

Two particle reduction for two-species Push TASEP with $12 \cdots 2$ initial order of particles

by

Aliaidar Alikhan

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Master of Science in Applied Mathematics

at the

NAZARBAYEV UNIVERSITY

April 28, 2026

© Nazarbayev University 2026. All rights reserved.

Author

Alikhan Aliaidar
Department of Mathematics
April 28, 2026

Certified by

Eunghyun Lee
Professor
Thesis Supervisor

Accepted by

Eunghyun Lee
Professor

Two particle reduction for two-species Push TASEP with $12 \cdots 2$ initial order of particles

by

Alikhan Aliaidar

Submitted to the Department of Mathematics
on April 28, 2026, in partial fulfillment of the
requirements for the degree of
Master of Science in Applied Mathematics

Abstract

The multi-species push TASEP is a stochastic particle system where each particle, denoted by a positive integer which is its species number, waits with an exponential distribution, suppose, with rate 1 before leaping one site to the right. When the target site is occupied, between species l and l' , the lower-numbered species ends up to the right of the site (empty sites are conventionally assigned a particle label of 0). This model was shown to be integrable in [2], and its transition probability admits a representation as a sum of $N!$ contour integrals indexed by permutations $\sigma \in S_N$. In this thesis, we specialise this representation to the initial species order $122 \cdots 2$. The computation is carried out explicitly for $N = 2, 3, 4$, and prove the general formula by mathematical induction.

Thesis Supervisor: Eunghyun Lee
Title: Professor

Contents

1	Introduction	1
1.1	Context and related work	1
1.2	Multi-species TASEP model	1
1.3	What this thesis does	2
2	Model and Integral Representation	3
2.1	The model in detail	3
2.2	The Bethe ansatz integral representation	4
2.3	Transposition matrices and the construction of $\mathbf{A}_\sigma$	4
2.4	Specialization to the multi-species push TASEP	5
2.5	Summary	6
3	The Summed Probability, Case by Case	7
3.1	Setup and target	7
3.2	The case $N = 2$	8
3.3	The case $N = 3$	9
3.3.1	The integral computation	11
3.3.2	Collecting the surviving residues	13
3.4	The case $N = 4$	14
3.4.1	The integral computation	15
3.4.2	Collecting the surviving residues	22
3.5	Matrix-element formula (Conjecture of A)	22
3.6	Integral cancellation (Lemma 1)	23
3.7	Surviving residues (Lemma 2)	24
3.8	Induction and the main theorem	24

List of Figures

1	Illustration of the two-species push TASEP. In (a), a species-2 particle jumps to a site occupied by a species-1 particle; the smaller-numbered particle ends up to the right of the collision. In (b), the arriving species-1 particle meets a species-2 particle, so the lower-species (species-1) arriver is pushed to the right; but that site is already occupied by another species-1 particle, so a second collision happens and the lower-species particle is pushed one step further. The whole cascade takes place in a single instant of continuous time.	2
2	The initial configuration $122\cdots 2$. The species-1 particle occupies the leftmost position y_1 , and species-2 particles occupy positions $y_2 < y_3 < \cdots < y_N$ to the right.	3

List of Tables

1	Matrix elements $[\mathbf{A}_\sigma]_{122,122}$ for $\sigma \in S_3$, with reduced-word decompositions.	11
2	Matrix elements $[A_\sigma]$ for $\sigma \in S_4$, with reduced-word decompositions.	14

1 Introduction

1.1 Context and related work

The asymmetric simple exclusion process (ASEP) is one of the most studied models in the theory of interacting particle systems. It was introduced by Spitzer [5]. The model is described as follows. Finitely many particles occupy sites of the integer line $\mathbb{Z}$, with at most one particle per site. Each particle carries an independent Poisson clock, when its clock rings, the particle attempts to jump one step to the right with probability p and one step to the left with probability $q = 1 - p$. The attempted jump succeeds only if the target site is empty; otherwise the particle stays where it is and waits for its next ring. The asymmetry $p \neq q$ is what gives the model its name. When $p = 1$ and $q = 0$, particles move only to the right and the model is called the totally asymmetric simple exclusion process (TASEP).

The fundamental object associated with such a process is its transition probability. Given an initial configuration $Y = (y_1, \dots, y_N)$ with $y_1 < \dots < y_N$ at time 0, the transition probability $P_Y(X, t)$ is the probability that the system is found in the configuration $X = (x_1, \dots, x_N)$ with $x_1 < \dots < x_N$ at time t . At first try, one might hope to write down the transition probability directly from the dynamics. In principle, it satisfies a system of linear differential equations called the master equation, which expresses the rate of change of each probability in terms of the probabilities of neighbouring configurations. In practice, however, the master equation couples many unknowns to each other, and there is no elementary way to solve it.

For some of interacting models, a remarkable phenomenon occurs: the transition probability can still be written in a closed form, this time as a sum of contour integrals indexed by permutations of the particles. Models with this property are called integrable, and the technique used to extract the closed-form expression is called the Bethe ansatz.

The Bethe ansatz was adapted by Schütz [4], and subsequently by Tracy and Widom [6], for the single-species ASEP. Their work showed that the transition probability of the N -particle ASEP can be written as a sum of $N!$ contour integrals, one for each permutation in the symmetric group S_N . For multi-species models, where particles carry species labels and interact according to their relative species, the formula becomes more complicated: the coefficients inside the integrals are no longer numbers but large matrices that encode which species sits where. Extracting a concrete number, namely the probability of a specific event, from the general formula requires extracting a specific matrix element, and this is the technical core of most work in the area, including ours. Several closely related models have been studied in the literature, and it is helpful to place the multi-species push TASEP among them.

The multi-species ASEP generalizes ASEP by attaching species labels to particles and prescribing that a particle can swap with a lower-numbered particle on an adjacent site. Its transition probability was worked out in full generality by Lee [1], who expressed it as a multiple contour integral weighted by permutation-indexed matrices. A striking simplification was found by Lee and Raimbekov [3] for four specific initial orders of species, namely $2 \dots 21$, $12 \dots 2$, $1 \dots 12$, and $21 \dots 1$, where the number of contour integrals collapses to at most $N!$, and the matrix elements admit a closed form in terms of inversions of the permutation. Their result is the direct precedent for this thesis.

1.2 Multi-species TASEP model

The multi-species push TASEP is the model studied in this thesis. It is a multi-species generalisation built on a different underlying dynamics: the pushing rule. It was introduced by Lee [2], who proved that it is integrable and wrote down its transition probability in the Bethe ansatz form. The dynamics we consider here is the $p = 1$ restriction of Lee's model, that is, the version in which particles only jump to the right.

In this thesis, we consider a stochastic particle model of two species with a finite number of particles on the infinite line $\mathbb{Z}$. This model is further constrained so that each point can have only one particle. Each particle is identified by a positive integer, known as its species. The state of the system is described by its particles and their positions. For example, a configuration with three particles at positions -2 , 1 , and 4 , with species 1 , 2 , and 2 respectively, is written as $(-2, 1, 4; 1, 2, 2)$, and indicates that the leftmost particle belongs to species 1 and the other two belong to species 2 .

The particles do not sit still. Each particle carries its own clock. The clock rings after a random waiting time that follows an exponential distribution with rate 1 , independently of all other clocks. When a particle's clock rings, the particle attempts to move one step to the right. If the target point is empty, the particle simply moves there. If the target point is already occupied by another particle, the two particles interact according to the following rule: the higher-numbered of the two particles ends up at the target point, and the lower-numbered is pushed one step further to the right. If that point is also occupied, the push continues, and so on. We call this model the multi species push TASEP.

The object of interest is the probability of observing the system in a given configuration at a given time, starting from a specified initial configuration. More precisely, if the system starts at time 0 in the configuration (Y, ν) and we wish to know the probability that at time t it is found in the configuration (X, π) , we write this probability as $P_{(Y, \nu)}(X, \pi, t)$.

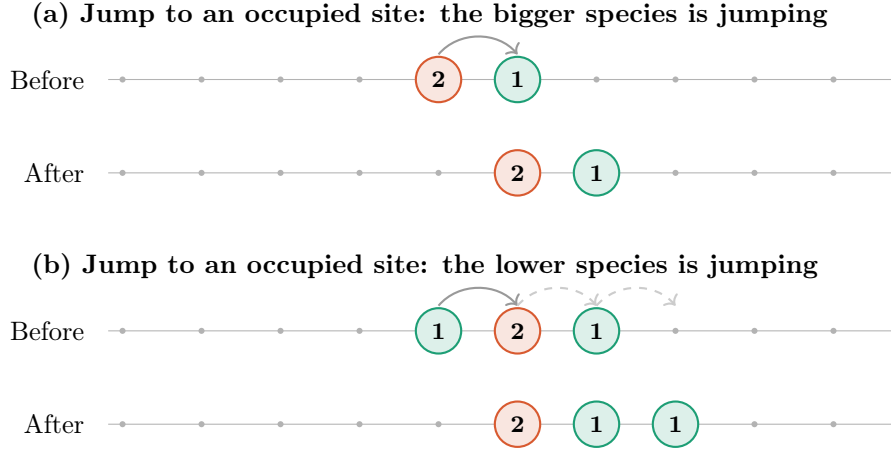


Figure 1: Illustration of the two-species push TASEP. In (a), a species-2 particle jumps to a site occupied by a species-1 particle; the smaller-numbered particle ends up to the right of the collision. In (b), the arriving species-1 particle meets a species-2 particle, so the lower-species (species-1) arriver is pushed to the right; but that site is already occupied by another species-1 particle, so a second collision happens and the lower-species particle is pushed one step further. The whole cascade takes place in a single instant of continuous time.

1.3 What this thesis does

We consider the two-species push TASEP with initial configuration in which the leftmost particle has species 1 and all other particles have species 2 . In symbols, the initial species order is $122 \cdots 2$, and the initial positions are $y_1 < y_2 < \cdots < y_N$. We ask for the probability that the species-1 particle is at a given position x at time t , with all other particles placed somewhere to its right.

Almost everything we need to set up this computation has already been worked out by other au-



Figure 2: The initial configuration $122 \cdots 2$. The species-1 particle occupies the leftmost position y_1 , and species-2 particles occupy positions $y_2 < y_3 < \cdots < y_N$ to the right.

thors, and we take their results as our starting point. The general integral representation of the transition probability for the multi-species push TASEP was derived by Lee and Raimbekov in [3]: that paper proved the model is integrable, wrote down the master equation for the transition probability, and expressed its solution as a sum of $N!$ N -fold contour integrals indexed by permutations in the symmetric group S_N . The Bethe ansatz framework that turns such an integrable model into an integral formula of this shape, together with the explicit construction of the permutation-indexed matrices $\mathbf{A}_\sigma$ from a simpler R -matrix, was formulated by Lee and Raimbekov in [3] for the multi-species ASEP. The same framework transfers to the multi-species push TASEP with only the R -matrix entries and the ε replaced by their multi-species push-TASEP counterparts. So in this thesis we do not re-derive the integral formula, the master equation, or the construction of $\mathbf{A}_\sigma$, we import them directly from [2] and [3]. The definitions of R -matrix and $\mathbf{A}_\sigma$ will be given later in the second section.

What this thesis does is calculate the probability of one specific case. Starting from the general integral formula of [2] specialised to the initial species order $122 \cdots 2$, we carry out the N -particle computation explicitly for $N = 2$, $N = 3$, and $N = 4$ by hand. A clear pattern emerges, the $N!$ terms in the sum over permutations group into $N!/2$ cancellation pairs at the N -fold level, and what remains collapses by one dimension to the answer for $N - 1$ particles. From the small cases we extract two lemmas, which we prove in general:

- an integral cancellation lemma, which states that the N -fold integrals from the two members of each pair cancel,
- a surviving residues lemma, which states that only the permutations fixing the last slot leave a non-zero $(N - 1)$ -fold residue.

Together with the base case $N = 2$, these lemmas give an induction on N . The induction shows that for every $N \geq 2$, the N -particle answer reduces to the $N = 2$ answer, independently of $y_3, y_4, \dots, y_N$. The final formula consists of a single double contour integral and two single contour integrals. This is the main theorem of the thesis.

2 Model and Integral Representation

In the introduction we described the multi-species push TASEP informally, with a few pictures and the phrase “the lower species is pushed.” In this section we make the definition precise, write down the equation that governs the probability over time, and state the integral representation of the transition probability that will be our starting point for the rest of the thesis. The material here is already in [2]; our job is only to fix notation and to collect the pieces we actually use.

2.1 The model in detail

Fix a positive integer N , the number of particles. A state of the system is a pair (X, π) , where $X = (x_1, \dots, x_N) \in \mathbb{Z}^N$ with $x_1 < x_2 < \cdots < x_N$ records the positions of the particles in increasing order, and $\pi = \pi(1)\pi(2) \cdots \pi(N)$ records their species. Here $\pi(i)$ is a positive integer, the species of the i^{th} leftmost particle. In our setting π is a permutation of a multi-set $M = [1, 2, \dots, 2]$.

Each particle carries its own clock. The clock rings after a random waiting time drawn from an exponential distribution with rate 1, independently of all other clocks. When the i^{th} particle's clock rings, that particle attempts to move one step to the right, from x_i to $x_i + 1$. What happens next depends on whether the target site is free:

- If $x_i + 1$ is unoccupied, the particle moves there, and no other particle is affected.
- If $x_i + 1$ is already occupied, the two particles collide. The rule is that the higher-numbered of the two ends up at the target site, and the lower-numbered is pushed one step further to the right. If $x_i + 2$ is also occupied, the pushed particle collides with its new right neighbour in the same way, and the push continues until a particle is pushed into an empty site. The whole sequence of pushes is instantaneous, meaning it takes place at a single instant in continuous time. See Figure 1.

Informally, in the multi-species push TASEP a particle never gives up a jump. It either lands on an empty site, or it displaces a lower-species neighbour, which in turn lands on an empty site or displaces its own neighbour, and so on down the line. The total number of particles and their species labels are conserved; only their positions and their order change. The two-species push TASEP is the special case in which the species labels are drawn from $\{1, 2\}$, so that every a_i is either 1 or 2. This is the only case we treat in this thesis.

2.2 The Bethe ansatz integral representation

The general form of this representation, valid for any multi-species particle model to which the Bethe ansatz applies, was written down by Lee and Raimbekov in their work on the multi-species ASEP [3]. We take their formula as our starting point, and for the construction of the matrices that appear in it we follow their conventions and notation without modification. The two ingredients that change when we move from their setting to ours are the explicit form of the underlying R -matrix and the ε ; both will be imported in Section 2.4 from Lee's paper on the multi-species push TASEP [2].

According to equation (1.1) of [3], for any initial state (Y, ν) and any final state (X, π) with $x_1 < x_2 < \dots < x_N$, the transition probability matrix admits the representation

$$\mathbf{P}_Y(X, t) = \oint_{C_N} \dots \oint_{C_1} \sum_{\sigma \in S_N} \mathbf{A}_\sigma \prod_{i=1}^N \left(\xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\varepsilon(\xi_i) t} \right) d\xi_1 \dots d\xi_N, \quad (1)$$

where $\oint_{C_i}$ means $(2\pi i)^{-1} \int_{C_i}$, the contour C_i is a positively oriented circle around the origin of radius greater than 1, the sum runs over all permutations σ of $\{1, \dots, N\}$, the ε will be specified in Section 2.4, and $\mathbf{A}_\sigma$ is an $N^N \times N^N$ matrix that depends on σ and on $\xi_1, \dots, \xi_N$, but not on X, Y , or t .

2.3 Transposition matrices and the construction of $\mathbf{A}_\sigma$

The description of $\mathbf{A}_\sigma$ that we use is the one given by Lee and Raimbekov in [3]. We restate it here so that our thesis is self-contained.

Let T_i denote the simple transposition in S_N that swaps the entries in the i^{th} and $(i+1)^{\text{st}}$ slots. It is a standard fact that any $\sigma \in S_N$ can be written as a product of simple transpositions, that is,

$$\sigma = T_{i_j}(\beta_j, \alpha_j) T_{i_{j-1}}(\beta_{j-1}, \alpha_{j-1}) \dots T_{i_1}(\beta_1, \alpha_1), \quad (2)$$

where the notation $T_{i_k}(\beta_k, \alpha_k)$ follows [3]: it denotes the simple transposition T_{i_k} , together with the extra bookkeeping that β_k and α_k are the two numbers actually being swapped at the k^{th} step. The expression in (2) is not unique: a given σ admits many different reduced words.

According to equation (1.4) of [3], to each simple transposition $T_i(\beta, \alpha)$ one associates an $N^N \times N^N$ matrix $\mathbb{T}_i(\beta, \alpha)$ defined by

$$\mathbb{T}_i(\beta, \alpha) = \underbrace{I_N \otimes \cdots \otimes I_N}_{(i-1) \text{ times}} \otimes R_{\beta\alpha} \otimes \underbrace{I_N \otimes \cdots \otimes I_N}_{(N-i-1) \text{ times}}, \quad (3)$$

where I_N is the $N \times N$ identity matrix and $R_{\beta\alpha}$ is an $N^2 \times N^2$ matrix whose rows and columns are labelled lexicographically by $11, 12, \dots, NN$. The matrix $\mathbf{A}_\sigma$ is then defined, following equation (1.6) of the same paper, by

$$\mathbf{A}_\sigma = \mathbb{T}_{i_j}(\beta_j, \alpha_j) \mathbb{T}_{i_{j-1}}(\beta_{j-1}, \alpha_{j-1}) \cdots \mathbb{T}_{i_1}(\beta_1, \alpha_1). \quad (4)$$

As Lee and Raimbekov note, although the reduced word in (2) is not unique, the right-hand side of (4) is well defined.

This is the end of what we import from [3]. The construction above is the same for the multi-species ASEP and for the multi-species push TASEP; the two models differ only in the explicit entries of the matrix $R_{\beta\alpha}$ and in the ε . The multi-species push-TASEP versions of both ingredients come from [2], and we now turn to them.

2.4 Specialization to the multi-species push TASEP

The matrix $R_{\beta\alpha}$ and the ε appropriate for the multi-species push TASEP were derived by Lee in [2]. We import the final formulas from that paper and use them, without re-deriving them, for the rest of the thesis.

According to equation (70) of [2], the ε for the general two-sided model is given by $\varepsilon(\xi) = p/\xi + q\xi - 1$, where p and q are the right- and left-jump probabilities. Our model is the $p = 1, q = 0$ restriction of Lee's model, because in the multi-species push TASEP particles only jump to the right. Specialising gives

$$\varepsilon(\xi) = \frac{1}{\xi} - 1, \quad (5)$$

which is consistent with the fact that a single particle in the multi-species push TASEP jumps one step to the right at rate 1.

The scalar factors $S_{\beta\alpha}$ and $T_{\beta\alpha}$. Lee, in equation (20) of [2], defines two scalar functions of ξ_α and ξ_β which in our notation read

$$S_{\beta\alpha} = -\frac{1 - 1/\xi_\alpha}{1 - 1/\xi_\beta} = -\frac{(\xi_\alpha - 1)\xi_\beta}{(\xi_\beta - 1)\xi_\alpha}, \quad T_{\beta\alpha} = \frac{1/\xi_\alpha - 1/\xi_\beta}{1 - 1/\xi_\beta} = \frac{\xi_\beta - \xi_\alpha}{\xi_\alpha(\xi_\beta - 1)}. \quad (6)$$

The second equalities are obtained by clearing denominators; the cleared form is the one that appears most often in the computations of Section 3. The sign of $S_{\beta\alpha}$ and the particular rational forms in (6) are the Bethe-ansatz fingerprint of the pushing rule, and they are what make the multi-species push TASEP factors different from those of the multi-species ASEP in [3].

The R -matrix. Lee gives the explicit form of $R_{\beta\alpha}$ for the multi-species push TASEP in equation (57) of [2] as

$$(R_{\beta\alpha})_{(ij, lm)} = \begin{cases} S_{\beta\alpha} & \text{if } ij = lm \text{ and } i \geq j, \\ -1 & \text{if } ij = lm \text{ and } i < j, \\ T_{\beta\alpha} & \text{if } ij = ml \text{ and } i > j, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Every row of $R_{\beta\alpha}$ has at most two non-zero entries: one on the diagonal, and at most one off the diagonal. All other entries vanish.

Example: the matrix $\mathbb{T}_1$ for $N = 3$ on the multi-set $[1, 2, 2]$. The smallest non-trivial case of the construction that combines (3) with (7) is $N = 3$. Consider the block of $\mathbb{T}_1(\beta, \alpha)$ with $(\beta, \alpha) = (2, 1)$ whose rows and columns are labelled by the three permutations of the multi-set $[1, 2, 2]$, namely $\pi^{(1)} = 122$, $\pi^{(2)} = 212$, and $\pi^{(3)} = 221$, where $\pi^{(k)}$ denotes the permutation that has species 1 in the k^{th} slot and species 2 in all other slots. Combining (3) with (7) gives

$$\mathbb{T}_1 = R_{(2,1)} \otimes I = \begin{bmatrix} -1 & 0 & 0 \\ T_{21} & S_{21} & 0 \\ 0 & 0 & S_{21} \end{bmatrix}. \quad (8)$$

The -1 on the diagonal corresponds to the case in which a species-1 particle sits at slot 1 and a species-2 particle sits at slot 2 (the pair $(i, j) = (1, 2)$ in (7)). The two S_{21} entries correspond to configurations in which the two swapped slots carry equal-species labels, or in which the lower species sits at slot 2. The off-diagonal T_{21} is the species-exchange term that couples $\pi^{(1)}$ to $\pi^{(2)}$.

General form on the multi-set $[1, 2, \dots, 2]$. The structure visible in (8) persists for arbitrary N . Restricting $\mathbb{T}_i(\beta, \alpha)$ to the N -dimensional subspace spanned by $\pi^{(1)}, \dots, \pi^{(N)}$ gives the following explicit description.

Definition 2.1 (General form of $\mathbb{T}_i$ on $[1, 2, \dots, 2]$). *For each $i \in \{1, \dots, N-1\}$ and each pair (β, α) , the $N \times N$ matrix $\mathbb{T}_i(\beta, \alpha)$ restricted to the basis $(\pi^{(1)}, \dots, \pi^{(N)})$ has entries*

$$[\mathbb{T}_i(\beta, \alpha)]_{\pi^{(k)}, \pi^{(l)}} = \begin{cases} -1 & \text{if } k = l = i, \\ S_{\beta\alpha} & \text{if } k = l \text{ and } l \neq i, \\ T_{\beta\alpha} & \text{if } k = i+1 \text{ and } l = i, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the restricted $\mathbb{T}_i$ is lower triangular, has $S_{\beta\alpha}$ on every diagonal entry except the i^{th} (which is -1), and carries a single off-diagonal entry $T_{\beta\alpha}$ at position $(i+1, i)$. All other entries vanish. Equation (8) is the case $N = 3$, $i = 1$, $(\beta, \alpha) = (2, 1)$ of Definition 2.1.

2.5 Summary

The probability that is the object of this thesis,

$$\sum_{i_1=1}^{\infty} \dots \sum_{i_{N-1}=1}^{\infty} P_{(y_1, \dots, y_N; 12 \dots 2)}^{(x, x+i_1, \dots, x+i_1+\dots+i_{N-1}; 12 \dots 2, t)},$$

has been reduced to an N -fold contour integral over $\xi_1, \dots, \xi_N$ of a sum over the $N!$ permutations in S_N . The general form of the integral, together with the construction of the matrices $\mathbf{A}_\sigma$ from transposition matrices $\mathbb{T}_i(\beta, \alpha)$, is imported from Lee and Raimbekov [3]. The entries of the underlying matrix $R_{\beta\alpha}$ that make the construction concrete for the multi-species push TASEP, together with the explicit form of the factors $S_{\beta\alpha}$, $T_{\beta\alpha}$, and the function ε , are imported from Lee [2].

What remains, and what the rest of this thesis carries out, is the explicit evaluation of the resulting N -fold contour integral for the initial species order $122 \dots 2$. This requires (i) a closed-form description of the matrix element $[\mathbf{A}_\sigma]_{12 \dots 2, 12 \dots 2}$ as a function of σ , which we conjecture and verify in small cases, and (ii) two integral lemmas which, combined with induction on N , reduce the N -particle answer to the $N = 2$ answer. The next section carries out this.

3 The Summed Probability, Case by Case

In Section 2 we reduced the quantity of interest to an N -fold contour integral of a sum over the $N!$ permutations in S_N . That is still a formidable expression. Before we try to prove anything in general, it is worth doing the computation by hand for $N = 2$, $N = 3$, and $N = 4$. The pattern that emerges from these three cases is what we will prove at the end of the section.

Throughout this section we fix both the initial and the final species order to $12 \cdots 2$: a single species-1 particle sits at the leftmost position, and the remaining $N - 1$ slots are filled by species-2 particles. We ask for the joint probability that at time t the species-1 particle has reached x and the species-2 particles have landed somewhere to its right, and we sum this probability over all allowed placements of the species-2 particles. The result is the marginal distribution of the species-1 particle's position.

3.1 Setup and target

Given an initial state $(Y, 12 \cdots 2)$ with $Y = (y_1, \dots, y_N)$ and $y_1 < \cdots < y_N$, we define the summed probability

$$F_N(x, t) := \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_{N-1}=1}^{\infty} P_{(Y, 12 \cdots 2)}^{(x, x+i_1, x+i_1+i_2, \dots, x+i_1+\cdots+i_{N-1}, 12 \cdots 2, t)}. \quad (9)$$

The final positions $x_1 = x$ and $x_k = x + i_1 + \cdots + i_{k-1}$ for $k \geq 2$ are strictly increasing because each $i_k \geq 1$, so exclusion is automatic, and summing over $i_1, \dots, i_{N-1}$ is the same as summing over all valid $(x_2, \dots, x_N)$ with $x < x_2 < \cdots < x_N$.

Using the integral representation (1) and extracting the $(\pi, \nu) = (12 \cdots 2, 12 \cdots 2)$ matrix element gives

$$F_N(x, t) = \sum_{i_1, \dots, i_{N-1}} \oint_{C_N} \cdots \oint_{C_1} \sum_{\sigma \in S_N} [\mathbf{A}_\sigma]_{12 \cdots 2, 12 \cdots 2} \prod_{k=1}^N \xi_{\sigma(k)}^{x_k - y_{\sigma(k)} - 1} e^{\varepsilon(\xi_k) t} d\xi_1 \cdots d\xi_N. \quad (10)$$

This is the object we will unpack case by case. For each N we start by computing the matrix elements $[\mathbf{A}_\sigma]_{12 \cdots 2, 12 \cdots 2}$ from the simple transposition matrices of Section 2, and then handle the integrals.

Throughout this section we take the contour C_i to be a positively oriented circle centered at the origin, of radius large enough that the pole at $\xi = 1$ lies inside C_i . We will also make use of a smaller contour c'_i , also centered at the origin, whose radius is strictly less than 1. For the N -fold integral in (10) we in fact need the contours for different variables to be nested: the ξ_i -contour is a positively oriented circle C_i centred at the origin with radius R_i , where

$$1 < R_1 < R_2 < \cdots < R_N,$$

equivalently $|\xi_1| < |\xi_2| < \cdots < |\xi_N|$ along these contours. This is the nesting under which the Bethe-ansatz integral is well-defined [2]: it keeps the poles of the factors $S_{\beta\alpha}$ off the integration contours, and it keeps every geometric series we are about to sum inside its radius of convergence. We write $\oint_{C_i}$ uniformly whenever the specific radius is not needed, and indicate the radius explicitly (for example, when shrinking a particular contour to c'_i to sum a geometric series) whenever the manipulation requires it.

Fubini's theorem. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ -finite measure spaces, and let $f : X \times Y \rightarrow \mathbb{C}$ be measurable. If*

$$\int_{X \times Y} |f(x, y)| d(\mu \otimes \nu)(x, y) < \infty,$$

then f is integrable on the product, and

$$\int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) d\mu(x) \right) d\nu(y) = \int_{X \times Y} f d(\mu \otimes \nu).$$

We apply this with $X = \mathbb{N}$ equipped with the counting measure μ_c , and $Y = C^N$ equipped with the arclength product measure $|d\xi_1| \cdots |d\xi_N|$. Both factors are σ -finite: the first as $\mathbb{N} = \bigcup_{n \geq 1} \{n\}$ with $\mu_c(\{n\}) = 1$, and the second because each contour C_i has finite arclength $2\pi R_i$, so C^N has finite total measure. With this choice

$$\int_X g(k) d\mu_c(k) = \sum_{k \geq 1} g(k),$$

so the iterated integral on $\mathbb{N} \times C^N$ is precisely the sum-integral pair we want to interchange.

Each application below has the same shape. After shrinking some ξ -contour to c'_i , we push a geometric sum $\sum_{k \geq 1} \xi^k$ inside the contour integral; the resulting summand is

$$f_k(\xi_1, \dots, \xi_N) = \xi^k [\mathbf{A}_\sigma]_{\pi, \nu} \prod_{j=1}^N \xi_j^{x_j - y_{\sigma(j)} - 1} e^{\varepsilon(\xi_j) t}, \quad (11)$$

where ξ denotes whichever contour variable has just been shrunk to c'_i . To verify the hypothesis of Fubini's theorem we bound $|f_k|$. The matrix element $[\mathbf{A}_\sigma]_{\pi, \nu}$, the monomials $\xi_j^{x_j - y_{\sigma(j)} - 1}$, and the exponentials $e^{\varepsilon(\xi_j) t}$ are jointly continuous in $(\xi_1, \dots, \xi_N)$ on the compact product of contours, so there is a constant $M < \infty$ that bounds their product uniformly on C^N . By construction $|\xi| \leq r < 1$ on c'_i , hence

$$\begin{aligned} \int_{\mathbb{N} \times C^N} |f_k(\xi_1, \dots, \xi_N)| d(\mu_c \otimes |d\xi|) &= \sum_{k \geq 1} \int_{C^N} |f_k(\xi_1, \dots, \xi_N)| |d\xi_1| \cdots |d\xi_N| \leq M \cdot \text{length}(C^N) \sum_{k \geq 1} r^k = \\ &= \frac{M \cdot \text{length}(C^N) \cdot r}{1 - r} < \infty. \end{aligned}$$

The integrand of (11) is therefore absolutely integrable on $\mathbb{N} \times C^N$, and Fubini's theorem permits the interchange

$$\sum_{k \geq 1} \int_{C^N} f_k(\xi) |d\xi_1| \cdots |d\xi_N| = \int_{C^N} \sum_{k \geq 1} f_k(\xi) |d\xi_1| \cdots |d\xi_N|.$$

The same bound covers every Fubini step in the cases $N = 2, 3, 4$ below as well as in the inductive argument of Section 3.8, so we shall not repeat the verification at each occurrence.

Every integrand we will write contains the same product of boundary weights and exponentials. Set

$$B := \prod_{k=1}^N \xi_k^{x - y_k - 1} e^{\varepsilon(\xi_k) t}. \quad (12)$$

3.2 The case $N = 2$

When $N = 2$ the symmetric group S_2 has only two elements, $\sigma = 12$ (the identity) and $\sigma = 21$, and there is only one summation variable, $i = i_1$.

The matrix elements we need are read off directly. For the identity, $\mathbf{A}_{12} = I$, so $[\mathbf{A}_{12}]_{12, 12} = 1$. For the single transposition, $\mathbf{A}_{21} = \mathbb{T}_1$, and restricted to the multi-set $[1, 2]$, the matrix is

$$\mathbb{T}_1 = \begin{bmatrix} -1 & 0 \\ T_{21} & S_{21} \end{bmatrix},$$

whose top-left entry is $[\mathbf{A}_{21}]_{12,12} = -1$.

Plugging these into (10) and using the shorthand (12),

$$\begin{aligned} \sum_{i_1=1}^{\infty} P_{(y_1, y_2, 12)}^{(x, x+i_1, 12, t)} &= \sum_{i_1=1}^{\infty} \oint_{C_2} \oint_{C_1} e^{t(\epsilon(\xi_1) + \epsilon(\xi_2))} ([A_{12}]_{12,12} \xi_1^{x-y_1-1} \xi_2^{x+i_1-y_2-1} \\ &+ [A_{21}]_{12,12} \xi_2^{x-y_2-1} \xi_1^{x+i_1-y_1-1}) d\xi_1 d\xi_2 = \sum_{i_1=1}^{\infty} \oint_{C_2} \oint_{C_1} B \xi_2^{i_1} d\xi_1 d\xi_2 - \sum_{i_1=1}^{\infty} \oint_{C_2} \oint_{C_1} B \xi_1^{i_1} d\xi_1 d\xi_2 \end{aligned}$$

The two monomials come from the two permutations: the identity gives $\xi_2^{i_1}$ (since $x_2 - x = i_1$ and the species-2 particle is already in slot 2), and $\sigma = 21$ gives $\xi_1^{i_1}$ with the minus sign from the matrix element.

Handling the geometric sum by the contour trick of Section 3.1: shrink the ξ_2 -contour from C_2 to c'_2 , use Fubini and $\sum_{i_1 \geq 1} \xi_2^{i_1} = \xi_2/(1 - \xi_2)$ inside c'_2 , and expand back to C_2 while picking up the residue at $\xi_2 = 1$:

$$\text{Res}_{\xi_2=1} \left[B \frac{\xi_2}{1 - \xi_2} \right] = -B|_{\xi_2=1} = -\xi_1^{x-y_1-1} e^{\epsilon(\xi_1)t}$$

(we used $\epsilon(1) = 0$ and $1^{x-y_2-1} = 1$). Hence

$$\sum_{i_1=1}^{\infty} \oint_{C_2} \oint_{C_1} B \xi_2^{i_1} d\xi_1 d\xi_2 = \oint_{C_2} \oint_{C_1} B \frac{\xi_2}{1 - \xi_2} d\xi_1 d\xi_2 + \oint_{C_1} \xi^{x-y_1-1} e^{\epsilon(\xi)t} d\xi.$$

The same calculation with ξ_1 in place of ξ_2 gives

$$\sum_{i_1=1}^{\infty} \oint_{C_2} \oint_{C_1} B \xi_1^{i_1} d\xi_1 d\xi_2 = \oint_{C_2} \oint_{C_1} B \frac{\xi_1}{1 - \xi_1} d\xi_1 d\xi_2 + \oint_{C_2} \xi^{x-y_2-1} e^{\epsilon(\xi)t} d\xi.$$

Subtracting,

$$F_2(x, t) = \oint_{C_2} \oint_{C_1} B \left(\frac{\xi_2}{1 - \xi_2} - \frac{\xi_1}{1 - \xi_1} \right) d\xi_1 d\xi_2 + \oint_{C_1} \xi^{x-y_1-1} e^{\epsilon(\xi)t} d\xi - \oint_{C_2} \xi^{x-y_2-1} e^{\epsilon(\xi)t} d\xi. \quad (13)$$

The bracketed expression simplifies. Combining fractions,

$$\frac{\xi_2}{1 - \xi_2} - \frac{\xi_1}{1 - \xi_1} = \frac{\xi_2 - \xi_1}{(1 - \xi_1)(1 - \xi_2)},$$

and comparing against the multi-species push-TASEP factor $T_{21} = (\xi_2 - \xi_1)/[\xi_1(\xi_2 - 1)]$ from (6),

$$\frac{\xi_2 - \xi_1}{(1 - \xi_1)(1 - \xi_2)} = T_{21} \frac{\xi_1}{\xi_1 - 1}.$$

Substituting into (13) yields

$$\boxed{F_2(x, t) = \oint_{C_2} \oint_{C_1} B T_{21} \frac{\xi_1}{\xi_1 - 1} d\xi_1 d\xi_2 + \oint_{C_1} \xi^{x-y_1-1} e^{\epsilon(\xi)t} d\xi - \oint_{C_2} \xi^{x-y_2-1} e^{\epsilon(\xi)t} d\xi.} \quad (14)$$

3.3 The case $N = 3$

For $N = 3$ we have six permutations in S_3 and two summation variables, i_1 and i_2 . We need the six matrix elements $[\mathbf{A}_\sigma]_{122, 122}$, and we compute them directly from the simple-transposition matrices of Section 2.

From the construction of Section 2, $\mathbf{A}_\sigma$ is a product of transposition matrices $\mathbb{T}_i$, with one factor for each simple transposition in some reduced word for σ . These matrices are $N^N \times N^N$, that is 27×27 for $N = 3$. On the multi-set $[1, 2, 2]$, however, only three basis vectors are populated, namely the three permutations of this multi-set:

$$\pi^{(1)} = 122, \quad \pi^{(2)} = 212, \quad \pi^{(3)} = 221.$$

The matrix element we want, $[\mathbf{A}_\sigma]_{122, 122}$, is the top-left entry of this restricted 3×3 block. Let us first write down the restricted simple-transposition matrices:

$$\begin{aligned} \mathbb{T}_1 &= R_{(2,1)} \otimes I = \begin{bmatrix} -1 & 0 & 0 \\ T_{21} & S_{21} & 0 \\ 0 & 0 & S_{21} \end{bmatrix}. \\ \mathbb{T}_2 &= I \otimes R_{(3,2)} = \begin{bmatrix} S_{32} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & T_{32} & S_{32} \end{bmatrix}. \end{aligned}$$

The identity. The identity permutation $\sigma = 123$ has the empty reduced word, so $\mathbf{A}_{123} = I$ and

$$[\mathbf{A}_{123}]_{122, 122} = 1.$$

The simple transpositions. Since $213 = T_1$ and $132 = T_2$ acting on the identity, we read the matrix elements straight off the top-left entries of $\mathbb{T}_1$ and $\mathbb{T}_2$:

$$[\mathbf{A}_{213}]_{122, 122} = [\mathbb{T}_1]_{1,1} = -1, \quad [\mathbf{A}_{132}]_{122, 122} = [\mathbb{T}_2]_{1,1} = S_{32}.$$

The products. The remaining three permutations 312, 231, and 321 require products of transposition matrices. Let us compute the products we will need:

1. $\mathbb{T}_1 \mathbb{T}_2$:

$$\mathbb{T}_1 \mathbb{T}_2 = \begin{bmatrix} -S_{32} & 0 & 0 \\ T_{31} S_{32} & -S_{31} & 0 \\ 0 & S_{31} T_{32} & S_{31} S_{32} \end{bmatrix}.$$

2. $\mathbb{T}_2 \mathbb{T}_1$:

$$\mathbb{T}_2 \mathbb{T}_1 = \begin{bmatrix} -S_{31} & 0 & 0 \\ -T_{21} & -S_{21} & 0 \\ T_{31} T_{21} & T_{31} S_{21} & S_{31} S_{21} \end{bmatrix}.$$

3. $\mathbb{T}_2 \mathbb{T}_1 \mathbb{T}_2$:

$$\mathbb{T}_2 \mathbb{T}_1 \mathbb{T}_2 = \begin{bmatrix} -S_{32} S_{21} & 0 & 0 \\ -T_{31} S_{32} & S_{31} & 0 \\ T_{31} T_{21} S_{32} & -T_{21} S_{31} + S_{21} S_{31} T_{32} & S_{32} S_{31} S_{21} \end{bmatrix}.$$

4. $\mathbb{T}_1 \mathbb{T}_2 \mathbb{T}_1$:

$$\mathbb{T}_1 \mathbb{T}_2 \mathbb{T}_1 = \begin{bmatrix} S_{31} & 0 & 0 \\ -(T_{32} S_{31} + S_{32} T_{21}) & -S_{32} S_{21} & 0 \\ S_{32} T_{31} T_{21} & S_{32} T_{31} S_{21} & S_{32} S_{31} S_{21} \end{bmatrix}.$$

Reading reduced words from right to left in the usual convention, the four products correspond to

$$312 = T_1 T_2, \quad 231 = T_2 T_1, \quad 321 = T_2 T_1 T_2 = T_1 T_2 T_1.$$

Reading off the top-left entries, we get

$$[\mathbf{A}_{312}]_{122,122} = -S_{32}, \quad [\mathbf{A}_{231}]_{122,122} = -S_{31}, \quad [\mathbf{A}_{321}]_{122,122} = -S_{32} S_{21}.$$

The longest permutation $\sigma = 321$ admits two reduced words, $T_2 T_1 T_2$ and $T_1 T_2 T_1$, which gave us the top-left entries $-S_{32} S_{21}$ and S_{31} respectively. These must agree, and a direct calculation using $S_{\beta\alpha} = -(\xi_\alpha - 1) \xi_\beta / [(\xi_\beta - 1) \xi_\alpha]$ confirms it:

$$-S_{32} S_{21} = -\frac{(\xi_1 - 1) \xi_3}{(\xi_3 - 1) \xi_1} = S_{31}.$$

The six matrix elements are collected in Table 1.

σ	$[\mathbf{A}_\sigma]_{122,122}$	Decomposition into simple transpositions
123	1	identity
132	S_{32}	T_2
213	-1	T_1
231	$-S_{31}$	$T_2 T_1$
312	$-S_{32}$	$T_1 T_2$
321	$-S_{32} S_{21}$	$T_2 T_1 T_2$

Table 1: Matrix elements $[\mathbf{A}_\sigma]_{122,122}$ for $\sigma \in S_3$, with reduced-word decompositions.

3.3.1 The integral computation

Plugging the six matrix elements into (10), using the shorthand (12), and tracking which $\xi_{\sigma(k)}$ sits in which slot, we get

$$F_3(x, t) = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \left(\xi_2^{i_1} \xi_3^{i_1+i_2} + S_{32} \xi_3^{i_1} \xi_2^{i_1+i_2} - \xi_1^{i_1} \xi_3^{i_1+i_2} \right. \\ \left. - S_{31} \xi_3^{i_1} \xi_1^{i_1+i_2} - S_{32} \xi_1^{i_1} \xi_2^{i_1+i_2} - S_{21} S_{32} \xi_2^{i_1} \xi_1^{i_1+i_2} \right) d\xi_1 d\xi_2 d\xi_3.$$

The six monomials split naturally into three pairs of values of this integral, (1, 2), (3, 4), and (5, 6), where in each pair the two permutations agree on the first slot and differ by swapping the last two entries. We will see that in each pair the triple integrals cancel, and only a double-integral residue survives.

First integral. Regrouping the exponents of ξ_3 pulls i_2 out of ξ_2 :

$$\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \xi_2^{i_1} \xi_3^{i_1+i_2} d\xi_1 d\xi_2 d\xi_3 = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B (\xi_2 \xi_3)^{i_1} \xi_3^{i_2} d\xi_1 d\xi_2 d\xi_3.$$

Let us shrink the ξ_3 -contour to c'_i with $|\xi_3| < 1$:

$$\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B (\xi_2 \xi_3)^{i_1} \xi_3^{i_2} d\xi_1 d\xi_2 d\xi_3 = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{c'_3} \oint_{C_2} \oint_{C_1} B (\xi_2 \xi_3)^{i_1} \xi_3^{i_2} d\xi_1 d\xi_2 d\xi_3.$$

Now, by Fubini we can push the i_2 -sum inside the integral:

$$\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{c'_3} \oint_{C_2} \oint_{C_1} B(\xi_2 \xi_3)^{i_1} \xi_3^{i_2} d\xi_1 d\xi_2 d\xi_3 = \sum_{i_1=1}^{\infty} \oint_{c'_3} \oint_{C_2} \oint_{C_1} B(\xi_2 \xi_3)^{i_1} \sum_{i_2=1}^{\infty} \xi_3^{i_2} d\xi_1 d\xi_2 d\xi_3.$$

Summing the geometric series and expanding the ξ_3 -contour back to C_3 , picking up the residue at $\xi_3 = 1$, we get

$$\begin{aligned} \sum_{i_1=1}^{\infty} \oint_{c'_3} \oint_{C_2} \oint_{C_1} B(\xi_2 \xi_3)^{i_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 &= \sum_{i_1=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B(\xi_2 \xi_3)^{i_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 \\ &+ \sum_{i_1=1}^{\infty} \oint_{C_2} \oint_{C_1} B \xi_2^{i_1} d\xi_1 d\xi_2. \end{aligned}$$

Second integral. The factor S_{32} brings $(\xi_2 - 1)$ in the numerator and $(\xi_3 - 1)$ in the denominator. The numerator factor will combine with the geometric sum and cancel cleanly. Expanding S_{32} :

$$\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{32} \xi_3^{i_1} \xi_2^{i_1+i_2} d\xi_1 d\xi_2 d\xi_3 = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2 - 1)\xi_3}{(\xi_3 - 1)\xi_2} \xi_3^{i_1} \xi_2^{i_1+i_2} d\xi_1 d\xi_2 d\xi_3.$$

Now let us shrink the ξ_2 -contour to c'_2 and sum the geometric series in i_2 :

$$\begin{aligned} \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2 - 1)\xi_3}{(\xi_3 - 1)\xi_2} \xi_3^{i_1} \xi_2^{i_1+i_2} d\xi_1 d\xi_2 d\xi_3 &= \\ \sum_{i_1=1}^{\infty} \oint_{C_3} \oint_{c'_2} \oint_{C_1} B \frac{-(\xi_2 - 1)\xi_3}{(\xi_3 - 1)\xi_2} (\xi_3 \xi_2)^{i_1} \frac{\xi_2}{1 - \xi_2} d\xi_1 d\xi_2 d\xi_3. \end{aligned}$$

The two $(\xi_2 - 1)$ factors cancel, and the resulting integrand is holomorphic at $\xi_2 = 1$, so expanding ξ_2 back to C_2 picks up no residue:

$$\sum_{i_1=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{\xi_3}{\xi_3 - 1} (\xi_3 \xi_2)^{i_1} d\xi_1 d\xi_2 d\xi_3.$$

The triple integrals from the first and second integrals are identical up to sign, and they cancel. Only the double-integral residue from the first integral survives.

Third integral. The same manipulation as for the first integral, with ξ_1 taking the role of ξ_2 :

$$- \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \xi_1^{i_1} \xi_3^{i_1+i_2} d\xi_1 d\xi_2 d\xi_3 = - \sum_{i_1=1}^{\infty} \oint_{c'_3} \oint_{C_2} \oint_{C_1} B (\xi_1 \xi_3)^{i_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3.$$

Expanding the ξ_1 -contour back to C_1 :

$$\begin{aligned} - \sum_{i_1=1}^{\infty} \oint_{c'_3} \oint_{C_2} \oint_{C_1} B (\xi_1 \xi_3)^{i_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 &= - \sum_{i_1=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B (\xi_1 \xi_3)^{i_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 \\ &- \sum_{i_1=1}^{\infty} \oint_{C_2} \oint_{C_1} B \xi_1^{i_1} d\xi_1 d\xi_2. \end{aligned}$$

Fourth integral. The same mechanism as for the second integral, with S_{31} in place of S_{32} :

$$-\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{31} \xi_3^{i_1} \xi_1^{i_1+i_2} d\xi_1 d\xi_2 d\xi_3 = -\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_3}{(\xi_3-1)\xi_1} \xi_3^{i_1} \xi_1^{i_1+i_2} d\xi_1 d\xi_2 d\xi_3.$$

Shrinking the ξ_1 -contour and summing the geometric series in i_2 :

$$-\sum_{i_1=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C'_1} B \frac{-(\xi_1-1)\xi_3}{(\xi_3-1)\xi_1} (\xi_3\xi_1)^{i_1} \frac{\xi_1}{1-\xi_1} d\xi_1 d\xi_2 d\xi_3 = -\sum_{i_1=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{\xi_3}{\xi_3-1} (\xi_3\xi_1)^{i_1} d\xi_1 d\xi_2 d\xi_3.$$

The triple integrals from the third and fourth integrals match up to sign and cancel; the surviving residue, this time with a minus sign, comes from the third integral.

Fifth integral. A different combination of variables, but the same pattern: the factor S_{32} meets the geometric sum in ξ_2 and produces a clean cancellation:

$$-\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{32} \xi_1^{i_1} \xi_2^{i_1+i_2} d\xi_1 d\xi_2 d\xi_3 = -\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \xi_1^{i_1} \xi_2^{i_1+i_2} d\xi_1 d\xi_2 d\xi_3.$$

Shrinking ξ_2 to C'_2 and summing in i_2 :

$$-\sum_{i_1=1}^{\infty} \oint_{C_3} \oint_{C'_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} (\xi_1\xi_2)^{i_1} \frac{\xi_2}{1-\xi_2} d\xi_1 d\xi_2 d\xi_3 = -\sum_{i_1=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{\xi_3}{\xi_3-1} (\xi_1\xi_2)^{i_1} d\xi_1 d\xi_2 d\xi_3.$$

Sixth integral. The product $S_{21}S_{32}$ expands into two factors whose (ξ_2-1) pieces cancel against each other before the geometric sum even enters:

$$-\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{21}S_{32} \xi_2^{i_1} \xi_1^{i_1+i_2} d\xi_1 d\xi_2 d\xi_3 = -\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \times \xi_2^{i_1} \xi_1^{i_1+i_2} d\xi_1 d\xi_2 d\xi_3.$$

Shrinking ξ_1 to C'_1 , summing the geometric series in i_2 , and expanding back:

$$-\sum_{i_1=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C'_1} B \frac{\xi_3}{(\xi_3-1)\xi_1} (\xi_2\xi_1)^{i_1} \frac{\xi_1}{1-\xi_1} d\xi_1 d\xi_2 d\xi_3 = \sum_{i_1=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{\xi_3}{\xi_3-1} (\xi_2\xi_1)^{i_1} d\xi_1 d\xi_2 d\xi_3.$$

The triple integrals of the fifth and sixth integrals agree up to sign and cancel; this third pair leaves no surviving residue.

3.3.2 Collecting the surviving residues

After all three pair cancellations, only the two double-integral residues from the first and third integrals remain:

$$\begin{aligned} F_3(x, t) &= \sum_{i_1=1}^{\infty} \oint_{C_2} \oint_{C_1} B \xi_2^{i_1} d\xi_1 d\xi_2 - \sum_{i_1=1}^{\infty} \oint_{C_2} \oint_{C_1} B \xi_1^{i_1} d\xi_1 d\xi_2 \\ &= \sum_{i_1=1}^{\infty} \oint_{C_2} \oint_{C_1} e^{t(\epsilon(\xi_1)+\epsilon(\xi_2))} \left(\xi_1^{x-y_1-1} \xi_2^{x+i_1-y_2-1} - \xi_2^{x-y_2-1} \xi_1^{x+i_1-y_1-1} \right) d\xi_1 d\xi_2 \\ &= \sum_{i_1=1}^{\infty} P_{(y_1, y_2, 12)}^{(x, x+i_1, 12, t)} = F_2(x, t). \end{aligned}$$

Therefore the answer is the same as in the case $N = 2$.

3.4 The case $N = 4$

For $N = 4$ there are 24 permutations in S_4 and three summation variables, i_1, i_2, i_3 . The 24 matrix elements $[\mathbf{A}_\sigma]_{1222, 1222}$ can be computed by the same procedure as for $N = 3$: pick a reduced word for each σ , multiply the corresponding restricted transposition matrices, and read off the top-left entry. The results are collected in Table 2.

σ	$[A_\sigma]$	Decomposition into simple transpositions
1234	1	identity
2134	-1	T_1
1324	S_{32}	T_2
1243	S_{43}	T_3
2314	$-S_{31}$	T_2T_1
3124	$-S_{32}$	T_1T_2
2143	$-S_{43}$	T_3T_1
1342	$S_{42}S_{32}$	T_3T_2
1423	$S_{42}S_{43}$	T_2T_3
3214	$-S_{21}S_{32}$	$T_1T_2T_1$
2341	$-S_{41}S_{31}$	$T_3T_2T_1$
3142	$-S_{42}S_{32}$	$T_3T_1T_2$
2413	$-S_{41}S_{43}$	$T_2T_3T_1$
4123	$-S_{42}S_{43}$	$T_1T_2T_3$
1432	$S_{32}S_{42}S_{43}$	$T_2T_3T_2$
3241	$-S_{41}S_{21}S_{32}$	$T_3T_1T_2T_1$
2431	$-S_{31}S_{41}S_{43}$	$T_2T_3T_2T_1$
3412	$-S_{41}S_{42}S_{32}$	$T_2T_3T_1T_2$
4213	$-S_{21}S_{42}S_{43}$	$T_1T_2T_3T_1$
4132	$-S_{32}S_{42}S_{43}$	$T_1T_2T_3T_2$
4231	$-S_{31}S_{21}S_{42}S_{43}$	$T_1T_2T_3T_2T_1$
3421	$-S_{21}S_{41}S_{42}S_{32}$	$T_2T_3T_1T_2T_1$
4312	$-S_{31}S_{32}S_{42}S_{43}$	$T_1T_2T_3T_1T_2$
4321	$-S_{21}S_{31}S_{32}S_{42}S_{43}$	$T_1T_2T_3T_1T_2T_1$

Table 2: Matrix elements $[A_\sigma]$ for $\sigma \in S_4$, with reduced-word decompositions.

The three restricted simple-transposition matrices used in these products are:

$$\mathbb{T}_1 = R_{(2,1)} \otimes I = \begin{bmatrix} -1 & 0 & 0 & 0 \\ T_{21} & S_{21} & 0 & 0 \\ 0 & 0 & S_{21} & 0 \\ 0 & 0 & 0 & S_{21} \end{bmatrix},$$

$$\mathbb{T}_2 = I \otimes R_{(3,2)} = \begin{bmatrix} S_{32} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & T_{32} & S_{32} & 0 \\ 0 & 0 & 0 & S_{32} \end{bmatrix},$$

$$\mathbb{T}_3 = I \otimes R_{(4,3)} = \begin{bmatrix} S_{43} & 0 & 0 & 0 \\ 0 & S_{43} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & T_{43} & S_{43} \end{bmatrix}.$$

$$\begin{aligned} & \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \sum_{i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \left(\right. \\ & \quad \xi_2^{i_1} \xi_3^{i_1+i_2} \xi_4^{i_1+i_2+i_3} - \xi_1^{i_1} \xi_3^{i_1+i_2} \xi_4^{i_1+i_2+i_3} + S_{32} \xi_3^{i_1} \xi_2^{i_1+i_2} \xi_4^{i_1+i_2+i_3} \\ & \quad + S_{43} \xi_2^{i_1} \xi_4^{i_1+i_2} \xi_3^{i_1+i_2+i_3} - S_{31} \xi_3^{i_1} \xi_1^{i_1+i_2} \xi_4^{i_1+i_2+i_3} - S_{32} \xi_1^{i_1} \xi_2^{i_1+i_2} \xi_4^{i_1+i_2+i_3} \\ & \quad - S_{43} \xi_1^{i_1} \xi_4^{i_1+i_2} \xi_3^{i_1+i_2+i_3} + S_{42} S_{43} \xi_4^{i_1} \xi_2^{i_1+i_2} \xi_3^{i_1+i_2+i_3} + S_{42} S_{32} \xi_3^{i_1} \xi_4^{i_1+i_2} \xi_2^{i_1+i_2+i_3} \\ & \quad - S_{21} S_{32} \xi_2^{i_1} \xi_1^{i_1+i_2} \xi_4^{i_1+i_2+i_3} - S_{41} S_{31} \xi_3^{i_1} \xi_4^{i_1+i_2} \xi_1^{i_1+i_2+i_3} - S_{42} S_{32} \xi_1^{i_1} \xi_4^{i_1+i_2} \xi_2^{i_1+i_2+i_3} \\ & \quad - S_{41} S_{43} \xi_4^{i_1} \xi_1^{i_1+i_2} \xi_3^{i_1+i_2+i_3} - S_{42} S_{43} \xi_1^{i_1} \xi_2^{i_1+i_2} \xi_3^{i_1+i_2+i_3} + S_{32} S_{42} S_{43} \xi_4^{i_1} \xi_3^{i_1+i_2} \xi_2^{i_1+i_2+i_3} \\ & \quad - S_{41} S_{21} S_{32} \xi_2^{i_1} \xi_4^{i_1+i_2} \xi_1^{i_1+i_2+i_3} - S_{31} S_{41} S_{43} \xi_4^{i_1} \xi_3^{i_1+i_2} \xi_1^{i_1+i_2+i_3} - S_{41} S_{42} S_{32} \xi_4^{i_1} \xi_1^{i_1+i_2} \xi_2^{i_1+i_2+i_3} \\ & \quad - S_{21} S_{42} S_{43} \xi_2^{i_1} \xi_1^{i_1+i_2} \xi_3^{i_1+i_2+i_3} - S_{32} S_{42} S_{43} \xi_1^{i_1} \xi_3^{i_1+i_2} \xi_2^{i_1+i_2+i_3} - S_{31} S_{21} S_{42} S_{43} \xi_2^{i_1} \xi_3^{i_1+i_2} \xi_1^{i_1+i_2+i_3} \\ & \quad \left. - S_{21} S_{41} S_{42} S_{32} \xi_4^{i_1} \xi_2^{i_1+i_2} \xi_1^{i_1+i_2+i_3} - S_{31} S_{32} S_{42} S_{43} \xi_3^{i_1} \xi_1^{i_1+i_2} \xi_2^{i_1+i_2+i_3} - S_{21} S_{31} S_{32} S_{42} S_{43} \xi_3^{i_1} \xi_2^{i_1+i_2} \xi_1^{i_1+i_2+i_3} \right) \\ & \quad d\xi_1 d\xi_2 d\xi_3 d\xi_4. \end{aligned}$$

Let us label these 24 monomials as integrals 1 through 24, in order of appearance. They split into twelve pairs of values of this integral, where in each pair the two permutations agree on the first two slots and differ by swapping the last two entries: In each pair the four-fold integrals will cancel against

$$\begin{aligned} & (1, 4), \quad (2, 7), \quad (3, 9), \quad (5, 11), \\ & (6, 12), \quad (8, 15), \quad (10, 16), \quad (13, 17), \\ & (14, 20), \quad (18, 22), \quad (19, 21), \quad (23, 24). \end{aligned}$$

each other, and only a three-fold residue at $\xi_4 = 1$ survives, when the first member of the pair has $\sigma(4) = 4$. We now work through all twelve pairs.

3.4.1 The integral computation

Pair (1, 4) The simplest pair: integral 1 has no S_{β_α} factors at all, and integral 4 has just one factor S_{43} .

First integral. Let us regroup the exponents as $\xi_2^{i_1} \xi_3^{i_1+i_2} \xi_4^{i_1+i_2+i_3} = (\xi_2 \xi_3 \xi_4)^{i_1} (\xi_3 \xi_4)^{i_2} \xi_4^{i_3}$, shrink the ξ_4 -contour to c'_4 , sum the geometric series in i_3 , and expand back to C_4 , picking up the residue at $\xi_4 = 1$:

$$\begin{aligned} & \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \xi_2^{i_1} \xi_3^{i_1+i_2} \xi_4^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ &= \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B (\xi_2 \xi_3 \xi_4)^{i_1} (\xi_3 \xi_4)^{i_2} \frac{\xi_4}{1 - \xi_4} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ &+ \sum_{i_1, i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B|_{\xi_4=1} (\xi_2 \xi_3)^{i_1} \xi_3^{i_2} d\xi_1 d\xi_2 d\xi_3. \end{aligned}$$

Fourth integral. Expanding $S_{43} = -(\xi_3 - 1)\xi_4/[(\xi_4 - 1)\xi_3]$, regrouping $\xi_2^{i_1} \xi_4^{i_1+i_2} \xi_3^{i_1+i_2+i_3} = (\xi_2 \xi_4 \xi_3)^{i_1} (\xi_4 \xi_3)^{i_2} \xi_3^{i_3}$, and running the contour trick on ξ_3 :

$$\begin{aligned} & \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{43} \xi_2^{i_1} \xi_4^{i_1+i_2} \xi_3^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ &= \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C'_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_3 - 1)\xi_4}{(\xi_4 - 1)\xi_3} (\xi_2 \xi_4 \xi_3)^{i_1} (\xi_4 \xi_3)^{i_2} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4. \end{aligned}$$

The two $(\xi_3 - 1)$ factors cancel, and the resulting integrand is holomorphic at $\xi_3 = 1$, so expanding ξ_3 back to C_3 picks up no residue:

$$\sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{\xi_4}{\xi_4 - 1} (\xi_2 \xi_4 \xi_3)^{i_1} (\xi_4 \xi_3)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

The four-fold integrals from integrals 1 and 4 match up to sign and cancel. Only the three-fold residue from integral 1 survives:

$$\sum_{i_1, i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B|_{\xi_4=1} (\xi_2 \xi_3)^{i_1} \xi_3^{i_2} d\xi_1 d\xi_2 d\xi_3.$$

Pair (2, 7) The same mechanism as Pair (1, 4): an S -factor expanded into the integrand combines with the geometric sum to cancel the partner's four-fold integral.

Second integral. Expanding S_{32} and regrouping $\xi_3^{i_1} \xi_2^{i_1+i_2} \xi_4^{i_1+i_2+i_3} = (\xi_3 \xi_2 \xi_4)^{i_1} (\xi_2 \xi_4)^{i_2} \xi_4^{i_3}$:

$$\begin{aligned} & \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{32} \xi_3^{i_1} \xi_2^{i_1+i_2} \xi_4^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ &= \sum_{i_1, i_2=1}^{\infty} \oint_{c'_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2 - 1)\xi_3}{(\xi_3 - 1)\xi_2} (\xi_3 \xi_2 \xi_4)^{i_1} (\xi_2 \xi_4)^{i_2} \frac{\xi_4}{1 - \xi_4} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ &= \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2 - 1)\xi_3}{(\xi_3 - 1)\xi_2} (\xi_3 \xi_2 \xi_4)^{i_1} (\xi_2 \xi_4)^{i_2} \frac{\xi_4}{1 - \xi_4} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ &+ \sum_{i_1, i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2 - 1)\xi_3}{(\xi_3 - 1)\xi_2} (\xi_3 \xi_2)^{i_1} \xi_2^{i_2} d\xi_1 d\xi_2 d\xi_3. \end{aligned}$$

Seventh integral. Expanding $S_{43} = -(\xi_3 - 1)\xi_4/[(\xi_4 - 1)\xi_3]$ and regrouping:

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{43} \xi_1^{i_1} \xi_4^{i_1+i_2} \xi_3^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C'_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_3 - 1)\xi_4}{(\xi_4 - 1)\xi_3} (\xi_1 \xi_4 \xi_3)^{i_1} (\xi_4 \xi_3)^{i_2} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{\xi_4}{\xi_4 - 1} (\xi_1 \xi_4 \xi_3)^{i_1} (\xi_4 \xi_3)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

The four-fold integrals from integrals 2 and 7 cancel, with no residue from integral 7.

Pair (3, 9) The same mechanism with different slot assignments.

Third integral. The same regrouping as in the previous pair, with ξ_4 shrunk again:

$$\begin{aligned}
& \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{32} \xi_3^{i_1} \xi_2^{i_1+i_2} \xi_4^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2 - 1)\xi_3}{(\xi_3 - 1)\xi_2} (\xi_3 \xi_2 \xi_4)^{i_1} (\xi_2 \xi_4)^{i_2} \frac{\xi_4}{1 - \xi_4} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& \quad + \sum_{i_1, i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B|_{\xi_4=1} \frac{-(\xi_2 - 1)\xi_3}{(\xi_3 - 1)\xi_2} (\xi_3 \xi_2)^{i_1} \xi_2^{i_2} d\xi_1 d\xi_2 d\xi_3.
\end{aligned}$$

Ninth integral. Now with the product $S_{42}S_{32}$:

$$\begin{aligned}
& \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{42}S_{32} \xi_3^{i_1} \xi_4^{i_1+i_2} \xi_2^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C'_2} \oint_{C_1} B \frac{-(\xi_2 - 1)\xi_4}{(\xi_4 - 1)\xi_2} \frac{-(\xi_2 - 1)\xi_3}{(\xi_3 - 1)\xi_2} (\xi_3 \xi_4 \xi_2)^{i_1} (\xi_4 \xi_2)^{i_2} \frac{\xi_2}{1 - \xi_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2 - 1)\xi_3}{(\xi_3 - 1)\xi_2} \frac{\xi_4}{\xi_4 - 1} (\xi_3 \xi_4 \xi_2)^{i_1} (\xi_4 \xi_2)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

The four-fold integrals from integrals 3 and 9 cancel.

Pair (5, 11) Replacing ξ_2 with ξ_1 in pair (3, 9) gives the same pattern.

Fifth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1 - 1)\xi_3}{(\xi_3 - 1)\xi_1} \xi_3^{i_1} \xi_1^{i_1+i_2} \xi_4^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1 - 1)\xi_3}{(\xi_3 - 1)\xi_1} (\xi_3 \xi_1 \xi_4)^{i_1} (\xi_1 \xi_4)^{i_2} \frac{\xi_4}{1 - \xi_4} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& \quad - \sum_{i_1, i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B|_{\xi_4=1} \frac{-(\xi_1 - 1)\xi_3}{(\xi_3 - 1)\xi_1} (\xi_3 \xi_1)^{i_1} \xi_1^{i_2} d\xi_1 d\xi_2 d\xi_3.
\end{aligned}$$

Eleventh integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{41} S_{31} \xi_3^{i_1} \xi_4^{i_1+i_2} \xi_1^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C'_1} B \frac{-(\xi_1-1)\xi_4}{(\xi_4-1)\xi_1} \frac{-(\xi_1-1)\xi_3}{(\xi_3-1)\xi_1} (\xi_3\xi_4\xi_1)^{i_1} (\xi_4\xi_1)^{i_2} \frac{\xi_1}{1-\xi_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_3}{(\xi_3-1)\xi_1} \frac{\xi_4}{\xi_4-1} (\xi_3\xi_4\xi_1)^{i_1} (\xi_4\xi_1)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

The four-fold integrals from integrals 5 and 11 cancel.

Pair (6, 12) Same mechanism, this time with S_{32} paired against $S_{42}S_{32}$.

Sixth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \xi_1^{i_1} \xi_2^{i_1+i_2} \xi_4^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} (\xi_1\xi_2\xi_4)^{i_1} (\xi_2\xi_4)^{i_2} \frac{\xi_4}{1-\xi_4} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& \quad - \sum_{i_1, i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B|_{\xi_4=1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} (\xi_1\xi_2)^{i_1} \xi_2^{i_2} d\xi_1 d\xi_2 d\xi_3.
\end{aligned}$$

Twelfth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{42} S_{32} \xi_1^{i_1} \xi_4^{i_1+i_2} \xi_2^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C'_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} (\xi_1\xi_4\xi_2)^{i_1} (\xi_4\xi_2)^{i_2} \frac{\xi_2}{1-\xi_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{\xi_4}{\xi_4-1} (\xi_1\xi_4\xi_2)^{i_1} (\xi_4\xi_2)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

The four-fold integrals from integrals 6 and 12 cancel.

Pair (8, 15) A more substantial pair: two factors on each side, and the cancellation now happens at the $\xi_3 = 1$ locus.

Eighth integral.

$$\begin{aligned}
& \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} \xi_4^{i_1} \xi_2^{i_1+i_2} \xi_3^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C'_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} (\xi_4\xi_2\xi_3)^{i_1} (\xi_2\xi_3)^{i_2} \frac{\xi_3}{1-\xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{\xi_4}{\xi_4-1} (\xi_4\xi_2\xi_3)^{i_1} (\xi_2\xi_3)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

Fifteenth integral.

$$\begin{aligned}
& \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{32} S_{42} S_{43} \xi_4^{i_1} \xi_3^{i_1+i_2} \xi_2^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
&= \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2'} \oint_{C_1} B \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} (\xi_4 \xi_3 \xi_2)^{i_1} (\xi_3 \xi_2)^{i_2} \frac{\xi_2}{1-\xi_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
&= \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} \frac{\xi_4}{\xi_4-1} (\xi_4 \xi_3 \xi_2)^{i_1} (\xi_3 \xi_2)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

The four-fold integrals from integrals 8 and 15 cancel.

Pair (10, 16)

Tenth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{21} S_{32} \xi_2^{i_1} \xi_1^{i_1+i_2} \xi_4^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
&= - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3'} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} (\xi_2 \xi_1 \xi_4)^{i_1} (\xi_1 \xi_4)^{i_2} \frac{\xi_4}{1-\xi_4} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
&= - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} (\xi_2 \xi_1 \xi_4)^{i_1} (\xi_1 \xi_4)^{i_2} \frac{\xi_4}{1-\xi_4} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
&\quad - \sum_{i_1, i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B|_{\xi_4=1} \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} (\xi_2 \xi_1)^{i_1} \xi_1^{i_2} d\xi_1 d\xi_2 d\xi_3.
\end{aligned}$$

Sixteenth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{41} S_{21} S_{32} \xi_2^{i_1} \xi_4^{i_1+i_2} \xi_1^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
&= - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1'} B \frac{-(\xi_1-1)\xi_4}{(\xi_4-1)\xi_1} \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} (\xi_2 \xi_4 \xi_1)^{i_1} (\xi_4 \xi_1)^{i_2} \frac{\xi_1}{1-\xi_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
&= - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{\xi_4}{\xi_4-1} (\xi_2 \xi_4 \xi_1)^{i_1} (\xi_4 \xi_1)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

The four-fold integrals from integrals 10 and 16 cancel.

Pair (13, 17)

Thirteenth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{41} S_{43} \xi_4^{i_1} \xi_1^{i_1+i_2} \xi_3^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
&= - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3'} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_4}{(\xi_4-1)\xi_1} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} (\xi_4 \xi_1 \xi_3)^{i_1} (\xi_1 \xi_3)^{i_2} \frac{\xi_3}{1-\xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
&= - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_4}{(\xi_4-1)\xi_1} \frac{\xi_4}{\xi_4-1} (\xi_4 \xi_1 \xi_3)^{i_1} (\xi_1 \xi_3)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

Seventeenth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{31} S_{41} S_{43} \xi_4^{i_1} \xi_3^{i_1+i_2} \xi_1^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C'_1} B \frac{-(\xi_1-1)\xi_3}{(\xi_3-1)\xi_1} \frac{-(\xi_1-1)\xi_4}{(\xi_4-1)\xi_1} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} (\xi_4\xi_3\xi_1)^{i_1} (\xi_3\xi_1)^{i_2} \frac{\xi_1}{1-\xi_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_3}{(\xi_3-1)\xi_1} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} \frac{\xi_4}{\xi_4-1} (\xi_4\xi_3\xi_1)^{i_1} (\xi_3\xi_1)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

The four-fold integrals from integrals 13 and 17 cancel.

Pair (14, 20)

Fourteenth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{42} S_{43} \xi_1^{i_1} \xi_2^{i_1+i_2} \xi_3^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C'_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} (\xi_1\xi_2\xi_3)^{i_1} (\xi_2\xi_3)^{i_2} \frac{\xi_3}{1-\xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{\xi_4}{\xi_4-1} (\xi_1\xi_2\xi_3)^{i_1} (\xi_2\xi_3)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

Twentieth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{32} S_{42} S_{43} \xi_1^{i_1} \xi_3^{i_1+i_2} \xi_2^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C'_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} (\xi_1\xi_3\xi_2)^{i_1} (\xi_3\xi_2)^{i_2} \frac{\xi_2}{1-\xi_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} \frac{\xi_4}{\xi_4-1} (\xi_1\xi_3\xi_2)^{i_1} (\xi_3\xi_2)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

The four-fold integrals from integrals 14 and 20 cancel.

Pair (18, 22)

Eighteenth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{41} S_{42} S_{32} \xi_4^{i_1} \xi_1^{i_1+i_2} \xi_2^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C'_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_4}{(\xi_4-1)\xi_1} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} (\xi_4\xi_1\xi_2)^{i_1} (\xi_1\xi_2)^{i_2} \frac{\xi_2}{1-\xi_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_4}{(\xi_4-1)\xi_1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{\xi_4}{\xi_4-1} (\xi_4\xi_1\xi_2)^{i_1} (\xi_1\xi_2)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

Twenty-second integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{21} S_{41} S_{42} S_{32} \xi_4^{i_1} \xi_2^{i_1+i_2} \xi_1^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C'_1} B \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_1-1)\xi_4}{(\xi_4-1)\xi_1} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} (\xi_4\xi_2\xi_1)^{i_1} (\xi_2\xi_1)^{i_2} \frac{\xi_1}{1-\xi_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{\xi_4}{\xi_4-1} (\xi_4\xi_2\xi_1)^{i_1} (\xi_2\xi_1)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

The four-fold integrals from integrals 18 and 22 cancel.

Pair (19, 21)

Nineteenth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{21} S_{42} S_{43} \xi_2^{i_1} \xi_1^{i_1+i_2} \xi_3^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C'_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} (\xi_2\xi_1\xi_3)^{i_1} (\xi_1\xi_3)^{i_2} \frac{\xi_3}{1-\xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{\xi_4}{\xi_4-1} (\xi_2\xi_1\xi_3)^{i_1} (\xi_1\xi_3)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

Twenty-first integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{31} S_{21} S_{42} S_{43} \xi_2^{i_1} \xi_3^{i_1+i_2} \xi_1^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C'_1} B \frac{-(\xi_1-1)\xi_3}{(\xi_3-1)\xi_1} \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} (\xi_2\xi_3\xi_1)^{i_1} (\xi_3\xi_1)^{i_2} \frac{\xi_1}{1-\xi_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} \frac{\xi_3}{\xi_3-1} (\xi_2\xi_3\xi_1)^{i_1} (\xi_3\xi_1)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

The four-fold integrals from integrals 19 and 21 cancel.

Pair (23, 24) The last pair has the longest products of factors: four in one member and five in the other.

Twenty-third integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{31} S_{32} S_{42} S_{43} \xi_3^{i_1} \xi_1^{i_1+i_2} \xi_2^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C'_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_3}{(\xi_3-1)\xi_1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} (\xi_3\xi_1\xi_2)^{i_1} (\xi_1\xi_2)^{i_2} \frac{\xi_2}{1-\xi_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_3}{(\xi_3-1)\xi_1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} \frac{\xi_4}{\xi_4-1} (\xi_3\xi_1\xi_2)^{i_1} (\xi_1\xi_2)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

Twenty-fourth integral.

$$\begin{aligned}
& - \sum_{i_1, i_2, i_3=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B S_{21} S_{31} S_{32} S_{42} S_{43} \xi_3^{i_1} \xi_2^{i_1+i_2} \xi_1^{i_1+i_2+i_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_1-1)\xi_3}{(\xi_3-1)\xi_1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} \\
& \quad (\xi_3 \xi_2 \xi_1)^{i_1} (\xi_2 \xi_1)^{i_2} \frac{\xi_1}{1-\xi_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
& = - \sum_{i_1, i_2=1}^{\infty} \oint_{C_4} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \frac{-(\xi_1-1)\xi_2}{(\xi_2-1)\xi_1} \frac{-(\xi_2-1)\xi_3}{(\xi_3-1)\xi_2} \frac{-(\xi_2-1)\xi_4}{(\xi_4-1)\xi_2} \frac{-(\xi_3-1)\xi_4}{(\xi_4-1)\xi_3} \frac{\xi_3}{\xi_3-1} (\xi_3 \xi_2 \xi_1)^{i_1} (\xi_2 \xi_1)^{i_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.
\end{aligned}$$

The four-fold integrals from integrals 23 and 24 cancel.

3.4.2 Collecting the surviving residues

All twelve pairs have cancelled at the four-fold level. The only contributions that survive are the three-fold residues from the six pairs whose first integral has $\sigma(4) = 4$, namely the pairs (1, 4), (2, 7), (3, 9), (5, 11), (6, 12), and (10, 16). The remaining six pairs have $\sigma(4) \neq 4$, and their would-be residues are killed by a $(\xi_{\sigma(4)} - 1)$ factor produced when we expand the S -factors, exactly as we saw in the cancellation calculations above.

The six surviving permutations, taken from the first members of these pairs, are

$$1234, 1324, 2134, 2314, 3124, 3214,$$

which are exactly the six permutations in S_4 that fix the last slot. Their three-fold residue contributions assemble into

$$\begin{aligned}
& \sum_{i_1, i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B \left(\xi_2^{i_1} \xi_3^{i_1+i_2} + S_{32} \xi_3^{i_1} \xi_2^{i_1+i_2} - \xi_1^{i_1} \xi_3^{i_1+i_2} \right. \\
& \quad \left. - S_{31} \xi_3^{i_1} \xi_1^{i_1+i_2} - S_{32} \xi_1^{i_1} \xi_2^{i_1+i_2} - S_{21} S_{32} \xi_2^{i_1} \xi_1^{i_1+i_2} \right) d\xi_1 d\xi_2 d\xi_3 \\
& = \sum_{i_1, i_2=1}^{\infty} \oint_{C_3} \oint_{C_2} \oint_{C_1} B|_{\xi_4=1} \sum_{\sigma \in S_3} [A_\sigma]_{122, 122} \prod_{k=1}^3 \xi_{\sigma(k)}^{x_k - y_{\sigma(k)} - 1} e^{\varepsilon(\xi_k) t} d\xi_1 d\xi_2 d\xi_3.
\end{aligned}$$

which, by (10) at $N = 3$, is exactly $F_3(x, t)$. We conclude

$$F_4(x, t) = F_3(x, t) = F_2(x, t).$$

The dependence on y_3 and y_4 has dropped out: the four-particle marginal of the species-1 particle's position reduces to the two-particle marginal, just as it did at $N = 3$.

3.5 Matrix-element formula (Conjecture of A)

To evaluate (10) for any particular N , we need the matrix element $[A_\sigma]_{12 \dots 2, 12 \dots 2}$ for each of the $N!$ permutations.

Matrix-element formula] For each $\sigma \in S_N$,

$$[A_\sigma]_{12 \dots 2, 12 \dots 2} = \begin{cases} \prod_{(\beta, \alpha) \in \text{Inv}(\sigma)} S_{\beta\alpha} & \text{if } \sigma(1) = 1, \\ - \prod_{(\beta, \alpha) \in \text{Inv}(\sigma) \setminus \{(k, 1)\}} S_{\beta\alpha} & \text{if } \sigma(1) = k, k \neq 1, \end{cases}$$

where $\text{Inv}(\sigma) = \{(\sigma(i), \sigma(j)) : i < j \text{ and } \sigma(i) > \sigma(j)\}$ is the inversion set of σ , and $S_{\beta\alpha}$ is the factor defined by (6).

3.6 Integral cancellation (Lemma 1)

The three cases share the same two structural features, and we isolate the first of them now.

Lemma 1 (Integral cancellation). *Fix $N \geq 2$ and let $\sigma, \sigma' \in S_N$ be a pair obtained by swapping the last two entries: σ with $\sigma(1) \cdots \sigma(N-1) \sigma(N)$ and σ' with $\sigma(1) \cdots \sigma(N) \sigma(N-1)$. Without loss of generality, $\sigma(N) > \sigma(N-1)$. Then the full N -fold integral contributions of σ and σ' to $F_N(x, t)$ cancel, and the total contribution of the pair reduces to a sum of $(N-1)$ -fold residues.*

Proof. The matrix element for σ' differs from the matrix element for σ by a single right-multiplication by the simple transposition that swaps the last two slots. At the level of the matrix element $[\mathbf{A}_\sigma]_{12 \dots 2, 12 \dots 2}$, this extra factor contributes exactly the scalar $S_{\sigma(N) \sigma(N-1)}$, all other factors are shared. Therefore

$$[\mathbf{A}_{\sigma'}]_{12 \dots 2, 12 \dots 2} = S_{\sigma(N) \sigma(N-1)} [\mathbf{A}_\sigma]_{12 \dots 2, 12 \dots 2}.$$

Let F denote everything in the two integrands except the last two slot factors: F is the prefactor B , the matrix element $[\mathbf{A}_\sigma]_{12 \dots 2, 12 \dots 2}$, and the $\xi_{\sigma(k)}^{i_1 + \dots + i_{k-1}}$ for $k = 1, \dots, N-2$. Then σ and σ' contribute

$$\begin{aligned} I_\sigma &= \sum_{i_1, \dots, i_{N-1}=1}^{\infty} \oint_{C_N} \cdots \oint_{C_1} F \xi_{\sigma(N-1)}^{i_1 + \dots + i_{N-2}} \xi_{\sigma(N)}^{i_1 + \dots + i_{N-1}} d\xi_1 \cdots d\xi_N, \\ I_{\sigma'} &= \sum_{i_1, \dots, i_{N-1}=1}^{\infty} \oint_{C_N} \cdots \oint_{C_1} F S_{\sigma(N) \sigma(N-1)} \xi_{\sigma(N)}^{i_1 + \dots + i_{N-2}} \xi_{\sigma(N-1)}^{i_1 + \dots + i_{N-1}} d\xi_1 \cdots d\xi_N. \end{aligned}$$

Write $a = i_1 + \dots + i_{N-2}$ and $b = i_{N-1}$. Regrouping,

$$\begin{aligned} \xi_{\sigma(N-1)}^a \xi_{\sigma(N)}^{a+b} &= (\xi_{\sigma(N-1)} \xi_{\sigma(N)})^a \xi_{\sigma(N)}^b, \\ \xi_{\sigma(N)}^a \xi_{\sigma(N-1)}^{a+b} &= (\xi_{\sigma(N-1)} \xi_{\sigma(N)})^a \xi_{\sigma(N-1)}^b. \end{aligned}$$

Sum over b using the contour trick. Shrink the $\xi_{\sigma(N)}$ -contour to $c'_{\sigma(N)}$ in I_σ and the $\xi_{\sigma(N-1)}$ -contour to $c'_{\sigma(N-1)}$ in $I_{\sigma'}$, apply Fubini and the geometric-series formula, expand back to $C_{\sigma(N)}$ picking up residues at $\xi_{\sigma(N)} = 1$ and $\xi_{\sigma(N-1)} = 1$ respectively. The main parts (non-residue pieces) are

$$\begin{aligned} I_\sigma^{\text{main}} &= \sum_{a=N-2}^{\infty} \oint_{C_N} \cdots \oint_{C_1} F (\xi_{\sigma(N-1)} \xi_{\sigma(N)})^a \frac{\xi_{\sigma(N)}}{1 - \xi_{\sigma(N)}} d\xi_1 \cdots d\xi_N, \\ I_{\sigma'}^{\text{main}} &= \sum_{a=N-2}^{\infty} \oint_{C_N} \cdots \oint_{C_1} F S_{\sigma(N) \sigma(N-1)} (\xi_{\sigma(N-1)} \xi_{\sigma(N)})^a \frac{\xi_{\sigma(N-1)}}{1 - \xi_{\sigma(N-1)}} d\xi_1 \cdots d\xi_N. \end{aligned}$$

The scalar factor attached to F in $I_{\sigma'}^{\text{main}}$ simplifies. Using $S_{\sigma(N) \sigma(N-1)} = -(\xi_{\sigma(N-1)} - 1) \xi_{\sigma(N)} / [(\xi_{\sigma(N)} - 1) \xi_{\sigma(N-1)}]$,

$$\begin{aligned} S_{\sigma(N) \sigma(N-1)} \frac{\xi_{\sigma(N-1)}}{1 - \xi_{\sigma(N-1)}} &= -\frac{(\xi_{\sigma(N-1)} - 1) \xi_{\sigma(N)}}{(\xi_{\sigma(N)} - 1) \xi_{\sigma(N-1)}} \cdot \frac{\xi_{\sigma(N-1)}}{1 - \xi_{\sigma(N-1)}} \\ &= \frac{\xi_{\sigma(N)}}{\xi_{\sigma(N)} - 1} = -\frac{\xi_{\sigma(N)}}{1 - \xi_{\sigma(N)}}, \end{aligned}$$

where the factor $-(\xi_{\sigma(N-1)} - 1)/(1 - \xi_{\sigma(N-1)}) = 1$. Therefore $I_{\sigma'}^{\text{main}} = -I_\sigma^{\text{main}}$, and the two main N -fold integrals cancel. What remains is a sum of two $(N-1)$ -fold residues: one at $\xi_{\sigma(N)} = 1$, from I_σ , and one at $\xi_{\sigma(N-1)} = 1$, from $I_{\sigma'}$. $\square$

3.7 Surviving residues (Lemma 2)

Lemma 1 leaves F_N as a sum of $(N - 1)$ -fold residues, two for each pair. The second lemma says that only half of these residues are actually non-zero.

Lemma 2 (Surviving residues). *We have the residue if and only if $\sigma(N) = N$, and the residue that we have only at $\xi_{\sigma(N)} = 1$.*

Proof. Suppose $\sigma(N) \neq N$. Since σ is a permutation of $\{1, \dots, N\}$ and $\sigma(N) < N$, the value N must appear at some earlier position $j < N$. In building $\mathbf{A}_\sigma$ as a product of simple transpositions, the value N must travel from position j to its final slot, which means at some point the simple transposition swapping N past $\sigma(N)$ is applied. That simple transposition contributes a factor $S_{N, \sigma(N)}$ to the matrix element. From the definition of the factor,

$$S_{N, \sigma(N)} = - \frac{(\xi_{\sigma(N)} - 1) \xi_N}{(\xi_N - 1) \xi_{\sigma(N)}}$$

contains $(\xi_{\sigma(N)} - 1)$ as a factor and vanishes at $\xi_{\sigma(N)} = 1$. The residue therefore vanishes.

If instead $\sigma(N) = N$, then no value greater than $\sigma(N)$ appears in the first $N - 1$ slots (there is none), so no factor of the form $S_{\beta, N}$ with $\beta > N$ is present. The residue is non-zero. $\square$

3.8 Induction and the main theorem

The two lemmas and the $N = 2$ base case combine into a one-step induction.

Theorem 1 (Main theorem). *For every $N \geq 2$, the summed probability $F_N(x, t)$ defined by (9) equals*

$$F_N(x, t) = F_2(x, t) = \oint_{C_2} \oint_{C_1} B T_{21} \frac{\xi_1}{\xi_1 - 1} d\xi_1 d\xi_2 + \oint_{C_1} \xi^{x-y_1-1} e^{\varepsilon(\xi)t} d\xi - \oint_{C_2} \xi^{x-y_2-1} e^{\varepsilon(\xi)t} d\xi,$$

independently of $y_3, \dots, y_N$.

Proof. Induction on N . The base case $N = 2$ is formula (14).

Assume $F_{N-1}(x, t) = F_2(x, t)$. Starting from (10),

$$F_N(x, t) = \sum_{i_1, \dots, i_{N-1}} \oint_{C_N} \cdots \oint_{C_1} \sum_{\sigma \in S_N} [\mathbf{A}_\sigma]_{12 \dots 2, 12 \dots 2} \prod_{k=1}^N \xi_{\sigma(k)}^{x_k - y_{\sigma(k)} - 1} e^{\varepsilon(\xi_k)t} d\xi_1 \cdots d\xi_N,$$

group the sum over $\sigma \in S_N$ into the $N!/2$ cancellation pairs of Lemma 1. Every pair contributes zero at the level of N -fold integrals and leaves only residues. By Lemma 2, in each pair only the member with $\sigma(N) > \sigma(N - 1)$ contributes, and even then only if $\sigma(N) = N$. The surviving permutations are exactly those fixing N in the last slot.

Each such surviving residue is a contribution of the form

$$\sum_{i_1, \dots, i_{N-2}} \oint_{C_N} \cdots \oint_{C_1} [\mathbf{A}_\sigma]_{12 \dots 2, 12 \dots 2} \Big|_{\xi_N=1} \prod_{k=1}^{N-1} \xi_{\sigma(k)}^{x_k - y_{\sigma(k)} - 1} e^{\varepsilon(\xi_k)t} d\xi_1 \cdots d\xi_{N-1}.$$

The evaluation $\xi_N = 1$ simplifies B to its $(N - 1)$ -particle form (since $1^{x-y_{N-1}-1} = 1$ and $e^{\varepsilon(1)t} = 1$), and the matrix element $[\mathbf{A}_\sigma]_{12 \dots 2, 12 \dots 2}$ for σ fixing N depends only on the restriction $\sigma|_{\{1, \dots, N-1\}}$, which ranges over all of S_{N-1} as σ ranges over the permutations in S_N fixing N . Summing the surviving residues,

$$F_N(x, t) = \sum_{i_1, \dots, i_{N-2}} \oint_{C_N} \cdots \oint_{C_1} \sum_{\sigma \in S_{N-1}} [\mathbf{A}_\sigma]_{12 \dots 2, 12 \dots 2} \prod_{k=1}^{N-1} \xi_{\sigma(k)}^{x_k - y_{\sigma(k)} - 1} e^{\varepsilon(\xi_k)t} d\xi_1 \cdots d\xi_{N-1} = F_{N-1}(x, t).$$

By the induction hypothesis, $F_{N-1}(x, t) = F_2(x, t)$, and the result follows. $\square$

The dependence on $y_3, y_4, \dots, y_N$ has disappeared entirely: at the level of the species-1 particle's marginal position, the initial locations of the species-2 particles (beyond the first two) play no role. This is the central phenomenon the thesis was built to expose, and it holds for any $N \geq 2$.

References

- [1] E. Lee. Exact formulas of the transition probabilities of the multi-species asymmetric simple exclusion process. *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)*, 16:139, 2020.
- [2] E. Lee. Integrability of the multi-species asymmetric simple exclusion processes with long-range jumps on $\mathbb{Z}$. *Symmetry*, 16(9):1164, 2024.
- [3] E. Lee and T. Raimbekov. Simplified forms of the transition probabilities of the two-species ASEP with some initial orders of particles. *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)*, 18:008, 2022.
- [4] G. M. Schütz. Exact solution of the master equation for the asymmetric exclusion process. *Journal of Statistical Physics*, 88(1-2):427–445, 1997.
- [5] F. Spitzer. Interaction of Markov processes. *Advances in Mathematics*, 5:246–290, 1970.
- [6] C. A. Tracy and H. Widom. Integral formulas for the asymmetric simple exclusion process. *Communications in Mathematical Physics*, 279(3):815–844, 2008.