Well-posedness of nonlinear Schrodinger-Airy type equation in Weighted Sobolev Spaces

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1 Introduction

Water waves are complex and important physical phenomena to explore in order to predict disastrous events. While the full equations for water wave motion are known, they are often too complicated to study or simulate directly, especially over long time scales. So, researchers have developed simplified models that capture the essential features of wave behavior in specific physical regimes.

One such model is the Dysthe equation, an extension of the well-known nonlinear Schrödinger (NLS) equation, that includes higher-order effects that are important for accurately modeling extreme wave events.

Despite its practical relevance, the Dysthe equation is mathematically more challenging than the NLS equation. In particular, understanding the well-posedness of its initial value problem — whether solutions exist, are unique, and depend continuously on the initial data — is essential for both theoretical and applied purposes.

In this work, we focus on the one-dimensional Dysthe equation and study its local well-posedness in Sobolev spaces. In [1], a similar equation was proved to be well-posed, but with a constraint on coefficients, and now we improve the result by removing the constraint, but prove the well-posedness in a less regular weighted Sobolev space.

The one-dimensional case of the Dysthe equation [2] is defined to be as follows.

$$\partial_t u + \frac{\omega}{2k_0}\partial_x u + i\frac{\omega}{8k_0^2}\partial_x^2 u - \frac{1}{16}\frac{\omega}{k_0^3}\partial_x^3 u \\ + \frac{i}{2}\omega k_0^2|u|^2 u - \frac{\omega k_0}{4}u^2\partial_x u + \frac{3}{2}\omega k_0|u|^2\partial_x u + i\frac{\omega k_0}{2}u\partial_x\mathcal{L}_h(|u|^2) = 0$$

where $\mathcal{L}_h$ is defined in Fourier variables by

$$\widehat{\mathcal{L}_h f}(\xi) = i\ \mathrm{sgn}(\xi)\coth(h|\xi|)\widehat{f}(\xi).$$

For simplicity, we consider an equation with independent real parameters and also set $h = +\infty$, which in applications refers to the analysis of waves in the deep water. The second term is usually dropped because we can do change of variables, so we have

$$\begin{cases} \partial_t u + ia\partial_x^2 u + b\partial_x^3 u + \\ \qquad +ic|u|^2 u + du^2\partial_x u + e|u|^2\partial_x u + ifu\partial_x\mathcal{L}_\infty(|u|^2) = 0 \\ u_0(x) = u(x,0) \end{cases} \tag{1}$$

And further we consider the similar Schrödinger-airy type equation

$$\begin{cases} \partial_t u + ia\partial_x^2 u + b\partial_x^3 u + \\ \qquad +ic|u|^2 u + du^2\partial_x u + e|u|^2\partial_x u = 0 \\ u_0(x) = u(x,0) \end{cases} \tag{2}$$

2 Preliminaries

For a function $f \in L^2(\mathbb{R})$, consider its Fourier transform

$$\mathcal{F}(f) = \widehat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx, \quad \xi \in \mathbb{R}$$

and its inverse Fourier transform by

$$\mathcal{F}^{-1}(f) = f^{\vee}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}.$$

In this work we use the inhomogeneous Sobolev space $H^s(\mathbb{R})$, of order $s \in \mathbb{R}$, defined as

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2},$$

which satisfy $H^{s'}(\mathbb{R}) \subset H^s(\mathbb{R})$ for $s \leq s'$, meaning,

$$\|f\|_{H^s} \lesssim \|f\|_{H^{s'}}.$$

To measure the regularity of functions defined in the space-time domain $\mathbb{R} \times [0, T]$, we introduce the mixed-norm Lebesgue spaces $L^p_x L^q_T$ or $L^q_T L^p_x$, $1 \leq p, q \leq \infty$, given respectively by the norms

$$\|f\|_{L^p_x L^q_T} := \left(\int_{\mathbb{R}} \left(\int_0^T |f(x,t)|^q\, dt \right)^{p/q} dx \right)^{1/p}$$

$$\|f\|_{L^q_T L^p_x} := \left(\int_0^T \left(\int_{\mathbb{R}} |f(x,t)|^p\, dx \right)^{q/p} dt \right)^{1/q}$$

with the standard modifications involving the essential supremum when p or q are equal to infinity.

For $\alpha \in \mathbb{C}$, we define the fractional derivative D^α_x as the Fourier multiplier given by

$$(D^\alpha_x f)^\wedge(\xi) = |\xi|^\alpha \hat{f}(\xi).$$

Analogously, we introduce the operator $(1 + D^2_x)^\alpha$ via

$$\left((1 + D^2_x)^\alpha f\right)^\wedge(\xi) = (1 + |\xi|^2)^\alpha \hat{f}(\xi).$$

Hence, the Plancherel identity allows us to write

$$\|f\|_{H^s} \sim \left\| (1 + D^2_x)^{s/2} f \right\|_{L^2} \lesssim \|f\|_{L^2} + \|D^s_x f\|_{L^2}.$$

Also, if we invoke the Hilbert transform $\mathcal{H}$ determined by

$$(\mathcal{H}f)^\wedge(\xi) = -i\mathrm{sgn}(\xi)\hat{f}(\xi),$$

we can relate D_x with the standard derivative ∂_x as $D_x = \mathcal{H}\partial_x$ or $\partial_x = \mathcal{H}D_x$.

The terms with L_∞ require some estimations. Since $\coth(h|\xi|) \longrightarrow 0$ as $h \longrightarrow \infty$,

$$\widehat{\mathcal{L}_\infty f}(\xi) = i\,\mathrm{sgn}(\xi)\hat{f}(\xi) \Rightarrow \mathcal{L}_\infty f(\xi) = -\mathcal{H}f(\xi) \tag{3}$$

where $\mathcal{H}$ is Hilbert transform. It satisfies the following property (see [4]) for $p, q \in (1, \infty)$

$$\|\mathcal{H}f\|_{L^p_x L^q_T} \lesssim \|f\|_{L^p_x L^q_T} \tag{4}$$

As for some useful estimates for the proofs in the work, we use the fractional Leibniz-rule type inequality. For $1 < p, p_1, p_2, q, q_2 < \infty$, $1 < q_1 \leq \infty$ satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$$

the following holds true (see theorem A.8 of [5])

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p L_T^q} \lesssim \|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}} \tag{5}$$

provided that $\alpha = \alpha_1 + \alpha_2$, $0 \leq \alpha_1, \alpha_1 \leq \alpha$. Using this inequality with $\alpha_1 = 0$ and Holder inequality, one can deduce

$$\|D_x^\alpha(fg)\|_{L_x^p L_T^q} \lesssim \|f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^\alpha g\|_{L_x^{p_2} L_T^{q_2}} + \|gD_x^\alpha f\|_{L_x^p L_T^q} \tag{6}$$

For some $\alpha > 0$ (see Lemma 4.7 in [8])

$$\|u\|_{L_T^5 L_x^\infty} \lesssim T^\alpha \left(\|u\|_{L_x^5 L_T^{10}} + \|D_x^{\frac{1}{4}} u\|_{L_x^5 L_T^{10}} \right) \tag{7}$$

3 Well posedness in Sobolev space

As mentioned in the introduction, we first prove that the one-dimensional Dysthe equation has a unique solution in a Sobolev space.

Theorem 1. *For $s \geq 3/4$ any $u_0 \in H^s(\mathbb{R})$ there exists an interval of time $[-T, T]$ such that $T = T(\|u_0\|_{H^{3/4}}) > 0$, with $T(\rho) \to \infty$ as $\rho \to 0$, and a unique strong solution $u(x,t)$ of the equation (1).*

Denote $N(u)$ or $N(u)(x,t)$ as the non-linear part of the equation. Applying Fourier transform and solving the differential equation w.r.t. t we get

$$\partial_t \widehat{u}(\xi, t) + (a\xi^2 + b\xi^3)\widehat{u}(\xi, t) = \widehat{N}(u)(\xi, t)$$

$$\widehat{u}(\xi, t) = e^{i(a\xi^2+b\xi^3)t}\widehat{u}_0(\xi) + \int_0^t e^{i(a\xi^2+b\xi^3)(t-\tau)}\widehat{N}(u)(\xi, \tau)d\tau$$

$$u(x,t) = S(t)u_0(x) + \int_0^t S(t-\tau)N(u)(x,\tau)d\tau \tag{8}$$

where $S(t) = \mathcal{F}^{-1}\left(e^{i(a\xi^2+b\xi^3)(t)}\right)$.

Consider the mapping

$$\Phi(u) := S(t)u_0(x) + \int_0^t S(t-\tau)N(u)(x,\tau)d\tau \tag{9}$$

and for some $\rho, T > 0$, that we will fix later, define the complete metric space

$$X_T^\rho := \{u \ : \ \|u\|_{X_T} \leq \rho\}$$

with the norm

$$\|u\|_{X_T} := \sum_{i=1}^{5} \mu_i^T(u) \tag{10}$$

where

$$\mu_1^T(u) := \|u\|_{L_T^\infty L_x^2} + \|D_x^{3/4}u\|_{L_T^\infty L_x^2}$$
$$\mu_2^T(u) := \|\partial_x u\|_{L_x^\infty L_T^2} + \|D_x^{3/4}u_x\|_{L_x^\infty L_T^2}$$
$$\mu_3^T(u) := \|\partial_x u\|_{L_x^{20} L_T^{5/2}}$$
$$\mu_4^T(u) := \|u\|_{L_x^5 L_T^{10}} + \|D_x^{3/4}u\|_{L_x^5 L_T^{10}}$$
$$\mu_5^T(u) := \|u\|_{L_x^4 L_T^\infty}$$

Our aim is to show that $\Phi(x,t)$ is a contraction on X_T^ρ, so the Banach fixed-point theorem will guarantee the existence and uniqueness of $u \in X_T^\rho$ with $u = \Phi(u)$, which in particular verifies (1).

Lemma 1. *(Theorem 4.1 of [8]) If $u \in H^s(\mathbb{R}), s \geq 3/4$, then*

$$\mu_i^T(S(t)u) \lesssim \|u\|_{H^s} \tag{11}$$

Lemma 2. *For some function $F(u)(x,t)$*

$$\sum_{i=1}^{5} \mu_i^T\left(\int_0^t S(t-\tau)F(u)(x,\tau)d\tau\right) \lesssim T^{\frac{1}{2}}\left(\|F(u)\|_{L_x^2 L_T^2} + \|D_x^{3/4}F(u)\|_{L_x^2 L_T^2}\right) \tag{12}$$

Proof. By Minkowski inequality, Lemma 1, and Holder inequality, respectively, we get

$$\sum_{i=1}^{5} \mu_i^T\left(\int_0^t S(t-\tau)F(u)(x,\tau)d\tau\right) \leq \sum_{i=1}^{5}\int_0^t \mu_i^T\left(S(t-\tau)F(u)(x,\tau)\right)d\tau \lesssim$$

$$\lesssim \int_0^t \|F(u)(x,\tau)\|_{H^s} d\tau = \int_0^t \| \left(F(u)(x,\tau)\right) \|_{L^2_x} d\tau + \int_0^t \|D_x^{3/4} F(u)(x,\tau)\|_{L^2_x} d\tau \lesssim$$
$$\lesssim T^{\frac{1}{2}} \left(\|F(u)\|_{L^2_T L^2_x} + \|D_x^{3/4} F(u)\|_{L^2_T L^2_x} \right)$$

□

Lemma 3. *For some $\alpha > 0$,*

$$\begin{aligned} \|uvw\|_{L^2_x L^2_T} + \|uvw_x\|_{L^2_x L^2_T} + \|\partial_x \mathcal{H}(uv) w\|_{L^2_x L^2_T} &\lesssim T^\alpha \|u\|_{X_T} \|v\|_{X_T} \|w\|_{X_T} \\ \|D_x^{3/4}(uvw)\|_{L^2_x L^2_T} + \|D_x^{3/4}(uvw_x)\|_{L^2_x L^2_T} + \|D_x^{3/4}(\partial_x \mathcal{H}(uv) w)\|_{L^2_x L^2_T} &\lesssim T^\alpha \|u\|_{X_T} \|v\|_{X_T} \|w\|_{X_T} \end{aligned} \tag{13}$$

Proof. Using Holder inequality and (7), we get

$$\begin{aligned} \|uvw\|_{L^2_x L^2_T} = \|uvw\|_{L^2_T L^2_x} &\lesssim \|u\|_{L^5_T L^\infty_x} \|v\|_{L_T^{\frac{10}{3}} L^\infty_x} \|w\|_{L^\infty_T L^2_x} \\ &\lesssim T^\alpha \|u\|_{L^5_T L^\infty_x} \|v\|_{L^5_T L^\infty_x} \|w\|_{L^\infty_T L^2_x} \lesssim \mu_4^T(u) \mu_4^T(v) \mu_1^T(w) \end{aligned}$$

$$\|uvw_x\|_{L^2_x L^2_T} \lesssim \|u\|_{L^4_x L^\infty_T} \|v\|_{L^4_x L^\infty_T} \|w_x\|_{L^\infty_x L^2_T} \lesssim \mu_5^T(u) \mu_5^T(v) \mu_1^T(w)$$

Applying (6) two times and (7) again gives

$$\begin{aligned} \|D_x^{3/4} uvw_x\|_{L^2_x L^2_T} &\lesssim \|u\|_{L^5_T L^\infty_x} \|v\|_{L_T^{\frac{10}{3}} L^\infty_x} \|D_x^{3/4} w\|_{L^\infty_x L^2_T} \\ &+ \|u\|_{L^5_T L^\infty_x} \|w\|_{L_T^{\frac{10}{3}} L^\infty_x} \|D_x^{3/4} v\|_{L^\infty_x L^2_T} \\ &+ \|w\|_{L^5_T L^\infty_x} \|v\|_{L_T^{\frac{10}{3}} L^\infty_x} \|D_x^{3/4} u\|_{L^\infty_x L^2_T} \lesssim T^\alpha \|u\|_{X_T} \|v\|_{X_T} \|w\|_{X_T} \end{aligned}$$

The same gives us the estimation with $\mu_2^T, \mu_3^T, \mu_4^T, \mu_5^T$ for the following

$$\begin{aligned} \|D_x^{3/4} uvw_x\|_{L^2_x L^2_T} &\lesssim \|u\|_{L^4_x L^\infty_T} \|v\|_{L^4_x L^\infty_T} \|D_x^{3/4} w_x\|_{L^\infty_x L^2_T} \\ &+ \|u\|_{L^4_x L^\infty_T} \|w_x\|_{L^{20}_x L_T^{\frac{5}{2}}} \|D_x^{3/4} v\|_{L^5_x L^{10}_T} \\ &+ \|v\|_{L^4_x L^\infty_T} \|w_x\|_{L^{20}_x L_T^{\frac{5}{2}}} \|D_x^{3/4} u\|_{L^5_x L^{10}_T} \lesssim \|u\|_{X_T} \|v\|_{X_T} \|w\|_{X_T} \end{aligned}$$

Since Hilbert transform commutes with partial derivative, is linear, and satisfies (4), we get

$$\begin{aligned} \|w \partial_x \mathcal{H}(uv)\|_{L^2_x L^2_T} &= \|w \mathcal{H}(\partial_x(uv))\|_{L^2_x L^2_T} \\ &\lesssim \|\mathcal{H}(\partial_x(uv))\|_{L^4_x L^2_T} \|w\|_{L^4_x L^\infty_T} \lesssim \|(\partial_x(uv))\|_{L^4_x L^2_T} \|w\|_{L^4_x L^\infty_T} \\ &\lesssim \|u\|_{L^4_x L^\infty_T} \|v_x\|_{L^\infty_x L^2_T} \|w\|_{L^4_x L^\infty_T} + \|u_x\|_{L^\infty_x L^2_T} \|v\|_{L^4_x L^\infty_T} \|w\|_{L^4_x L^\infty_T} \\ &\lesssim \|u\|_{X_T} \|v\|_{X_T} \|w\|_{X_T} \end{aligned}$$

And again by (6)

$$\begin{aligned} \|D_x^{3/4} \left(w \partial_x \mathcal{H}(uv) \right) \|_{L^2_x L^2_T} &\lesssim \|w\|_{L^4_x L^\infty_T} \|D_x^{3/4} (\partial_x \mathcal{H}(uv)\|_{L^4_x L^2_T} + \|\partial_x \mathcal{H}(uv)\|_{L_x^{\frac{10}{3}} L_T^{\frac{5}{2}}} \|D_x^{3/4} w\|_{L^5_x L^{10}_T} \\ &\lesssim \|w\|_{L^4_x L^\infty_T} \|\mathcal{H}(D_x^{3/4} (\partial_x(uv))\|_{L^4_x L^2_T} + \|\mathcal{H}(\partial_x(uv))\|_{L_x^{\frac{10}{3}} L_T^{\frac{5}{2}}} \|D_x^{3/4} w\|_{L^5_x L^{10}_T} \\ &\lesssim \|w\|_{L^4_x L^\infty_T} \|D_x^{3/4} (\partial_x(uv)\|_{L^4_x L^2_T} + \|(\partial_x(uv))\|_{L_x^{\frac{10}{3}} L_T^{\frac{5}{2}}} \|D_x^{3/4} w\|_{L^5_x L^{10}_T} \\ &\lesssim \|w\|_{L^4_x L^\infty_T} \|v_x\|_{L^{20}_x L_T^{\frac{5}{2}}} \|D_x^{3/4} u\|_{L^5_x L^{10}_T} + \|w\|_{L^4_x L^\infty_T} \|u\|_{L^4_x L^\infty_T} \|D_x^{3/4} v_x\|_{L^\infty_x L^2_T} \\ &\quad + \|w\|_{L^4_x L^\infty_T} \|u_x\|_{L^{20}_x L_T^{\frac{5}{2}}} \|D_x^{3/4} v\|_{L^5_x L^{10}_T} + \|w\|_{L^4_x L^\infty_T} \|v\|_{L^4_x L^\infty_T} \|D_x^{3/4} u_x\|_{L^\infty_x L^2_T} \\ &\quad + \|v\|_{L^4_x L^\infty_T} \|u_x\|_{L^{20}_x L_T^{\frac{5}{2}}} \|D_x^{3/4} w\|_{L^5_x L^{10}_T} + \|u\|_{L^4_x L^\infty_T} \|v_x\|_{L^{20}_x L_T^{\frac{5}{2}}} \|D_x^{3/4} w\|_{L^5_x L^{10}_T} \\ &\lesssim \|u\|_{X_T} \|v\|_{X_T} \|w\|_{X_T} \end{aligned}$$

□

Corollary 1. *For some $\alpha > 0$*

$$\|N(u)\|_{L^2_T L^2_x} + \|D_x^{3/4} N(u)\|_{L^2_T L^2_x} \lesssim T^\alpha \|u\|^3_{X_T} \tag{14}$$

Proof. Considering (3), the terms of $N(u)$ are a particular case of ter+ms in Lemma 3, so the proof is straightforward. □

Proof of Theorem 1. Φ is well-defined. Meaning the goal is to prove that for $u \in X_T^\rho$

$$\|\Phi(u)\|_{X_T} \leq \rho \tag{15}$$

$$\|\Phi(u)\|_{X_T} = \sum_{i=1}^5 \mu_i^T (\Phi(u)) \leq \sum_{i=1}^5 \mu_i^T (S(t)u_0) + \sum_{i=1}^5 \mu_i^T \left(\int_0^t S(t-\tau) N(u)(x,\tau) d\tau \right)$$

using Lemma 1 and Lemma 2 with (14) for the first and second term respectively, we get that

$$\|\Phi(u)\|_{X_T} \leq C\|u_0\|_{H^{3/4}} + CT^\alpha \|u\|^3_{X_T} \leq C\|u_0\|_{H^{3/4}} + CT^\alpha \rho^3 \tag{16}$$

for some $C, \alpha > 0$. Then, taking

$$\rho := 2C\|u_0\|_{H^{3/4}}$$

and $T > 0$ sufficiently small satisfying

$$\frac{\rho}{2} + CT^\alpha \rho^3 \leq \rho$$

we deduce (15).

Φ is a contraction. Equivalently, for $u, v \in X_T^\rho$, and some $0 < K < 1$,

$$\|\Phi(u) - \Phi(v)\|_{X_T} \leq K\|u - v\|_{X_T} \tag{17}$$

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|_{X_T} &= \left\| \int_0^t S(t-\tau) N(u)(x,\tau) d\tau - \int_0^t S(t-\tau) N(v)(x,\tau) d\tau \right\|_{X_T} = \\
&= \left\| \int_0^t S(t-\tau) \Big[N(u)(x,\tau) - N(v)(x,\tau) \Big] d\tau \right\|_{X_T} \lesssim \\
&\lesssim T^{\frac{1}{2}} \left(\|N(u) - N(v)\|_{L^2_x L^2_T} + \|D_x^{3/4} (N(u) - N(v))\|_{L^2_x L^2_T} \right)
\end{aligned}$$

by Lemma 2. We break down the expression and consider each term separately.

Easy to verify that

$$u^2 u_x - v^2 v_x = u^2 (u-v)_x + v_x (u+v)(u-v)$$

Since this difference can be expressed as a sum of terms of form uvw_x, we can apply Lemma 3 to see

$$\begin{aligned}
&\|u^2 u_x - v^2 v_x\|_{L^2_x L^2_T} + \|D_x^{3/4}(u^2 u_x - v^2 v_x)\|_{L^2_x L^2_T} \lesssim \\
&\qquad \lesssim T^\alpha \Big[\|u\|^2_{X_T} \|u - v\|_{X_T} + \|u + v\|_{X_T} \|v\|_{X_T} \|u - v\|_{X_T} \Big] \lesssim T^\alpha \rho^2 \|u - v\|_{X_T}
\end{aligned} \tag{18}$$

for some $\alpha > 0$. Easy to verify that we can express other terms similarly

$$|u|^2 u_x - |v|^2 v_x = u_x \bar{u}(u - v) + u_x v \overline{(u - v)}_x + |v|^2 (u - v)_x$$

$$|u|^2 u - |v|^2 v = u\bar{u}(u - v) + uv\overline{(u - v)} + |v|^2 (u - v)$$

$$u\partial_x \mathcal{L}_\infty(|u|^2) - v\partial_x \mathcal{L}_\infty(|v|^2) = -\partial_x \mathcal{H}(|u|^2)(u - v) + v\partial_x \mathcal{H}(\bar{u}(u - v)) + v\partial_x \mathcal{H}(v\overline{(u - v)})$$

So, repeating estimations as in (18) to conclude that

$$\|\Phi(u) - \Phi(v)\|_{X_T} \leq MT^\alpha \rho^2 \|u - v\|_{X_T}$$

for some $M > 0$. Choosing T small enough verifies (17) finishes the proof.

□

4 Well posedness in weighted Sobolev

Since the solution of such equations describes water waves, we want to account for their features, one of those being bounded in space. So, we want to find a solution that decays at $x \to \pm\infty$. That is why we search for the well-posedness in the weighted Sobolev space. However, such a problem is more complicated, and in this capstone project, for the weighted part, we consider a simpler equation. It was proved to be well-posed in the weighted Sobolev space in [1], but with a certain constraint on coefficients. The following theorem states that it's well-posed in a less regular space, but without the additional conditions.

Theorem 2. *For $s \geq 3/4$ any $u_0 \in H^s(\mathbb{R}) \cap L^2(|x|^{2m}dx)$ with $0 < m < 3/8$ there exists $T > 0$ and a unique solution u of (2) such that*

$$u(\cdot, t) \in H^{3/4}(\mathbb{R}) \cap L^2(|x|^{2m}dx)$$

For this proof, we start with the same mapping (9). Similarly, for some ρ, T that will be fixed later, define the metric space:

$$Y_T^\rho := u : \|u\|_{Y_T} \leq \rho$$

with the norm

$$\|u\|_{Y_T} := \|u\|_{X_T} + \sum_{i=1}^{2} \lambda_i^T(u) + \||x|^m u\|_{L_T^\infty L_x^2},$$

where, $\|u\|_{X_T}$ is defined as in (10), and

$$\lambda_1^T(u) = \left\{ \sum_j \sup_{|t|<T} \sup_x \left|\chi_{j/N}(x)u\right|^2 \right\}^{1/2}$$

$$\lambda_2^T(u) = \left\{ \sup_j \int_0^T \int \left|\chi_{j/N}(x)\partial_x u\right|^2 dxdt \right\}^{1/2}$$

$$\lambda_3^T(u) = \lambda_2^T(|x|^m u)$$

where $\chi(x) \in C_c^\infty(\mathbb{R})$ is a cut-off function such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $[0,1]$, supp $\chi \subset (-1,3)$ and set

$$\chi_{j/N}(x) := \chi\left(\frac{x-j}{N}\right)$$

Our aim again is to show that Φ is a contraction on Y_T^ρ to apply Banach fixed-point theorem to prove the existence and uniqueness of the solution.

First we show that Φ is well-defined, so for $u \in Y_T^\rho$ we want

$$\|\Phi(u)\|_{Y_T} \leq \rho$$

Since $\|\Phi(u)\|_{X_T}$ is bounded, according to (16), we consider the left components of the norm.

$$\|x^m \Phi(u)\|_{L_T^\infty L_x^2} \leq \|x^m S(t)u_0\|_{L_T^\infty L_x^2} + \left\| x^m \int_0^t S(t-\tau)N(u)d\tau \right\|_{L_T^\infty L_x^2} =: L + NL$$

The proof of inequalities (19) and (20) are shown in [1]. The goal here is to "transfer" $|x|^m$ to the inside of the integral.

$$L \lesssim (1+T)\|u_0\|_{H^{1/4}} + \||x|^m u_0\|_{L_T^\infty L_x^2} \tag{19}$$

$$NL \lesssim T^\theta \|u\|_{Y_T} + \left\| \int_0^t S(t-\tau)|x|^m \left(N(u) - id|u|^2 u\right) d\tau \right\|_{L_T^\infty L_x^2} \tag{20}$$

Now we break down $N(u)$ and consider simpler integrals.

$$\left\|\int_0^t \|S(t-\tau)|x|^m u^2\partial_x u\|_{L^2_x} d\tau\right\|_{L^\infty_T} \leq \int_0^T \|\sup_t S(t-\tau)|x|^m u^2 \partial_x u\|_{L^2_x} d\tau$$

$$\text{Plancherel's Theorem (twice)} \lesssim \int_0^T \||x|^m u^2 u_x\|_{L^2_x} d\tau$$

$$\text{since } \chi^3_{j/N} \text{ at a point sum to 1} \lesssim \int_0^T \left\|\sum_j \chi^3_{j/N}|x|^m u^2 \partial_x u\right\|_{L^2_x} d\tau \lesssim \sum_j \int_0^T \left\|\chi^3_{j/N}|x|^m u^2 u_x\right\|_{L^2_x} d\tau$$

$$\text{by Holder inequality} \lesssim \sum_j T^{1/2}\left(\int_0^T \left\|\chi^3_{j/N}|x|^m u^2 u_x\right\|^2_{L^2_x} d\tau\right)^{1/2}$$

$$= T^{1/2}\sum_j \left\|(\chi_{j/N}u)^2(\chi_{j/N}|x|^m u_x)\right\|_{L^2_T L^2_x}$$

$$\lesssim T^{1/2} \underbrace{\sum_j \sup_t \sup_x |\chi_{j/N}u|^2}_{\lambda^T_1(u)^2} \cdot \sup_j \|\chi_{j/N}|x|^m u_x\|_{L^2_T L^2_x} \tag{21}$$

Now we introduce $\psi_{\sim 0} \in C_c^\infty(\mathbb{R})$ is a cut-off function such that $0 \leq \psi_{\sim 0} \leq 1$, $\psi_{\sim 0} \equiv 1$ on $[-1, 1]$, supp $\psi_{\sim 0} \subset (-2, 2)$, and $\phi_1 = 1 - \phi_{\sim 0}$. Using this, we breakdown the norm in (21) :

$$\sup_j \|\chi_{j/N}|x|^m u_x\|_{L^2_T L^2_x} \lesssim \sup_j \|\chi_{j/N}\psi_{\sim 0}|x|^m u_x\|_{L^2_T L^2_x} + \sup_j \|\chi_{j/N}\psi_1|x|^m u_x\|_{L^2_T L^2_x} \tag{22}$$

When the first term is non-zero, $|x|^m \leq 1$, so

$$\sup_j \|\chi_{j/N}\psi_{\sim 0}|x|^m u_x\|_{L^2_T L^2_x} \leq \sup_j \|\chi_{j/N}u_x\|_{L^2_T L^2_x} = \lambda^T_2(u) \tag{23}$$

When the second term is non-zero, $\frac{1}{x} < 1$, so considering that $m < 1$,

$$\begin{aligned}\sup_j \|\chi_{j/N}\psi_1|x|^m u_x\|_{L^2_T L^2_x} &= \sup_j \left\|\chi_{j/N}\psi_1\left(\partial_x(|x|^m u) - \frac{m}{|x|^{1-m}}u\right)\right\|_{L^2_T L^2_x} \\ &\lesssim \sup_j \left\|\chi_{j/N}\psi_1 \partial_x(|x|^m u)\right\|_{L^2_T L^2_x} + \sup_j \left\|\chi_{j/N}\psi_1 \frac{m}{|x|^{1-m}}u\right\|_{L^2_T L^2_x} \\ &\lesssim \lambda^T_3(u) + \sup_j \left\|\chi_{j/N}u\right\|_{L^2_T L^2_x} \\ &\lesssim \lambda^T_3(u) + \|u\|_{L^\infty_T L^2_x} = \|u\|_{X_T} + \lambda^T_3(u)\end{aligned} \tag{24}$$

Combining (22), (23), (24) to (21) gives us

$$\left\|\int_0^t S(t-\tau)|x|^m u^2 \partial_x u d\tau\right\|_{L^\infty_T L^2_x} \lesssim \|u\|^3_{Y_T} \tag{25}$$

Notice that this inequality also holds for the $|u|^2 u_x$ term.

Next we prove the same for

$$\lambda^T_1(\Phi(u)) + \lambda^T_2(\Phi(u)) + \lambda^T_3(\Phi(u))$$

By [6] (2.25 and 2.26), we have the inequalities for the linear part:

$$\begin{aligned}\lambda^T_1(S(t)u_0) &\lesssim \|u_0\|_{H^{3/4}} \\ \lambda^T_2(S(t)u_0) &\lesssim \|u_0\|_{L^2}\end{aligned} \tag{26}$$

This bounds the linear parts of the two norms. Next, from [3] (1.8), and [7] (Lemma 2.8), the following statement is true for operators $U(t) = \mathcal{F}^{-1}\left(e^{i(b\xi^3)(t)}\right)$ and $W(t) = \mathcal{F}^{-1}\left(e^{i(a\xi^2)(t)}\right)$, so combining them we have the following true for our operator also:

$$
\begin{aligned}
|x|^m S(t)u_0 &= S(t)(|x|^m u_0) + S(t)\left[\Psi_{t,m}\widehat{u_0}\right]^\vee \\
\|S(t)\left(\Psi_{t,m}\widehat{u_0}\right)^\vee\|_{L^2_x} &\lesssim (1+t)\|u_0\|_{L_2} + \|D_x^{2m}u_0\|_{L_2}
\end{aligned}
\tag{27}
$$

Applying this and (26) for the last norm,

$$
\lambda_3^T(S(t)u_0) = \lambda_2^T(|x|^m S(t)u_0) = \lambda_2^T(S(t)|x|^m u_0) + \lambda_2^T(S(t)\left(\Psi_{t,m}\widehat{u_0}\right)^\vee) \lesssim \||x|^m u_0\|_{L_2} + \|u_0\|_{H^{2m}}
$$

Now for the nonlinear part,

$$
\begin{aligned}
\lambda_1^T\left(\int_0^t S(t-\tau)N(u)d\tau\right)^2 &\lesssim \sum_j \left\|\int_0^T \left|\chi_{j/N}S(t-\tau)N(u)\right| d\tau\right\|^2_{L_T^\infty L_x^2} \\
&\lesssim \int_0^T \sum_j \left\|\chi_{j/N}S(t-\tau)N(u)\right\|^2_{L_T^\infty L_x^2} d\tau \\
&= \int_0^T \lambda_1^T\Big(S(t-\tau)N(u)\Big)^2 d\tau = \int_0^T \lambda_1^T\Big(S(t)S(-\tau)N(u)\Big)^2 d\tau \\
\text{by (26) and Lemma 3} &\lesssim \int_0^T \Big\|S(-\tau)N(u)\Big\|^2_{H^1} d\tau \le \|N(u)\|^2_{L_T^2 H^1} \lesssim \|u\|^6_{X^T}
\end{aligned}
$$

Moving to the next norm, analogically,

$$
\begin{aligned}
\lambda_2^T\left(\int_0^t S(t-\tau)N(u)d\tau\right) &\lesssim \sup_j \left\|\int_0^t \left|\chi_{j/N}\partial_x\left[S(t-\tau)N(u)\right]\right| d\tau\right\|_{L_T^2 L_x^2} \\
&\lesssim \int_0^T \sup_j \left\|\chi_{j/N}\partial_x\left[S(t-\tau)N(u)\right\| d\tau\right\|_{L_T^2 L_x^2} \\
&= \int_0^T \lambda_2^T\Big(S(t-\tau)N(u)\Big)d\tau \lesssim \int_0^T \|N(u)\|_{L_x^2} d\tau \le T^{1/2}\|N(u)\|_{L_T^2 L_x^2} \lesssim T^{1/2}\|u\|^3_{X_T}
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\lambda_3^T\left(\int_0^t S(t-\tau)N(u)d\tau\right) =&\lambda_2^T\left(|x|^m\int_0^t S(t-\tau)N(u)d\tau\right) \\
\text{by (27) } \lesssim&\lambda_2^T\left(\int_0^t S(t-\tau)|x|^m N(u)d\tau\right) + \lambda_2^T\left(\int_0^t S(t-\tau)\left[\Psi_{t,m}\widehat{N(u)}\right]^\vee d\tau\right) \\
\text{similarly to (28) } \lesssim&\||x|^m N(u)\|_{L_T^2 L_x^2} + (1+t)\|N(u)\|_{L_2} + \|D_x^{2m}N(u)\|_{L_2} \lesssim T^\alpha\|u\|^3_{Y_T}
\end{aligned}
$$

And the last inequality is true for $0 < m < 3/8$. The proof is finished analogously to the previous section.

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