Nazarbayev University

Math 499 Capstone project

Actuarial Applications of a two-parameter generalized Logistic model

Nargiz Alekberova

Research Supervisor:

Dr. Dongming Wei

Second Reader:

Dr. Anastasios Bountis

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Abstract

This project presents a generalization of the classical logistic and the Gompertz model by using two parameter power law exponent.The suggested generalized with two parameters models was shown by numerical example to be potentially a better choice for fitting a certain data. The solutions of these models are used in terms of force of mortality function in actuarial math as insurance contingency models or as hazard function for quality control in science and engineering. The complexity of solution is presented in the form of hypergeometric function and corresponding to it nonlinear implicit regression analysis. Due to the mentioned complexity we develop the approximation of the actual solution and consider only its special cases.

1 Introduction

Let $\widetilde{p}(t)$ be a growth rate of some population and r_t be the relative growth rate at time t. The rate-state general ordinary differential equation is given by [1]

$$\frac{d\widetilde{p}(t)}{dt} = r(t)f(t,\widetilde{p}(t)) \tag{1}$$

Depending on the form of $r(t)$, different representations of the growth rate functions can be derived. This includes the common solutions of the rate-state general ODE (1), such as the Gompertz and logistic. Both models have found numerous applications for modelling and forecasting the behaviour of such processes as the life cycle of technological products, population growth, financial and actuarial development [4], [6]. The Gompertz and logistic's similar property of "Sigmoidal (S) - shaped" curve is suitable when describing a slow initial growth followed by rapid exponential increase with a certain saturation level (upper asymptote). Their fundamental difference, however, is that while Gompertz is a symmetrical function, Logistic is asymmetrical [4].

Throughout the history the Gompertz model has been in longer use than the logistic model. The model was first implemented by the British mathematician Benjamin Gompertz in 1825 [5]. He initially introduced this model as the one describing the relationship between the death rate and age. Soon the insurance industry started to apply the introduced by Gompertz model to evaluate death risk and construct actuarial tables [5], [8]. However, it was only after Makeham's addition of the third component in 1860, so called the age-independent constant, the Gompertz model became applicable in various process descriptions such as the demand in goods and services, biological growth of different organisms, growth of tumour and cancer cells [5], [9].

The logistic model was developed by Pierre-Francois Verhulst in 1838, and is often called the Verhulst model [7]. Verhulst developed his logistic model to describe the biological population dynamics having a limiting environment. This model later has found applications in fitting the behaviour of natural growth, the changes in socio-technological and economical systems [10].

As discussed above, both the logistic and Gompertz models became widespread in analysing and forecasting growth of various systems. However, inappropriate use of those "S-shaped" models frequently leads to the results not very coherent with a given data and its further prediction [10]. Thus, some improvements can be made in terms of generalization of variables to enable the better fit and long-term forecast of a certain sample.

In this paper, the generalization with power law exponent of the classical logistic and Gomepertz model is studied as the further application in actuarial math used as the force of mortality or hazard functions. In Section 2 we reveal some classical population models such as the logistic and the Gompertz models. Then, in Section 3 we introduce one and two parameter generalized logistic models with power law exponent. In Section 4 we try to do regression analysis of the suggested models and see the arising difficulties within. That is why, in Section 5, we come up with the approximation of model by using Taylor series expansion. Following this approximation, in Section 6 we develop least squares estimation used to numerically find necessary parameters. Finally, in Section 7, we apply the resulting suggested model to compute the force of mortality used in actuarial math.

2 The Population Growth Models

In this section we look at the derivations of the Gompertz and logistic models from the general ODE (1) based on the representations of $r(t)$.

2.1 Gompertz Model

The Gompertz model can be derived from the ODE (1) using the given rate function $r(t) = k\ ln\left(\frac{A}{\widetilde{p}(t)}\right)$, where A represents the upper asymptote, also called the carrying capacity, i.e. the maximum number of population, and k - the rate of growth, which affects the slope at an inflection of S-curve [1], [7]. Substituting this into original differential equation (1) we get the following:

$$\frac{d\widetilde{p}(t)}{dt} = k\widetilde{p}(t)\ ln\left(\frac{N}{\widetilde{p}(t)}\right) \tag{2}$$

The equation is separable, and we have:

$$\frac{d\widetilde{p}(t)}{\widetilde{p}(t)\ ln\left(\frac{A}{\widetilde{p}(t)}\right)} = kdt$$

$$\frac{\left(\frac{-A}{\widetilde{p}^2(t)}\right) d\widetilde{p}(t)}{\frac{A}{\widetilde{p}(t)}\ ln\left(\frac{A}{\widetilde{p}(t)}\right)} = -kdt\ \rightarrow$$

$$ln\left(ln\left(\frac{A}{\widetilde{p}(t)}\right)\right) = -kt + C\ \rightarrow$$

$$ln\left(\frac{A}{\widetilde{p}(t)}\right) = exp(-kt + c)$$

This can be re-parametrized using $B = e^C$:

$$ln\left(\frac{A}{\widetilde{p}(t)}\right) = Be^{-kt}\ \rightarrow$$

$$\frac{A}{\widetilde{p}(t)} = exp(exp(-kt + c))$$

or

$$\frac{A}{\widetilde{p}(t)} = exp(-Bexp(-kt))$$

So, the resulting equations of the growth rate with different types of parametrisations are the following:

$$\widetilde{p}(t) = Aexp(-exp(-kt + c)) \tag{3}$$

or

$$\widetilde{p}(t) = Aexp(-Bexp(-kt)) \tag{4}$$

Both forms of derivations (3) and (4) represent the Gompertz model. The choice of the form of the Gompertz model defines the interpretations of the parameters itself, and usually can be converted between those types of models. This can be illustrated for parameters c and B from the equations (3) and (4): $B = exp(c)$ and so $c = ln(B)$. Moreover, when $t = 0$ $\widetilde{p}(0) = Aexp(-B) = A_0$, with A_0 representing the initial number of a certain population. So, we have $A_0 = Ae^{-B}$, or $B = ln(\frac{A}{A_0})$.

The Gompertz equation can be further simplified by normalization of the population rate. This is obtained by rewriting population as the fraction of population value per its maximum number, i.e. $p(t) = \dfrac{\widetilde{p}(t)}{A}$.

The following are normalized equations of the Gompertz model with two types of parametrizations (3) and (4) are:

$$p(t) = exp(-exp(-kt + c)) \tag{5}$$

or

$$p(t) = exp(-Bexp(-kt)) \tag{6}$$

2.2 Classical Logistic Model

Similarly, the classical logistic model, also called Pierre-Francois Verhulst's logistic equation, can be derived from ODE (1) with the given growth rate of $r(t) = r\left(\frac{A-\widetilde{p}(t)}{A}\right)$, where A is the maximum number of population and r is the growth rate :

$$\frac{d\widetilde{p}(t)}{dt} = r\widetilde{p}(t)\left(\frac{N - \widetilde{p}(t)}{N}\right) \tag{7}$$

Rearranging this separable differential equation:

$$\frac{d\widetilde{p}(t)}{\widetilde{p}(t)\frac{A-\widetilde{p}(t)}{A}} = rdt$$

$$\left(\frac{-A}{(A - \widetilde{p}(t))\widetilde{p}(t)}\right) d\widetilde{p}(t) = -rdt$$

Using partial fractions we have the next equation:

$$\left[\frac{-1}{(A - \widetilde{p}(t))} - \frac{1}{\widetilde{p}(t)}\right] d\widetilde{p}(t) = -rdt \quad \rightarrow$$

$$ln\left(\frac{A-\widetilde{p}(t)}{\widetilde{p}(t)}\right) = -rt + C \quad \rightarrow$$

$$\frac{A-\widetilde{p}(t)}{\widetilde{p}(t)} = e^{-rt+c}$$

Similarly to the Gompertz model derivation, B parameter can be introduced as $B = e^c$:

$$\frac{A-\widetilde{p}(t)}{\widetilde{p}(t)} = Be^{-rt}$$

Thus, two versions of the equations (with c or B parameters) are derived:

$$\widetilde{p}(t) = \frac{A}{1+e^{-rt+c}} \tag{8}$$

or

$$\widetilde{p}(t) = \frac{A}{1+Be^{-rt}} \tag{9}$$

The obtained forms (or re-parametrizations) represent the classical logistic model of Verhulst. Note that c-parameter can be derived from B though $c = ln(B)$, and it is more useful when finding the location of inflection point directly [7]. Moreover, as $t = 0 \quad \widetilde{p}(0) = \frac{A}{1+B} = A_0$, where A_0 is the initial number of a certain population. So, we have $A_0 = \frac{A}{1+B}$, or $B = \frac{A-A_0}{A_0}$.

The classical logistic model can also be simplified by normalizing the population data. Following the same substitution as for Gompertz equation, i.e. $p(t) = \frac{\widetilde{p}(t)}{A}$ we obtain the following normalized forms of classical logistic model (8) and (9) :

$$p(t) = \frac{1}{1+Be^{-rt}} \tag{10}$$

or

$$p(t) = \frac{1}{1+e^{-rt+c}} \tag{11}$$

3 Generalized Logistic Models

Pierre-Francois Verhulst's logistic equation is used to describe various peocesses beginning from growth rate of a bacteria to the human population. Although the classical logistic equation is one of the most commonly applied by statisticians, it is sometimes used incorrectly and fails to closely fit a given data. Therefore, some improvements can be done to the general equation of the classical logistic model. Yao Zheng, 2010, in his paper suggests to modify the model by adding the power law exponent parameter [2]. In this section we will consider Yao Zheng's method of applying the power law exponent and develop it further by using one more exponent parameter.

3.1 Generalized Logistic Model with one parameter

Recalling the ODE of the classical logistic model (7), further simplification is done in terms of normalization of the data, i.e. by diving the equation to the maximum number of population such as $p(t) = \dfrac{\widetilde{p}(t)}{A}$:

$$\frac{dp}{dt} = r(1-p)p \tag{12}$$

Yao Zheng proposes the idea of extending the growth rate by the power law exponent. This is done by rewriting the rate as rp^{α} [2]. Then the generalized version of the classical logistic model with α parameter is:

$$\frac{dp}{dt} = r(1-p)p^{\alpha}, \quad \alpha > 0 \tag{13}$$

The case with $\alpha = 1$ is the classical logistic model of Pierre-Francois Verhulst itself (11). Thus, the cases with $\alpha > 0$ can be used for describing more complex instances of data sets. Solving further the generalized by Yao Zheng one parameter logistic equation (13) we obtain the following:

$$\int \frac{dp}{(1-p)p^{\alpha}} = \int r dt$$

$$\frac{-p^{1-\alpha}{}_2F_1\left(1, 1-\alpha; 2-\alpha; p\right)}{\alpha - 1} = rt + C$$

The obtained equation is the hypergeometric function ${}_2F_1$ with $\alpha > 0$. So the resulting generalized model with one parameter is:

$$p(t) = \left[\frac{(1-\alpha)(rt+C)}{{}_2F_1\left(1, 1-\alpha; 2-\alpha; p\right)}\right]^{\frac{1}{1-\alpha}} \tag{14}$$

or when taking $B = C(1-\alpha)$

$$p(t) = \left[\frac{rt(1-\alpha)+B}{{}_2F_1\left(1, 1-\alpha; 2-\alpha; p\right)}\right]^{\frac{1}{1-\alpha}}$$

To analyze the performance of the introduced model (14) we will consider different cases of the α parameter values and compare them with the Gompertz and classical logistic models. All the simulations in this section will be based on the data represents the relationship between time (in minutes) and Chloride ion concentration(in %) (for further details see the appendix A1). It is important to mention that the considered example of Chloride ion concentration is similar to general tendency of population growth. To construct the models that will be a good fit for the given data the appropriate parameters should be introduced. In case of the Gompertz and the classical logistic models the parameters were optimized through regression analysis. Since the given data follows a non-linear trend, the non-linear regression analysis was in preference. The analysis was done using nls-function in R. More detailed description of the R-code used in this section can be found in Appendix A2.

For the Gompertz model (5) non-linear regression testing gives the following optimized parameters:

Formula: conc ∼ A * exp(-exp(-k * time + c))

Parameters: Estimate

A: 35.35977 k: 0.26223 c: 0.31454

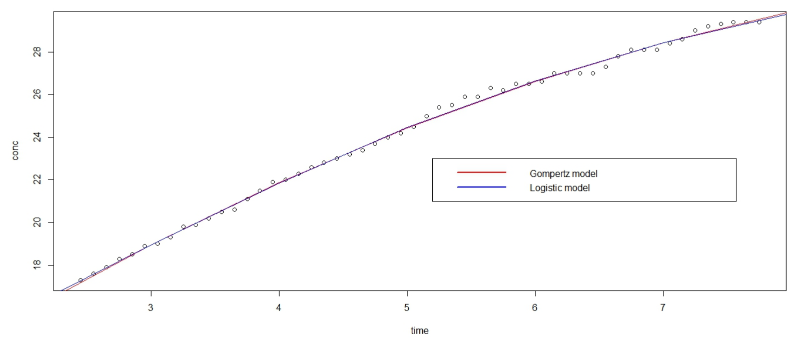
The same procedure for the classical logistic model (11) provides the following parameters:

Formula: conc ~ A/(1 + exp(-r * time + c))

Parameters: Estimate

A: 33.46121 c: 0.83141 r: 0.36597

Figure 1 shows the S-curves of the Gompertz and logistic models constructed under the obtained optimized parameters. As it can be seen from the Figure 1, both graphs with estimated conditions behave in the similar, almost the same manner and closely fit the data.



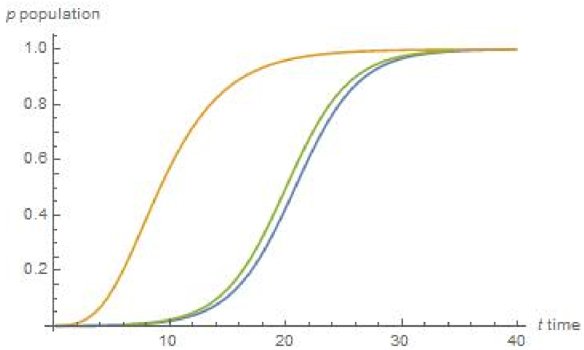
The graphs of the Gompertz and logistic models fitting the Chloride data in A1 using non-linear regression analysis.

The optimized parameters from the Chloride data (see Appendix A1) will be used in the simulation process in order to compare the curves of the Gompertz and classical logistic models and the generalized logistic models with different α-values. In this simulation, the initial number of population is $p(0) = 0.0005$. Next, we find time t derived from each level of normalized population in the interval $[0, 0.995]$ starting from 0 with increment of 0.005. From the non-linear regression analysis we take growth rate for Gompertz model as $k = 0.26223$, and for the classical logistic model as $r = 0.36597$. Other parameters are obtained from the normalized form of the Gompertz model (6) and the classical logistic model (11). Particularly, we derive from the Gompertz model B parameter, which equals to $B = ln\left(\frac{1}{p(0)}\right) = -ln(p(0)) = 7.601$, and from the classical logistic model c value, that is $c = ln((1 - p(0))/p(0)) = 7.600$. Below is the summary of the resulting parameters and initial conditions that will be used for the simulations:

the Gompertz model:	p(0)=0.0005	B=7.601	k=0.26223
the logistic model:	p(0)=0.0005	c=7.600	r=0.36597

We begin the simulation of the generalized logistic model (14) with the closest to the classical logistic model value of $\alpha = 0.99$. The Figure 2 shows the graphs of these two models in comparison with the Gompertz model. It can be observed that the introduced model with $\alpha = 0.99$ is indeed close to the behaviour of the classical logistic model.

Following the same simulation procedures we construct the graphs of the generalized logistic model with $\alpha = 0.95, 0.75, 0.50, 0.25$ with the classical and the Gompertz model, all shown in the Figure 3. For some values of α, the representation of the generalized logistic function is actually simpler, and does not require to solve the hypergeometric function ${}_2F_1$. This is the case,

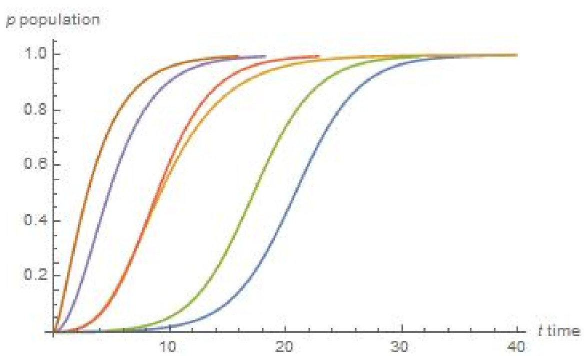


The simulated graphs of the generalized logistic($\alpha = 0.99$), the Gompertz and classical logistic models

for example, of $\alpha = 0.25$:

$$\int \frac{dp}{(1-p)p^{0.25}} = \int r dt$$

$$-log(1 - p^{0.25}) + log(1 + p^{0.25}) - 2\ tan^{-1}(p^{0.25}) = rt + C$$

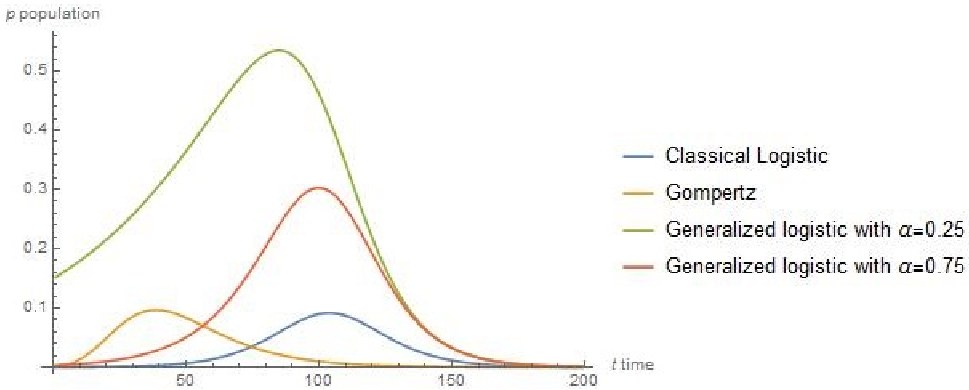


The simulated graphs of generalized logistic with α parameters, the Gompertz and classical logistic models

From the Figure 3 we observe that generalized logistic models with α close to 0 take a sharper increase compared with models having α-values close to 1, which have a slow growth at both start and end-tails. To demonstrate this tendency further we plot the growth rates of the Gompertz (5), the classical logistic (11) and generalized with one parameter logistic (14) models. Recall that the growth rate of the model is defined by the derivative of the population growth equations, i.e. $\frac{dp}{dt}$. Below presented are the growth rates of the three considered models:

the Gompertz model:	$\frac{dp}{dt} = Bkexp(-kt + Bexp(-kt))$
the logistic model:	$\frac{dp}{dt} = \frac{rexp(-rt+c)}{(1+exp(-rt+c))^2} = \frac{r(1-p(t))}{p(t)}$
the generalized logistic model:	$\frac{dp}{dt} = r(1 - p(t))p(t)$

The Figure 4 shows the growth rate patterns of the Gompertz, the classical logistic and the generalized logistic with parameters $\alpha = 0.25$ and $\alpha = 0.75$. Among three models the Gompertz model has the most asymmetrical curve of the growth rate. Both generalized and classical logistic model have similar growth rate pattern of the symmetrical decrease towards both tails after reaching the peak level. However, compared with classical the generalized logistic model shows higher increase rate at the same low level of infected population.



The plots of growth rates of the generalized logistic (with $\alpha = 0.25$ and $\alpha = 0.75$), the Gompertz and classical logistic models

3.2 Generalized Logistic Model with two parameters

The generalized logistic model with one parameter gave us the idea of extending the model even further. We introduce a more general method, that is the two parameter generalized logistic model. The model is introduced by adding two parameters α and β to the normalized ODE of the classical logistic equation (7):

$$\frac{dp}{dt} = r(1-p)^{\beta} p^{\alpha}, \quad with \quad \alpha, \beta > 0 \tag{15}$$

After solving the ODE (15), we obtain the following:

$$\int \frac{dp}{(1-p)^{\beta} p^{\alpha}} = \int r dt$$

$$\frac{-p^{1-\alpha}{}_2F_1\left(1-\alpha, \beta; 2-\alpha; p\right)}{\alpha - 1} = rt + C$$

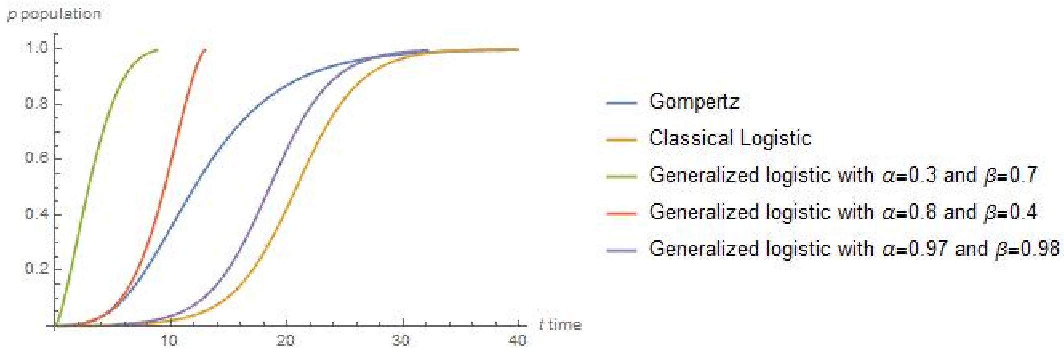
$$p(t) = \left[\frac{(1-\alpha)(rt + C)}{{}_2F_1\left(1-\alpha, \beta; 2-\alpha; p\right)}\right]^{\frac{1}{1-\alpha}} \tag{16}$$

or when taking $B = C(1-\alpha)$:

$$p(t) = \left[\frac{rt(1-\alpha) + B}{{}_2F_1\left(1-\alpha, \beta; 2-\alpha; p\right)}\right]^{\frac{1}{1-\alpha}}$$

The resulting equation (16) is again the hypergeometric function ${}_2F_1$, but with two parameters $\alpha, \beta > 0$.

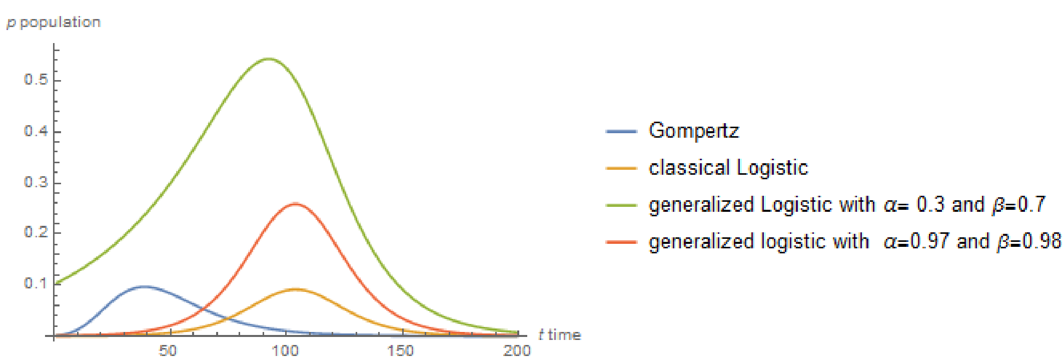
The Figure 5 illustrates the behaviour of the generalized with two parameters logistic model in comparison with the classical logistic and the Gompertz models. The generalized with two parameters logistic models is represented by three cases: α and β close to 1, α close to 0 and



The comparison of the generalized logistic with two parameters, the Gompertz and classical logistic models

β close to 1, and conversely. In this suggested type of generalization, as it can be seen from the Figure 5, we can variate both tails of growth curve. This means that the starting and ending phases can be varied just by specific α and β values. Moreover, the fixation of the end tail can allow more accurately to predict future values based on the previous trend. This modification to the classical logistic model has potential to be a better description of different data types such as diseases, insurance and different population growth rates. Therefore, further analysis of the behaviour of the generalized logistic model under different α and β parameters should be evaluated by finding real-world data models.

The Figure 6 shows the growth rate patterns of the Gompertz, the classical logistic and the generalized logistic with parameters α and β parameters. As it was for the one parameter case, two parameter generalized logistic model shows higher increase rate than the classical. Moreover, compared with one parameter model two parameter model had more adjustment variation in the tails.



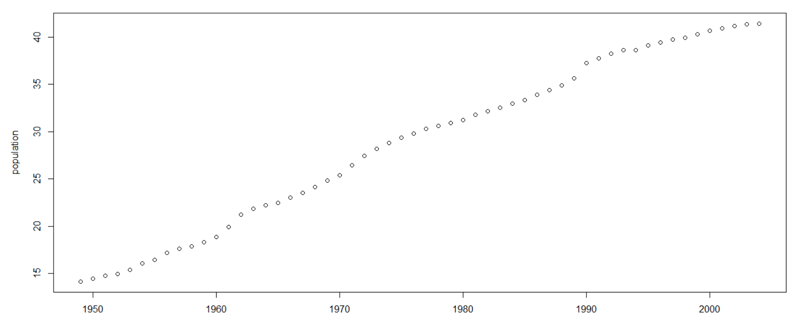
The plots of growth rates of the generalized logistic with two parameters, the Gompertz and classical logistic models

4 Regression Modelling

As it was discussed on the previous section, there is a potential that two-parameter generalization of the classical logistic model fits some data better than classical and one parameter models. To prove this statistically we firstly have to find the best optimal parameters describing some given data. The obtained equations of the generalized logistic model with one and two parameters(14) and (16) include hypergeometric functions, which are non-linear. Therefore, we need to develop some algorithms for performing non-linear regression analysis. In the following section we will develop strategies to find better prediction of parameters by doing regression modelling in R. Moreover, the comparison of classical and improved logistic models will be done when using linear and nonlinear regression. In case of non-linear regression *nls*-function in R will be used to find better estimates for the parameters [2].

4.1 Linear Regression Modelling

As a simple example of how advanced logistic model is a better fitter than classical, population data from Yao-Zheng's paper will be considered[2]. Particularly, the data from Chinese province Wu-Gong County,from 1949 through 2004. The data is shown in the figure below:
>plot(population year,data=data) The *lm* function is based on the simple linear least-square



The plot of the Wu-Gong County population

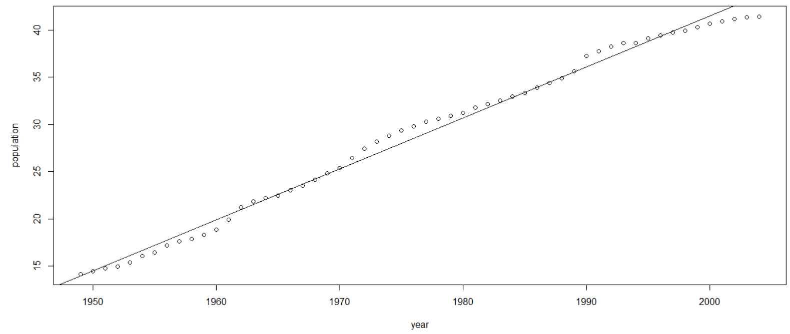
regression model.The straight line added to the initial plot is clearly not a good match for modelling these growth rate mode.

>abline(lm(population year,data=data))

From graph we suggest that the maximum number of the Wu-Gong County's population is 55. This number is then used to plot the linear model of this population using *lm* function. Below is a quick summary of the *lm*-based model:

>lm(logit(population/55) year,data)

Call:
lm(formula = logit(population/55) year, data = data)

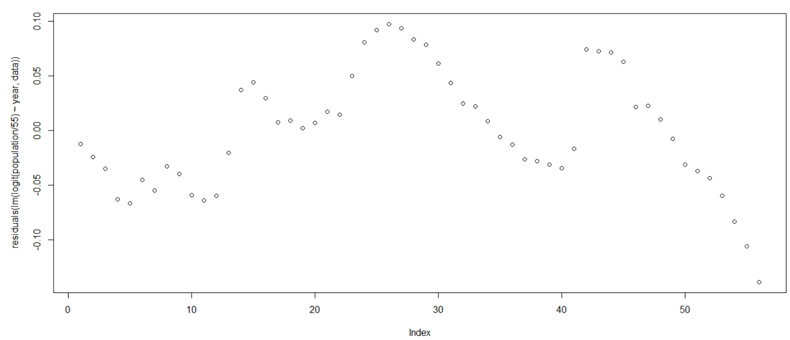


The linear least-squares model of Wu-Gong population

Coefficients:
(Intercept) year -82.61521 0.04185

Finally, the residuals graph presented on the Figure 9 presents a certain of pattern, thus, showing that the *lm*-based model needs the significant improvements:

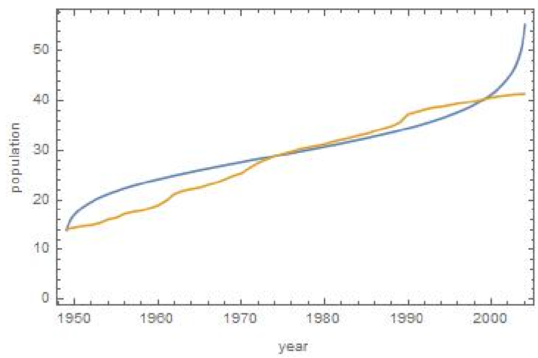
> plot(residuals(lm(logit(population/55) year,data)))



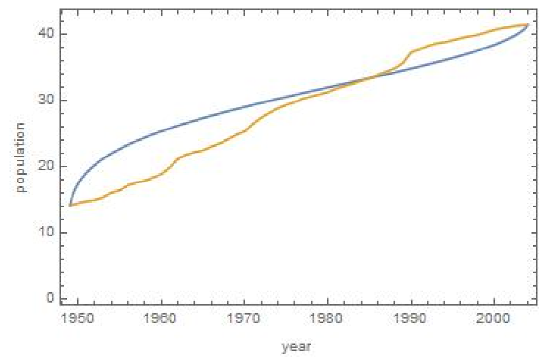
The residuals of the linear least-squares model of Wu-Gong population

4.2 Nonlinear Regression Modelling

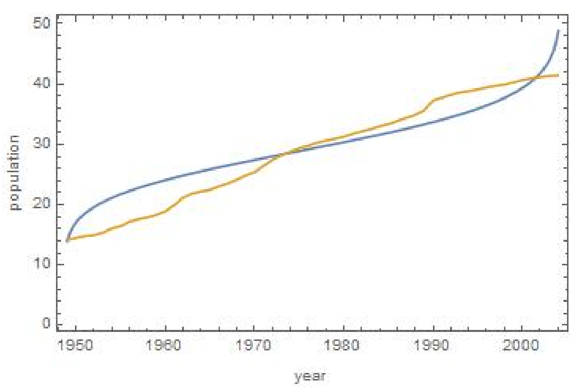
For the case of the Wu-Gong County's population, generalized logistic model with two parameters seems to be a better fit to the given data. This is shown in the Figure 10 with different parameters of α and β. The α and β parameters in the Figure 10(b) visually appear to be the closest fit from both tails to the original data.



(a) Generalized logistic with $\alpha = 0.8$, $\beta = 1$ and $R = 0.2$



(b) Generalized logistic with $\alpha = 0.75$, $\beta = 0.5148$ and $R = 0.15$



(c) Generalized logistic with $\alpha = 0.8$, $\beta = 0.9$ and $R = 0.2$

The generalized logistic model of Wu-Gong's population (blue) and the original data plot of the Wu-Gong census (orange)

5 Approximation of the Generalized Logistic Models

In the Section 3 we obtained the following equations of the generalized logistic models with one and two parameters:

$$p(t) = \left[\frac{rt(1-\alpha) + B}{{}_2F_1\left(1, 1-\alpha; 2-\alpha; p\right)} \right]^{\frac{1}{1-\alpha}} \tag{17}$$

$$p(t) = \left[\frac{rt(1-\alpha) + B}{{}_2F_1\left(1-\alpha, \beta; 2-\alpha; p\right)} \right]^{\frac{1}{1-\alpha}} \tag{18}$$

where ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function, which has the series form of

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{19} $$

defined for $a, b > 0$ and $|z| < 1$ [12].

5.1 Linear Transformations on one-parameter generalized logistic model

Missov et al.[9] in their paper on calculating life expectancies use approximations on the hypergeometric functions introduced in the "Handbook of Mathematical Functions" [12]. Following the same handbook, we apply linear transformations and properties of the Gaussian hypergeometric functions. For the hypergeometric function ${}_2F_1\left(1, 1-\alpha; 2-\alpha; p\right)$ in the one-parameter generalized logistic model (17) under Abramowitz and Stegun's (15.3.8) formulation [12] the following transformation is obtained:

$$ {}_2F_1\left(1, 1-\alpha; 2-\alpha; p\right) = (1-p)^{-1} \frac{\Gamma(2-\alpha)\Gamma(-\alpha)}{\Gamma(1-\alpha)\Gamma(1-\alpha)} {}_2F_1\left(1, 1; 1+\alpha; \frac{1}{1-p}\right) $$

$$ +(1-p)^{-(1-\alpha)} \frac{\Gamma(2-\alpha)\Gamma(\alpha)}{\Gamma(1)\Gamma(1)} {}_2F_1\left(1-\alpha, 1-\alpha; 1-\alpha; \frac{1}{1-p}\right) $$

The hypergeometric function ${}_2F_1\left(1-\alpha, 1-\alpha; 1-\alpha; \frac{1}{1-p}\right)$ is then transformed under (15.1.8) and the following result is obtained:

$$ {}_2F_1\left(1, 1-\alpha; 2-\alpha; p\right) = (1-p)^{-1} \frac{\Gamma(2-\alpha)\Gamma(-\alpha)}{\Gamma^2(1-\alpha)} {}_2F_1\left(1, 1; 1+\alpha; \frac{1}{1-p}\right) $$

$$ +(-p)^{-(1-\alpha)} \Gamma(2-\alpha)\Gamma(\alpha) $$

Note that in cases when $\alpha \to 1$, ${}_2F_1\left(1, 1; 1+\alpha; \frac{1}{1-p}\right) \to {}_2F_1\left(1, 1; 2; \frac{1}{1-p}\right)$, and by Abramowitz and Stegun's (15.1.3)

$$ {}_2F_1\left(1, 1; 2; \frac{1}{1-p}\right) = -\left(\frac{1}{1-p}\right)^{-1} ln\left(1 - \frac{1}{1-p}\right) $$

The resulting transformed one-parameter generalized logistic model (17) is

$$ p(t) = \left[\frac{rt(1-\alpha) + B}{(1-p)^{-1} \frac{\Gamma(2-\alpha)\Gamma(-\alpha)}{\Gamma^2(1-\alpha)} {}_2F_1\left(1, 1; 1+\alpha; \frac{1}{1-p}\right) + (-p)^{-(1-\alpha)} \Gamma(2-\alpha)\Gamma(\alpha)} \right]^{\frac{1}{1-\alpha}} \tag{20} $$

5.2 Linear Transformations on two-parameter generalized logistic model

In a similar way from the handbook's (15.3.8) [12] hypergeometric transformation we obtain the following identity for the function ${}_2F_1\left(1-\alpha, \beta; 2-\alpha; p\right)$ from the equation (18):

$$ {}_2F_1\left(1-\alpha, \beta; 2-\alpha; p)\right) = (1-p)^{-(1-\alpha)} \frac{\Gamma(2-\alpha)\Gamma(\beta+\alpha-1)}{\Gamma(\beta)\Gamma(1)} {}_2F_1\left(1-\alpha, 2-\alpha-\beta; 2-\alpha-\beta; \frac{1}{1-p}\right) $$

$$+(1-p)^{-\beta}\frac{\Gamma(2-\alpha)\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(2-\alpha-\beta)}{}_2F_1\left(\beta,1;\beta+\alpha;\frac{1}{1-p}\right)$$

The Gaussian hypergeometric function ${}_2F_1\left(1-\alpha,2-\alpha-\beta;2-\alpha-\beta;\frac{1}{1-p}\right)$ under (15.1.8) can be further transformed to $\left(1-\frac{1}{1-p}\right)^{-(1-\alpha)}$, which is expressed in

$${}_2F_1\left(1-\alpha,\beta;2-\alpha;p\right)=(1-p)^{-(1-\alpha)}\frac{\Gamma(2-\alpha)\Gamma(\beta+\alpha-1)}{\Gamma(\beta)}\left(1-\frac{1}{1-p}\right)^{-(1-\alpha)}$$

$$+(1-p)^{-\beta}\frac{\Gamma(2-\alpha)\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(2-\alpha-\beta)}{}_2F_1\left(\beta,1;\beta+\alpha;\frac{1}{1-p}\right)$$

Note that, similarly to the one-parameter case, when both $\alpha\to 1$ and $\beta\to 1$

$${}_2F_1\left(\beta,1;\beta+\alpha;\frac{1}{1-p}\right)\to {}_2F_1\left(1,1;2;\frac{1}{1-p}\right)=-\left(\frac{1}{1-p}\right)^{-1}ln\left(1-\frac{1}{1-p}\right)$$

The resulting transformation on the two-parameter generalized logistic model (18) obtained in the following form:

$$p(t)=\left[\frac{rt(1-\alpha)+B}{\frac{\Gamma(2-\alpha)\Gamma(\beta+\alpha-1)}{\Gamma(\beta)}(-p)^{-(1-\alpha)}+\frac{\Gamma(2-\alpha)\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(2-\alpha-\beta)}(1-p)^{-\beta}{}_2F_1\left(\beta,1;\beta+\alpha;\frac{1}{1-p}\right)}\right]^{\frac{1}{1-\alpha}} \tag{21}$$

5.3 Taylor Series Approximation of generalized logistic model

Eq.(15) can be rewritten in the following form:

$$\int\frac{p^{-\alpha}dp}{(1-p)^{\beta}}=\int rdt \tag{22}$$

Then, the left-hand side denominator of the eq.(22)can be approximated by using Taylor series expansion:

$$(1-p)^{-\beta}=1+\beta p+\frac{\beta(\beta+1)}{2!}p^2+\frac{\beta(\beta+1)(\beta+2)}{3!}p^3+...$$

So the eq.(22) when left with three terms of Taylor expansion takes the following form:

$$\int p^{-\alpha}\left(1+\beta p+\frac{\beta(\beta+1)}{2!}p^2+\frac{\beta(\beta+1)(\beta+2)}{3!}p^3\right)=rt+C \tag{23}$$

$$p^{1-\alpha}\left(\frac{\beta(\beta^2+3\beta+2)}{6(4-\alpha)}p^3+\frac{\beta(\beta+1)}{2(3-\alpha)}p^2+\frac{\beta}{(2-\alpha)}p+\frac{1}{(1-\alpha)}\right)=rt+C \tag{24}$$

Similarly, approximated with two terms the nonlinear model eq.(22)is

$$\int p^{-\alpha}(1+\beta p+\frac{\beta(\beta+1)}{2!}p^2)=rt+C \tag{25}$$

$$p^{1-\alpha}\left(\frac{\beta(\beta+1)}{2(3-\alpha)}p^2+\frac{\beta}{(2-\alpha)}p+\frac{1}{(1-\alpha)}\right)=rt+C \tag{26}$$

Approximated with one term the same nonlinear model eq.(22)is

$$\int p^{-\alpha}(1+\beta p) = rt + C \tag{27}$$

$$p^{1-\alpha}\left(\frac{\beta}{(2-\alpha)}p + \frac{1}{(1-\alpha)}\right) = rt + C \tag{28}$$

$$p = \left[\frac{rt+C}{\frac{\beta(\beta+1)}{2(3-\alpha)}p^2 + \frac{\beta}{(2-\alpha)}p + \frac{1}{(1-\alpha)}}\right]^{\frac{1}{1-\alpha}}$$

6 Least Squares Estimation in Nonlinear Regression

6.1 Using Linear Transformations

In Section 4 we have seen that introduced two-parameter generalized logistic model can be a better fit for the example of the Wu-Gong County's population than the classical models. Moreover, diagnosis on the obtained linearly regressed model has shown that some transformations or nonlinear regression should be considered for the better result. In this section we will develop the least squares criterion for the purpose of applying it in nonlinear regression analysis. The previously obtained and transformed two-parameter logistic model (21) will be used in this least squares estimation. The leat squares criterion Q using formulat (13.15) from Kutner et al. [13] is

$$Q = \sum_{i=1}^{n}[p_i - f(\mathbf{t}_i, \boldsymbol{\gamma})]^2 \tag{29}$$

where $f(\mathbf{t}_i, \boldsymbol{\gamma})$ is the mean response for the ith case according to the nonlinear response function $f(\mathbf{t}, \boldsymbol{\gamma})$. As Kutner et al. [13] explains, the criterion Q is obtained by minimizing it with respect to nonlinear regression parameters. Generally, the least squares estimates can be derived through normal equations or iterative numerical procedures.

The partial derivative of Q with respect to the model's parameter γ_k is given by Kutner et al.'s (15.16) [13]:

$$\frac{\partial Q}{\partial \gamma_k} = \sum_{i=1}^{n} -2[p_i - f(\mathbf{t}_i, \boldsymbol{\gamma})]\left[\frac{\partial f(\mathbf{t}_i, \boldsymbol{\gamma})}{\partial \gamma_k}\right] \tag{30}$$

Our introduced generalized model (21) has three independent parameters α, β, r. Note that parameter B can be directly derived by using above mentioned parameters through the identity

$$B = p_0^{(1-\alpha)}{}_2F_1\left(1-\alpha, \beta; 2-\alpha; p\right) \tag{31}$$

where p_0 is the initial population at time $t = 0$.

After three partial derivatives are each set to 0, we have the following equations:

$$\sum_{i=1}^{n} p_i \left[\frac{\partial f(\mathbf{t}_i, \boldsymbol{\gamma})}{\partial \alpha}\right]_{\boldsymbol{\gamma}=\mathbf{g}} - \sum_{i=1}^{n} f(\mathbf{t}_i, \mathbf{g})\left[\frac{\partial f(\mathbf{t}_i, \boldsymbol{\gamma})}{\partial \alpha}\right]_{\boldsymbol{\gamma}=\mathbf{g}} = 0 \tag{32}$$

$$\sum_{i=1}^{n} p_i \left[\frac{\partial f(\mathbf{t}_i, \boldsymbol{\gamma})}{\partial \beta}\right]_{\boldsymbol{\gamma}=\mathbf{g}} - \sum_{i=1}^{n} f(\mathbf{t}_i, \mathbf{g}) \left[\frac{\partial f(\mathbf{t}_i, \boldsymbol{\gamma})}{\partial \beta}\right]_{\boldsymbol{\gamma}=\mathbf{g}} = 0 \tag{33}$$

$$\sum_{i=1}^{n} p_i \left[\frac{\partial f(\mathbf{t}_i, \boldsymbol{\gamma})}{\partial r}\right]_{\boldsymbol{\gamma}=\mathbf{g}} - \sum_{i=1}^{n} f(\mathbf{t}_i, \mathbf{g}) \left[\frac{\partial f(\mathbf{t}_i, \boldsymbol{\gamma})}{\partial r}\right]_{\boldsymbol{\gamma}=\mathbf{g}} = 0 \tag{34}$$

where $\boldsymbol{\gamma}$ is the vector of the model parameters and $\mathbf{g}$ is the vector of the least squares estimates, such that

$$\boldsymbol{\gamma} = \begin{bmatrix} \alpha \\ \beta \\ r \end{bmatrix}, \text{ and } \mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix}.$$

Consider one of partial derivatives of $f(\mathbf{t}_i, \boldsymbol{\gamma})$:

$$\frac{\partial f(\mathbf{t}_i,\boldsymbol{\gamma})}{\partial \beta} = (rt(1-\alpha)+B)^{\frac{1}{1-\alpha}} *$$

$$* \frac{\partial}{\partial \beta}\left[\frac{\Gamma(2-\alpha)\Gamma(\beta+\alpha-1)}{\Gamma(\beta)}(-p)^{-(1-\alpha)} + \frac{\Gamma(2-\alpha)\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(2-\alpha-\beta)}(1-p)^{-\beta}{}_2F_1\left(\beta, 1; \beta+\alpha; \frac{1}{1-p}\right)\right]^{\frac{-1}{1-\alpha}}$$

$$= \left(\frac{-1}{1-\alpha}\right)(rt(1-\alpha)+B)^{\frac{1}{1-\alpha}} *$$

$$* \frac{\partial}{\partial \beta}\left[\frac{\Gamma(2-\alpha)\Gamma(\beta+\alpha-1)}{\Gamma(\beta)}(-p)^{-(1-\alpha)} + \frac{\Gamma(2-\alpha)\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(2-\alpha-\beta)}(1-p)^{-\beta}{}_2F_1\left(\beta, 1; \beta+\alpha; \frac{1}{1-p}\right)\right]^{(\frac{-1}{1-\alpha}-1)}$$

Note that the derivative of the Gamma function is expressed as the following:

$$\frac{\Gamma(\beta)}{d\beta} = \frac{d}{d\beta}\int_0^{\infty} x^{\beta-1}e^{-x}dx = \int_0^{\infty} x^{\beta-1}ln(x)e^{-x}dx$$

Since our suggested generalized model (21) has an implicit form, the normal equations (32) - (34) can only be solved numerically, and in some cases by using some of approximation methods on Gaussian hypergeometric functions introduced in Section 5.

6.2 Using Taylor Series Approximation

6.2.1 One-term Taylor Series Approximation

We rewrite least squares criterion formula (29) as the function of α, β, r and c :

$$F(\alpha, \beta, r, c) = \sum_{i=1}^{n} |p(t_i, \alpha, \beta, r, c) - p_i|^2 \tag{35}$$

Following the same minimisation technique discussed in Section 6.1 we set the following system of partial derivatives corresponding to $\bigtriangledown F(\alpha, \beta, r, c) = 0$:

$$\begin{cases} \frac{\partial F(\alpha,r,c)}{\partial\alpha} = 0 \\ \frac{\partial F(\alpha,r,c)}{\partial\beta} = 0 \\ \frac{\partial F(\alpha,r,c)}{\partial r} = 0 \\ \frac{\partial F(\alpha,r,c)}{\partial c} = 0 \end{cases} \tag{36}$$

The first equation of the system (36) can be written in the form (30) introduced earlier in the Section 6.1:

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial\alpha} = \sum_{i=1}^{n} 2|p(t_i,\alpha,\beta,r,c) - p_i|\frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial\alpha} \tag{37}$$

In the derivation below let $p' = \frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial\alpha}$. Then

$$\frac{\partial p(\alpha)^{1-\alpha}}{\partial\alpha} = \frac{\partial e^{(1-\alpha)lnp(\alpha)}}{\partial\alpha} = e^{(1-\alpha)lnp(\alpha)}\frac{\partial}{\partial\alpha}\left[(1-\alpha)lnp(\alpha)\right] = p^{1-\alpha}\left[-lnp + (1-\alpha)\frac{p'}{p}\right]$$

$$\frac{\partial p(\alpha)^{2-\alpha}}{\partial\alpha} = p^{2-\alpha}\frac{\partial}{\partial\alpha}\left[(2-\alpha)lnp(\alpha)\right] = p^{2-\alpha}\left[-lnp + (2-\alpha)\frac{p'}{p}\right]$$

We then find p' from the one term Taylor series approximation (28):

$$\frac{\partial}{\partial\alpha}\left[\frac{\beta}{(2-\alpha)}p^{(2-\alpha)} + \frac{1}{1-\alpha}p^{(1-\alpha)}\right] = \frac{\partial}{\partial\alpha}\left[rt + c\right]$$

$$\frac{\beta}{(2-\alpha)^2}p^{2-\alpha} + \frac{\beta}{(2-\alpha)}p^{2-\alpha}\left[-lnp + (2-\alpha)\frac{p'}{p}\right] + \frac{1}{(1-\alpha)^2}p^{1-\alpha} + \frac{1}{(1-\alpha)}p^{1-\alpha}\left[-lnp + (1-\alpha)\frac{p'}{p}\right] = 0$$

$$p^{1-\alpha}\left[\frac{\beta}{(2-\alpha)^2}p + \frac{1}{(1-\alpha)^2}\right] - p^{1-\alpha}lnp\left[\frac{\beta}{2-\alpha}p + \frac{1}{1-\alpha}\right] + p'p^{-\alpha}(\beta p + 1) = 0$$

$$p' = \frac{p^{1-\alpha}\left[lnp\left(\frac{\beta}{2-\alpha}p + \frac{1}{1-\alpha}\right) - \left(\frac{\beta}{(2-\alpha)^2}p + \frac{1}{(1-\alpha)^2}\right)\right]}{p^{-\alpha}(\beta p + 1)}$$

$$p' = \frac{p\left[lnp\left(\frac{\beta}{2-\alpha}p + \frac{1}{1-\alpha}\right) - \left(\frac{\beta}{(2-\alpha)^2}p + \frac{1}{(1-\alpha)^2}\right)\right]}{(\beta p + 1)}$$

So, the equation (37) can now be written as

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial\alpha} = \sum_{i=1}^{n} 2|p(t_i,\alpha,\beta,r,c) - p_i|\frac{p\left[lnp\left(\frac{\beta}{2-\alpha}p + \frac{1}{1-\alpha}\right) - \left(\frac{\beta}{(2-\alpha)^2}p + \frac{1}{(1-\alpha)^2}\right)\right]}{(\beta p + 1)} \tag{38}$$

Following the same steps we derive the second equation of the system (36)

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial\beta} = \sum_{i=1}^{n} 2|p(t_i,\alpha,r,c) - p_i|\frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial\beta} \tag{39}$$

Let $p' = \frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial \beta}$. Then

$$\frac{\partial p^{1-\alpha}}{\partial \beta} = \frac{\partial e^{(1-\alpha)lnp(\alpha)}}{\partial \beta} = e^{(1-\alpha)lnp(\alpha)}\frac{\partial}{\partial \beta}\left[(1-\alpha)lnp(\alpha)\right] = (1-\alpha)p^{1-\alpha}\frac{p'}{p} = (1-\alpha)p^{-\alpha}p'$$

$$\frac{\partial p^{2-\alpha}}{\partial \beta} = (2-\alpha)p^{2-\alpha}\frac{p'}{p} = (2-\alpha)p^{1-\alpha}p'$$

We similarly find p' from the one term Taylor series approximation (28):

$$\frac{\partial}{\partial \beta}\left[\frac{\beta}{(2-\alpha)}p^{(2-\alpha)} + \frac{1}{1-\alpha}p^{(1-\alpha)}\right] = \frac{\partial}{\partial \beta}\left[rt+c\right]$$

$$\frac{1}{(2-\alpha)}p^{(2-\alpha)} + \frac{\beta}{(2-\alpha)}(2-\alpha)p^{(1-\alpha)}p' + \frac{1}{1-\alpha}(1-\alpha)p^{-\alpha}p' = 0$$

$$p'p^{-\alpha}\left(\beta p + 1\right) + \frac{1}{2-\alpha}p^{2-\alpha} = 0$$

$$p' = \frac{-\frac{1}{(2-\alpha)}p^{(2-\alpha)}}{p^{-\alpha}\left(\beta p + 1\right)}$$

$$p' = \frac{-p^2}{(2-\alpha)\left(\beta p + 1\right)}$$

So, the equation (46) can now be written as

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial \beta} = \sum_{i=1}^{n} 2|p(t_i,\alpha,\beta,r,c) - p_i|\frac{-p^2}{(2-\alpha)\left(\beta p + 1\right)} \tag{40}$$

The third equation of the system (36) has the form:

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial r} = \sum_{i=1}^{n} 2|p(t_i,\alpha,\beta,r,c) - p_i|\frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial r} \tag{41}$$

Let $p' = \frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial r}$. Then

$$\frac{\partial p^{1-\alpha}}{\partial r} = (1-\alpha)p^{-\alpha}p'$$

$$\frac{\partial p^{2-\alpha}}{\partial r} = (2-\alpha)p^{1-\alpha}p'$$

We similarly find p' from the one term Taylor series approximation (28):

$$\frac{\partial}{\partial r}\left[\frac{\beta}{(2-\alpha)}p^{(2-\alpha)} + \frac{1}{1-\alpha}p^{(1-\alpha)}\right] = \frac{\partial}{\partial r}\left[rt+c\right]$$

$$\frac{\beta}{(2-\alpha)}(2-\alpha)p^{(1-\alpha)}p' + \frac{1}{1-\alpha}(1-\alpha)p^{-\alpha}p' = t$$

$$p^{-\alpha}p'\left(\beta p + 1\right) = t$$

$$p' = \frac{tp^{\alpha}}{(\beta p + 1)}$$

Finally, the equation (48) can now be written as

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial r} = \sum_{i=1}^{n} 2|p(t_i,\alpha,\beta,r,c) - p_i| \frac{tp^{\alpha}}{(\beta p+1)} \tag{42}$$

The fourth equation of the system (36) has the form:

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial c} = \sum_{i=1}^{n} 2|p(t_i,\alpha,\beta,r,c) - p_i| \frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial c} \tag{43}$$

Let $p' = \frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial c}$. Then

$$\frac{\partial p^{1-\alpha}}{\partial c} = (1-\alpha)p^{-\alpha}p'$$
$$\frac{\partial p^{2-\alpha}}{\partial c} = (2-\alpha)p^{1-\alpha}p'$$

We similarly find p' from the one term Taylor series approximation (28):

$$\frac{\partial}{\partial c}\left[\frac{\beta}{(2-\alpha)}p^{(2-\alpha)} + \frac{1}{1-\alpha}p^{(1-\alpha)}\right] = \frac{\partial}{\partial c}\left[rt+c\right]$$

$$\frac{\beta}{(2-\alpha)}(2-\alpha)p^{(1-\alpha)}p' + \frac{1}{1-\alpha}(1-\alpha)p^{-\alpha}p' = 1$$

$$p^{-\alpha}p'\left(\beta p+1\right) = 1$$

$$p' = \frac{1}{p^{-\alpha}\left(\beta p+1\right)}$$

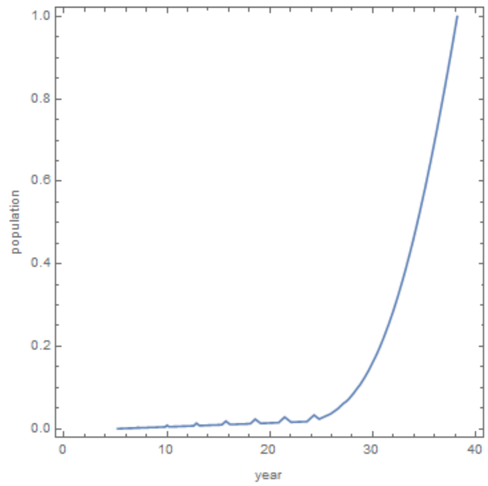
Finally, the equation (50) can now be written as

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial c} = \sum_{i=1}^{n} 2|p(t_i,\alpha,\beta,r,c) - p_i| \frac{p^{\alpha}}{\beta p+1} \tag{44}$$

We now use numerical methods computation in Python (see A3) to find the solution satisfying the system of equations (45),47, (51),(49). The derived parameters are the following:

$$\begin{aligned} \alpha &= 0.887528508 \\ \beta &= 0.999999968 \\ r &= 0.296686111 * 10^{-9} \\ c &= -1.54443853 \end{aligned}$$

Note that the numerical methods heavily depend on initial guess, since it can compute only local minimums. In our case the following initial conditions were entered: $\alpha = 0.9, \beta = 0.5, r = 0.2, c = -5$. Since the Wu-Gong population model follows the growing tendency, the value of the growth rate r should be positive and significantly bigger than 0. However, the least squares method derived in Python gave the result close to 0. That is why we rewrite the obtained value of growth rate as $r = 0.296686111$. After several modifications we now plot the population growth graph:



The graph of model approximated with one term Taylor series expansion(r=0.296686111)

6.2.2 Two-term Taylor Series Approximation

We now add the second term of Taylor series expansion and follow the same procedure for estimation of parameters through least-squares methods.

In the derivation below let $p' = \frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial \alpha}$. Then

$$\frac{\partial p(\alpha)^{3-\alpha}}{\partial \alpha} = p^{3-\alpha}\left[-lnp + (3-\alpha)\frac{p'}{p}\right]$$

We then find p' from the two term Taylor series approximation (26):

$$\frac{\partial}{\partial \alpha}\left[\frac{\beta(\beta+1)}{2(3-\alpha)}p^{3-\alpha} + \frac{\beta}{(2-\alpha)}p^{(2-\alpha)} + \frac{1}{1-\alpha}p^{(1-\alpha)}\right] = \frac{\partial}{\partial \alpha}\left[rt+c\right]$$

$$\frac{\beta(\beta+1)}{2(3-\alpha)^2}p^{3-\alpha} + \frac{\beta(\beta+1)}{2(3-\alpha)}p^{3-\alpha}\left[-lnp + (3-\alpha)\frac{p'}{p}\right] + \frac{\beta}{(2-\alpha)^2}p^{2-\alpha} + \frac{\beta}{(2-\alpha)}p^{2-\alpha}\left[-lnp + (2-\alpha)\frac{p'}{p}\right] +$$

$$+\frac{1}{(1-\alpha)^2}p^{1-\alpha} + \frac{1}{(1-\alpha)}p^{1-\alpha}\left[-lnp + (1-\alpha)\frac{p'}{p}\right] = 0$$

$$p^{1-\alpha}\left[\frac{\beta(\beta+1)}{2(3-\alpha)^2}p^2 + \frac{\beta}{(2-\alpha)^2}p + \frac{1}{(1-\alpha)^2}\right] - p^{1-\alpha}lnp\left[\frac{\beta(\beta+1)}{2(3-\alpha)}p^2 + \frac{\beta}{2-\alpha}p + \frac{1}{1-\alpha}\right] +$$

$$+p'p^{-\alpha}\left(\frac{\beta(\beta+1)}{2}p^2 + \beta p + 1\right) = 0$$

$$p' = \frac{p^{1-\alpha}\left[\left(\frac{\beta(\beta+1)}{2(3-\alpha)^2}p^2 + \frac{\beta}{(2-\alpha)^2}p + \frac{1}{(1-\alpha)^2}\right) - lnp\left(\frac{\beta(\beta+1)}{2(3-\alpha)}p^2 + \frac{\beta}{2-\alpha}p + \frac{1}{1-\alpha}\right)\right]}{p^{-\alpha}\left(\frac{\beta(\beta+1)}{2}p^2 + \beta p + 1\right)}$$

$$p' = \frac{p\left[\left(\frac{\beta(\beta+1)}{2(3-\alpha)^2}p^2 + \frac{\beta}{(2-\alpha)^2}p + \frac{1}{(1-\alpha)^2}\right) - lnp\left(\frac{\beta(\beta+1)}{2(3-\alpha)}p^2 + \frac{\beta}{2-\alpha}p + \frac{1}{1-\alpha}\right)\right]}{\frac{\beta(\beta+1)}{2}p^2 + \beta p + 1}$$

So, the equation (37) can now be written as

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial \alpha}=\sum_{i=1}^{n}2|p(t_i,\alpha,\beta,r,c)-p_i|\frac{p\left[\left(\frac{\beta(\beta+1)}{2(3-\alpha)^2}p^2+\frac{\beta}{(2-\alpha)^2}p+\frac{1}{(1-\alpha)^2}\right)-lnp\left(\frac{\beta(\beta+1)}{2(3-\alpha)}p^2+\frac{\beta}{2-\alpha}p+\frac{1}{1-\alpha}\right)\right]}{\frac{\beta(\beta+1)}{2}p^2+\beta p+1} \tag{45}$$

Following the same steps we derive the second equation of the system (36)

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial \beta}=\sum_{i=1}^{n}2|p(t_i,\alpha,r,c)-p_i|\frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial \beta} \tag{46}$$

Let $p'=\frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial \beta}$. Then

$$\frac{\partial p^{3-\alpha}}{\partial \beta}=(3-\alpha)p^{3-\alpha}\frac{p'}{p}=(3-\alpha)p^{2-\alpha}p'$$

We similarly find p' from the one term Taylor series approximation (28):

$$\frac{\partial}{\partial \beta}\left[\frac{\beta(\beta+1)}{2(3-\alpha)}p^{3-\alpha}+\frac{\beta}{(2-\alpha)}p^{(2-\alpha)}+\frac{1}{1-\alpha}p^{(1-\alpha)}\right]=\frac{\partial}{\partial \beta}\left[rt+c\right]$$

$$\frac{2\beta+1}{2(3-\alpha)}p^{3-\alpha}+\frac{\beta(\beta+1)}{2(3-\alpha)}(3-\alpha)p^{2-\alpha}p'+\frac{1}{(2-\alpha)}p^{(2-\alpha)}+\frac{\beta}{(2-\alpha)}(2-\alpha)p^{1-\alpha}p'+\frac{1}{1-\alpha}(1-\alpha)p^{-\alpha}p'=0$$

$$p'p^{-\alpha}\left(\frac{\beta(\beta+1)}{2}p^2+\beta p+1\right)+p^{2-\alpha}\left[\frac{2\beta+1}{2(3-\alpha)}p+\frac{1}{2-\alpha}\right]=0$$

$$p'=\frac{-p^{2-\alpha}\left[\frac{2\beta+1}{2(3-\alpha)}p+\frac{1}{2-\alpha}\right]}{p^{-\alpha}\left(\frac{\beta(\beta+1)}{2}p^2+\beta p+1\right)}$$

$$p'=\frac{-p^{2}\left[\frac{2\beta+1}{2(3-\alpha)}p+\frac{1}{2-\alpha}\right]}{\frac{\beta(\beta+1)}{2}p^2+\beta p+1}$$

So, the equation (46) can now be written as

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial \beta}=\sum_{i=1}^{n}2|p(t_i,\alpha,\beta,r,c)-p_i|\frac{-p^{2}\left[\frac{2\beta+1}{2(3-\alpha)}p+\frac{1}{2-\alpha}\right]}{\left(\frac{\beta(\beta+1)}{2}p^2+\beta p+1\right)} \tag{47}$$

The third equation of the system (36) has the form:

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial r}=\sum_{i=1}^{n}2|p(t_i,\alpha,\beta,r,c)-p_i|\frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial r} \tag{48}$$

Let $p'=\frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial r}$. Then

$$\frac{\partial p^{3-\alpha}}{\partial r}=(3-\alpha)p^{2-\alpha}p'$$

We similarly find p' from the one term Taylor series approximation (28):

$$\frac{\partial}{\partial r}\left[\frac{\beta(\beta+1)}{2(3-\alpha)}p^{3-\alpha}+\frac{\beta}{(2-\alpha)}p^{(2-\alpha)}+\frac{1}{1-\alpha}p^{(1-\alpha)}\right]=\frac{\partial}{\partial r}\left[rt+c\right]$$

$$\frac{\beta(\beta+1)}{2(3-\alpha)}(3-\alpha)p^{2-\alpha}p'+\frac{\beta}{(2-\alpha)}(2-\alpha)p^{1-\alpha}p'+\frac{1}{1-\alpha}(1-\alpha)p^{-\alpha}p'=t$$

$$p^{-\alpha}p'\left(\frac{\beta(\beta+1)}{2}p^2+\beta p+1\right)=t$$

$$p'=\frac{tp^{\alpha}}{\left(\frac{\beta(\beta+1)}{2}p^2+\beta p+1\right)}$$

Finally, the equation (48) can now be written as

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial r}=\sum_{i=1}^{n}2|p(t_i,\alpha,\beta,r,c)-p_i|\frac{tp^{\alpha}}{\left(\frac{\beta(\beta+1)}{2}p^2+\beta p+1\right)} \tag{49}$$

The fourth equation of the system (36) has the form:

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial c}=\sum_{i=1}^{n}2|p(t_i,\alpha,\beta,r,c)-p_i|\frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial c} \tag{50}$$

Let $p'=\frac{\partial p(t_i,\alpha,\beta,r,c)}{\partial c}$. Then

$$\frac{\partial p^{3-\alpha}}{\partial c}=(3-\alpha)p^{2-\alpha}p'$$

We similarly find p' from the one term Taylor series approximation (28):

$$\frac{\partial}{\partial c}\left[\frac{\beta(\beta+1)}{2(3-\alpha)}p^{3-\alpha}+\frac{\beta}{(2-\alpha)}p^{(2-\alpha)}+\frac{1}{1-\alpha}p^{(1-\alpha)}\right]=\frac{\partial}{\partial c}\left[rt+c\right]$$

$$\frac{\beta(\beta+1)}{2(3-\alpha)}(3-\alpha)p^{2-\alpha}p'+\frac{\beta}{(2-\alpha)}(2-\alpha)p^{1-\alpha}p'+\frac{1}{1-\alpha}(1-\alpha)p^{-\alpha}p'=1$$

$$p^{-\alpha}p'\left(\frac{\beta(\beta+1)}{2}p^2+\beta p+1\right)=1$$

$$p'=\frac{p^{\alpha}}{\left(\frac{\beta(\beta+1)}{2}p^2+\beta p+1\right)}$$

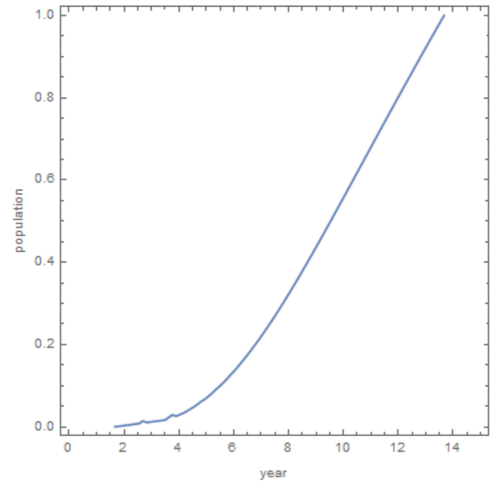
Finally, the equation (50) can now be written as

$$\frac{\partial F(\alpha,\beta,r,c)}{\partial c}=\sum_{i=1}^{n}2|p(t_i,\alpha,\beta,r,c)-p_i|\frac{p^{\alpha}}{\left(\frac{\beta(\beta+1)}{2}p^2+\beta p+1\right)} \tag{51}$$

We now use numerical methods computation in Python (see A4) to find the solution satisfying the system of equations (45),(47), (51),(49). The derived parameters are the following:

$$\begin{aligned}\alpha &= 0.62269113\\ \beta &= 0.81494785\\ r &= 0.29618363\\ c &= -0.49396155\end{aligned}$$

The obtained parameters satisfy the expected interval and thus we the corresponding plot is the following:



The graph of model approximated with two term Taylor series expansion

7 Actuarial Notations for the Generalized Logistic Models

Out introduced generalized logistic models (17) and (18) can be applied in the calculation of actuarial quantities such as the survival functions and mortality rate (also called hazard function in biostatistics and epidemiology). On the example of the Gompertz-Makeham model we will show how to correlate the population growth function with such actuarial quantities as the survival functions and mortality rate (also called hazard function in biostatistics and epidemiology).

7.1 Actuarial functions for the Gompertz-Makeham model

Recall from Dickson et al. [14] equations (2.12) and (3.1) that the survival function $S_x(t)$, i.e. the probability of (x) survival for at least t years is given by the following property:

$$S_x(t) = {}_tp_x = \frac{l_{x+t}}{l_x} \tag{52}$$

where l_x is the the expected number of survivors at age x. The expected number of survivors can be calculated from a model which predicts the population number within given predictor of time t (in this case age). The Gompertz model (4) discussed in the section 2.1 is one of those models, commonly used in the insurance industry. The force of mortality at age $x+t$ is then found by using Dickson et al.'s equation (2.16) [14] as

$$\mu_{x+t} = -\frac{1}{{}_tp_x}\frac{d}{dx}{}_tp_x = -\frac{d}{dx}ln\left({}_tp_x\right) = -\frac{d}{dx}ln\left(\frac{l_{x+t}}{l_x}\right) \tag{53}$$

$$\left[\frac{(r(x+t)+C)\left(\frac{\beta(\beta+1)}{2(3-\alpha)}p(x)^2+\frac{\beta}{(2-\alpha)}p(x)+\frac{1}{(1-\alpha)}p(x)\right)}{(rx+C)\left(\frac{\beta(\beta+1)}{2(3-\alpha)}p(x+t)^2+\frac{\beta}{(2-\alpha)}p(x+t)+\frac{1}{(1-\alpha)}p(x+t)\right)}\right]^{\frac{1}{1-\alpha}}$$

$$\left[\frac{rx+C}{\frac{\beta(\beta+1)}{2(3-\alpha)}p(x)^2+\frac{\beta}{(2-\alpha)}p(x)+\frac{1}{(1-\alpha)}p(x)}\right]^{\frac{1}{1-\alpha}}$$

For the Gompertz model (4) the force of mortality is obtained by the following equations:

$$\mu_{x+t}=-\frac{d}{dx}ln\left(ae^{-be^{-k(x+t)}}\right)=-bke^{-k(x+t)} \tag{54}$$

Let $B=-bk$ and $a=-k$. The resulting force of mortality at age $(x+t)$ is the given by

$$\mu_{x+t}=Be^{a(x+t)} \tag{55}$$

Along with equation (55) Gompertz (1825) also introduces another formulation of the force of mortality by taking $c=lna$:

$$\mu_{x+t}=Bc^{(x+t)} \tag{56}$$

Makeham (1860) added the extra parameter A that considers the force of accidental death. The resulting equation is called the Gompertz-Makeham force of mortality and defined by

$$\mu_{x+t}=A+Bc^{(x+t)} \tag{57}$$

The survival function can be obtained directly from the force of mortality by the using following equation [14]

$$S_x(t)=exp\bigg(-\int_0^t\mu_{x+s}ds\bigg) \tag{58}$$

The Gompertz-Makeham survival function is then equal to

$$S_x(t)=exp\bigg(-\int_0^t A+Bc^{(x+s)}ds\bigg)=exp\left[-At-\frac{B}{lnc}c^x\left(c^t-1\right)\right] \tag{59}$$

7.2 Actuarial functions for the one-parameter generalized logistic model

We will now use our generalized logistic models (17) and (18) for the calculations of the expected number of survivors (l_x) at age x. Following the same procedure as for Gompertz-Makeham model we obtain our own force of mortality and associated survival function. We begin with the derivations of actuarial notation for the one-parameter generalized logistic model(17). First, we obtain the following force of mortality based on the population model (17):

$$\mu_{x+t}=-\frac{d}{dx}ln\left[\frac{r(t+x)(1-\alpha)+B}{rx(1-\alpha)+B}\frac{{}_2F_1\big(1,1-\alpha;2-\alpha;p(x)\big)}{{}_2F_1\big(1,1-\alpha;2-\alpha;p(x+t)\big)}\right]^{\frac{1}{1-\alpha}} \tag{60}$$

$$\mu_{x+t}=-\frac{d}{dx}ln\left[\frac{(r(x+t)+C)\left(\frac{1}{(3-\alpha)}p(x)^2+\frac{1}{(2-\alpha)}p(x)+\frac{1}{(1-\alpha)}p(x)\right)}{(rx+C)\left(\frac{1}{(3-\alpha)}p(x+t)^2+\frac{1}{(2-\alpha)}p(x+t)+\frac{1}{(1-\alpha)}p(x+t)\right)}\right]^{\frac{1}{1-\alpha}}$$

Then, the survival function is obtained by using equation (58).

7.3 Actuarial functions for the two-parameter generalized logistic model

Similarly to the one-parameter generalized logistic model we obtain the force of mortality for the two-parameter case:

$$\mu_{x+t} = -\frac{d}{dx} ln \left[\frac{r(t+x)(1-\alpha)+B}{rx(1-\alpha)+B} \frac{{}_2F_1\left(1-\alpha,\beta;2-\alpha;p(x)\right)}{{}_2F_1\left(1-\alpha,\beta;2-\alpha;p(x+t)\right)} \right]^{\frac{1}{1-\alpha}} \tag{61}$$

$$\mu_{x+t} = -\frac{d}{dx} ln \left[\frac{(r(x+t)+C)\left(\frac{\beta(\beta+1)}{2(3-\alpha)}p(x)^2 + \frac{\beta}{(2-\alpha)}p(x) + \frac{1}{(1-\alpha)}p(x)\right)}{(rx+C)\left(\frac{\beta(\beta+1)}{2(3-\alpha)}p(x+t)^2 + \frac{\beta}{(2-\alpha)}p(x+t) + \frac{1}{(1-\alpha)}p(x+t)\right)} \right]^{\frac{1}{1-\alpha}}$$

Conclusion

The project describes the generalized version of the classical logistic and the Gompertz models.The solution of these models is complex, since requires nonlinear implicit regression analysis. To find fitting parameters Taylor series approximation with several terms was used. The future work is expected to expand number of terms in the approximation series to make it closer to the original model.The suggested model was shown to have application in actuarial math as the force of mortality or hazard function.

Appendix A.

Data Sets and Mathematical Computations.

A1. Data set used in the Example 1 on page 6.

Chloride ion concentration versus time.

Time (min)	Conc. (%)	Time (min)	Conc. (%)	Time (min)	Conc. (%)
2.45	17.3	4.25	22.6	6.05	26.6
2.55	17.6	4.35	22.8	6.15	27.0
2.65	17.9	4.45	23.0	6.25	27.0
2.75	18.3	4.55	23.2	6.35	27.0
2.85	18.5	4.65	23.4	6.45	27.0
2.95	18.9	4.75	23.7	6.55	27.3
3.05	19.0	4.85	24.0	6.65	27.8
3.15	19.3	4.95	24.2	6.75	28.1
3.25	19.8	5.05	24.5	6.85	28.1
3.35	19.9	5.15	25.0	6.95	28.1
3.45	20.2	5.25	25.4	7.05	28.4
3.55	20.5	5.35	25.5	7.15	28.6
3.65	20.6	5.45	25.9	7.25	29.0
3.75	21.1	5.55	25.9	7.35	29.2
3.85	21.5	5.65	26.3	7.45	29.3
3.95	21.9	5.75	26.2	7.55	29.4
4.05	22.0	5.85	26.5	7.65	29.4
4.15	22.3	5.95	26.5	7.75	29.4

Reproduced from J. Sredni, "Problems of Design, Estimation, and Lack of Fit in Model Building," Ph.D. Thesis, University of Wisconsin Madison, 1970, with permission of the author [11].

A2. R-code used in the Example on page 6 for the data set A1.

> "Chloride"< − structure(.Data = list(time = c(2.45, 2.55, 2.65, 2.75, 2.85, 2.95, 3.05, 3.15, 3.25, 3.35, 3.45, 3.55, 3.65, 3.75, 3.85, 3.95, 4.05, 4.15, 4.25, 4.35, 4.45, 4.55, 4.65, 4.75, 4.85, 4.95, 5.05, 5.15, 5.25, 5.35, 5.45, 5.55, 5.65, 5.75, 5.85, 5.95, 6.05, 6.15, 6.25, 6.35, 6.45, 6.55, 6.65, 6.75, 6.85, 6.95, 7.05, 7.15, 7.25, 7.35, 7.45, 7.55, 7.65, 7.75), conc = c(17.3, 17.6, 17.9, 18.3, 18.5, 18.9, 19, 19.3, 19.8, 19.9, 20.2, 20.5, 20.6, 21.1, 21.5, 21.9, 22, 22.3, 22.6, 22.8, 23, 23.2, 23.4, 23.7, 24, 24.2, 24.5, 25, 25.4, 25.5, 25.9, 25.9, 26.3, 26.2, 26.5, 26.5, 26.6, 27, 27, 27, 27, 27.3, 27.8, 28.1, 28.1, 28.1, 28.4, 28.6, 29, 29.2, 29.3, 29.4, 29.4, 29.4)), row.names = c("1", "2", "3", "4", "5", "6", "7", "8", "9", "10", "11", "12", "13", "14", "15", "16", "17", "18", "19", "20", "21", "22", "23", "24", "25", "26", "27", "28", "29", "30", "31", "32", "33", "34", "35", "36", "37", "38", "39", "40", "41", "42", "43", "44", "45", "46", "47", "48", "49", "50", "51", "52", "53", "54"), class = "data.frame", reference = "A1.9, p. 274")

> plot(Chloride)

> lm(log(-log(conc/40))∼time,Chloride)

Call:

lm(formula = log(-log(conc/40))∼ time, data = Chloride)

Coefficients:

(Intercept)	time
0.2774	-0.1940

> pop.mod< −nls(conc∼A*exp(-exp(-k*time+c)),start=list(A=40,k=0.2774,c=-0.194),

+data=Chloride,trace=TRUE)

3519.411 :	40.0000	0.2774	-0.1940
23.26283 :	35.5525109	0.2357100	0.2689087
1.851928 :	35.1739845	0.2635205	0.3107111
1.735086 :	35.3602509	0.2621978	0.3145172
1.735054 :	35.3597718	0.2622279	0.3145367

> summary(pop.mod)

Formula: conc ∼ A * exp(-exp(-k * time + c))

Parameters:

	Estimate	Std. Error	t value	Pr(> |t|)
A	35.35977	0.51230	69.02	<2e-16 ***
k	0.26223	0.01038	25.25	<2e-16 ***
c	0.31454	0.01251	25.14	<2e-16 ***

− − −

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1844 on 51 degrees of freedom

Number of iterations to convergence: 4

Achieved convergence tolerance: 3.112e-06

> with(Chloride,lines(seq(1,60,by=1),predict(pop.mod,data.frame(time=seq(1,60,by=1))),col="red"))

> lm(logit(conc/40)~time,Chloride)

Call:

lm(formula = logit(conc/40) ~ time, data = Chloride)

Coefficients:

(Intercept)	time
-0.8214	0.2480

> pop.mod2< −nls(conc~A/(1+exp(-r*time+c)),start=list(A=40,c=-0.8214,r=0.248),

+data=Chloride,trace=TRUE)

6932.902 :	40.0000	-0.8214	0.2480
801.2023 :	35.6078627	1.0908199	0.2737039
218.9281 :	29.7959812	0.6464326	0.3563680
2.898035 :	33.5059654	0.8526074	0.3645205
1.693147 :	33.4612307	0.8312563	0.3658883
1.69305 :	33.4612124	0.8314142	0.3659651

> summary(pop.mod2)

Formula: conc ~ A/(1 + exp(-r * time + c))

Parameters:

	Estimate	Std. Error	t value	Pr(> |t|)
A	33.46121	0.34452	97.12	<2e-16 ***
c	0.83141	0.01577	52.72	<2e-16 ***
r	0.36597	0.01068	34.28	<2e-16 ***

− − −

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1822 on 51 degrees of freedom

Number of iterations to convergence: 5 Achieved convergence tolerance: 5.808e-06

> with(Chloride,lines(seq(1,60,by=1),predict(pop.mod2,data.frame(time=seq(1,60,by=1))),

+col="blue"))

```
> legend(5.2,23, c("Gompertz model","Logistic model"),lty=c(1,1), lwd=c(2.5,2.5),
+col=c("red","blue"))
```

A3. Python Code for one-term Taylor Series approximation

```
In [61]: import numpy as np
         import pandas as pd
         import scipy.optimize as sc
         import math
```

```
In [62]: data = pd.read_csv("Data_WuGong.csv",sep=',')
         data["year"] = np.arange(0,56,1)
         data[0:5]
```

```
Out[62]:    year  population
         0     0       14.16
         1     1       14.48
         2     2       14.81
         3     3       14.96
         4     4       15.38
```

```
In [63]: def func(p):
             x1, x2, x3, x4, x5 = p # x1 - p, x2 - alpha, x3 - beta, x4 -r, x5 - c
             f = []
             f1 = 0
             f2 = 0
             f3 = 0
             f4 = 0
             def p_fun(n):
                 return (x3/(2-x2))*(x1**(2-x2)) + (1/(1-x2))*(x1**
                 (1-x2)) - x4*data["year"][n] - x5

             for i in range(56):

                 f1 += 2*abs(p_fun(i) - data["population"][i])* (p_fun(i)) * (
                      math.log(abs(p_fun(i)))*((x3*(p_fun(i))/(2-x2))+(1/ (
                1-x2)))-(x3*(p_fun(i))/((2-x2)**2) + (1/((1-x2)**2)) ))/( x3*p_fun(i) + 1 )
                 f2 += 2*abs(p_fun(i) - data["population"][i]
                 ) * ((p_fun(i))**2) / ((x3*(p_fun(i)) + 1)*(x2-2))
                 f3 += 2*abs(p_fun(i) - data["population"][i]) * (data["year"
                 ][i]*((p_fun(i))**x2)) /( x3*p_fun(i) + 1)
                 f4 += 2*abs(p_fun(i) - data["population"][
                 i]) * ((p_fun(i))**x2) /(x3*p_fun(i) + 1)
                 f.append(f1)
                 f.append(f2)
                 f.append(f3)
                 f.append(f4)
             return tuple(f)
```

```
        x0 = np.array([1.,0.9,0.7,0.2, -5])
        x = sc.least_squares(func, x0, bounds = ((0,0,0,0,-np.inf),(np.inf,1,1,np.inf,np.inf))
```

```
In [64]: x
```

```
Out[64]:  active_mask: array([ 0,  0,  0, -1,  0])
                 cost: 4189059707.1885786
                  fun: array([ 2.54137930e+01, -4.59725563e+02,0.00000000e+00,1.43328421e+01,
                                 [-4.47372437e+01,  3.65309982e+02, -6.16362328e+02,
                  1.56221715e+03,  2.06107699e+01]])
              message: '`ftol` termination condition is satisfied.'
                 nfev: 29
                 njev: 18
           optimality: 3197003957.261415
               status: 2
              success: True
                    x: array([ 8.19953051e-01,  8.87528508e-01,  9.99999968e-01,
                    2.96686111e-10, -1.54443853e+01])
```

A3. Python Code for two-term Taylor Series approximation

```
In [18]: import numpy as np
         import pandas as pd
         import scipy.optimize as sc
         import math
```

```
In [19]: data = pd.read_csv("Data_WuGong.csv",sep=',')
         data["year"] = np.arange(0,56,1)
         data[0:5]
```

Out[19]:

	year	population
0	0	14.16
1	1	14.48
2	2	14.81
3	3	14.96
4	4	15.38

```
In [20]: def func(p):
             x1, x2, x3, x4, x5 = p # x1 - p, x2 - alpha, x3 - beta, x4 -r, x5 - c
             f = []
             f1 = 0
             f2 = 0
             f3 = 0
             f4 = 0
             def p_fun(n):
                 return (x3*(x3+1)/(2*(3-x2)))*(x1**(3-x2))+(x3/(2
                 -x2))*(x1**(2-x2)) + (1/(1-x2))*(x1**(1-x2)) - x4*data["year"][n] - x5

             for i in range(56):

                 f1 += 2*abs(p_fun(i) - data["population"][i])* (p_fun(i)) * (math
                 .log(abs(p_fun(i)))*((x3*(x3+1)*((p_fun(i))**2)/(2*
                 (3-x2)))+(x3*(p_fun(i))/(2-x2))+(1/(1-x2)))-
                 (x3*(x3+1)*((p_fun(i))**2)/(2*((3-x2)**2))+x3
                 *(p_fun(i))/((2-x2)**2) + (1/((1-x2)**2)) ))/
                 ((x3*(x3+1)*((p_fun(i))**2)/2)+x3*p_fun(i) + 1)
                 f2 += 2*abs(p_fun(i) - data["population"][
                 i]) * ((-1)*(p_fun(i))**2)*((2*x3+1)*(p_fun(i))/
                 (2*(3-x2))+1/(1-x2)) / ((x3*(x3+1)*(p_fun(i)**
                 2))/2+x3*(p_fun(i)) + 1)
                 f3 += 2*abs(p_fun(i) - data["population"][
                 i]) * (data["year"][i]*((p_fun(i))
                 **x2)) /((x3*(x3+1)*((p_fun(i))**
                 2)/2)+x3*(p_fun(i)) + 1)
                 f4 += 2*abs(p_fun(i) - data["population"][
```

```
                i]) * ((p_fun(i))**x2) /((x3*(x3+1)*
                ((p_fun(i))**2)/2)+x3*(p_fun(i)) + 1)
                f.append(f1)
                f.append(f2)
                f.append(f3)
                f.append(f4)
            return tuple(f)

        x0 = np.array([1.,0.9,0.7,0.2, -5])
        x = sc.least_squares(func, x0, bounds = ((0,0,0,0,-
        np.inf),(np.inf,1,1,np.inf,np.inf)))
```

```
In [21]: x
```

```
Out[21]:  active_mask: array([0, 0, 0, 0, 0])
                 cost: 2979227649.920868
                  fun: array([ 7.11741831e+01, -6.26180288e+01, 0.00000000e+00,1.15021127e-01,
                 1.20459938e+02, -1.06438053e+02,  8.37281178e-02,  1.98749245e-01,
                                 [ 7.43572231e+04, -4.82534539e+04,  2.13608439e+05,
                             nan, -1.29231747e+04]])
              message: '`xtol` termination condition is satisfied.'
                 nfev: 35
                 njev: 15
           optimality: nan
               status: 3
              success: True
                    x: array([ 3.83952961,  0.62269113,  0.81494785,  0.29618363, -0.49396155])
```

References

[1] Koya, R.P., and Goshu A.T. *"Solutions of Rate-state Equation Describing Biological Growths"*. 2013. American Journal of Mathematics and Statistics, 3(6):305-311. DOI: 10.5923/j.ajms.20130306.02

[2] Yao Zheng. The Power Law Population Growth Model. 2007

[3] Fox, J., and Weisberg, S. Nonlinear Regression and Nonlinear Least Squares in R. An Appendix to An R Companion to Applied Regression. 2010, second edition.

[4] Nguimkeu, Pierre. A simple selection test between the Gompertz and Logistic growth models. 2014. Technological Forecasting and Social Change, 88: 98-105, ISSN 0040-1625. http://dx.doi.org/10.1016/j.techfore.2014.06.017. (http://www.sciencedirect.com/science/article/pii/S0040162514002170)

[5] Tjørve KMC, Tjørve E. 2017. The use of Gompertz models in growth analyses, and new Gompertz-model approach: An addition to the Unified-Richards family. PLoS ONE 12(6): e0178691. https://doi.org/10.1371/journal.pone.0178691

[6] Trappey, Charles V and Wu, Hsin-ying. "An Evaluation of the Extended Logistic, Simple Logistic, and Gompertz Models for Forecasting Short Lifecycle Products and Services". 2008. Advanced Engineering Mathematics, 22 (4): 421-430, ISSN 1474-0346. http://dx.doi.org/10.1016/j.aei.2008.05.007. (http://www.sciencedirect.com/science/article/pii/S1474034608000372)

[7] Even Tjørve, Kathleen M.C. Tjørve, A unified approach to the Richards-model family for use in growth analyses: Why we need only two model forms, Journal of Theoretical Biology, Volume 267, Issue 3, 2010, Pages 417-425, ISSN 0022-5193, http://dx.doi.org/10.1016/j.jtbi.2010.09.008. (http://www.sciencedirect.com/science/article/pii/S0022519310004741)

[8] Berger, R.D. Comparison of the Gompertz and Logistic Equations to Describe Plant Disease Progress. Phytopathology 71:716-719. Accepted for publication 13 August 1980. Copyright The American Phytopathological Society. DOI: 10.1094/Phyto-71-716.

[9] Missov, T.I., Lenart, A., and Vaupel, J.W. Gompertz-Makeham Life Expectancies: Analytical Solutions, Approximations, and Inferences. 2012. Paper presented at Population Association of America 2012 Conference, San Francisco, United States. (http://paa2012.princeton.edu/papers/121013)

[10] Dmitry Kucharavy, Roland De Guio. "Application of Logistic Growth Curve". Procedia Engineering, Volume 131, 2015, Pages 280-290, ISSN 1877-7058, http://dx.doi.org/10.1016/j.proeng.2015.12.390. (http://www.sciencedirect.com/science/article/pii/S1877705815042745)

[11] Douglas M. Bates, Donald G. Watts.Nonlinear Regression Analysis and Its Applications.Published Online: 27 MAY 2008.Online ISBN: 9780470316757. DOI: 10.1002/9780470316757.refs (http://www.stat.wisc.edu/courses/st849-bates/slides/nraia2.pdf)

[12] Abramowitz, M. and I.Stegun. 1965. "Handbook of Mathematical Functions." Chapter 15: Hypergeometric Funcstions, p. 556-566. Washington, DC:US Government Printing Office

[13] Michael Kutner, Christopher J. Nachtsheim, John Neter, William Li. 2004. "Applied Linear Statistical Models". McGraw-Hill/Irwin. ISBN 13: 9780073108742

[14] Dickson, D., Hardy, M., Waters, H. (2009). Actuarial Mathematics for Life Contingent Risks (International Series on Actuarial Science). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511800146