Transition Probability of the PushTASEP with one parameter

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Abstract

We consider a n-particle PushTASEP, a model introduced by Alimohammadi et al. In their work they obtained the exact solution for the master equation of PushTASEP with boundary conditions. In this paper we use the technique outlined by Tracy and Widom to extend their result and find a solution which satisfies the initial condition of the model.

1 Introduction

The n-particle totally asymmetric simple exclusion process with drop-push dynamics (PushTASEP) is an interacting particle system in which n particles move to the right along an integer line. If the place to the right of the particle x is empty, it jumps to the right with rate 1. If two particles are adjacent, particle x jumps to the right and pushes the neighbouring particle to the right with rate r_1. For k adjacent particles, particle x pushes all neighbouring particles to the right with rate r_k. Any position can be occupied by at most one particle at time t [1].

If there is at least one empty position between the particles, the master equation of the model is

$$\begin{aligned}\frac{d}{dt}P(x_1,x_2,...,x_n,t) = P(x_1-1,x_2,...,x_n,t)+P(x_1,x_2-1,...,x_n,t)+...\\ +P(x_1,x_2,...,x_n-1,t)-nP(x_1,x_2,...,x_n,t),\end{aligned} \tag{1}$$

where the function $P(x_1,x_2,...,x_n,t)$ is the probability of finding n particles at positions $x_1,x_2,...,x_n$ with $x_1 < x_2 < ... < x_n$ at time t. The initial positions $y_1,y_2,...,y_n$ of the particles at time $t=0$ are omitted from notation for simplicity. Alimohammadi et al. [1] have shown the exact solution of the master equation (1) with boundary conditions

$$\begin{aligned}P(x_1,...,x_{i-1},x,x,x_{i+2},...,x_n,t) = r_1P(x_1,...,x_{i-1},x-1,x,x_{i+2},...,x_n,t)\\ +(1-r_1)P(x_1,...,x_{i-1},x,x+1,x_{i+2},...,x_n,t),\end{aligned} \tag{2}$$

where i=1,2,...,n.

The main objective of this paper is to find the transition probability of the PushTASEP by solving the equation (1) with boundary conditions (2) and initial condition

$$P(y_1,y_2,...,y_n,t=0) = \begin{cases}1, \text{if } x_k = y_k \text{ for all } k=1,...,n,\\ 0, \text{otherwise.}\end{cases} \tag{3}$$

Following the results of Alimohammadi et al. [1], in Section 2 we will show that all boundary conditions of the PushTASEP model are implied by boundary conditions (2). In Section 3, we use the method given by Tracy and Widom [2] to obtain the transition probability of the model. We introduce variables $\xi_1, \xi_2, ..., \xi_n, \eta \in \mathbb{C}$ and $\gamma = \frac{1}{\xi_1} + ... + \frac{1}{\xi_n} - n$, and show that the formula for the transition probability is

$$P(x_1, x_2, ..., x_n, t) = \sum_{\sigma \in S_n} \Big(\Big(\frac{1}{2\pi i}\Big)^n e^{\gamma t} \oint ... \oint \Big(\prod_{\alpha,\beta} S_{\alpha,\beta}\Big) \Big(\prod_{k=1}^{n} \xi_{\sigma(k)}^{x_k - y_{\sigma(k)} - 1 + \sigma(k) - k}\Big) d\xi_1 d\xi_2 ... d\xi_n \Big), \tag{4}$$

where $k = 1, 2, ..., n$. The sum in (4) is taken over all permutations σ in S_n, with (α, β) all possible inversions of the permutation σ such that $\alpha > \beta$ and the coefficient $S_{\alpha,\beta} := -\frac{r+(1-r)\xi_\alpha \xi_\beta - \xi_\beta}{r+(1-r)\xi_\alpha \xi_\beta - \xi_\alpha}$. The integration is along simple closed contours around $\xi_1 = 0, \xi_2 = 0, ..., \xi_n = 0$ respectively, with radii $R < \frac{\eta}{r+(1-r)\eta}$ and small enough to avoid other singularities in (4). In the Appendix, we give the solution for the three-particle case.

2 Redundancy of the boundary conditions

2.1 Boundary conditions of a three-particle model

The master equation for a three-particle PushTASEP, if all of the particles are separated by at least one empty space, is

$$\frac{d}{dt} P(x_1, x_2, x_3, t) = P(x_1 - 1, x_2, x_3, t) + P(x_1, x_2 - 1, x_3, t) + P(x_1, x_2, x_3 - 1, t) - 3P(x_1, x_2, x_3, t). \tag{5}$$

Consider the particle configuration $x_1 = x, x_2 = x+1, x_3$ where two out of three particles are adjacent. Master equation for this configuration is

$$\begin{aligned} \frac{d}{dt} P(x, x+1, x_3, t) = P(x-1, x+1, x_3, t) + r_1 P(x-1, x, x_3, t) + P(x, x+1, x_3 - 1, t) \\ -(2 + r_1) P(x, x+1, x_3, t), \end{aligned} \tag{6}$$

where r_1 is the rate with which a particle pushes one particle to the right when jumping. If we set $x_1 = x, x_2 = x+1$ in (5) and equate to (6), we get the following boundary condition:

$$P(x, x, x_3, t) = r_1 P(x-1, x, x_3, t) + (1 - r_1) P(x, x+1, x_3, t). \tag{7}$$

Similarly, for $x_1, x_2 = x, x_3 = x+1$ we have the boundary condition

$$P(x_1, x, x, t) = r_1 P(x_1, x-1, x, t) + (1 - r_1) P(x_1, x, x+1, t). \tag{8}$$

For the configuration $x_1 = x, x_2 = x+1, x_3 = x+2$ where all three particles are adjacent, we have the following master equation:

$$\begin{aligned} \frac{d}{dt} P(x, x+1, x+2, t) = P(x-1, x+1, x+2, t) + r_1 P(x-1, x, x+2, t) \\ + r_2 P(x-1, x, x+1, t) - (1 + r_1 + r_2) P(x, x+1, x+2, t), \end{aligned} \tag{9}$$

where r_2 is the rate with which a particle pushes two particles to the right when jumping. Set $x_1 = x, x_2 = x+1, x_3 = x+2$ in (5) and equate it to (9) to get the boundary condition

$$P(x, x+1, x+1, t) = r_2 P(x-1, x, x+1, t) + (1 - r_2) P(x, x+1, x+2, t). \tag{10}$$

By (8), $P(x,x+1,x+1,t)=r_1P(x,x,x+1,t)+(1-r_1)P(x,x+1,x+2,t)$. Express $P(x,x,x+1,t)$ from (7) and substitute:

$$P(x,x+1,x+1,t)=\frac{r_1^2}{1-r_1(1-r_1)}P(x-1,x,x+1,t)+\frac{1-r_1}{1-r_1(1-r_1)}P(x,x+1,x+2,t). \tag{11}$$

So if

$$r_2=\frac{r_1^2}{1-r_1(1-r_1)}=\left(1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu^2}\right)^{-1},$$

where $\mu=r_1,\lambda=1-r_1$, boundary condition (10) is implied by boundary conditions (7), (8).

2.2 Proof of redundancy of the boundary conditions in a n-particle system

Similar to the three-particle case, if we consider a particle configuration with $n+2$ adjacent particles $x_1=x,x_2=x+1,...,x_{n+2}=x+n+1$, $n\geq 1$, we get boundary conditions different from those given by (2). Following the results of Alimohammadi et al. [1], we show that all these boundary conditions of a n-particle system are implied by boundary conditions (2).

Lemma 1. *If $r_n=\left[1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu^2}+...+\frac{\lambda^n}{\mu^n}\right]^{-1}$, where $\mu=r_1,\lambda=1-r_1$, and $n\geq 1$, then boundary condition*

$$\begin{aligned}P(x,x+1,...,x+n-1,x+n,x+n,t)&=r_{n+1}P(x-1,x,...,x+n-2,x+n-1,x+n,t)\\&+(1-r_{n+1})P(x,x+1,...,x+n-1,x+n,x+n+1,t),\end{aligned} \tag{12}$$

which arises from particle configuration with $n+2$ adjacent particles $x_1=x,x_2=x+1,...,x_{n+2}=x+n+1$ is redundant.

Proof. We proceed by induction [1]. We have shown in Section 2.1, that in a three-particle system the boundary condition arising from particle placement with all three particles being adjacent is redundant.

Similarly, in a $n+2$ particle system, if $r_2=\left(1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu^2}\right)^{-1}$, boundary conditions

$$\begin{aligned}P(x_1,...,x_{i-1},x,x+1,x+1,x_{i+3},...,x_{n+1},x_{n+2},t)&=r_2P(x_1,...,x_{i-1},x-1,x,x+1,x_{i+3},...,x_{n+1},x_{n+2},t)\\&+(1-r_2)P(x_1,...,x_{i-1},x,x+1,x+2,x_{i+3},...,x_{n+1},x_{n+2},t),\end{aligned} \tag{13}$$

are implied by boundary conditions

$$\begin{aligned}P(x_1,...,x_{i-1},x,x,x_{i+2},...,x_{n+1},x_{n+2},t)&=r_1P(x_1,...,x_{i-1},x-1,x,x_{i+2},...,x_{n+1},x_{n+2},t)\\&+(1-r_1)P(x_1,...,x_{i-1},x,x+1,x_{i+2},...,x_{n+1},x_{n+2},t),\end{aligned} \tag{14}$$

where i=1,2,...,n, n+1.

Suppose that if $r_k=\left[1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu^2}+...+\frac{\lambda^k}{\mu^k}\right]^{-1}$, then boundary condition from $k+1$ adjacent particles configuration is implied by (14) for any $k\in 2,3,...,n$. In particular, if $r_n=\left[1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu^2}+...+\frac{\lambda^n}{\mu^n}\right]^{-1}$, then boundary conditions from $n+1$ adjacent particles configurations

$$\begin{aligned}P(x,x+1,...,x+n-1,x+n-1,x_{n+2},t)&=r_nP(x-1,x,...,x+n-2,x+n-1,x_{n+2},t)\\&+(1-r_n)P(x,x+1,...,x+n-1,x+n,x_{n+2},t)\end{aligned} \tag{15}$$

and

$$\begin{aligned}P(x_1,x,x+1,...,x+n-2,x+n-1,x+n-1,t)=r_nP(x_1,x-1,x,...,x+n-3,\\x+n-2,x+n-1,t)+(1-r_n)P(x_1,x,x+1,...,x+n-2,x+n-1,x+n,t)\end{aligned} \tag{16}$$

are implied by (14).

Consider one of the boundary conditions in (14), with i = n+1:

$$\begin{aligned} P(x_1,x_2,...,x_n,x+n,x+n,t) &= r_1 P(x_1,x_2,...,x_n,x+n-1,x+n,t) \\ &+(1-r_1)P(x_1,x_2,...,x_n,x+n,x+n+1,t), \end{aligned} \tag{17}$$

and set $x_1 = x, x_2 = x+1, ..., x_n = x+n-1$.

In (15) set $x_{n+2} = x+n$, express $P(x,x+1,...,x+n-1,x+n-1,x+n,t)$ and substitute to (17):

$$\begin{aligned} P(x,x+1,...,x+n-1,x+n,x+n,t) &= \frac{r_1 r_n}{1-r_1(1-r_n)} P(x-1,x,...,x+n-2, \\ x+n-1,x+n,t) &+ \frac{(1-r_1)}{1-r_1(1-r_n)} P(x,x+1,...,x+n-1,x+n,x+n+1,t). \end{aligned} \tag{18}$$

If $r_{n+1} = \frac{r_1 r_n}{1-r_1(1-r_n)} = \left[1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu^2}+...+\frac{\lambda^{n+1}}{\mu^{n+1}}\right]^{-1}$, (18) is the same as (12), that is (12) is implied by (15) and (17). Given that (15) follows from (14), we conclude that all boundary conditions of the model are implied by boundary conditions (14).

□

3 Transition probability of the one-parameter PushTASEP

As we have shown in Section 2, the n-particle PushTASEP has the following master equation and $n-1$ boundary conditions:

$$\begin{aligned} \frac{d}{dt}P(x_1,x_2,...,x_n,t) &= P(x_1-1,x_2,...,x_n,t) + P(x_1,x_2-1,...,x_n,t) + ... \\ &+P(x_1,x_2,...,x_n-1,t) - nP(x_1,x_2,...,x_n,t), \end{aligned} \tag{1}$$

$$\begin{aligned} P(x_1,...,x_{i-1},x,x,x_{i+2},...,x_n,t) &= rP(x_1,...,x_{i-1},x-1,x,x_{i+2},...,x_n,t) \\ &+(1-r)P(x_1,...,x_{i-1},x,x+1,x_{i+2},...,x_n,t), \end{aligned} \tag{2}$$

where i=1,2,...,n, and notation for the rate r_1 from Section 2 was replaced by r.

3.1 Separation of variables

We use separation of variables to solve the master equation (1) [2]. Suppose the transition probability can be written as $P(x_1,x_2,...,x_n,t) = X(x_1,x_2,...,x_n)T(t)$, where $X(x_1,x_2,...,x_n) \neq 0$ and $T(t) \neq 0$. Substitute it into equation (1):

$$\frac{T'(t)}{T(t)} = \frac{X(x_1-1,x_2,...,x_n)+X(x_1,x_2-1,...,x_n)+...+X(x_1,x_2,...,x_n-1)-nX(x_1,x_2,...,x_n)}{X(x_1,x_2,...,x_n)} = \gamma.$$

The spatial part can be solved by substituting $X(x_1,x_2,...,x_n) = \xi_1^{x_1}\xi_2^{x_2}...\xi_n^{x_n}$ [2]. Any other expression of this form with indexes from 1 to n permuted, is also a solution, so solution to the spatial part is a linear combination of solutions:

$$X(x_1,x_2,...,x_n) = \sum_{\sigma\in S_n} A_\sigma \prod_{k=1}^{n} \xi_{\sigma(k)}^{x_k},$$

where A_σ is the weight of each term and $\sigma(k)$ is the k^{th} element of the permutation σ.

Solution to the time part is

$$T(t) = A(\xi_1, \xi_2, ..., \xi_n)e^{\gamma t},$$

where $\gamma = \frac{1}{\xi_1} + ... + \frac{1}{\xi_n} - n$.

Then, solution to the master equation (1) is

$$P(x_1, x_2, ..., x_n, t) = A(\xi_1, \xi_2, ..., \xi_n)e^{\gamma t} \sum_{\sigma \in S_n} A_\sigma \prod_{k=1}^{n} \xi_{\sigma(k)}^{x_k}. \tag{19}$$

3.2 Solution with the boundary conditions

Substitute the solution (19) into (2), cancel the time part and rearrange:

$$\sum_{\sigma \in S_n} A_\sigma \xi_{\sigma(1)}^{x_1} \cdots \xi_{\sigma(i)}^{x} \xi_{\sigma(i+1)}^{x} \cdots \xi_{\sigma(n)}^{x_n} - \sum_{\sigma \in S_n} (1-r) A_\sigma \xi_{\sigma(1)}^{x_1} \cdots \xi_{\sigma(i)}^{x} \xi_{\sigma(i+1)}^{x+1} \cdots \xi_{\sigma(n)}^{x_n}$$
$$- \sum_{\sigma \in S_n} r A_\sigma \xi_{\sigma(1)}^{x_1} \cdots \xi_{\sigma(i)}^{x-1} \xi_{\sigma(i+1)}^{x} \cdots \xi_{\sigma(n)}^{x_n} = 0.$$

$$\sum_{\sigma \in S_n} A_\sigma \prod_{\substack{k=1,\\ k \neq i, i+1}}^{n} \xi_{\sigma(k)}^{x_k} [\xi_{\sigma(i)}^{x} \xi_{\sigma(i+1)}^{x} - (1-r) \xi_{\sigma(i)}^{x} \xi_{\sigma(i+1)}^{x+1} - r \xi_{\sigma(i)}^{x-1} \xi_{\sigma(i+1)}^{x}] = 0.$$

Permutations in S_n can be grouped into two: even and odd. Let σ be an even permutation, $\sigma \in A_n$, and $T_i(\sigma)$ be an odd permutation obtained by transposition of the i^{th} and $(i+1)^{th}$ element in σ:

$$\sum_{\sigma \in A_n} \Big(A_\sigma \prod_{\substack{k=1,\\ k \neq i, i+1}}^{n} \xi_{\sigma(k)}^{x_k} [\xi_{\sigma(i)}^{x} \xi_{\sigma(i+1)}^{x} - (1-r) \xi_{\sigma(i)}^{x} \xi_{\sigma(i+1)}^{x+1} - r \xi_{\sigma(i)}^{x-1} \xi_{\sigma(i+1)}^{x}]$$
$$+ A_{T_i(\sigma)} \prod_{\substack{k=1,\\ k \neq i, i+1}}^{n} \xi_{T_i(\sigma)(k)}^{x_k} [\xi_{T_i(\sigma)(i)}^{x} \xi_{T_i(\sigma)(i+1)}^{x} - (1-r) \xi_{T_i(\sigma)(i)}^{x} \xi_{T_i(\sigma)(i+1)}^{x+1}$$
$$- r \xi_{T_i(\sigma)(i)}^{x-1} \xi_{T_i(\sigma)(i+1)}^{x}] \Big) = 0.$$

Since $T_i(\sigma)$ only interchanges positions of the i^{th} and $(i+1)^{th}$ element in σ, $T_i(\sigma)(i) = \sigma(i+1)$ and $T_i(\sigma)(i+1) = \sigma(i)$. For all other elements where $k \neq i, i+1$, $T_i(\sigma)(k) = \sigma(k)$:

$$\sum_{\sigma \in A_n} \prod_{\substack{k=1,\\ k \neq i, i+1}}^{n} \xi_{\sigma(k)}^{x_k} \Big(A_\sigma [\xi_{\sigma(i)}^{x} \xi_{\sigma(i+1)}^{x} - (1-r) \xi_{\sigma(i)}^{x} \xi_{\sigma(i+1)}^{x+1} - r \xi_{\sigma(i)}^{x-1} \xi_{\sigma(i+1)}^{x}]$$
$$+ A_{T_i(\sigma)} [\xi_{\sigma(i+1)}^{x} \xi_{\sigma(i)}^{x} - (1-r) \xi_{\sigma(i+1)}^{x} \xi_{\sigma(i)}^{x+1} - r \xi_{\sigma(i+1)}^{x-1} \xi_{\sigma(i)}^{x}] \Big) = 0. \tag{20}$$

Set each term of the sum in (20) to be equal to 0, then if

$$A_{T_i(\sigma)} = -\frac{r + (1-r)\xi_{\sigma(i)}\xi_{\sigma(i+1)} - \xi_{\sigma(i)}}{r + (1-r)\xi_{\sigma(i)}\xi_{\sigma(i+1)} - \xi_{\sigma(i+1)}} \frac{\xi_{\sigma(i+1)}}{\xi_{\sigma(i)}} A_\sigma, \text{ for all } \sigma \in A_n, \tag{21}$$

the boundary conditions (2) are satisfied [2].

3.3 Lemma for weights A_σ

Any permutation can be written as a product of transpositions. Then it follows from (21) that for any permutation σ in S_n, A_σ can be expressed in terms of the weight for identity permutation, $A_{123\ldots n} = A_I$:

$$A_\sigma = \prod_{\alpha,\beta}\Big(-\frac{r+(1-r)\xi_\alpha\xi_\beta-\xi_\beta}{r+(1-r)\xi_\alpha\xi_\beta-\xi_\alpha}\Big)\frac{\xi_\alpha}{\xi_\beta}A_I = \prod_{\alpha,\beta}S_{\alpha,\beta}\frac{\xi_\alpha}{\xi_\beta}A_I, \tag{22}$$

where (α,β) are all possible inversions of permutation σ, such that $\alpha>\beta$. We define a coefficient $S_{\alpha,\beta} := -\frac{r+(1-r)\xi_\alpha\xi_\beta-\xi_\beta}{r+(1-r)\xi_\alpha\xi_\beta-\xi_\alpha}$. Since there are no inversions in the identity permutation, for $\sigma = I$, set $\prod_{\alpha,\beta}S_{\alpha,\beta} := 1$.

The product of ratios of ξ's in (22) can be simplified as shown in the following lemma.

Lemma 2. *For given $\sigma \in S_n$,*

$$\prod_{\alpha,\beta}\frac{\xi_\alpha}{\xi_\beta} = \prod_{k=1}^{n}\xi_{\sigma(k)}^{\sigma(k)-k}, \tag{23}$$

where $k \in \{1,2,\ldots,n\}$.

Proof. For $\sigma(k)$, k^{th} element of the given permutation σ, we have $\xi_{\sigma(k)}^{l-m}$, where:

- l is number of inversions of the type (k,j) such that $k<j$ and $\sigma(j)<\sigma(k)$, and
- m is number of inversions of the type (j,k) such that $j<k$ and $\sigma(j)>\sigma(k)$.

These m inversions occur in the first $(k-1)$ positions. Then in the first $(k-1)$ positions there are m numbers greater than $\sigma(k)$ and $k-1-m$ numbers less than $\sigma(k)$. In general, if we consider n numbers from 1 to n and $1 \le \sigma(k) \le n$, there are $n-\sigma(k)$ numbers greater than $\sigma(k)$ and $\sigma(k)-1$ numbers less than $\sigma(k)$.

So in the remaining $(n-k)$ positions of the permutation σ:

- $n-\sigma(k)-m$ numbers greater than $\sigma(k)$, and
- $l = \sigma(k)-1-(k-1-m)$ numbers less than $\sigma(k)$.

Hence $\xi_{\sigma(k)}^{l-m} = \xi_{\sigma(k)}^{\sigma(k)-1-(k-1-m)-m} = \xi_{\sigma(k)}^{\sigma(k)-k}$. Then if we consider all possible inversions and elements of σ, we get:

$$\prod_{\alpha,\beta}\frac{\xi_\alpha}{\xi_\beta} = \prod_{k=1}^{n}\xi_{\sigma(k)}^{\sigma(k)-k},$$

which is the same result as in (23). □

3.4 Solution with the initial condition

To solve with the initial condition (3), we will use the fact that for all integer k, $z \in \mathbb{C}$, integral along a simple closed contour around $z=0$ is

$$\Big(\frac{1}{2\pi i}\Big)\oint z^k dz = \begin{cases} 1, \text{if } k=-1, \\ 0, \text{otherwise.} \end{cases} \tag{24}$$

Using Lemma 2, we can express A_σ in terms of A_I and rewrite (19):

$$P(x_1,x_2,\ldots,x_n,t) = A(\xi_1,\xi_2,\ldots,\xi_n)e^{\gamma t}\sum_{\sigma\in S_n}\Big(\prod_{\alpha,\beta}S_{\alpha,\beta}\Big)\Big(\prod_{k=1}^{n}\xi_{\sigma(k)}^{x_k+\sigma(k)-k}\Big)A_I.$$

Let $A(\xi_1,\xi_2,...,\xi_n)=\prod_{k=1}^{n}\xi_{\sigma(k)}^{-y_{\sigma(k)}-1}$, $A_I=\left(\frac{1}{2\pi i}\right)^n$, and take n contour integrals with respect to $\xi_1,\xi_2,...,\xi_n$ along simple closed contours around $\xi_1=0,\xi_2=0,...,\xi_n=0$ respectively [2]:

$$P(x_1,x_2,...,x_n,t)=\oint...\oint\Big(\sum_{\sigma\in S_n}\Big(\frac{1}{2\pi i}\Big)^n\Big(\prod_{\alpha,\beta}S_{\alpha,\beta}\Big)\Big(\prod_{k=1}^{n}\xi_{\sigma(k)}^{x_k-y_{\sigma(k)}-1+\sigma(k)-k}\Big)e^{\gamma t}\Big)d\xi_1 d\xi_2...d\xi_n=$$
$$\sum_{\sigma\in S_n}\Big(\Big(\frac{1}{2\pi i}\Big)^n e^{\gamma t}\oint...\oint\Big(\prod_{\alpha,\beta}S_{\alpha,\beta}\Big)\Big(\prod_{k=1}^{n}\xi_{\sigma(k)}^{x_k-y_{\sigma(k)}-1+\sigma(k)-k}\Big)d\xi_1 d\xi_2...d\xi_n\Big). \quad (4)$$

At time $t=0$ the time part vanishes:

$$P(x_1,x_2,...,x_n,0)=\sum_{\sigma\in S_n}\Big(\Big(\frac{1}{2\pi i}\Big)^n\oint...\oint\Big(\prod_{\alpha,\beta}S_{\alpha,\beta}\Big)\Big(\prod_{k=1}^{n}\xi_{\sigma(k)}^{x_k-y_{\sigma(k)}-1+\sigma(k)-k}\Big)d\xi_1 d\xi_2...d\xi_n\Big). \quad (25)$$

In (25), consider the term of the sum corresponding to $\sigma=I$. Using (24) and the fact that for the identity permutation $\prod_{\alpha,\beta}S_{\alpha,\beta}:=1$, $\sigma(k)=k$ and $y_{\sigma(k)}=y_k$:

$$\Big(\frac{1}{2\pi i}\Big)^n\oint...\oint\prod_{k=1}^{n}\xi_k^{x_k-y_k-1}d\xi_1 d\xi_2...d\xi_n=\prod_{k=1}^{n}\Big(\frac{1}{2\pi i}\oint\xi_k^{x_k-y_k-1}d\xi_k\Big)=\begin{cases}1, \text{if } x_k=y_k,\\ 0, \text{otherwise.}\end{cases} \quad (26)$$

For all terms of the sum in (25) corresponding to the permutations $\sigma\neq I$, there exists at least one inversion $(\sigma(\alpha^*),\sigma(\beta^*))$, such that $\alpha^*<\beta^*$ and $\sigma(\alpha^*)>\sigma(\beta^*)$:

$$\Big(\frac{1}{2\pi i}\Big)^n\oint...\oint\Big(\prod_{\substack{k=1,\\ k\neq\alpha^*,\beta^*}}^{n}\xi_{\sigma(k)}^{x_k-y_{\sigma(k)}-1+\sigma(k)-k}\Big)\Big[\oint\oint\Big(\prod_{\substack{(\alpha,\beta),\\ (\alpha,\beta)\neq(\sigma(\alpha^*),\sigma(\beta^*))}}S_{\alpha,\beta}\Big)S_{\sigma(\alpha^*),\sigma(\beta^*)}$$
$$\xi_{\sigma(\alpha^*)}^{x_{\alpha^*}-y_{\sigma(\alpha^*)}-1+\sigma(\alpha^*)-\alpha^*}\xi_{\sigma(\beta^*)}^{x_{\beta^*}-y_{\sigma(\beta^*)}-1+\sigma(\beta^*)-\beta^*}d\xi_{\sigma(\alpha^*)}d\xi_{\sigma(\beta^*)}\Big]d\xi_1 d\xi_2...d\xi_n. \quad (27)$$

Consider only the integrals with respect to $\xi_{\sigma(\alpha^*)}$ and $\xi_{\sigma(\beta^*)}$ in (27):

$$\oint\oint\Big(\prod_{\substack{(\alpha,\beta),\\ (\alpha,\beta)\neq(\sigma(\alpha^*),\sigma(\beta^*))}}S_{\alpha,\beta}\Big)S_{\sigma(\alpha^*),\sigma(\beta^*)}\xi_{\sigma(\alpha^*)}^{x_{\alpha^*}-y_{\sigma(\alpha^*)}-1+\sigma(\alpha^*)-\alpha^*}\xi_{\sigma(\beta^*)}^{x_{\beta^*}-y_{\sigma(\beta^*)}-1+\sigma(\beta^*)-\beta^*}d\xi_{\sigma(\alpha^*)}d\xi_{\sigma(\beta^*)}=$$
$$\oint\oint\Big(\prod_{\substack{(\alpha,\beta),\\ (\alpha,\beta)\neq(\sigma(\alpha^*),\sigma(\beta^*))}}S_{\alpha,\beta}\Big)\frac{r+(1-r)\xi_{\sigma(\alpha^*)}\xi_{\sigma(\beta^*)}-\xi_{\sigma(\beta^*)}}{r+(1-r)\xi_{\sigma(\alpha^*)}\xi_{\sigma(\beta^*)}-\xi_{\sigma(\alpha^*)}}\xi_{\sigma(\alpha^*)}^{x_{\alpha^*}-y_{\sigma(\alpha^*)}-1+\sigma(\alpha^*)-\alpha^*}\xi_{\sigma(\beta^*)}^{x_{\beta^*}-y_{\sigma(\beta^*)}-1+\sigma(\beta^*)-\beta^*}d\xi_{\sigma(\alpha^*)}d\xi_{\sigma(\beta^*)}.$$

Let $\xi_{\sigma(\alpha^*)}=\frac{\eta}{\xi_{\sigma(\beta^*)}}$, $d\xi_{\sigma(\alpha^*)}=\frac{d\eta}{\xi_{\sigma(\beta^*)}}$ [2]:

$$\oint\eta^{x_{\alpha^*}-y_{\sigma(\alpha^*)}-1+\sigma(\alpha^*)-\alpha^*}\Big(\oint\Big(\prod_{\substack{(\alpha,\beta),\\ (\alpha,\beta)\neq(\sigma(\alpha^*),\sigma(\beta^*))}}S_{\alpha,\beta}\Big)\frac{r+(1-r)\eta-\xi_{\sigma(\beta^*)}}{r+(1-r)\eta-\frac{\eta}{\xi_{\sigma(\beta^*)}}}$$
$$\xi_{\sigma(\beta^*)}^{x_{\beta^*}-y_{\sigma(\beta^*)}+\sigma(\beta^*)-\beta^*-x_{\alpha^*}+y_{\sigma(\alpha^*)}-\sigma(\alpha^*)+\alpha^*}\frac{1}{\xi_{\sigma(\beta^*)}}d\xi_{\sigma(\beta^*)}\Big)d\eta=$$
$$\oint\eta^{x_{\alpha^*}-y_{\sigma(\alpha^*)}-1+\sigma(\alpha^*)-\alpha^*}\Big(\oint\Big(\prod_{\substack{(\alpha,\beta),\\ (\alpha,\beta)\neq(\sigma(\alpha^*),\sigma(\beta^*))}}S_{\alpha,\beta}\Big)\frac{r+(1-r)\eta-\xi_{\sigma(\beta^*)}}{r\xi_{\sigma(\beta^*)}+(1-r)\eta\xi_{\sigma(\beta^*)}-\eta}$$
$$\xi_{\sigma(\beta^*)}^{x_{\beta^*}-y_{\sigma(\beta^*)}+\sigma(\beta^*)-\beta^*-x_{\alpha^*}+y_{\sigma(\alpha^*)}-\sigma(\alpha^*)+\alpha^*}d\xi_{\sigma(\beta^*)}\Big)d\eta. \quad (28)$$

Since $x_{\beta^*} - x_{\alpha^*} \geq \beta^* - \alpha^*$ and $y_{\sigma(\alpha^*)} - y_{\sigma(\beta^*)} \geq -\sigma(\beta^*) + \sigma(\alpha^*)$, the power of $\xi_{\sigma(\beta^*)}$,

$$x_{\beta^*} - x_{\alpha^*} - \beta^* + \alpha^* + y_{\sigma(\alpha^*)} - y_{\sigma(\beta^*)} + \sigma(\beta^*) - \sigma(\alpha^*) \geq 0. \tag{29}$$

Then the integrand in (28) has singularities at $\xi_{\sigma(\beta^*)} = \frac{\eta}{r+(1-r)\eta}$ and at points where the denominator of coefficients $S_{\alpha,\beta}$ are zero. Then for a simple closed contour around $\xi_{\sigma(\beta^*)} = 0$ with radius $R < \frac{\eta}{r+(1-r)\eta}$ and small enough to avoid other singularities, the integral with respect to $\xi_{\sigma(\beta^*)}$ in (28) is always zero.

This implies that for all x_k, the expression in (28) is equal to zero, and consequently the value of (27) is equal to zero. So terms of the sum (25) for all permutations $\sigma \neq I$ are always equal to zero. Then (4) satisfies the initial condition (3).

References

[1] M. Alimohammadi, V. Karimipour, and M. Khorrami. "Exact solution of a one-parameter family of asymmetric exclusion processes". In: *Phys. Rev. E* 57 (6 June 1998), pp. 6370–6376. DOI: 10.1103/PhysRevE.57.6370.

[2] C.A. Tracy and H. Widom. "Integral Formulas for the Asymmetric Simple Exclusion Process". In: *Commun. Math. Phys.* 279 (Mar. 2008), pp. 815–844. DOI: 10.1007/s00220-008-0443-3.

Appendix

As we have shown in the Section 2.1, for a three-particle model we have the master equation (5) with boundary conditions (7) and (8). The initial condition (3) for $n = 3$ is

$$P(y_1, y_2, y_3, t = 0) = \begin{cases} 1, \text{if } x_k = y_k \text{ for all } k = 1,2,3, \\ 0, \text{otherwise.} \end{cases} \tag{30}$$

Separation of variables

Suppose the transition probability can be written as $P(x_1,x_2,x_3,t) = X(x_1,x_2,x_3)T(t)$, where $X(x_1,x_2,x_3) \neq 0$ and $T(t) \neq 0$. Substitute it into equation (5):

$$X(x_1,x_2,x_3)T'(t) = X(x_1-1,x_2,x_3)T(t) + X(x_1,x_2-1,x_3)T(t) + X(x_1,x_2,x_3-1)T(t) - 3X(x_1,x_2,x_3)T(t),$$

$$\frac{T'(t)}{T(t)} = \frac{X(x_1-1,x_2,x_3) + X(x_1,x_2-1,x_3) + X(x_1,x_2,x_3-1) - 3X(x_1,x_2,x_3)}{X(x_1,x_2,x_3)} = \gamma.$$

The spatial part can be solved by a substitution $X(x_1,x_2,x_3) = \xi_1^{x_1}\xi_2^{x_2}\xi_3^{x_3}$:

$$\xi_1^{x_1-1}\xi_2^{x_2}\xi_3^{x_3} + \xi_1^{x_1}\xi_2^{x_2-1}\xi_3^{x_3} + \xi_1^{x_1}\xi_2^{x_2}\xi_3^{x_3-1} = (3+\gamma)\xi_1^{x_1}\xi_2^{x_2}\xi_3^{x_3},$$

$$\frac{1}{\xi_1} + \frac{1}{\xi_2} + \frac{1}{\xi_3} - 3 = \gamma.$$

Substituting any other expression of this form with indexes from 1 to 3 permuted we get the same γ, so all these substitutions are solutions. Then solution to the spatial part is a weighted linear combination of solutions.

Solution to the time part $\frac{T'(t)}{T(t)} = \gamma$ is

$$T(t) = A(\xi_1,\xi_2,\xi_3)e^{\gamma t},$$

where function $A(\xi_1, \xi_2, \xi_3)$ is arbitrary .

Then full solution to the master equation (5) is

$$\begin{aligned} P(x_1, x_2, x_3, t) = A(\xi_1, \xi_2, \xi_3) e^{\gamma t} (A_{123} \xi_1^{x_1} \xi_2^{x_2} \xi_3^{x_3} + A_{132} \xi_1^{x_1} \xi_3^{x_2} \xi_2^{x_3} \\ + A_{213} \xi_2^{x_1} \xi_1^{x_2} \xi_3^{x_3} + A_{231} \xi_2^{x_1} \xi_3^{x_2} \xi_1^{x_3} + A_{312} \xi_3^{x_1} \xi_1^{x_2} \xi_2^{x_3} + A_{321} \xi_3^{x_1} \xi_2^{x_2} \xi_1^{x_3}). \end{aligned} \tag{31}$$

Solution with the boundary conditions

Substitute the solution (31) into the boundary condition (7), cancel the time part, and rearrange:

$$\begin{aligned} \xi_1^x \xi_2^x \xi_3^{x_3} [A_{123} - (1-r_1)\xi_2 A_{123} - \frac{r_1}{\xi_1} A_{123} + A_{213} - (1-r_1)\xi_1 A_{213} - \frac{r_1}{\xi_2} A_{213}] + \\ \xi_1^x \xi_3^x \xi_2^{x_3} [A_{312} - (1-r_1)\xi_1 A_{312} - \frac{r_1}{\xi_3} A_{312} + A_{132} - (1-r_1)\xi_3 A_{132} - \frac{r_1}{\xi_1} A_{132}] + \\ \xi_2^x \xi_3^x \xi_1^{x_3} [A_{231} - (1-r_1)\xi_3 A_{231} - \frac{r_1}{\xi_2} A_{231} + A_{321} - (1-r_1)\xi_2 A_{321} - \frac{r_1}{\xi_3} A_{321}] = 0. \end{aligned}$$

For the the boundary condition (7) to be satisfied, we need this sum to be equal to 0. So if we set each term in this sum to be equal to 0, the boundary condition (7) is satisfied. We know that $\xi_1, \xi_2, \xi_3 \neq 0$ by assumption, hence we set the expressions in each of the square brackets to be equal to 0. Notice that this allows us to express the weights corresponding to odd permutations in terms of the weights for even permutations:

$$\begin{aligned} A_{213} &= -\frac{r_1 + (1-r_1)\xi_1\xi_2 - \xi_1}{r_1 + (1-r_1)\xi_1\xi_2 - \xi_2} \frac{\xi_2}{\xi_1} A_{123}, \\ A_{132} &= -\frac{r_1 + (1-r_1)\xi_1\xi_3 - \xi_3}{r_1 + (1-r_1)\xi_1\xi_3 - \xi_1} \frac{\xi_1}{\xi_3} A_{312}, \\ A_{321} &= -\frac{r_1 + (1-r_1)\xi_2\xi_3 - \xi_2}{r_1 + (1-r_1)\xi_2\xi_3 - \xi_3} \frac{\xi_3}{\xi_2} A_{231}. \end{aligned} \tag{32}$$

Following the same steps for the boundary condition (8), we get a similar result:

$$\begin{aligned} A_{132} &= -\frac{r_1 + (1-r_1)\xi_2\xi_3 - \xi_2}{r_1 + (1-r_1)\xi_2\xi_3 - \xi_3} \frac{\xi_3}{\xi_2} A_{123}, \\ A_{213} &= -\frac{r_1 + (1-r_1)\xi_1\xi_3 - \xi_3}{r_1 + (1-r_1)\xi_1\xi_3 - \xi_1} \frac{\xi_1}{\xi_3} A_{231}, \\ A_{321} &= -\frac{r_1 + (1-r_1)\xi_1\xi_2 - \xi_1}{r_1 + (1-r_1)\xi_1\xi_2 - \xi_2} \frac{\xi_2}{\xi_1} A_{312}. \end{aligned} \tag{33}$$

Let the even permutation be $\sigma = (...\alpha\beta...)$ where α is the i^{th} element and β is the $(i+1)^{th}$, then the odd permutation obtained by transposition of the i^{th} and $(i+1)^{th}$ element is $T_i(\sigma) = (...\beta\alpha...)$. Then if weights above adhere to the following pattern for $i = 1, 2$, $\sigma = 123, 231, 312$:

$$A_{T_i(\sigma)} = -\frac{r_1 + (1-r_1)\xi_{\sigma(i)}\xi_{\sigma(i+1)} - \xi_{\sigma(i)}}{r_1 + (1-r_1)\xi_{\sigma(i)}\xi_{\sigma(i+1)} - \xi_{\sigma(i+1)}} \frac{\xi_{\sigma(i+1)}}{\xi_{\sigma(i)}} A_\sigma,$$

boundary conditions (7), (8) are satisfied.

Solution with the initial condition

Defining $S_{\alpha,\beta}$ as in (22) and using (32), (33), we can write A_{231} as follows:

$$A_{231} = \frac{r_1+(1-r_1)\xi_1\xi_3-\xi_1}{r_1+(1-r_1)\xi_1\xi_3-\xi_3}\frac{r_1+(1-r_1)\xi_1\xi_2-\xi_1}{r_1+(1-r_1)\xi_1\xi_2-\xi_2}\frac{\xi_3}{\xi_1}\frac{\xi_2}{\xi_1}A_{123} = S_{3,1}S_{2,1}\frac{\xi_2\xi_3}{\xi_1^2}A_{123}. \tag{34}$$

Similarly, we can express all other weights in terms of A_{123}, and then the solution (31) is

$$\begin{aligned}P(x_1,x_2,x_3,t) = A(\xi_1,\xi_2,\xi_3)e^{\gamma t}A_{123}(&\xi_1^{x_1}\xi_2^{x_2}\xi_3^{x_3} + S_{3,2}\xi_1^{x_1}\xi_3^{x_2+1}\xi_2^{x_3-1} + S_{2,1}\xi_2^{x_1+1}\xi_1^{x_2-1}\xi_3^{x_3} \\ &+S_{3,1}S_{2,1}\xi_2^{x_1+1}\xi_3^{x_2+1}\xi_1^{x_3-2} + S_{3,1}S_{3,2}\xi_3^{x_1+2}\xi_1^{x_2-1}\xi_2^{x_3-1} + S_{3,2}S_{3,1}S_{2,1}\xi_3^{x_1+2}\xi_2^{x_2}\xi_1^{x_3-2}).\end{aligned} \tag{35}$$

Let $A(\xi_1,\xi_2,\xi_3) = \prod_{k=1}^{3}\xi_{\sigma(k)}^{-y_{\sigma(k)}-1} = \xi_1^{-y_1-1}\xi_2^{-y_2-1}\xi_3^{-y_3-1}$, $A_I = \left(\frac{1}{2\pi i}\right)^3$, and take contour integrals with respect to ξ_1,ξ_2,ξ_3 along simple closed contours around $\xi_1=0,\xi_2=0,\xi_3=0$ respectively:

$$\begin{aligned}P(x_1,x_2,x_3,0) &= \left(\frac{1}{2\pi i}\right)^3\oint\oint\oint \xi_1^{x_1-y_1-1}\xi_2^{x_2-y_2-1}\xi_3^{x_3-y_3-1}d\xi_1d\xi_2d\xi_3 \\ &+\left(\frac{1}{2\pi i}\right)^3\oint\oint\oint S_{3,2}\xi_1^{x_1-y_1-1}\xi_3^{x_2-y_3}\xi_2^{x_3-y_2-2}d\xi_1d\xi_2d\xi_3 + \left(\frac{1}{2\pi i}\right)^3\oint\oint\oint S_{2,1}\xi_2^{x_1-y_2}\xi_1^{x_2-y_1-2}\xi_3^{x_3-y_3-1}d\xi_1d\xi_2d\xi_3 \\ &+\left(\frac{1}{2\pi i}\right)^3\oint\oint\oint S_{3,1}S_{2,1}\xi_2^{x_1-y_2}\xi_3^{x_2-y_3}\xi_1^{x_3-y_1-3}d\xi_1d\xi_2d\xi_3 + \left(\frac{1}{2\pi i}\right)^3\oint\oint\oint S_{3,1}S_{3,2}\xi_3^{x_1-y_3+1}\xi_1^{x_2-y_1-2}\xi_2^{x_3-y_2-2}d\xi_1d\xi_2d\xi_3 \\ &+\left(\frac{1}{2\pi i}\right)^3\oint\oint\oint S_{3,2}S_{3,1}S_{2,1}\xi_3^{x_1-y_3+1}\xi_2^{x_2-y_2-1}\xi_1^{x_3-y_1-3}d\xi_1d\xi_2d\xi_3.\end{aligned} \tag{36}$$

Consider the first term of the sum in (36) and use (24):

$$\begin{aligned}\left(\frac{1}{2\pi i}\right)^3\oint\oint\oint \xi_1^{x_1-y_1-1}\xi_2^{x_2-y_2-1}\xi_3^{x_3-y_3-1}d\xi_1d\xi_2d\xi_3 &= \left(\frac{1}{2\pi i}\oint\xi_1^{x_1-y_1-1}d\xi_1\right)\left(\frac{1}{2\pi i}\oint\xi_2^{x_2-y_2-1}d\xi_2\right) \\ \left(\frac{1}{2\pi i}\oint\xi_3^{x_3-y_3-1}d\xi_3\right) &= \begin{cases}1, \text{if } x_1=y_1,x_2=y_2,x_3=y_3,\\ 0, \text{otherwise.}\end{cases}\end{aligned} \tag{37}$$

Consider the term of the sum in (36) corresponding to the permutation $\sigma = 231$ and let $\xi_2 = \frac{\eta}{\xi_1}$:

$$\begin{aligned}\left(\frac{1}{2\pi i}\right)^3\oint\oint\oint S_{3,1}S_{2,1}\xi_2^{x_1-y_2}\xi_3^{x_2-y_3}\xi_1^{x_3-y_1-3}d\xi_1d\xi_2d\xi_3 &= \left(\frac{1}{2\pi i}\right)^3\oint\xi_3^{x_2-y_3}\oint\eta^{x_1-y_2} \\ \oint\frac{r_1+(1-r_1)\xi_1\xi_3-\xi_1}{r_1+(1-r_1)\xi_1\xi_3-\xi_3}&\frac{r_1+(1-r_1)\eta-\xi_1}{r_1\xi_1+(1-r_1)\eta\xi_1-\eta}\xi_1^{x_3-y_1+y_2-x_1-3}d\xi_1d\eta d\xi_3.\end{aligned} \tag{38}$$

Since $x_3-x_1\geq 2$ and $y_2-y_1\geq 1$, the power of ξ_1 in (38),

$$x_3-x_1+y_2-y_1-3\geq 0.$$

Then for a contour around $\xi_1=0$ with radius $R<\frac{\eta}{r_1+(1-r_1)\eta}$ and $R<\frac{\xi_3-r_1}{(1-r_1)\xi_3}$, the integral in (38) is zero. Similarly, we can show that all terms of the sum in (36) corresponding to other permutations $\sigma\neq I$ are zero. Hence

$$P(x_1,x_2,x_3,t) = \sum_{\sigma\in S_3}\left[\left(\frac{1}{2\pi i}\right)^3 e^{\gamma t}\oint\oint\oint\prod_{\alpha,\beta}S_{\alpha,\beta}\prod_{k=1}^{3}\xi_{\sigma(k)}^{x_k-y_{\sigma(k)}-1+\sigma(k)-k}d\xi_1d\xi_2d\xi_3\right] \tag{39}$$

is the solution to the master equation (5) with boundary conditions (7), (8) and initial condition (30).