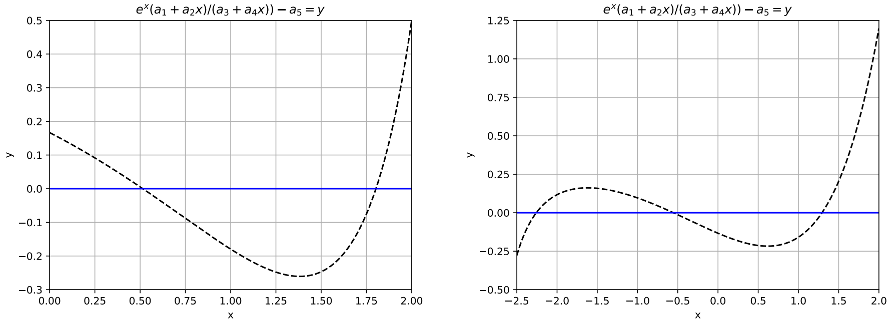

Figure 2: Equation with parameters, which have 2 and 3 roots

3.2 Quadratic Lambert function

Quadratic Lambert function $W(a;b)$ at the point x is a real or complex number that satisfies the equation:

$$W(a;b)e^{aW(a;b)^2+W(a;b)} = b; \tag{6}$$

This equation was studied by Mező et al., 2020 for the Einstein-Maxwell field equation. Previously, we showed that we can derive that equation from: $e^x(a_1 + a_2\sqrt{x}) = a_3$. Depending on a_1, a_2, a_3, we could get 0 to 3 solutions.

3.2.1 Transformation of a transcendental equation: $e^x(a_1+a_2\sqrt{x}) = a_3$

Our transcendental equation is:

$$e^x(a_1 + a_2\sqrt{x}) = a_3\ ; \tag{7}$$

The goal is to transform it to a $ye^{(y^2+y)} = x$, so we could find it's solution. First we will substitute $y = \sqrt{x} + \frac{a_1}{a_2}$, so we could obtain:

$$\begin{aligned} &e^{\left(y-\frac{a_1}{a_2}\right)^2}\left(a_1 + a_2\left(y - \frac{a_1}{a_2}\right)\right) = a_3 \Rightarrow \\ &\Rightarrow ye^{\left(y-\frac{a_1}{a_2}\right)^2} = \frac{a_3}{a_2} \Rightarrow \\ &\Rightarrow ye^{\left(y^2-2y\frac{a_1}{a_2}+\frac{a_1^2}{a_2^2}\right)} = \frac{a_3}{a_2}\ ; \end{aligned}$$

Then we substitute $z = -2y\frac{a_1}{a_2}$, to get a familiar equation (6):

$$\begin{aligned} &\frac{a_2}{2a_1}ze^{\left(\frac{a_2^2}{4a_1^2}z^2+z+\frac{a_1^2}{a_2^2}\right)} = \frac{a_3}{a_2} \Rightarrow \\ &\Rightarrow ze^{\left(\frac{a_2^2}{4a_1^2}z^2+z+\frac{a_1^2}{a_2^2}\right)} = -\frac{2a_1a_3}{a_2^2} \Rightarrow \\ &\Rightarrow ze^{\left(\frac{a_2^2}{4a_1^2}z^2+z\right)} = -\frac{2a_1a_3}{a_2^2}e^{-\frac{a_1^2}{a_2^2}}\ ; \end{aligned}$$

By substituting $a = \frac{a_2^2}{4a_1^2}$ and $b = -\frac{2a_1a_3}{a_2^2}e^{-\frac{a_1^2}{a_2^2}}$, we get $ze^{az^2+z} = b$. Since the solution to this equation is the quadratic Lambert function: $W(a;b)$, we are deriving one of the solutions for (7).

$$x = \left(-\frac{a_2}{2a_1}W\left(\frac{a_2^2}{4a_1^2}, -\frac{2a_1a_3}{a_2^2}e^{-\frac{a_1^2}{a_2^2}}\right) - \frac{a_1}{a_2}\right)^2\ ; \tag{8}$$

3.2.2 Taylor Series approximation for quadratic Lambert function

In order to find the value of quadratic Lambert function at some point, we will use Taylor Series approximation for quadratic Lambert function $W(a,b)$ around $b = 0$, that was derived by Mező et al., 2020, which is:

$$W(a;b) = \frac{1}{a}\sum_{n=1}^{\infty}\left(\frac{(ab)^n}{n}\sum_{m=\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1}\frac{\left(-\frac{n}{a}\right)^m}{(n-1-m)!\,(2m-n+1)!}\right)\ ; \tag{9}$$

3.2.3 Radius of convergence of Taylor series

However, we also would like to know the radius of convergence for the Taylor series approximation for the quadratic Lambert function. Obviously, for $\frac{(ab)^n}{n}$, the biggest term will be with the highest power of n, thus:

$$lim_{n\to\infty}\frac{(ab)^n}{n}\sum_{m=\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1}\frac{\left(-\frac{n}{a}\right)^m}{(n-1-m)!\,(2m-n+1)!}\;;$$

We need to find such b, after which this limit goes to infinity. Let us analyze our sum series. Since $n\to\infty$, we can use the Stirling approximation in our factorials.

$$lim_{n\to\infty}\frac{\left(\frac{n}{a}\right)^x}{2\pi\sqrt{(n-1-x)\,(2x-n+1)}\left(\frac{(n-1-x)}{e}\right)^{(n-1-x)}\left(\frac{(2x-n+1)}{e}\right)^{(2x-n+1)}}\;;$$

As $n\to\infty$, only one term from this series will matter, since all other terms relatively will be insignificant. Thus, we will find derivative of this function, and find when it's equal 0. Let our function be $f(x)$. Then:

$$lim_{n\to\infty}\frac{d}{dx}f(x)=0\Rightarrow lim_{n\to\infty}\frac{Q}{4\pi\sqrt{(n-x-1)\,(-n+2x+1)}}=0\,; \qquad (10)$$

where

$$\begin{aligned}Q=\;&e^x\left(\frac{n}{a}\right)^x(n-x-1)^{x-n}\,(-n+2x+1)^{n-2(x+1)}\\&\cdot\Bigg(\left(n^2-n\,(3x+2)+2x^2+3x+1\right)\\&\cdot\left(-2\ln\!\left(\frac{n}{a}\right)-2\ln(n-x-1)+4\ln(-n+2x+1)\right)-3n+4x+3\Bigg)\;;\end{aligned}$$

After some simplifications, we get:

$$\begin{aligned}&lim_{n\to\infty}\;2\left(2x^2+(3-3n)\,x+(n-1)^2\right)\ln\left(\frac{(-n+2x+1)^2}{n\,(n-x-1)}\right)+\\&+2\ln(a)\left(2x^2+(3-3n)\,x+(n-1)^2\right)-3n+4x+3=0\,;\end{aligned}$$

Since $n \to \infty$, $\ln\left(\frac{(-n+2x+1)^2}{n(n-x-1)}\right) = \ln(1) = 0$.
The same applies to $\ln(a)\left(2x^2 + (3-3n)x + (n-1)^2\right)$. It will be insignificant to $-3n+4x+3$, due to $\ln(a)$, thus the largest term of our sum will be when $m = \frac{3}{4}n$. Now we need to find such x when function below diverge:

$$lim_{n\to\infty} \ x^{n-1}\frac{\left(-\frac{n}{a}\right)^m}{(n-1-m)!\,(2m-n+1)!} \ ;$$

After some substitution and simplification, we get:

$$lim_{n\to\infty} \ \frac{1}{nx\pi\sqrt{2}} \cdot \left(\frac{2xe^{\frac{3}{4}}}{a^{\frac{3}{4}}}\right)^n \ ; \tag{11}$$

From here we can find out that this function will diverge if $x > \frac{a^{\frac{3}{4}}}{2e^{\frac{3}{4}}}$.

Thus, the Taylor series approximation for the quadratic Lambert function around $b = 0$ will diverge when $b > \frac{a^{\frac{3}{4}}}{2e^{\frac{3}{4}}}$. However, it's worth noting, that since we assume $n \to \infty$, Taylor series will only diverge for very large n. And for relatively small values, it will still converge.

3.2.4 Newton's Method in quadratic Lambert function

Depending on our parameters, the quadratic Lambert function can have up to 3 real solutions, each of them found in different branches. However, with the Taylor series we could find only one solution. In order to find others, we will use Newton's Method to numerically approximate these solutions:

$$x_{n+1} = x_n - \frac{e^{x_n}\left(a_1 + a_2\sqrt{x_n}\right) - a_3}{e^{x_n}\left(a_1 + a_2\sqrt{x_n} + \frac{1}{2}\frac{a_2}{\sqrt{x_n}}\right)} \ ; \tag{12}$$

3.2.5 The branch structure of $e^x(a_1 + a_2\sqrt{x}) = a_3$

Since we want to properly use Newton's method to find other real solutions, we will look at the branch structure of the quadratic Lambert function analyzed by Mező et al., 2020. Assume $x = b$. After some transformations and substitutions, let $\alpha = \frac{\left(\sqrt{a_1^2 - 2a_2^2} - a_1\right)^2}{4a_2^2}$ and $\beta = \frac{\left(\sqrt{a_1^2 - 2a_2^2} + a_1\right)^2}{4a_2^2}$. Since $a = \frac{a_2^2}{4a_1^2}$, it means $a > 0, \forall a$. Then:

1. If $0 < a < \frac{1}{8}$ then we have only three real branches:

   $W_0(a,x) : (-\infty, \beta]$ is a strictly increasing function.

   $W_1(a,x) : [\beta, \alpha]$ is a strictly decreasing function.

$W_2(a,x) : [\alpha, +\infty)$ is a strictly increasing function.

2. If $a = \frac{1}{8}$ then we have only two real branches:

   $W_0(a,x) : (-\infty, -2]$ is also a strictly increasing function.

   $W_1(a,x) : [-2, +\infty)$ is a strictly increasing function.

3. If $a > \frac{1}{8}$ then we have only one real branch:

   $W_0(a,x) : (-\infty, +\infty)$ is a strictly increasing function.

3.2.6 Number of solutions of the equation: $e^x(a_1 + a_2\sqrt{x}) = a_3$

Depending on the parameters, the number of real branches varies from 0 to 3. These graphs show examples of equations that have all possible numbers of solutions:

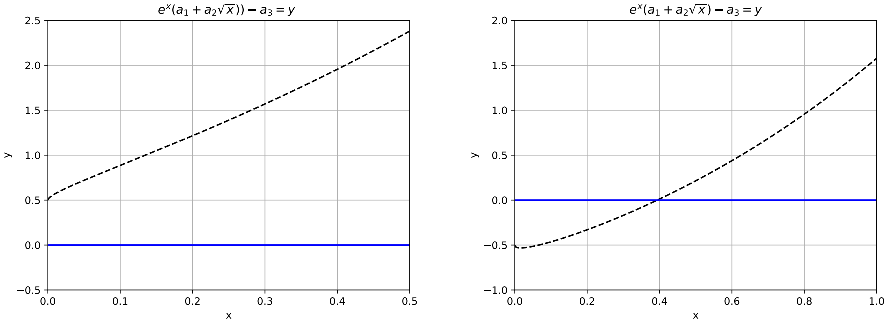

Figure 3: Equation with parameters, which have 0 and 1 roots

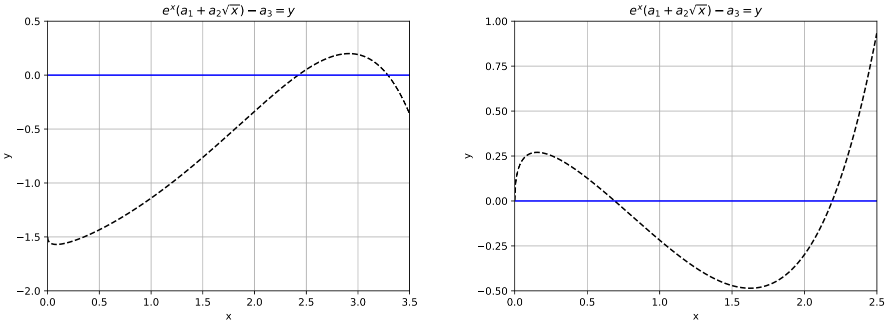

Figure 4: Equation with parameters, which have 2 and 3 roots

3.3 Transcendental Equation: $e^x\left(\frac{a_1+a_2\sqrt{x}}{a_3+a_4\sqrt{x}}\right) = a_5$

Furthermore, we have the third function that we wanted to study. It is a combination of the previous two ones: $e^x\frac{a_1+a_2x}{a_3+a_4x} = a_5$ and $e^x(a_1 + a_2\sqrt{x}) = a_3$.

$$e^x\left(\frac{a_1 + a_2\sqrt{x}}{a_3 + a_4\sqrt{x}}\right) = a_5\ ; \tag{13}$$

So far we didn't find any methods to find analytical solution to at least one of the roots of this equation. Thus, in order to find it's solutions, we must approach numerical approximations. However, we can identify the number of real branches that it has and the points at which they change.

3.3.1 Newton's Method

Depending on our parameters, the third function can have up to 3 real solutions, each of them found in different branches. In order to find them, we will use Newton's Method to numerically approximate these solutions:

$$x_{n+1} = x_n - \frac{e^{x_n}\left(\frac{a_1+a_2\sqrt{x_n}}{a_3+a_4\sqrt{x_n}}\right) - a_5}{e^{x_n}\left(\frac{a_1+a_2\sqrt{x_n}}{a_3+a_4\sqrt{x_n}} + \frac{1}{2\sqrt{x_n}}\left(\frac{a_2a_3-a_1a_4}{(a_3+a_4\sqrt{x_n})^2}\right)\right)}\ ; \tag{14}$$

3.3.2 The branch structure of $e^x\left(\frac{a_1+a_2\sqrt{x}}{a_3+a_4\sqrt{x}}\right)=a_5$

We first will find derivative for our equation, since it's roots are the points where branches changes.

$$\frac{d}{dx}\left(e^x\frac{a_1+a_2\sqrt{x}}{a_3+a_4\sqrt{x}}-a_5\right)\Rightarrow$$

$$\Rightarrow e^x\frac{a_1+a_2\sqrt{x}}{a_3+a_4\sqrt{x}}+e^x\frac{\frac{a_2}{2\sqrt{x}}(a_3+a_4\sqrt{x})-\frac{a_4}{2\sqrt{x}}(a_1+a_2\sqrt{x})}{(a_3+a_4\sqrt{x})^2}=0\,;\Rightarrow$$

$$\Rightarrow \frac{a_1+a_2\sqrt{x}}{a_3+a_4\sqrt{x}}+\frac{\frac{a_2}{2\sqrt{x}}(a_3+a_4\sqrt{x})-\frac{a_4}{2\sqrt{x}}(a_1+a_2\sqrt{x})}{(a_3+a_4\sqrt{x})^2}=0\,;\Rightarrow$$

$$\Rightarrow (a_1+a_2\sqrt{x})(a_3+a_4\sqrt{x})+\frac{a_2}{2\sqrt{x}}(a_3+a_4\sqrt{x})-\frac{a_4}{2\sqrt{x}}(a_1+a_2\sqrt{x})=0;\Rightarrow$$

$$\Rightarrow 2\sqrt{x}(a_1+a_2\sqrt{x})(a_3+a_4\sqrt{x})+a_2(a_3+a_4\sqrt{x})-a_4(a_1+a_2\sqrt{x})=0\,;\Rightarrow$$

$$\Rightarrow 2a_1a_3\sqrt{x}+2a_2a_3x+2a_1a_4x+2a_2a_4x\sqrt{x}+a_2a_3-a_1a_4=0\,;\Rightarrow$$

$$\Rightarrow (a_2a_3-a_1a_4)+2a_1a_3\sqrt{x}+(2a_2a_3+2a_1a_4)x+2a_2a_4x\sqrt{x}=0\,;$$

In the end, we get a cubic equation with $\sqrt{x}$ as a variable. That means that as we change a_1, a_2, a_3, a_4, a_5, we get from 2 to 4 real branches. And the roots of this cubic equation will be points, where we change real branches.

3.3.3 Solutions of the equation $e^x\left(\frac{a_1+a_2\sqrt{x}}{a_3+a_4\sqrt{x}}\right)=a_5$

The number of real branches varies from 2 to 4 depending on the parameters. These graphs show examples of equations that have all possible numbers of solutions:

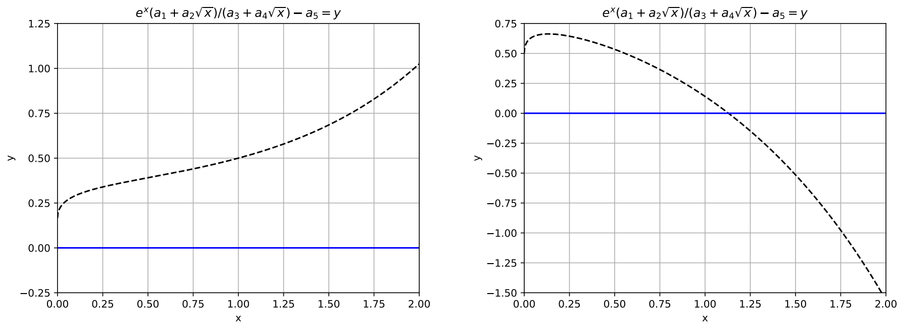

Figure 5: Equation with parameters, which have 0 and 1 roots

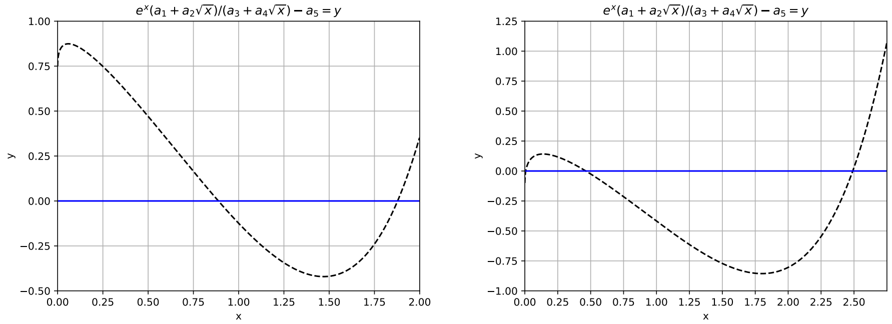

Figure 6: Equation with parameters, which have 2 and 3 roots

4 Application of Lambert functions

4.1 Lambert W function in Chemical Engineering

In the topic of Dead-core phenomena, there is a transcendental equation derived from the convection-diffusion reaction equation, which can be solved using the Lambert W function Amirali, 2022. The aim is to find the mass Peclet number Pe_m from the equation : $\frac{1-e^{\frac{-Pe_m}{2}}}{Pe_m} = \frac{1}{\varphi^2}$.
Let $x = Pe_m$ and $y = \varphi$. Then, solve the equation: $\frac{x}{1-e^{\frac{-x}{2}}} = y^2$

$$\frac{x}{1-e^{\frac{-x}{2}}} = y^2 \Rightarrow \frac{x - xe^{\frac{-x}{2}}}{1-e^{\frac{-x}{2}}} + \frac{xe^{\frac{-x}{2}}}{1-e^{\frac{-x}{2}}} = y^2 \Rightarrow x + \frac{xe^{\frac{-x}{2}}}{1-e^{\frac{-x}{2}}} = y^2 \ ;$$

Multiply both sides by $-\frac{1}{2}$ and take the exponent:

$$e^{-\frac{1}{2}x} * e^{\frac{-\frac{1}{2}xe^{\frac{-x}{2}}}{1-e^{\frac{-x}{2}}}} = e^{-\frac{1}{2}y^2} \Rightarrow \frac{-\frac{1}{2}xe^{\frac{-x}{2}}}{1-e^{\frac{-x}{2}}} * e^{\frac{-\frac{1}{2}xe^{\frac{-x}{2}}}{1-e^{\frac{-x}{2}}}} = -\frac{1}{2}y^2 e^{-\frac{1}{2}y^2} \ ; \tag{15}$$

Using Lambert W function for Eq. (15) we get:

$$\frac{-\frac{1}{2}xe^{\frac{-x}{2}}}{1-e^{\frac{-x}{2}}} = W\left(-\frac{1}{2}y^2e^{-\frac{1}{2}y^2}\right) \Rightarrow \frac{-xe^{\frac{-x}{2}}}{1-e^{\frac{-x}{2}}} = 2W\left(-\frac{1}{2}y^2e^{-\frac{1}{2}y^2}\right) \Rightarrow \frac{-xe^{\frac{-x}{2}}}{1-e^{\frac{-x}{2}}} +$$
$$+ \frac{x}{1-e^{\frac{-x}{2}}} = 2W\left(-\frac{1}{2}y^2e^{-\frac{1}{2}y^2}\right) + y^2 \Rightarrow \frac{x - xe^{\frac{-x}{2}}}{1-e^{\frac{-x}{2}}} = 2W\left(-\frac{1}{2}y^2e^{-\frac{1}{2}y^2}\right) + y^2 \Rightarrow$$
$$\Rightarrow x = 2W\left(-\frac{1}{2}y^2e^{-\frac{1}{2}y^2}\right) + y^2 \Rightarrow P_{em} = 2W\left(-\frac{1}{2}\varphi^2e^{-\frac{1}{2}\varphi^2}\right) + \varphi^2 \ ; \tag{16}$$

Again, we use the principal branch W_0 in the solution, because if we use another branch W_{-1}, then $Pe_m = 2W(-\frac{1}{2}\varphi^2 e^{-\frac{1}{2}\varphi^2})+\varphi^2 = 2(-\frac{1}{2}\varphi^2)+\varphi^2 = 0$. However, Pe_m should be positive.

4.2 Quadratic Lambert function in Magnetic Micro-Electro-Mechanical-Structures

The application of the quadratic Lambert function in magMEMS is found in our article (Abdildayev et al., 2025). There is also another transcendental equation used in the Magnetic Micro-Electro-Mechanical-Structures model for obtaining dynamic pull-in which has a solution of Lambert W function (He et al., 2021). However, we are more interested in applications of generalized Lambert functions.

The operation of Magnetic Micro-Electro-Mechanical-Structures using filament wires of finite length can be characterized by:

$$\ddot{x} + x - K\left[\frac{\xi^2(1-x)}{\sqrt{\xi^2(1-x)^2+1}} + \frac{1}{(1-x)\sqrt{\xi^2(1-x)^2+1}} - \xi\right] = 0\,; \tag{17}$$

For the case of $\xi = 0$, the existence of periodic solutions is ensured if equation $-s^2 - 2Kln|1-s| = 0$ has a root in $(0,1)$. Solve this equation:

$$-s^2 - 2Kln|1-s| = 0 \Rightarrow s^2 = -2Kln(1-s)\,;$$

Let $1-s = e^{-x}, x > 0$. Then, $s^2 = (1-e^{-x})^2$. Substitute this to the equation:

$$\left(1-e^{-x}\right)^2 = 2Kx \Rightarrow e^{-x} = 1-\sqrt{2Kx} \Rightarrow e^{x}\left(1-\sqrt{2Kx}\right) = 1\,; \tag{18}$$

Unfortunately, it is impossible to get this equation to the form: $xe^x = a$. However, the reduced equation is the equation of the form $e^x(a_1 + a_2\sqrt{x}) = a_3$ with $a_1 = 1$, $a_2 = -\sqrt{2K}$ and $a_3 = 1$ which can be solved using Quadratic Lambert function.

5 Conclusion

As a result, three transcendental equations were considered and analyzed: $e^x\left(\frac{a_1x+a_2}{a_3x+a_4}\right) = a_5$, $e^x(a_1 + a_2\sqrt{x}) = a_3$, and $e^x\left(\frac{a_1+a_2\sqrt{x}}{a_3+a_4\sqrt{x}}\right) = a_5$.

The Lambert W function is well understood, so there was no problem with a full analysis of this function. Branches and approximations via the Taylor series were derived, but our contribution was to transform the equations into the form $xe^x = y$. In addition, we have determined in which branch the solution will be located.

For the equation $e^x \left(\frac{a_1x+a_2}{a_3x+a_4}\right) = a_5$, a connection with the equation whose solution is the r-Lambert function was shown, and one of its solutions and its Taylor approximation were found. A formula for the Newton method was derived for the approximation of roots. In addition, the structure of branches and graphs with all possible numbers of solutions was shown.

For the equation $e^x(a_1 + a_2\sqrt{x}) = a_3$, a transformation into an equation having a solution in the form of a quadratic Lambert function was shown. In addition to the similar p-Lambert function analysis, a radius of convergence for the Taylor approximation for the quadratic Lambert function was found.

For the equation $e^x \left(\frac{a_1+a_2\sqrt{x}}{a_3+a_4\sqrt{x}}\right) = a_5$, only graphs with all possible numbers of solutions were shown. Also, an equation was derived, the study of which will further allow us to determine the structure of branches of this generalization.

In the future, we can go deeper into the study of these three equations, since not all solutions have been found, and Taylor approximations have not been found or analyzed in all considered generalizations of the Lambert function. Perhaps these equations will be analyzed in aspects that have not even been touched upon in this paper. In addition, we can study other transcendental equations containing polynomial, rational, and exponential functions (for example, $e^x(a_1 + a_2\sqrt{x})(a_3 + a_4\sqrt{x}) = a_5$). The problem with studying new equations is that they are even more difficult to analyze. Perhaps we will be able to discover new formulas and methods for studying these equations. In addition, we can study these and other transcendental equations that have solutions in the form of generalized Lambert functions on the complex plane.

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