Finite Time and Fixed Time Synchronization of Shunting Inhibitory Memristive Neural Networks via Sliding Mode Control

by

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Abstract

This thesis investigates the finite time synchronization and fixed time synchronization of shunting inhibitory memristive neural networks with time-varying delays via sliding mode control. First, a new terminal sliding mode surface is designed and its reachability is analysed. Then, a unique sliding mode controller is constructed suitable for both the finite time and fixed time synchronization and its stability analysis is done by using Lyapunov functionals. Finally, numerical examples with the estimated settling times are provided to show the effectiveness of our results.

Thesis Supervisor: Ardak Kashkynbayev
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Chapter 1

Introduction and preliminaries

1.1 Shunting inhibitory cellular neural networks

In 1988, a novel class of information processing systems, named cellular neural network (CNN), was proposed by Chua and Yang [1]. It consists of regularly spaced units, called *cells*, each of which can be considered as an electrical circuit. Unlike in neural networks, the cells in CNN architecture are connected with their neighbouring cells only. While not neighbouring cells may affect each other indirectly because of the continuous time propagation of CNN dynamics. A two-dimensional cellular neural network is given in figure 1-1. Theoretically, one can construct CNNs of any dimension, but the model we consider in this thesis work is two-dimensional model.

Since their introduction, CNNs have been extensively applied mostly in image processing, pattern recognition and simulation of physical and biological systems [2, 3]. In this study, we focus on the special type of CNNs, called shunting inhibitory cellular neural network, which was first derived by Bouzerdoum and Pinter in 1990s [4]. SICNNs are biologically inspired neural networks where the interactions between the

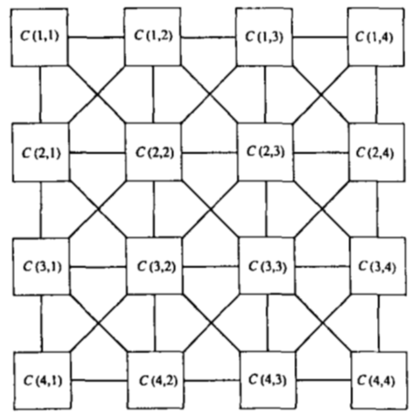

Figure 1-1: A two-dimensional cellular neural network

neighbouring cells are of the shunting type only. That is, the cells or neurons are provided by gain control mechanism so that they function as adaptive nonlinear filters [5]. SICNNs have shown great potential in many classification and function approximation tasks. For instance, SICNNs were applied in the colour image enhancement and were used to improve the image quality and sharpness [6]. For the last few decades, the existence and uniqueness of stable solutions were studied [8, 9, 10, 11]. Kashkynbayev et al. [12] analysed the global Lagrange stability of SICNNs with time-varying delays. Yang investigated the stability of the periodic solution of Cohen-Grossberg SICNNs with delays and impulse [13].

1.2 The memristor

Memristor, the fourth basic circuit element after the capacitor, resistor and inductor, was introduced by Chua in 1971, which resulted in the derivation of memristive neural networks (MNNs) [14]. Due to the fact that the memristor memorizes the applied voltage history, it plays an important role in simulating neuronal activities. Particularly, it can remember the past actions such as the direction flow of the electric charges. The main property of the dynamic equation of the MNN model is that it depends on the piecewise function of the state of the cell. MMNs have been widely applied in many fields such as reconstruction of images and other machine learning problems [15, 16, 17, 18]. Qi et al. [15]investigated the stability of general memristive neural networks with time-varying impulses. Hu et al. [19] illustrated the applications of memristive multilayer cellular neural networks in image processing. Pershin and Di Ventra [20] demonstrated the use of memristive neural networks in the production of associative memory devices. To enrich the memory property of memristance in the cell communication of CNN, Lin and Chaoling [23] imposed that memristor-based SICNN with leakage delays and proved a unique almost periodic solution that is globally exponentially stable. Due to the discontinuity of the memristive switching sense of neighbourhood cell transmissions of SICNN, there are few studies on the dynamics of SIMNN model in the available literature. It is common in nature when implementing neural networks, the more realistic models should include some past history of the states of the system. This led to the involvement of time delays in the neural networks which were introduced in [21].

1.3 Delay differential equations

The simplest delay differential equation is given by

$$u'(t) = -u(t - \tau) \tag{1.1}$$

where $\tau > 0$ is called the delay and the negative sign on the right indicate negative feedback.

When $\tau = 0$, we recover the simple ODE

$$u'(t) = -u(t), \tag{1.2}$$

whose general solution, $u(t) = u(0)e^{-t}$ decays to 0.

If we prescribe $u(t)$ for $-\tau \leqslant t \leqslant 0$, then the Eq. (6.7) should have a unique solution for $t > 0$. Suppose we set

$$u(t) = 1, -\tau \leqslant t \leqslant 0, \tag{1.3}$$

as "initial data" for equation 1.1. Then, on the interval $0 \leqslant t \leqslant \tau$ the argument of u on the right side satisfies $t - \tau \leqslant 0$ so

$$u'(t) = -u(t-\tau) = -1,$$

and therefore

$$u(t) = u(0) + \int_0^t (-1)\, ds = 1 - t,\ 0 \leqslant t \leqslant \tau \tag{1.4}$$

On $\tau \leqslant t \leqslant 2\tau$, we have $0 \leqslant t - \tau \leqslant \tau$ so by equation 1.4 we have

$$u'(t) = -u(t-\tau) = -[1 - (t-\tau)],$$

and thus

$$\begin{aligned} u(t) &= u(\tau) + \int_\tau^t -[1-(s-\tau)]\, ds \\ &= 1 - \tau + [-s + \frac{1}{2}(s-\tau)^2|_{s=\tau}^{s=t} \\ &= 1 - t + (t-\tau)^2/2,\ \tau \leqslant t \leqslant 2\tau, \end{aligned} \tag{1.5}$$

$$u(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{[t-(k-1)\tau]^k}{k!},\ (n-1)\tau \leqslant t \leqslant n\tau,\ n \geqslant 1, \tag{1.6}$$

$$u(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{[t-(k-1)\tau]^k}{k!},\ (n-1)\tau \leqslant t \leqslant n\tau,\ n \geqslant 1. \tag{1.7}$$

Thus, $u(t)$ is a polynomial of degree n on each subinterval $(n-1)\tau \leqslant t \leqslant n\tau$. It follows that $u(t)$ is a smooth function except at each $n\tau, n \geqslant 0$.

a) $u'(0-) = 0$ and $u'(0+) = -1$ so u' has a jump discontinuity at $t = 0$

b) $u''(\tau-) = 0$ and $u''(\tau+) = -1$ so u' has a jump discontinuity at $t = \tau$

1.3.1 Existence and uniqueness of solutions

Now let's consider the nonlinear delay differential equation

$$x'(t) = f(t, x(t), x(t-\tau)), \tag{1.8}$$

with a single delay $\tau > 0$. Assume that $f(t, x, y)$ and $f_x(t, x, y)$ are continuous on $\mathbb{R}^3$. Let $s \in \mathbb{R}$ be given and let $\phi : [s-\tau, s] \to \mathbb{R}$ be continuous. We seek a solution $x(t)$ of equation 1.8 satisfying

$$x(t) = \phi(t),\ s - \tau \leqslant t \leqslant s \tag{1.9}$$

and satisfying 1.8 on $s \leqslant t < s + \sigma$ for some $\sigma > 0$.

Theorem. (Existence and Uniqueness). Let $f(t, x, y)$ and $f_x(t, x, y)$ be continuous on $\mathbb{R}^3$, and let $\phi : [s-\tau, s] \to \mathbb{R}$ be continuous. Then, there exists $\sigma > s$ and a unique solution of the initial value problem (1.8)-(1.9) on $[s-\tau, \sigma]$ [22].

1.3.2 Lyapunov functionals

Consider

$$x'(t) = f(t, x_t) \tag{1.10}$$

where $f : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ is continuous, $f(t, 0) = 0$ and $x_t(\theta, \phi) = x(t + \theta, \phi(t + \theta))$ for $\theta \in [-\tau, 0]$. Let $V : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ be continuous and $x(\sigma, \phi)$ be the solution of (1.10) through (σ, ϕ). We denote

$$V' = V'(t, \phi) = \lim_{h \to 0+} \frac{1}{h}[V(t + h, x_{t+h}(t, \phi)) - V(t, \phi)].$$

Theorem. Let $u(s), v(s), w(s) : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and $u(s) > 0, v(s) > 0$ and $w(s) > 0$ for $s > 0$. Then the following statements are true:

(i) If there is a $V : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ such that

$$u(|\phi(0)|) \leqslant V(t, \phi) \leqslant v(||\phi||),$$

$$V'(t, \phi) \leqslant -w(|\phi(0)|),$$

then $x = 0$ is uniformly stable.

(ii) If, in addition to (i), $w(s) > 0$ for $s > 0$, then $x = 0$ is uniformly asymptotically stable.

Example 1. Consider the scalar equation

$$x'(t) = -a(t)x(t) - b(t)x(t - \tau(t)) \tag{1.11}$$

where $a(t), b(t)$ and $\tau(t)$ are bounded continuous functions, $a(t) > 0, \tau(t) > 0, \tau'(t) < 1$.

If $b(t) = 0$, then (1.11) becomes an ordinary differential equation; a trivial Lyapunov function is $V_1(x(t)) = \frac{x^2(t)}{2}$. Its derivative is

$$\begin{aligned} V_1'(x(t)) &= x(t)[-a(t)x(t) - b(t)x(t-\tau(t))] \\ &= -a(t)x^2(t) - b(t)x(t)x(t-\tau(t)). \end{aligned} \tag{1.12}$$

We cannot determine the sign of V_1', since we do not know the sign of $x(t)x(t-\tau(t))$. In order to find a Lyapunov functional V, we want the term like $-x^2(t-\tau(t))$ in V'. We try

$$V(t, x(t)) = \frac{1}{2}x^2(0) + \alpha \int_{-\tau(t)}^{0} x^2(\theta)d\theta, \qquad \alpha > 0\, is\, a\, constant$$

or equivalently,

$$V(x_t) = V(t, x_t) := \frac{1}{2}x^2(t) + \alpha \int_{-\tau(t)}^{0} x^2(t+\theta)d\theta.$$

We have

$$\begin{aligned} V'(x_t) &= -(a-\alpha)x^2(t) - b(t)x(t)x(t-\tau(t)) \\ &= -\alpha(1-\tau'(t))x^2(t-\tau(t)), \end{aligned} \tag{1.13}$$

since $\int_{-\tau(t)}^{0} x^2(t+\theta)d\theta = \int_{t-\tau(t)}^{t} x^2(\theta)d\theta$. Clearly, if

$$b^2(t) < 4(a(t)-\alpha)(1-\tau'(t))\alpha \tag{1.14}$$

then $V'(x_t) < 0$. Let $\tau(t) \leqslant \tau$, where τ is a positive constant; $u(s) = s^2/2; v(s) = ((1/2) + \alpha\tau)s^2$. Then,

$$u(|\phi(0)|) \leqslant V(t, \phi) \leqslant v(||\phi||)$$

If $\alpha > 0$ satisfies 1.14, then there may be a positive $\epsilon > 0$ such that

$$V'(x_t) \leqslant -\epsilon x^2(t).$$

Thus, we take $w(s) = \epsilon s^2$. By theorem, we know $x = 0$ is uniformly asymptotically stable. Indeed, since (1.11) is linear, we see that all solutions of (1.11) tend to $x = 0$ if (1.14) is true for some positive constant α.

When a, b and τ are constants, 1.14 reduces to

$$b^2 < 4(a - \alpha)\alpha \leqslant a^2$$

which implies that if $|b| < a$, then $x = 0$ is globally asymptotically stable, i.e.,

$$\lim_{t \to +\infty} x_t(\phi) = 0$$

for $\phi \in C$. Note that the length of delay τ is not restricted [22].

1.4 Synchronization

Synchronization refers to a control problem whose basic idea is to control the response system so that its state trajectories follow the drive system states asymptotically. Synchronization has gained considerable attention from researchers since it is widely used in spatiotemporal systems, secure communication, image processing, and recognition tasks [24, 25]. There are many types of synchronization methods involving adaptive, phase, lag and complete synchronization [32]-[35]. However, these synchronization schemes only guarantee synchronization in an infinite time. To reach synchronization in a finite time, the alternative method is finite time synchronization which is more practical [36, 37]. However, the time whereby the synchronization occurs depends on the initial conditions which may not be available in some cases. The fixed time synchronization is an effective method to tackle this problem, where the initial conditions are not needed to calculate the settling time at which the synchronization occurs [38]. Analysis on both the finite time and fixed time synchronization of memristive neural networks via feedback controller can be found in [39]. Also, in our previous work, we constructed state-feedback controllers to reach finite time and fixed time synchronization of SIMNNs [40].

On the parallel perspective, to ensure synchronization in memristive neural networks, different control methods were suggested such as output feedback control [26], event-triggered control [27], impulsive control [28], intermittent control [29]. A more recent one is a sliding mode control (SMC) which changes the response system dynamics by adding a discontinuous control which makes the system to track the trajectories of the drive system. For the last half-century, the SMC has been studied due to its simplicity and resilience to parameter variations and perturbations. It was successfully applied in synchronization of neural networks by designing a suitable sliding surface and a discontinuous controller function which forces the slave system states to track the fixed sliding manifold [30, 31]. To the best of our knowledge, there are no available literature on the sliding mode control techniques applied in the synchronization problem of SIMNNs which is the main contribution of this paper. First, a new terminal sliding mode surface is designed and its reachability is analysed. Then, a suitable sliding mode controller is constructed and the stability of the sliding mode is proved.

Chapter 2

Sliding mode control (SMC)

2.1 What is SMC?

In modelling real-life phenomena, there will usually be errors between the ideal plant and the mathematical model. It might be caused by unmodelled factors, parameter variations or external disturbances. Our task is to control such plant-model mismatches to achieve the required performance levels. In the seek to solve this problem, several robust control methods were proposed in the recent years. One of the robust controller design approaches is the so called sliding mode control method.

Sliding mode control method is a specific type of variable structure control system (VSCS) which was proposed by Soviet scientists Filippov and Popovski in 1960s and was exploited further by Emelyanov, Utkin and Neymark [41, 42, 43, 44, 45]. In VSCSs, the control parameters may vary to maintain the system trajectories in the desired hypersurface chosen a priori. After reaching that hypersurface, the system is insensitive to external disturbances and parameter variations. Emelyanov and Utkin proved that this type of control works for both stabilization and trajectory tracking problems. However, because of the discontinuity of the control law, this control method was criticized for its so called *chattering effect*. There were proposed several improvements for this problem such as equivalent control, generalized control and higher order controls [46, 47, 48].

The SMC involves two parts: first designing a purposed sliding surface and second constructing a discontinuous sliding mode controller to guide the solution trajectories of a chaotic system to a sliding manifold. Since SMC method is effective and guarantees the sliding motion and synchronization of master-slave systems, it was succesfully applied in the synchronization of chaotic system [49, 50, 51, 52].

2.2 Sliding mode notion

Consider a dynamical system given by the following system of differential equations:

$$\frac{dx}{dt} = f(x,t) + g(x,t)u(x,t), \tag{2.1}$$

where

- $t \in \mathbb{R}$ is the time
- $x \in \mathbb{R}^n$ is the state vector
- $u \in \mathbb{R}^n$ is the control vector
- f and g are vector fields

Suppose that the control vector function $u(x,t)$ has a discontinuity on a sliding surface $S(x) = 0$:

$$u(x,t) = \begin{cases} u^{+}(x,t) \;\; if \;\; S(x) > 0, \\ \\ u^{-}(x,t) \;\; if \;\; S(x) < 0. \end{cases} \tag{2.2}$$

Definition 2.2.1. *[53] The system defined by (2.1) and (2.2) is called a variable structure system. Further if the control function $u(x,t)$ satisfies the attractiveness condition $S(x)S'(x) < 0$, the control is said to be in sliding mode.*

Definition 2.2.2. *[53] A sliding mode exists on $S(x) = 0$ if and only if the phase trajectory is on the sliding surface and the attractiveness condition is verified, i.e.*

$$x(t) \in \{x|S(x) = 0\} \quad and \quad S(x)S'(x) < 0$$

*The sliding mode is said **ideal** if $S(x) = 0$ and $S'(x) = 0$. In the case where only the attractiveness condition is verified $(S(x)S'(x) < 0)$, the sliding is said **real**.*

Remark. The equivalent control is constructed so that the designed sliding mode surface $S(x)$ satisfied the conditions $S(x) = 0$ and $S'(x) = 0$, i.e. ideal sliding mode.

2.3 Reaching phase and sliding phase

There are two phases in the evolution of system state trajectories:

- The reaching phase is from the initial states to the intersection with the sliding surface.
- The sliding phase is from the intersection with the sliding surface to the origin.

Both phases are illustrated in figure 2-1 in the case of simple second order system.

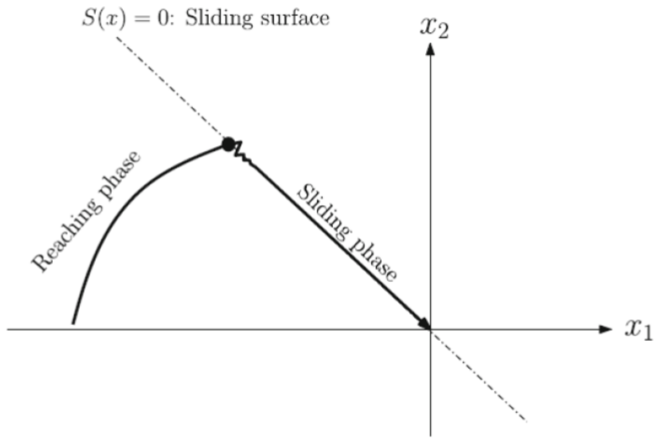

Figure 2-1: Reaching phase and sliding phase

During the reaching phase, the attractiveness condition of the sliding surface $S(x)S'(x) < 0$ is verified. This analysis is called *a reachability analysis*. But it does not guarantee a sliding mode in a finite time.

To ensure a finite sliding time, the condition becomes:

$$S(x)S'(x) < -\epsilon \ \ if \ \ S(x) \neq 0 \ \ and \ \ \epsilon > 0$$

During sliding phase ($S(x) = 0 \ \ and \ \ S'(x)S(x) < 0$), the discontinuous controller function is activated to make the system trajectories slide along the predefined sliding surface $S(x)$.

2.4 An illustrative example: Ritikate system

The design of a control law for the synchronization of master-slave system will be illustrated in the example of Ritikate system [54]. The master-slave Ritikate systems are modelled by the following equations:

Master system:

$$
\begin{aligned}
\dot{x}_m &= -1.2x_m + z_m y_m \\
\dot{y}_m &= -1.2y_m + (z_m - 3)x_m \\
\dot{z}_m &= 1 - x_m y_m
\end{aligned}
\tag{2.3}
$$

Slave system:

$$
\begin{aligned}
\dot{x}_s &= -1.2x_s + z_s y_s + u + d(t) \\
\dot{y}_s &= -1.2y_s + (z_s - 3)x_s \\
\dot{z}_s &= 1 - x_s y_s
\end{aligned}
\tag{2.4}
$$

The initial conditions are $x_m(0) = 1$, $y_m(0) = 1$, $z_m(0) = 1$, $x_s(0) = 0.5$, $y_s(0) = 0.5$, $z_s(0) = 0.5$ and $u = 0$ is a synchronous controller, $d(t) = 0.1sint(t)$ is external interference. Define errors as $e_1(t) = x_s(t) - x_m(t)$, $e_2(t) = y_s(t) - y_m(t)$, $e_3(t) = z_s(t) - z_m(t)$.

Then the error dynamics can be written as:

$$
\begin{aligned}
\dot{e}_1 &= \dot{x}_s - \dot{x}_m = -1.2e_1 + z_s y_s - z_m y_m + u + d(t) \\
\dot{e}_2 &= \dot{y}_s - \dot{y}_m = -1.2e_2 + (z_s - 3)x_s - (z_m - 3)x_m \\
\dot{e}_3 &= \dot{z}_s - \dot{z}_m = -x_s y_s + x_m y_m
\end{aligned}
\tag{2.5}
$$

The controller u is defined as

$$
u = u_{eq} + u_{sw} \tag{2.6}
$$

and the switching surface is defined as

$$
s = c_1 e_1 + c_2 e_2 + c_3 e_3 \tag{2.7}
$$

where c_1, c_2, c_3 are constant coefficients and the equivalent control u_{eq} is obtained by $\dot{s} = 0$. However, u_{eq} cannot obtain the sliding mode if the initial state is not on the switching surface. Thus, additional switching function u_{sw} must be designed satisfying the reaching condition of sliding mode, $(s(t)\dot{s}(t) < 0)$.

During the ideal sliding mode, the system must satisfy the following condition:

$$s(t) = 0 \ \ and \ \ \dot{s} = 0 \tag{2.8}$$

Taking derivative of 2.7 with respect to time, and substituting 2.5 we get:

$$\begin{aligned}\dot{s} &= c_1\dot{e}_1 + c_2\dot{e}_2 + c_3\dot{e}_3 \\ &= c_1(-1.2e_1 + z_s y_s - z_m y_m + u_{eq} + d(t)) \\ &+ c_2(-1.2e_2 + (z_s - 3)x_s - (z_m - 3)x_m) \\ &+ c_3(-x_s y_s + x_m y_m) = 0\end{aligned} \tag{2.9}$$

From 2.9, the equivalent control u_{eq} is obtained and is given by:

$$\begin{aligned}u_{eq} = [&-c_1(-1.2e_1 + z_s y_s - z_m y_m + d(t)) \\ &- c_2(-1.2e_2 + (z_s - 3)x_s - (z_m - 3)x_m) \\ &- c_3(-x_s y_s + x_m y_m)]/c_1\end{aligned} \tag{2.10}$$

Now, design the switching control u_{sw} as:

$$u_{sw} = -\lambda \cdot sign(s) \tag{2.11}$$

where $\lambda > 0$ is a constant.

Since the external interference $d(t)$ is bounded and unknown, the overall sliding mode controller u is designed as follows:

$$
\begin{aligned}
u &= u_{eq} + u_{sw} \\
&= [-c_1(-1.2e_1 + z_s y_s - z_m y_m) \\
&- c_2(-1.2e_2 + (z_s - 3)x_s - (z_m - 3)x_m) \\
&- c_3(-x_s y_s + x_m y_m)]/c_1 \\
&- \lambda sign(s)
\end{aligned} \tag{2.12}
$$

Now, we need to prove that the sliding mode dynamics 2.5 is asymptotically stable. Define the Lyapunov function:

$$V = \frac{1}{2}s^2 \tag{2.13}$$

Differentiating $V(t)$ w.r.t. time along the trajectories of 2.7 yields:

$$
\begin{aligned}
\dot{V} &= s\dot{s} \\
&= s[c_1\dot{e_1} + c_2\dot{e_2} + c_3\dot{e_3}] \\
&= s[c_1(-1.2e_1 + z_sy_s - z_my_m + u + d(t)) \\
&+ c_2(-1.2e_2 + (z_s - 3)x_s - (z_m - 3)x_m) \\
&+ c_3(-x_sy_s + x_my_m)] \\
&= s\{c_1[-1.2e_1 + z_sy_s - z_my_m \\
&+ (-c_1(-1.2e_1 + z_sy_s - z_my_m) \\
&- c_2(-1.2e_2 + (z_s - 3)x_s - (z_m - 3)x_m) \\
&- c_3(-x_sy_s + x_my_m))/c_1 - \lambda sign(s) + d(t)] \\
&+ c_2(-1.2e_2 + (z_s - 3)x_s - (z_m - 3)x_m) \\
&+ c_3(-x_sy_s + x_my_m)\} \\
&= s[c_1d(t) - \lambda \cdot sign(s)] \\
&= s[d(t) - \lambda \cdot sign(s)] \\
&= s[d(t)] - \lambda \cdot |s|
\end{aligned}
\tag{2.14}
$$

Let $d(t)$ is bounded by $|d(t)| \leq \gamma$. Then,

$$
\begin{aligned}
\dot{V} &\leq s[d(t)] - \lambda \cdot |s| \\
&\leq |s| \cdot [|d(t)|] - \lambda \cdot |s| \\
&\leq |s| \cdot (\gamma - \lambda)
\end{aligned}
\tag{2.15}
$$

For arbitrary constants $\lambda > \gamma$, we get $\dot{V} < 0$. From the Lyapunov stability theory, the error system 2.5 is asymptotically stable under the controller $u(t)$ in 2.12.

Chapter 3

Synchronization analysis

Let us consider the delayed shunting inhibitory memristive neural networks (SIMNNs) as in the following initial value problem form:

$$\frac{dx_{ij}(t)}{dt} = -\alpha_{ij}(t)x_{ij}(t) - \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}(x_{ij}(t))f(x_{hl}(t-\lambda_{hl}(t)))x_{ij}(t) + I_{ij}(t), \quad (3.1)$$

with initial conditions are of the form $x_{ij}(t_0) = \phi_{ij}(t_0)$, $t_0 \in [-\lambda_{hl}, 0]$ for $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, where $\phi_{ij}(\cdot)$ denotes real-valued bounded continuous function defined on $(-\infty, 0]$; $x_{ij}(t)$ denotes the voltage of the cell $C_{ij}(x_{ij}(t))$; $I_{ij}(t)$ is the external input to $C_{ij}(x_{ij}(t))$ such that $|I_{ij}(t)| \leq \overline{I}_{ij}$; here, a two-dimensional grid of processing cell C_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, denote the cell at the (i, j) position of the lattice; the $r-$neighborhood of $C_{ij}(x_{ij}(t))$ is defined as

$$N_r(i, j) = \{C_{hl}(x_{ij}(t)) : \max\{|h-i|, |l-j|\} \leq r,\ 1 \leq h \leq m,\ 1 \leq l \leq n\}.$$

Further, the passive decay rate of the cell activity is defined by a positive definite function $\alpha_{ij}(t)$; $C_{ij}^{hl}(x_{ij}(t)) \geq 0$ is the connection or coupling strength of the postsynaptic activity of the cell $C_{hl}(x_{ij}(t))$ transmitted to the cell $C_{ij}(x_{ij}(t))$, and it can be constrained by the following switching mode from the favor of feature of memristor and current-voltage characteristics:

$$C_{ij}^{hl}(x_{ij}(t)) = \begin{cases} \acute{C}_{ij}^{hl}, |x_{ij}(t)| \leq \Theta_{ij}, \\ \grave{C}_{ij}^{hl}, |x_{ij}(t)| > \Theta_{ij}, \end{cases}$$

with the switching jumps Θ_{ij}, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ and $\acute{C}_{ij}$, and $\grave{C}_{ij}$ are known constants with respect to memristors. Denote $\overline{C}_{ij}^{hl} = max\left\{|\acute{C}_{ij}^{hl}|, |\grave{C}_{ij}^{hl}|\right\}$.

In the system (3.1), the activation function $f(\cdot)$ is a positive continuous function representing the output or firing rate of the cell $C_{hl}(t)$. Set $x = \big(x_{11}, \dots, x_{1n}, \dots, x_{m1} \dots, x_{mn}\big)^T$, $f(x) = \big(f_{11}(x), \dots, f_{1n}(x), \dots, f_{m1}(x)$
$\dots, f_{mn}(x)\big)^T$; the continuous function $\lambda_{hl}(t)$ corresponds to the transmission delay along the axon of the (h, l)th cell from the (i, j)th cell which satisfies $0 \leq \lambda_{hl}(t) \leq \lambda_{hl}$.

The following assumptions are made for the nonlinear activation function to reach the desired synchronization criteria.

(A1) The neural activation function $f(\cdot)$ satisfy the Lipschitz condition, i.e., there exists a positive number L^f such that for all $x \neq y$, $|f(x) - f(y)| \leq L^f|x - y|$.

(A2) The neural activation function $f(\cdot)$ is bounded, i.e., there exits a scalar $M > 0$ such that $|f(\cdot)| \leq M$.

Next, for the drive system (3.1) the corresponding response system is formulated as

$$\frac{dy_{ij}(t)}{dt} = -\alpha_{ij}(t)y_{ij}(t) - \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}(y_{ij}(t))f(y_{hl}(t-\lambda_{hl}(t)))y_{ij}(t) + w_{ij}(t) + I_{ij}(t) \quad (3.2)$$

where $w_{ij}(t)$ is a sliding mode controller to be constructed later.

Denote the synchronization error $e_{ij}(t) = y_{ij}(t) - x_{ij}(t)$, then the error dynamics of the proposed synchronization problem yields.

$$\begin{aligned}\frac{de_{ij}(t)}{dt} = & -\alpha_{ij}e_{ij}(t) - \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}(y_{ij}(t))f(y_{hl}(t-\lambda_{hl}(t)))y_{ij}(t) \\ & + \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}(x_{ij}(t))f(x_{hl}(t-\lambda_{hl}(t)))x_{ij}(t) + w_{ij}(t). \qquad (3.3)\end{aligned}$$

Definition 3.0.1. *[55] If there exists a constant $t^* > 0$ which depends on the initial vector value that satisfies*

$$\lim_{t\to t^*} |e_{ij}(t)| = 0; \ |e_{ij}(t)| = 0, \ t \geq t^*,$$

then the drive-response system (3.1) *and* (3.2) *is said to achieve finite time synchronization with the settling time t^*.*

Definition 3.0.2. *[55] The drive-response system* (3.1) *and* (3.2) *is said to achieve fixed time synchronization if it satisfies Definition 3.0.1 and there exits a positive constant T_{max} such that t^* is always less than T_{max} for any $e(t_0)$, where $e(t_0) = (e_{11}(t_0), \ldots, e_{1n}(t_0), \ldots, e_{m1}(t_0) \ldots, e_{mn}(t_0))^T$, where $t_0 \in [-\lambda_{hl}, 0]$.*

In what follows, the following lemmas will be useful in obtaining the main results.

Lemma 1. *[56] Assume that a continuous, positive definite function $V(t)$ satisfies the following inequality:*

$$V'(t) \leq -P(t)V^{\eta}(t),$$

where $P(t)$ is a positive definite function, $0 < \eta < 1$ and there is a constant $\lambda > 0$ satisfying

$$\int_{t_0}^{t} P(s)ds \geq \lambda(t - t_0), \ \ for \ all \ \ t \geq t_0.$$

Then, $V(t) = 0$ *for all* $t \geq T$ *with* $T = t_0 + \frac{V^{1-\eta}(t)}{(1-\eta)\lambda}$

Lemma 2. *[56] Assume that a continuous, positive definite function* $V(t)$ *satisfies the following inequality:*

$$V'(t) \leq -P(t)V^{\eta_1}(t) - Q(t)V^{\eta_2}(t),$$

for some positive definite functions $P(t), Q(t)$, $\eta_1 > 1$, $0 < \eta_2 < 1$ *and there are* $\lambda > 0$ *and* $l > 0$ *s.t. for any* $t, t' \in [t_0, \infty)$ *satisfying*

$$\int_{t'}^{t} P(s)ds \geq \lambda(t - t'), \;\; for \;\; all \;\; t \geq t',$$
$$\int_{t'}^{t} Q(s)ds \geq l(t - t'), \;\; for \;\; all \;\; t \geq t'.$$

Then, $V(t) = 0$ *for all* $t \geq T_{max}$ *with the settling time given by* $T_{max} = \frac{1}{(1-\eta_1)\lambda} + \frac{1}{(\eta_2-1)l}$.

3.1 Main Results

The finite time and fixed time synchronization for the master-slave system (3.1) and (3.2) are analysed in this section. By introducing a unique control function and considering corresponding Lyapunov functionals for each case, the rechability of a sliding mode surface and stability of SIMNNs were proved.

3.2 Sliding mode surface and sliding mode controller

The sliding mode surface is designed as follows:

$$S = \{e(t) \,|\, S(e(t)) = 0\} \tag{3.4}$$

The state variables of sliding surface are described by:

$$S_{ij}(t) = \beta_{ij}(t)e_{ij}(t) + \int_0^t (|e_{ij}(s)|^{k_1} + |e_{ij}(s)|^{k_2})sgn(e_{ij}(s))ds \tag{3.5}$$

where $\beta_{ij}(t)$ are positive definite functions for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, $0 \le k_1 \le 1$, $k_2 > 0$ and $sgn(\cdot)$ is a signum function.

By finding the time derivative of equation (3.5), we can reach the motion of the state variable of sliding surface as in the following form;

$$\dot{S}_{ij}(t) = \beta_{ij}(t)\dot{e}_{ij}(t) + \dot{\beta}_{ij}(t)e_{ij}(t) + (|e_{ij}(t)|)^{k_1} + |e_{ij}(t)|^{k_2})sgn(e_{ij}(t)). \tag{3.6}$$

3.2.1 Reachability analysis

Let us assume that the state variables reach the sliding surface (3.4) in finite time t^*. That is,

$$S(t) = \dot{S}(t) = 0, \text{for all } t \ge t^*.$$

From the reachablity property of the sliding surface ($\dot{S}_{ij}(t) = 0$) and from equation (3.6), one has

$$\dot{e}_{ij}(t) \quad = \quad -\frac{1}{\beta_{ij}(t)}[\dot{\beta}_{ij}(t)e_{ij}(t) + (|e_{ij}(t)|)^{k_1} + |e_{ij}(t)|^{k_2})sgn(e_{ij}(t)] \tag{3.7}$$

The state variables of system (3.2) will converge to the sliding surface (3.4) in a finite time t^*. After reaching the surface, the system (3.7) is activated and its state variables tend to the origin in another fixed time T_{max}.

From the systems (3.2) and (3.7), we have

$$-\frac{1}{\beta_{ij}(t)}[\dot{\beta}_{ij}(t)e_{ij}(t)+(|e_{ij}(t)|)^{k_1}+|e_{ij}(t)|^{k_2})sgn(e_{ij}(t))] = -\alpha_{ij}(t)e_{ij}(t)-\Phi(x_{ij}(t), y_{ij}(t))+w_{ij}(t)$$

where $\Phi(x_{ij}(t), y_{ij}(t)) = \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}(y_{ij}(t))f(y_{hl}(t-\lambda_{hl}(t)))y_{ij}(t) - \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}(x_{ij}(t))f(x_{hl}(t-\lambda_{hl}(t)))x_{ij}(t)$

3.3 Finite time synchronization

In this subsection, the finite time synchronization of the master-slave system (3.1) and (3.2) is proved.

Theorem 3. *Suppose that the assumptions (**A1**)-(**A2**) are satisfied. Consider the following controller:*

$$\begin{aligned} w_{ij}(t) &= -\frac{1}{\beta_{ij}(t)}(\dot{\beta}_{ij}(t)e_{ij}(t) + |e_{ij}(t)|)^{k_1} + |e_{ij}(t)|^{k_2})sgn(e_{ij}(t)) + \alpha_{ij}e_{ij}(t) + \Phi(x_{ij}(t), y_{ij}(t)) - \\ &\quad -(\lambda + |S_{ij}(t)|^{k_2})sgn(S_{ij}(t) \end{aligned} \tag{3.8}$$

where constant $\lambda > 0$. *Then, the master-slave system* (3.1) *and* (3.2) *achieves finite time synchronization under controller* (3.8) *with the setting time given by* $t^* = \dfrac{\left(\sum_{i,j}|S_{ij}(0)|\right)^{1-k_2}}{\beta(1-k_2)}$, *where a constant* $\beta = inf\{\beta_{ij}(t)\}$.

Proof. To achieve finite time synchronization we construct the following Lyapunov function. $V_1(t) = \sum_{i,j} |S_{ij}(t)|$. Computing the derivative yields

$$
\begin{aligned}
\dot{V}_1(t) &= \sum_{i,j} sgn(S_{ij}(t))\frac{dS_{ij}(t)}{dt} \\
&= \sum_{i,j} sgn(S_{ij}(t))\Bigg[\dot{\beta}_{ij}(t)e_{ij}(t) + \beta_{ij}(t)\dot{e}_{ij}(t) + (|e_{ij}(t)|)^{k_1} + |e_{ij}(t)|^{k_2})sgn(e_{ij}(t))\Bigg] \\
&= \sum_{i,j} sgn(S_{ij}(t)\Bigg[\dot{\beta}_{ij}(t)e_{ij}(t) - \alpha_{ij}\beta_{ij}(t)e_{ij}(t) - \beta_{ij}(t)\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}(y_{ij}(t))f(y_{hl}(t-\lambda_{hl}(t)))y_{ij}(t) \\
&+ \beta_{ij}(t)\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}(x_{ij}(t))f(x_{hl}(t-\lambda_{hl}(t)))x_{ij}(t) + \beta_{ij}(t)w_{ij}(t) + (|e_{ij}(t)|)^{k_1} \\
&+ |e_{ij}(t)|^{k_2})sgn(e_{ij}(t))\Bigg] \\
&= \sum_{i,j} sgn(S_{ij}(t)\Bigg[\dot{\beta}_{ij}(t)e_{ij}(t) - \alpha_{ij}\beta_{ij}(t)e_{ij}(t) - \beta_{ij}(t)\Phi(x_{ij}(t),y_{ij}(t)) - \dot{\beta}_{ij}(t)e_{ij}(t) - \\
&- (|e_{ij}(t)|)^{k_1} + |e_{ij}(t)|^{k_2})sgn(e_{ij}(t)) + \alpha_{ij}\beta_{ij}(t)e_{ij}(t) + \beta_{ij}(t)\Phi(x_{ij}(t),y_{ij}(t)) - \\
&- \beta_{ij}(t)(\lambda + |S_{ij}(t)|^{k_2})sgn(S_{ij}(t) + (|e_{ij}(t)|)^{k_1} + |e_{ij}(t)|^{k_2})sgn(e_{ij}(t))\Bigg] \\
&= \sum_{i,j} -sgn(S_{ij}(t)\beta_{ij}(t)(\lambda + |S_{ij}(t)|^{k_2})sgn(S_{ij}(t) \\
&\le \sum_{i,j} -\beta|S_{ij}(t)|^{k_2} \le -\beta\left(\sum_{i,j}|S_{ij}(t)|\right)^{k_2} = -\beta V_1^{k_2}(t)
\end{aligned}
$$

Thus, by lemma (1) we get $V_1(t) = 0$ for all $t^* = \dfrac{\left(\sum_{i,j}|S_{ij}(0)|\right)^{1-k_2}}{\beta(1-k_2)}$. In other words, $|S_{ij}(t)| = 0$ for all $t \ge t^*$ which implies the error system states reach zero in finite time t^*. Hence, the master-slave system (3.1) and (3.2) synchronizes in a finite time t^* by Definition (3.0.1). □

3.4 Fixed time synchronization

In this subsection, the fixed time synchronization of the master-slave system (3.1) and (3.2) is analysed.

Theorem 4. *Suppose that the assumptions (**A1**)-(**A2**) are correct. Then, the master-slave system* (3.1) *and* (3.2) *achieves fixed time synchronization under controller* (3.8) *within* $T_{max} = \dfrac{1}{\beta(1-k_2)} + \dfrac{1}{\beta\lambda}$, *where a constant* $\beta = inf\{\beta_{ij}(t)\}$.

Proof. Consider the following Lyapunov function. $V_2(t) = \sum_{i,j} S_{ij}^2(t)$. Differentiating $V_2(t)$ we get

$$
\begin{aligned}
\dot{V_2}(t) =& \sum_{i,j} S_{ij}(t)\frac{dS_{ij}(t)}{dt} \\
=& \sum_{i,j} S_{ij}(t)\Bigg[\dot{\beta}_{ij}(t)e_{ij}(t) + \beta_{ij}(t)\dot{e}_{ij}(t) + (|e_{ij}(t)|)^{k_1} + |e_{ij}(t)|^{k_2})sgn(e_{ij}(t))\Bigg] \\
=& \sum_{i,j} S_{ij}(t)\Bigg[\dot{\beta}_{ij}(t)e_{ij}(t) - \alpha_{ij}\beta_{ij}(t)e_{ij}(t) - \beta_{ij}(t)\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}(y_{ij}(t))f(y_{hl}(t-\lambda_{hl}(t)))y_{ij}(t) \\
& + \beta_{ij}(t)\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}(x_{ij}(t))f(x_{hl}(t-\lambda_{hl}(t)))x_{ij}(t)+ \\
& +\beta_{ij}(t)w_{ij}(t) + (|e_{ij}(t)|)^{k_1} + |e_{ij}(t)|^{k_2})sgn(e_{ij}(t))\Bigg] \\
=& \sum_{i,j} S_{ij}(t)\Bigg[\dot{\beta}_{ij}(t)e_{ij}(t) - \alpha_{ij}\beta_{ij}(t)e_{ij}(t) - \beta_{ij}(t)\Phi(x_{ij}(t),y_{ij}(t)) - \dot{\beta}_{ij}(t)e_{ij}(t) \\
& - (|e_{ij}(t)|)^{k_1} + |e_{ij}(t)|^{k_2})sgn(e_{ij}(t)) + \alpha_{ij}\beta_{ij}(t)e_{ij}(t) + \beta_{ij}(t)\Phi(x_{ij}(t),y_{ij}(t)) \\
& - \beta_{ij}(t)(\lambda + |S_{ij}(t)|^{k_2})sgn(S_{ij}(t) + (|e_{ij}(t)|)^{k_1} + |e_{ij}(t)|^{k_2})sgn(e_{ij}(t))\Bigg] \\
=& \sum_{i,j} S_{ij}(t)\Bigg[-\beta_{ij}(t)(\lambda + |S_{ij}(t)|^{k_2})\Bigg]sgn(S_{ij}(t)) \\
\leq& \sum_{i,j} -\beta\Bigg[\lambda|S_{ij}(t)| + |S_{ij}(t)|^{k_2+1}\Bigg] \leq -\beta\lambda\sum_{i,j}|S_{ij}(t)| - \beta\sum_{i,j}|S_{ij}(t)|^{k_2+1} \\
\leqslant& -2\beta V_2(t)^{\frac{k_2+1}{2}} - 2\beta\lambda V_2(t)^{\frac{1}{2}}
\end{aligned}
$$

where $\frac{k_2+1}{2} > 1$. Correspondingly, $V_2(t) = 0$ for all $t \geq T_{max} = \dfrac{1}{\beta(1-k_2)} + \dfrac{1}{\beta\lambda}$ by Lemma (2). Hence, the master-slave system (3.1) and (3.2) reaches synchronization in a fixed time by definition (3.0.2). $\square$

Chapter 4

Numerical simulations

In this section, a numerical example with its simulations are presented to validate the performance of proposed SMC to attain the desired finite-/fixed time synchronization of the considered derive-response system model.

Example 4.1 Let us consider the SIMCNN based derive system (3.1) in which the parameters are assumed as $m = 2$, $n = 3$, $r = 1$, and

$$\begin{cases} \frac{dx_{11}(t)}{dt} = & -0.5x_{11}(t) - [0.5f(x_{12}(t-\lambda_{12}(t))) + 0.3f(x_{21}(t-\lambda_{21}(t))) \\ & +0.2f(x_{22}(t-\lambda_{22}(t)))]x_{11}(t) + 0.7; \\ \frac{dx_{12}(t)}{dt} = & -0.5x_{12}(t) - [0.5f(x_{11}(t-\lambda_{11}(t))) + 0.5f(x_{13}(t-\lambda_{13}(t))) + 0.3f(y_{21}(t-\lambda_{21}(t))) \\ & +0.2f(x_{22}(t-\lambda_{22}(t))) + 0.5f(x_{23}(t-\lambda_{23}(t)))]x_{12}(t) + 0.5; \\ \frac{dx_{13}(t)}{dt} = & -0.5x_{13}(t) - [0.5f(x_{12}(t-\lambda_{12}(t))) + 0.3f(x_{22}(t-\lambda_{22}(t))) \\ & +0.2f(x_{23}(t-\lambda_{23}(t)))]x_{13}(t) + 0.4; \\ \frac{dx_{21}(t)}{dt} = & -0.5x_{21}(t) - [0.5f(x_{11}(t-\lambda_{11}(t))) + 0.3f(x_{12}(t-\lambda_{12}(t))) \\ & +0.2f(x_{22}(t-\lambda_{22}(t)))]x_{21}(t) + 0.3; \\ \frac{dx_{22}(t)}{dt} = & -0.5x_{22}(t) - [0.5f(x_{11}(t-\lambda_{11}(t))) + 0.5f(x_{12}(t-\lambda_{12}(t))) + 0.5f(x_{13}(t-\lambda_{13}(t))) + \\ & +0.3f(x_{21}(t-\lambda_{21}(t))) + 0.5f(x_{23}(t-\lambda_{23}(t)))]x_{22}(t) + 0.5; \\ \frac{dx_{23}(t)}{dt} = & -0.5x_{23}(t) - [0.5f(x_{12}(t-\lambda_{12}(t))) + 0.3f(x_{13}(t-\lambda_{13}(t))) \\ & +0.2f(x_{22}(t-\lambda_{22}(t)))]x_{23}(t) + 0.4; \end{cases}$$

its corresponding response system is

$$\begin{cases} \frac{dy_{11}(t)}{dt} = & -0.5y_{11}(t) - [0.5f(y_{12}(t-\lambda_{12}(t))) + 0.3f(y_{21}(t-\lambda_{21}(t))) \\ & +0.2f(y_{22}(t-\lambda_{22}(t)))]y_{11}(t) + 0.7 + w_{11}(t); \\ \frac{dy_{12}(t)}{dt} = & -0.5y_{12}(t) - [0.5f(y_{11}(t-\lambda_{11}(t))) + 0.5f(y_{13}(t-\lambda_{13}(t))) + 0.3f(y_{21}(t-\lambda_{21}(t))) + \\ & +0.2f(y_{22}(t-\lambda_{22}(t))) + 0.5f(y_{23}(t-\lambda_{23}(t)))]y_{12}(t) + 0.5 + w_{12}(t); \\ \frac{dy_{13}(t)}{dt} = & -0.5y_{13}(t) - [0.5f(y_{12}(t-\lambda_{12}(t))) + 0.3f(y_{22}(t-\lambda_{22}(t))) \\ & +0.2f(y_{23}(t-\lambda_{23}(t)))]y_{13}(t) + 0.4 + w_{13}(t); \\ \frac{dy_{21}(t)}{dt} = & -0.5y_{21}(t) - [0.5f(y_{11}(t-\lambda_{11}(t))) + 0.3f(y_{12}(t-\lambda_{12}(t))) \\ & +0.2f(y_{22}(t-\lambda_{22}(t)))]y_{21}(t) + 0.3 + w_{21}(t); \\ \frac{dy_{22}(t)}{dt} = & -0.5y_{22}(t) - [0.5f(y_{11}(t-\lambda_{11}(t))) + 0.5f(y_{12}(t-\lambda_{12}(t))) + 0.5f(y_{13}(t-\lambda_{13}(t))) + \\ & +0.3f(y_{21}(t-\lambda_{21}(t))) + 0.5f(y_{23}(t-\lambda_{23}(t)))]y_{22}(t) + 0.5 + w_{22}(t); \\ \frac{dy_{23}(t)}{dt} = & -0.5x_{23}(t) - [0.5f(y_{12}(t-\lambda_{12}(t))) + 0.3f(y_{13}(t-\lambda_{13}(t))) \\ & +0.2f(y_{22}(t-\lambda_{22}(t)))]y_{23}(t) + 0.4 + w_{23}(t). \end{cases}$$

The initial conditions are taken as $x_{11}(t) = -0.025$, $x_{12}(t) = 0.036, x_{13}(t) = -0.014, x_{21}(t) = 0.012, x_{22}(t) = -0.021, x_{23}(t) = 0.042$ and $y_{11}(t) = 0.5, y_{12}(t) = 0.8, y_{13}(t) = 0.12, y_{21}(t) = -3, y_{22}(t) = -1.2, y_{23}(t) = 0.4, \forall t \in [-1, 0]$. The activation functions $f(\cdot) = tanh(\cdot) + 1.1$ and the time-varying delay term is $\lambda_{hl}(t) = |sin(t)| + 1$. In the response system, the control function $w_{ij}(t)$ is given as in Theorem 3 for both the finite time and fixed time synchronization analysis, where constants $k_1 = k_2 = 0.5$ and $\lambda = 0.5$.

One can see from 4-1, for two different initial conditions, the drive-response system solutions behave chaotically until end time without controllers applied. While synchronization is observed approximately at time $t = 0.25$ when the sliding mode

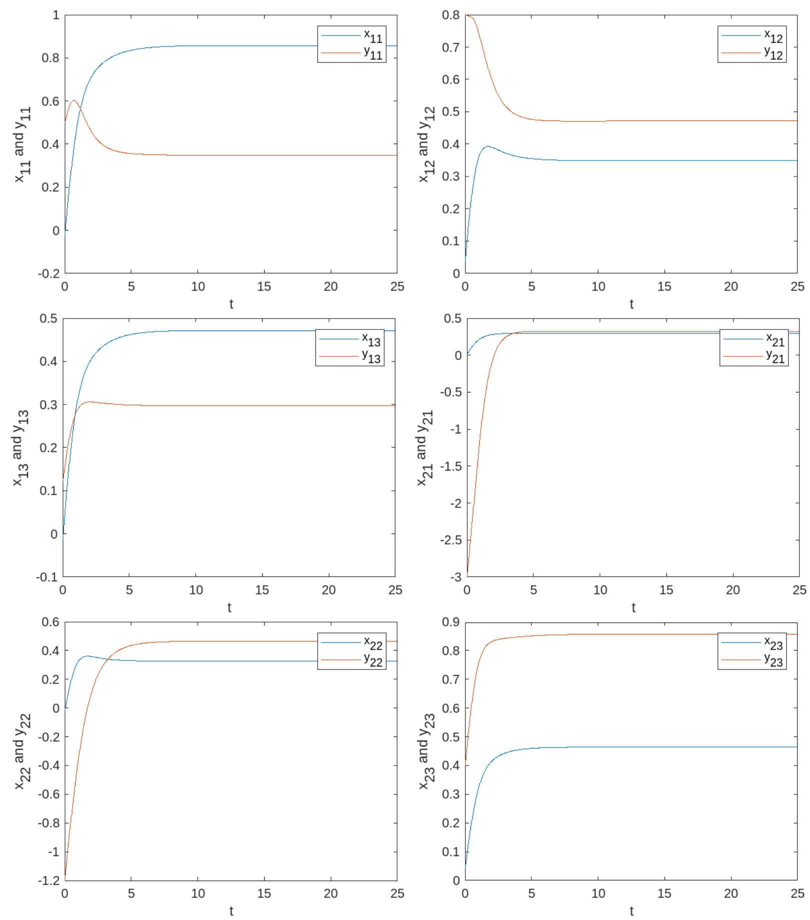

Figure 4-1: State trajectories of the master and slave system without controller.

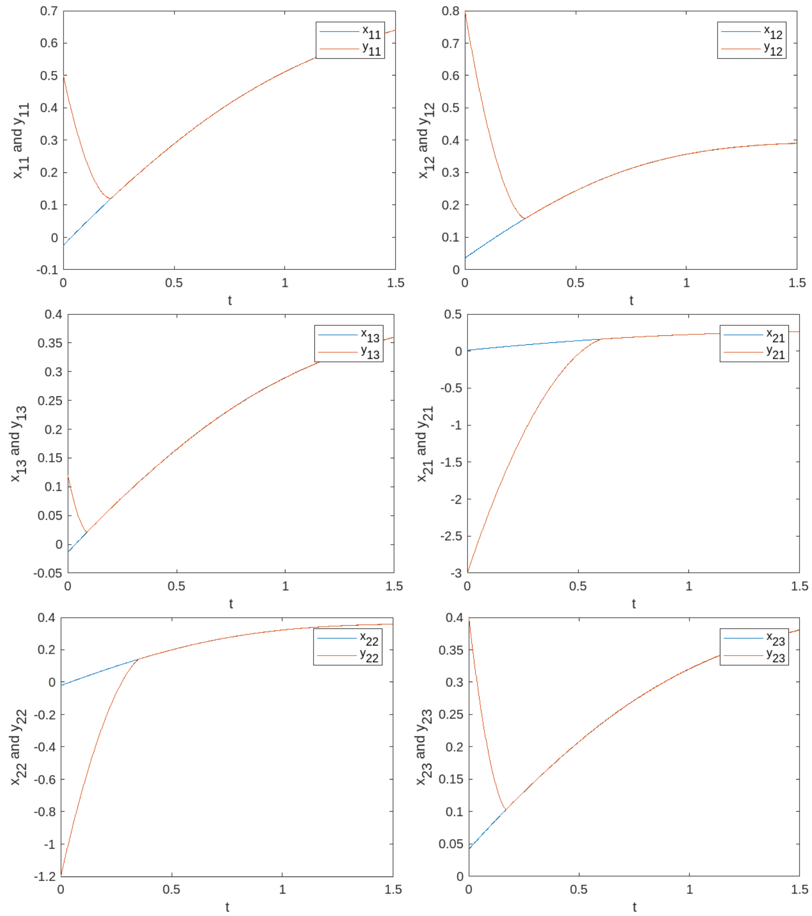

Figure 4-2: State trajectories of the master and slave system with controller $w_{ij}(t)$ applied.

controller is applied to the response system (4-2).

By Theorem 3, the settling time for the finite time synchronization t^* can be calculated as $t^* = \frac{\left(\sum_{i,j} |S_{ij}(0)|\right)^{1-k_2}}{\beta(1-k_2)} = \frac{2.6^{0.5}}{0.25} \approx 6.4498$ seconds. By Theorem 4, the reaching time for the fixed time synchronization is $T_{max} = \frac{1}{\beta(k_2-1)} + \frac{1}{\beta\lambda} = \frac{1}{0.5(1-0.5)} + \frac{1}{0.5 \cdot 0.5} = 8$ seconds. From the Figure 2, one can see that the drive-response system reached finite time and fixed time synchronization before t^* and T_{max} which verifies the theoretical findings in Theorems 3 and 4. Compared with the results of our previous work, the synchronization is achieved faster with SMC than with the feedback controller constructed in [40].

Chapter 5

Conclusion

To conclude, the finite time and fixed time synchronization of the shunting inhibitory memristive neural networks with time varying delays were studied via sliding mode control method. First, a suitable sliding surface was designed on which the synchronization error is zero. Then, by constructing a unified sliding mode controller and applying Lyapunov functionals, sufficient conditions were derived for the synchronization at a finite and a fixed time. Numerical simulations supported the theoretical results and it was observed that the settling time for the finite time synchronization was less than that of the fixed time synchronization. Comparing with our previous work, the sliding mode control technique showed a better performance than the state-feedback control method in the synchronization analysis of SIMNNs in terms of practical application, fast convergence and resilience to disturbances.

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