

Capstone Project

Abuzer Abuov

March 2026

Abstract

This paper investigates the Cohen-Macaulay (CM) property of the coordinate ring associated with the variety of commuting matrices. While the case for $n = 2$ is well-established, the construction of suitable systems of parameters (s.o.p.) for $n \geq 3$ remains a central challenge. We analyze three candidate sequences proposed for the $n = 3$ case using the computer algebra system `Macaulay2`. Our computational results indicate that Guess 1 fails for $p = 11$ and $p = 19$. Furthermore, Guess 2 and Guess 3, while valid for a large range of primes, fail specifically at $p = 3$ and $p = 2$, respectively. In the latter case, a similar system of parameters can be used to fix the case $p = 2$ specifically. Finally, we discuss the reduction of the problem to Macaulay's resultant, noting that the resulting $\binom{16}{7} = 11,440$ matrix size currently precludes a definitive computational proof.

1 Preliminaries

Definition 1. A commutative ring R with unity is called Noetherian if every ideal in R is finitely generated.

From now on we will assume that all rings in question are Noetherian and commutative with 1.

Definition 2. A ring R is called *local* if it has a unique maximal ideal $\mathfrak{m}$. We will denote such local rings as $(R, \mathfrak{m})$.

Definition 3. Let R be a ring and $S \subseteq R$ be a multiplicatively closed set containing 1. The localization of R at S , denoted by $S^{-1}R$, is a ring of fractions $\{\frac{r}{s} \mid r \in R, s \in S\}$ modulo the equivalence relation

$$\frac{r_1}{s_1} \sim \frac{r_2}{s_2} \iff \exists t \in S : t(r_1 s_2 - r_2 s_1) = 0$$

Definition 4. For a ring R and a prime ideal P , the set $S = R \setminus P$ is multiplicatively closed. The localization of R at P , denoted as R_P , is then the ring $S^{-1}R$.

Definition 5. For a ring R , the *Krull Dimension* of R , denoted $\dim R$, is the maximal number n such that there exist prime ideals $P_0, P_1, \dots, P_n$ in R such that

$$P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n$$

Definition 6. For a ring R , a regular R -sequence is a sequence of elements in R , $x_1, x_2, \dots, x_n$ so that $(x_1, \dots, x_n) \neq R$, x_1 is not a zero divisor in R , and x_i is not a zero divisor in $R/(x_1, \dots, x_{i-1})$ for $i = 2, \dots, n$.

Definition 7. A system of parameters (s.o.p.) for a local ring $(R, \mathfrak{m})$ is a sequence of elements $x_1, \dots, x_n$ such that $\dim R = n$ and $\dim R/(x_1, \dots, x_n) = 0$.

Definition 8. For I an ideal in R , $\text{depth } I$ is the length of the longest regular sequence in I . If $(R, \mathfrak{m})$ is a local ring, we also denote $\text{depth } R = \text{depth } \mathfrak{m}$.

Definition 9. Let $(R, \mathfrak{m})$ be a local ring. R is Cohen-Macaulay (CM) if $\dim R = \text{depth } R$.

There are also several equivalent definitions of Cohen-Macaulay rings that we are going to state as a proposition.

Proposition 1. For a local ring $(R, \mathfrak{m})$, the following conditions are equivalent:

- (1) $(R, \mathfrak{m})$ is CM.
- (2) Some s.o.p. in R is a regular sequence.
- (3) Every s.o.p. in R is a regular sequence.

(4) $\text{depth } P = \text{ht } P$ for all prime ideals P in R .

(5) $\text{depth } I = \text{ht } I$ for all ideals I in R .

A crucial part of this paper is the following fact

Theorem 1 ([2], p. 122). Let $(R, \mathfrak{m})$ be a local ring and suppose $x \in \mathfrak{m}$ is a non-zero divisor. Then

$$\dim A/(x) = \dim A - 1$$

Notice that if there is a regular sequence $x_1, \dots, x_n$ in a local ring $(R, \mathfrak{m})$ of the length equal to $\dim R = n$. Then from the theorem above we get that

$$\begin{aligned} \dim R/(x_1, \dots, x_n) &= \dim R/(x_1, \dots, x_{n-1}) - 1 = \dots \\ \dots &= \dim R/(x_1) - (n-1) = \dim R - n = 0 \end{aligned}$$

Which makes the sequence $x_1, \dots, x_n$ a system of parameters, proving that the ring R is CM. So in order to show that a certain local ring is CM, it is enough to find a regular sequence of the length equal to the dimension of the ring.

Definition 10. A ring R is CM if for every maximal ideal $\mathfrak{m}$ of R , $R_{\mathfrak{m}}$ is CM.

2 Introduction

Let k be a field and $X = (x_{i,j})_{1 \leq i,j \leq n}, Y = (y_{i,j})_{1 \leq i,j \leq n}$ be matrices of n^2 indeterminates over k each. Suppose also that k is algebraically closed without loss of generality (otherwise we can replace k by its algebraic closure). Let $R = k[X, Y]$ be a polynomial ring in $2n^2$ indeterminates. Denote $C = XY - YX$ the commutator matrix and let I be the ideal in R generated by the entries of C . This ideal then defines an affine algebraic set of commuting matrices $\mathbb{V}(I)$ in $\mathbb{A}_k^{2n^2}$. The study of this set has a long history. Gerstenhaber proved that V is irreducible in 1961 [5]. Motzkin and Taussky also proved that it is irreducible and of dimension $n^2 + n$ [8]. It was conjectured by M. Artin and M. Hochster that the coordinate ring of V is Cohen-Macaulay domain in 1981. A positive answer was given by Mary Thompson in her PhD thesis for the case $n = 3$ in 1985 [9]. Whether the result holds for higher dimensions still remains an open question. A central challenge in this area is the construction of suitable systems of parameters for the rings under consideration.

3 Case $n = 2$

The case when $n = 2$ is well understood and can be considered essentially resolved. In general, by looking at the entries of the commutator matrix, we can notice that it is generated by the maximal minors of the matrix

$$M = \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{n,n} \\ y_{1,1} & y_{1,2} & \dots & y_{n,n} \end{bmatrix}$$

Denote this ideal as $I_2(M)$. According to [3], $R/I_2(M)$ is CM of dimension $n^2 + 1$ and there exists a maximal regular sequence consisting of

$$s_{i,j} = \begin{cases} x_{i,j} - y_{i,j+1} & \text{if } j < n \\ x_{i,j} - y_{i+1,1} & \text{if } j = n \text{ and } i \neq n \end{cases}$$

for all $1 \leq i, j \leq n$, together with $x_{n,n}$. This regular sequence can better be understood as replacing the original matrix by the matrix

$$M' = \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{n,n-1} & 0 \\ 0 & x_{1,1} & x_{1,2} & \dots & x_{n,n-1} \end{bmatrix}$$

This will be brought up again in the next section when discussing the case $n = 3$. For now, if $n = 2$, M and M' are equal to

$$M = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{2,1} & x_{2,2} \\ y_{1,1} & y_{1,2} & y_{2,1} & y_{2,2} \end{bmatrix} \quad \text{and} \quad M' = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{2,1} & 0 \\ 0 & x_{1,1} & x_{1,2} & x_{1,2} \end{bmatrix}$$

which gives us the sequence $x_{1,1} - y_{1,2}, x_{1,2} - y_{2,1}, x_{2,1} - y_{2,2}, x_{2,2} - y_{1,1}, x_{2,2}$.

Notice that the given sequence is also a regular sequence of length 5 in R/I . Since $\dim R/I = 6$, it remains to add one more element so that the obtained sequence will remain regular, and $y_{2,2}$ will do. To justify this, note that the $n \times n$

identity matrix I_n and any of its scalar multiples commute with every $n \times n$ matrix. Therefore, matrices X and Y commute if and only if matrices $X - x_{2,2}I_2$ and $Y - y_{2,2}I_2$ also commute. Indeed, suppose that $XY = YX$, then

$$\begin{aligned}(X - x_{2,2}I_2)(Y - y_{2,2}I_2) &= XY - x_{2,2}Y - Xy_{2,2} + x_{2,2}y_{2,2}I_2 \\ &= YX - Yx_{2,2} - y_{2,2}X + x_{2,2}y_{2,2}I_2 \\ &= (Y - y_{2,2}I_2)(X - x_{2,2}I_2)\end{aligned}$$

and the converse is proven in an analogous way. This means we can assume that $x_{2,2} = y_{2,2} = 0$ without loss of generality. Since we've already identified $x_{2,2}$ with 0 in the previous step, this adds exactly one element to the regular sequence, as we desired. This leaves us with the following regular sequence of length 6

$$x_{1,1} - y_{1,2}, x_{1,2} - y_{2,1}, x_{2,1} - y_{2,2}, x_{2,2} - y_{1,1}, x_{2,2}, y_{2,2}$$

4 On the Search for Systems of Parameters

At present, there is no general systematic method for producing systems of parameters, and most approaches rely on heuristic or computational techniques.

In this section, we present several candidate sequences conjectured to be systems of parameters for the case $n = 3$, originally proposed by Freyja Hreinsdóttir and Zhibek Kadyrsizova. Using computational tools such as Macaulay2, one can verify that these sequences are indeed systems of parameters in characteristic zero. However, their validity in arbitrary prime characteristic remains unproven.

First two guesses are presented by Freyja H. [7]. What is interesting about this work in particular is that the author presents two sequences that they claim to be systems of parameters, but

- Firstly, they never mention the characteristic of the field that they are working in and never present the proof of such claims. We are guessing that the characteristic is the default, which is 0, and that the proof was done using computational tools.
- Secondly, which is more interesting, is that after running both of the candidates through a computer algebra system, we found out that the first candidate is not a system of parameters when the characteristic of the field is 11 or 19, and the second candidate is not a system of parameters in characteristic 3.

This will be discussed further in the next section. For now, below are the candidates.

Guess 1

First, we have a regular sequence described by identifying the matrix Y by the following

$$\begin{bmatrix} x_{1,1} & x_{2,2} & x_{3,3} \\ x_{1,3} & x_{2,1} & x_{3,2} \\ x_{1,2} & x_{2,3} & x_{3,1} \end{bmatrix}$$

meaning that the sequence is

$$\begin{aligned}y_{1,1} - x_{1,1}, & \quad y_{1,2} - x_{2,2}, & \quad y_{1,3} - x_{3,3}, \\ y_{2,1} - x_{1,3}, & \quad y_{2,2} - x_{2,1}, & \quad y_{2,3} - x_{3,2}, \\ y_{3,1} - x_{1,2}, & \quad y_{3,2} - x_{2,3}, & \quad y_{3,3} - x_{3,1}\end{aligned}$$

Freyja reports that adding x_9, x_8 , and x_1 to the sequence gives a maximal regular sequence of length 12. They also claim that in general, replacing the matrix Y by the matrix

$$\begin{bmatrix} x_{1,1} & x_{2,2} & x_{3,3} & \dots & x_{n,n} \\ x_{1,n} & x_{2,1} & x_{3,2} & \dots & x_{n,n-1} \\ x_{1,n-1} & x_{2,n} & x_{3,1} & \dots & x_{n,n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1,2} & x_{2,3} & x_{3,4} & \dots & x_{n,1} \end{bmatrix}$$

gives a regular sequence of length n^2 , and n more guesses need to be done in order to find a maximal regular sequence.

Guess 2

As mentioned in section 3, the ideal I is contained in the ideal $I_2(M)$ generated by the 2×2 minors of the matrix

$$M = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{n,n} \\ y_{1,1} & y_{1,2} & \cdots & y_{n,n} \end{bmatrix}$$

and which has a quotient ring of dimension $n^2 + 1$ with a maximal regular sequence defined by replacing the matrix above by the matrix

$$M' = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{n,n-1} & 0 \\ 0 & x_{1,1} & x_{1,2} & \cdots & x_{n,n-1} \end{bmatrix}$$

This motivated the author's attempt at checking this sequence for the ring defined by the commutator matrix. Dividing by this sequence amounts to replacing X and Y by the matrices

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & x_{11} & x_{12} \\ x_{13} & x_{21} & x_{22} \\ x_{23} & x_{31} & x_{32} \end{bmatrix}$$

respectively. Dividing further by $x_{3,2}$ and $x_{1,1} + \cdots + x_{3,1}$ is conjectured to give a maximal regular sequence.

Guess 3

The final candidate, provided by Zhibek Kadyrsizova, is the following

$$\begin{aligned} x_{1,1} - y_{1,1}, & \quad x_{1,2} - y_{2,2}, & \quad x_{1,3} - y_{3,3}, \\ x_{2,3} - y_{3,2}, & \quad x_{2,2} - y_{2,1}, & \quad x_{2,1} - y_{3,1}, \\ x_{3,1} - y_{1,2}, & \quad x_{3,2} - y_{1,3}, & \quad x_{3,3} - y_{2,3}, \\ x_{1,2}, & \quad x_{1,1}, & \quad x_{1,3} - x_{2,1} \end{aligned}$$

However, this sequence is not an s.o.p. when characteristic of the field is 2, therefore we consider one more sequence that is going to solve this particular case:

$$\begin{aligned} x_{1,1} - y_{1,1}, & \quad x_{1,2} - y_{2,2}, & \quad x_{1,3} - y_{3,3}, \\ x_{2,3} - y_{3,2}, & \quad x_{2,2} - y_{2,1}, & \quad x_{2,1} - y_{3,1}, \\ x_{3,1} - y_{1,2}, & \quad x_{3,2} - y_{1,3}, & \quad x_{3,3} - y_{2,3}, \\ x_{1,2}, & \quad x_{1,1}, & \quad x_{2,1} - x_{3,2} \end{aligned}$$

Note that the only difference is in the last element of the sequence. One of the surprising things perhaps is the fact that one would expect that just manually proving that these are indeed regular sequences would be, even if tedious, a rather simple task, that would require only some number of algebraic manipulations. However, this problem proved to be a lot harder than was initially expected after some time of unsuccessful attempts.

5 Reducing the problem

Even though proving that above examples indeed constitute systems of parameters is a problem that is not solved practically, it is at least solved in some philosophical sense. Indeed, consider the following theorem

Theorem 2 (Weak homogeneous Nullstellensatz). Let I be a homogeneous ideal in the polynomial ring $k[x_0, x_1, \dots, x_n]$ over an algebraically closed field k . Then

$$\mathbb{V}(I) = \{(0, \dots, 0)\} \iff \sqrt{I} = (x_0, \dots, x_n)$$

Now denote the ideal $\mathfrak{m} = (x_0, \dots, x_n)$ and suppose $\sqrt{I} = \mathfrak{m}$. This means that the only prime ideal containing I is $\mathfrak{m}$, which implies that there is only one prime ideal in R/I , that is $\mathfrak{m}/I$. This gives us that $\dim R/I = 0$.

So in order to prove that the above sequences are indeed systems of parameters, it is enough to show that the system of equations defined by the generators of the ideal I (after taking the quotient by the said s.o.p.) has only the trivial solution. In this section, we are going to present a way to do exactly that.

5.1 Sylvester Matrix

Let

$$P(x) = a_m x^m + \dots a_1 x + a_0$$

and

$$Q(x) = b_n x^n + \dots b_1 x + b_0$$

be polynomials in $K[x]$. Then the system of equations

$$\begin{cases} P(x) = 0 \\ Q(x) = 0 \end{cases}$$

has a non-trivial solution if and only if they have a non-zero common factor, which happens if and only if there don't exist non-zero polynomials $U, V \in K[x]$ with $\deg U < n$ and $\deg V < m$ such that

$$U(x)P(x) + V(x)Q(x) = 0$$

Denote

$$R(x) = U(x)P(x) + V(x)Q(x) = r_{m+n}x^{m+n} + \dots + r_0 =$$

and the coefficients of U and V as $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$ respectively (note that $u_n = v_m = 0$). Then the vector $r = (r_{m+n}, \dots, r_0)$ is given by

$$r = (S_{P,Q})^T \begin{pmatrix} u \\ v \end{pmatrix}$$

Where $S_{P,Q}$ is a $(m+n) \times (m+n)$ matrix obtained in the following way

- First n rows: coefficients of P shifted progressively.
- Following m rows: coefficients of Q shifted progressively.

For example, if $P(x) = a_2 x^2 + a_1 x + a_0$ and $Q(x) = b_3 x^3 + b_2 x^2 + b_1 x + b_0$, then

$$S_{P,Q} = \begin{pmatrix} a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 \end{pmatrix}$$

The matrix $S_{P,Q}$ is called Sylvester Matrix [1]. The above condition is now reduced to this matrix having a trivial kernel, which happens if and only if

$$\det S_{P,Q} = 0$$

5.2 Macaulay's resultant

The above procedure can be generalized to n homogeneous polynomials over n variables and the result is due to Macaulay [4].

Given n homogeneous polynomials $f_1, f_2, \dots, f_n$ in $K[x_1, \dots, x_n]$ with degrees $d_1, \dots, d_n$ respectively, construct a matrix M in the following way:

1. Let $D = d_1 + \dots d_n - n + 1$.
2. For each f_i , multiply it by all monomials of degree $D - d_i$.
3. Stack coefficients of all these products as rows in a matrix.

The greatest common divisor of all maximal minors of this matrix is what Macaulay calls the resultant of the system of polynomials and is denoted as

$$\text{Res}(f_1, \dots, f_n)$$

Macaulay proved the following result

Theorem 3. The system of homogeneous polynomial equations

$$f_1 = f_2 = \cdots = f_n = 0$$

has a non-trivial root if and only if

$$\text{Res}(f_1, \dots, f_n) = 0$$

Notice that for the case $n = 2$, this procedure is equivalent to Sylvester's Matrix discussed earlier.

Since $\text{tr}(XY - YX) = 0$ one of the entries of C can be omitted without any effect on the generated ideal, therefore there are 8 generators of I that are homogeneous polynomials of degree 2. We also identify all the y 's with some of the x 's in all of the examples, and we identify at least one of the x 's by 0, so we are always left with at most 8 variables. Which makes the method of Macaulay's resultant applicable. However, in this particular case, the number of the columns in the constructed matrix will be

$$\binom{16}{7} = 11,440$$

and computing the determinant of a $11,440 \times 11,440$ matrix is practically impossible for now.

All of this means that showing that examples showed earlier constitute a system of parameters for the corresponding ring is a task that can be done in a finite amount of time. However, the amount of computations required for this task is too big to actually execute it.

6 Checking examples for some prime characteristics

To gain insight into the problem, we perform explicit computations for several prime characteristics using computer algebra systems. These calculations, while limited by computational resources, provide partial evidence regarding the conjectured systems of parameters.

At present, the results neither confirm nor refute the conjecture, but they suggest that the proposed sequences may indeed have the desired properties.

Below is the code written in the language of a computer algebra system Macaulay2 [6].

```
testSOP = (p) -> (
  K = GF(p);
  S = K[x_{1,1}..x_{3,3}, y_{1,1}..y_{3,3}];

  X = genericMatrix(S, x_{1,1}, 3, 3);
  Y = genericMatrix(S, y_{1,1}, 3, 3);
  C = X * Y - Y * X;
  I = ideal(C);
  R = S / I;

  params = ideal(...); --Put the sequence that is going to be tested to be an s.o.p.

  return (dim R == numgens params) and (dim (R / params) == 0);
);

primesUpTo = N -> select(toList(2..N), isPrime);

for p in primesUpTo 100000 do (
  print("p  =" | toString p | "<u->u" | toString testSOP(p));
);
```

Running this code for 3 candidates presented in the section 4 gives the following results:

1. As mentioned earlier, the first candidate works for all checked characteristics except when $p = 11$ and $p = 19$.
2. The second candidate doesn't work for $p = 3$ and works for all the other characteristics up to 100,000.
3. The last candidate works for all characteristics up to 100,000 except for $p = 2$, in which case the second option can be used instead.

7 Nearly Commuting matrices

Despite being simple to define, the variety of commuting matrices proved to be a complicated object to research. One can instead look at pairs of matrices that have a commutator matrix that have 0 as off-diagonal entries. Such matrices are called nearly commuting and the coordinate ring of this set is obtained by taking the quotient of R by the ideal J generated by all of the off-diagonal entries of the matrix C .

Theorem 4 ([10]). The set of nearly commuting matrices is a complete intersection of dimension with two irreducible components, one of which is the variety of commuting matrices.

This means that

1. J is generated by a regular sequence.
2. $\mathbb{V}(J) = \mathbb{V}(I) \cup \mathbb{V}(Q)$ where Q is a prime ideal in R . $\mathbb{V}(Q)$ is called the skew-component.

A very natural next step then would be to describe the ideal Q .

Denote $P = \sqrt{I}$, it is known that

$$P = J : Q \quad \text{and} \quad Q = J : P$$

and it was initially conjectured that Q is generated by the $n \times n$ minors of the matrix

$$\begin{bmatrix} u_0 & u_1 & \dots & u_{n-1} & v_0 & v_1 & \dots & v_{n-1} \end{bmatrix}$$

where u_i is the column consisting of the diagonal entries of X^i and v_i is the column consisting of the diagonal entries of Y^i . However, while this seems to work for the case $n = 3$, it is probably not true for $n \geq 4$ and that more generators are needed.

8 Conclusion

The study of commuting varieties highlights the gap between theoretical algebraic foundations and practical computational limits. For the $n = 2$ case, a regular sequence can be identified, and the ring R/I is known to be CM. However, our investigation into the $n = 3$ case reveals that proposed systems of parameters are often sensitive to the field's characteristic. Specifically, computational testing up to $p = 100,000$ shows that:

- Guess 1 is functions as an s.o.p. for most characteristics but fails at $p = 11$ and $p = 19$.
- Guess 2 functions as an s.o.p. for most characteristics but fails at $p = 3$.
- Guess 3 is valid for characteristics up to 100,000 except for the case $p = 2$, which is solved using another sequence.

While the problem is “philosophically solved” through its reduction to homogeneous polynomial equations and Macaulay’s resultant, the required matrix of size $11,440 \times 11,440$ remains a significant computational barrier. Consequently, finding a characteristic-independent system of parameters for higher dimensions remains an open and challenging question.

References

- [1] Panagiotis S. Vigklas Alkiviadis G. Akritas, Gennadi I. Malaschonok, *Sturm sequences and modified subresultant polynomial remainder sequences*, Serdica Journal of Computing (2014).
- [2] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.
- [3] W. Bruns and U. Vetter, *Determinantal rings*, Springer-Verlag, 1988.
- [4] Macaulay F.S, *The algebraic theory of modular systems*, Cambridge University Press, 1916.
- [5] M. Gerstenhaber, *On dominance and varieties of commuting matrices*, Ann. Math. **73** (1961), no. 2, 324–348.
- [6] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <https://macaulay2.com> (2023).

- [7] Freyja Hreinsdóttir, *Miscellaneous results and conjectures on the ring of commuting matrices*, Analele Științifice ale Universității Ovidius Constanța **14** (2006), no. 2, 45–60.
- [8] Theodore S. Motzkin and Olga Taussky, *Pairs of matrices with property L II*, Transactions of the American Mathematical Society **80** (1955), no. 2, 387–401.
- [9] M. Thompson, *Topics in the ideal theory of commutative Noetherian rings*, Ph.D. thesis, University of Michigan, 1985.
- [10] H.-W.V. Young, *On matrix pairs with diagonal commutators*, J. Algebra **570** (2021), 437–451.