Pointwise Estimates for Polyharmonic Green's Function

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0.1 Abstract

This thesis focuses on the polyharmonic Green's function, one of fundamental concepts in mathematical analysis and partial differential equations. The Green's function plays a crucial role in solving problems related to the behavior of harmonic functions, and has a wide range of applications in physics and engineering. The introduction section starts with an overview of the notation used throughout the thesis. A brief historical and literature review follows to provide context and a better understanding of the importance of the topic. The preliminary section then lays out the necessary background knowledge required for further proofs of the main theorems. The thesis then proceeds into a detailed discussion of known pointwise estimates proofs, which are crucial to understanding the Green's function's behavior. The main contribution of this thesis is a partial solution to an open problem from Mazya's list of "Seventy Five (thousand) unsolved problems in analysis and partial differential equations" from [27].

Chapter 1

Introduction

1.1 Notations

Let $x = (x_1, x_2, x_3, ..., x_n) \in \mathbb{R}^n$. As usual, the Laplacian Δu of a smooth function u is the sum of the second partial derivatives with respect to each component x_i:

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

Our interest is iteration of the above Laplace operator (or the Laplacian), that is, the so-called polyharmonic operator. It is defined inductively by:

$$(-\Delta)^m u = -\Delta((-\Delta)^{m-1} u), \qquad \text{where } m = 1, 2, \ldots.$$

It is clear to see:

$$(-\Delta)^m u = \sum_{l_1+\ldots+l_n=m} \frac{m!}{l_1!\ldots l_n!} \frac{\partial^{2m} u}{\partial x_1^{2l_1} \ldots \partial x_n^{2l_n}}.$$

Throughout this thesis $B_r \subset \mathbb{R}^n$ denotes a ball with radius r centered at the origin.

Let's define the distance function between given x and the boundary $\partial\Omega$ of a given domain $\Omega \subset R^n$ as the following:

$$d(x) := \operatorname{dist}(x, \partial\Omega) = \min_{y \in \partial\Omega} |x - y|, \quad x \in \Omega,$$

where Ω is a smooth and bounded domain.

The estimates of the Green function will be often presented in the unit ball in $\mathbb{R}^n$, for that purpose the following notations will be used:

For $x, y \in \bar{B}$, where $\bar{B} = B \cup \partial B$, we write

$$[XY] := \sqrt{|x|^2|y|^2 - 2x \cdot y + 1} = \left| |x|y - \frac{x}{|x|} \right| = \left| |y|x - \frac{y}{|y|} \right| . \tag{1.1.1}$$

$[XY]$ is the distance from $|y|x$ to the projection of y on the unit sphere, which is larger than $|x-y|$. Indeed,

$$\left[XY\right|^2 - |x-y|^2 = \left(1-|x|^2\right)\left(1-|y|^2\right) > 0 \text{ for } x, y \in B \, .$$

$1 - |x| = d(x)$ for $x \in \bar{B}$.

We say $f(t) \simeq g(t)$ if there exists $C > 0$ such that for all t: $\frac{1}{C} f(t) \leq g(t) \leq C f(t), f, g \geq 0$

We define derivatives of u as the following:

$$D^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} ... \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u \quad \text{for } \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k.$$

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded set with boundary $\partial\Omega$. Consider the polyharmonic Dirichlet problem:

$$(-\Delta_x)^m G_{m,n}(x,y) = \delta(x-y), \quad x, y \in \Omega, \tag{1.1.2}$$

with the homogeneous Dirichlet boundary conditions:

$$\frac{\partial^i}{\partial n_x^i} G_{m,n}|_{x \in \partial\Omega} = 0, \quad i = 0, ..., m-1, \tag{1.1.3}$$

where δ is the Dirac delta function. The solution $G_{m,n}(x,y)$ of (1.1.2)-(1.1.3) is called the Green function for the polyharmonic Dirichlet problem. For basic properties of δ, please see the next section.

1.2 Short historical and literature review

Higher order linear elliptic equations have a wide range of applications in science, describing models of different phenomena. The first attempt to use polyharmonic operators was made by Jacob Bernoulli II. He tried to model the vibrations of plate by the fourth order operator $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$. His attempts were inspired by physicist Chlandni who presented nodal line patterns of vibrating plates around 1800. However, his model didn't describe vibrations precisely since it was not rotationally symmetric. Sophie Germain then used Δ^2 in 1811 to model an elastic plate. The bi-Laplacian operator Δ^2 appears in problems of linear elasticity (see, e.g. [12]). An example of such problem is the clamped plate problem:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\frac{\partial u}{\partial n}$ is the outer unit normal on the boundary $\partial\Omega$ of $\Omega \subset \mathbb{R}^n$.

George Green was the first to introduce the approach of using Green's functions to solve many physics problems. Green's function's concept can be discussed by considering the dynamics of some particle with zero initial velocity and under the force $F(t)$. First, a short impulse of a force is considered. One chooses an impulse in such a way, so that a unit time momentum is induced at some time t'. The Green function $G(t, t')$ is defined to be a displacement of a particle $s(t)$ in some later time t. A force $F(t')$ acting for an infinitesimal time $\Delta t'$ creates an impulse with a magnitude $F(t')\Delta t'$. Force that is applied continuously to the particle is considered as an impulse generating force. The motion of particle can be found by summing the effects of impulses in the time interval from t_0 to t. Thus

$$s(t) = \int_{t_0}^{t} G(t, t')F(t')dt',$$

with initial conditions: $s(0) = \frac{ds}{dt}(0) = 0$. Hence, knowing Green's function, the response of a system to any force can be easily calculated. For Green's technique's details we refer to [4].

Originally, Green worked on solving problems in electrostatics in a bounded region. The Green function $G(r, r')$ in that problem is the potential at r which is produced by a point charge at r'. Analogous to an impulse acting at an instant in time, a point charge acts at a single point in space.

Shortly, the Green function technique is as following:

Let $\mathcal{L}_x$ be a linear differential operator, where $x = (x_1, x_2, ..., x_n)$, then Green's function $G(x, y)$ of a linear operator $\mathcal{L}_x$ satisfies the following expression:

$$\mathcal{L}_x G(x, y) = \delta(x - y),$$

where $\delta(x - y)$ is the Dirac delta function. Let us briefly recall the definition of the Dirac delta function: Dirac delta function (δ), is the generalized function over $\mathbb{R}^n$, whose value is zero everywhere except 0, and integral over a real line is 1. It is also known as the impulse function. Mathematically defined as the following:

$$\delta(x) = \begin{cases} \infty, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases}$$

with $\int_{-\infty}^{\infty} \delta(x)dx = 1$.

Once we know the Green function $G(x, y)$, we can solve any differential equation containing $\mathcal{L}_x$.

Suppose, we are given a problem:

$$\mathcal{L}_x \phi(x) = f(x).$$

The solution is :

$$\phi(x) = \int G(x, y) f(y) dy.$$

Verifying the answer:

$$\mathcal{L}_x \phi(x) = \int \mathcal{L}_x G(x, y) f(y) dy = \int \delta(x - y) f(y) dy = f(x).$$

In the 19th century, the method of Green's functions was widely used to solve partial differential equations in thermal, electrical, and mechanical physics, as well as problems in magnetism. Green's function is also used in formulating the theory of wave scattering. In the 1950-1960s scientists started to use Green's function method in many-body interactions theory of a condensed matter physics. Nowadays, physicists use Green's function technique in the areas that were not even known in the times of Green. And it is highly likely that Green's function will continue to be used in whatever is going to be developed in the future. Green's function is very powerful technique that allows to solve differential equations in different areas, however constructing and solving for Green's function is not a trivial problem. It is a challenging procedure and requires rigorous mathematics. Going back to the problems with polyharmonic Green's function (1.1.2), the first explicit solution for polyharmonic problem was calculated by Boggio. Boggio gave a representation of polyharmonic Green's function in a ball. Boggio's formula in an integral form is a classical tool to construct an explicit solution to the polyharmonic equation. More precisely, Boggio [3] derived an explicit formula for $G_{-\Delta^m,B_1}$

$$\begin{cases} (-\Delta)^m u = f & \text{in } B_1, \\ u = \frac{\partial^j}{\partial n^j} u = 0 \quad \text{on } \partial B_1 \text{ with} & j = 1, 2, ..., m-1, \end{cases}$$

where B_1 is the unit ball centered at the origin, n is the unit normal at the boundary ∂B_1.

The solution of a problem is the following expression:

$$u(x) = \int_{B_1} G_{-\Delta^m,B_1}(x,y) f(y) dy.$$

The representation of Green's function by Boggio mentioned in [12]:

$$G_{m,n,B_1}(x,y) = k_{m,n} |x-y|^{2m-n} \int_1^{\left||x|y - \frac{x}{|x|}\right|/|x-y|} (v^2-1)^{m-1} v^{1-n} dv.$$

The positive constants $k_{m,n}$ are defined by

$$k_{m,n} = \frac{1}{n e_n 4^{m-1}((m-1)!)^2}, \quad e_n = \frac{\pi^n}{\Gamma(1+n/2)}.$$

The representation of Green's function above holds true for the domain B_1.
Let us consider the biharmonic case ($m = 2$) to understand the nature of the problem:

$$\begin{cases} (-\Delta)^2 u = f & \text{in } B_1, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B_1, \end{cases}$$

Dirichlet boundary value problem with biharmonic operator with $n = 2$

$$\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

appears in many fields of applied mathematics, for example, in fluid mechanics and elasticity theory. In the latter, $u(x, y)$ defines Airy stress function, in the theory of thin plates-displacement of a plate under the external force.

For the case when $n = 2 = m$, the Green function

$$G_{-\Delta^2, B_1}(x, y) = \frac{1}{8\pi}|x - y|^2 \int_1^{\left| |x|y - \frac{x}{|x|} \right| / |x-y|} \frac{v^2 - 1}{v} dv$$

is strictly positive [12]. One can show that the polyharmonic Green's function in a ball is always positive. In these dimensions, it can be seen that the problem models a plate of a circular shape, f is an acting force, and the u is a deflection of a plate.

The positivity of Green's function is an important question that can help to solve other physical problems. For example, the positivity of Green's function helps to answer the question presented by Boggio in [3]: "If a clamped plate of circular shape will be pushed upwards, will the clamped plate bend upwards everywhere, too?". The answer to the problem above is yes, since Green's function $G_{(-\Delta)^2, B_1}(x, y)$ in a given domain and $f(x)$ are positive.

In some general domain Ω

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \frac{\partial^j}{\partial n^j} u = 0 \quad \text{on } \partial\Omega \quad \text{with} & j = 1, 2, ..., m - 1, \end{cases} \tag{1.2.1}$$

Green's function is not necessarily positive, it can be sign-changing. There are many papers that study positivity of Green's function. For example, in the problem mentioned above, Hadamarad and Boggio [3] conjectured that the Green's function $G_{-\Delta^2,\Omega}$ is positive for convex two-dimensional domains. Garabedian [11] provided a counterexample for smooth convex domains, showing that Green's function is sign-changing in ellipse. Another counterexample is Duffin's work [9], where the author shows that the Green's function changes the sign in a long rectangle. Moreover, the Boggio-Hadamard's conclusion in not true for dimensions other than two.

For dimensions $m \geq 2, n \geq 2$, Kozlov et al. [24] provided a strictly convex domain where Green's function is sign-changing.

For the constant right-hand side of (1.2.1), Grunau and Sweers [16] found domains for $m = 2$ where Green's function changes sign.

As mentioned above, we cannot claim that Green's function is positive in domains other than ball. Therefore, much work has been done to identify the domains where the Green's function's positivity holds. Grunau and Sweers [14] considered two-dimensional domains which are close to balls. Dal'Acqua and Sweers [8] gave an example of a non-convex domain for $m = 2, n = 2$. For the case when $m = 2$ and higher dimensions, Grunau and Roberts [18] proved positivity in the domain close to the ball. The result is true for the dimensions where $n \geq 2m - 1$.

Grunau and Sweers in [15] used a method suggested by Nehari [32] for the general case to find the regions where Green's function is positive. They found a constant $\delta_{m,n}$ which doesn't depend on the domain Ω such that for every x, y in Ω, and

$$|x - y| < \delta_{m,n} \max\{d(x), d(y)\},$$

where $d(x) := \text{dist } (x, \partial\Omega) = \inf_{x* \in \partial\Omega} |x - x^*|$ so that

$$G_{-\Delta^m,\Omega}(x, y) > 0.$$

The authors proved this for dimensions $n > 2m$. Kokritz [23] proved it for the case $n = 2m$. Thus, the negative part is uniformly bounded when x and y are uniformly distanced from the boundary $\partial\Omega$. For example, for the biharmonic case, minimal distance $\delta = \delta(\Omega)$ can be identified such that for all $x, y \in \Omega$ with $x \neq y$,

$$|x - y| < \delta \text{ implies that } G_{(-\Delta)^2,\Omega}(x, y) > 0.$$

The result of local positivity was obtained by Grunau and Robert [18] for the case when $n \geq 3$ and for $n = 2$ by Dall'Aaqua et al. in [6].
Eichmann and Schätzle [10] considered the positivity of Green's function in a clamped plate problem with high tension. The problem is presented by $\Delta^2 u - \gamma\Delta u = f$ with clamped boundary conditions. The authors showed that with a given upward pushing force f, there exists such γ_0, so that for all $\gamma \geq \gamma_0$ the bending u is always positive.
Let us give a short overview of what is done for characterization of Green's function.

In general, for the second-order equations with dimension $m = 1$, the Green function's estimates from both sides are known for sufficiently smooth domains. For higher dimensions, Gruter [17] and Widman [31] identified estimates from above. Zhao [33],[34] found estimates of the Green function from two sides.

Liu and Dai [26] considered a system of polyharmonic equations with Dirichlet boundary conditions in a half-space and proved nonexistence of nonnegative nontrivial classical solutions for the problem.

Ancona [1], Cranston et al. [5], Hueber and Sievking [19] provided estimates for second-order differential equations in sufficiently smooth domains. Estimates in non-smooth domains were presented by Maz'ya and Mayboroda [28].

Karachik [21] provided the Green function for the biharmonic Dirichlet problem and found integral representation of the solution in terms of Green's function. Also, Karachik [22] found polynomial solution for Dirichlet problem with polynomial boundary data and right-hand side in unit ball.

Gazzola and Sperone [13] discussed radial properties (symmetry and monotonicity) of radial solutions of semilinear higher order elliptic equations. The result yielded in formula for computing partial derivatives of solutions of polyharmonic problems.

Aroua and Bellassoued [2] considered an inverse boundary value problem for polyharmonic operator, moreover the authors discussed stability estimate for the inverse problem.

1.3 Structure

In Chapter 2, we introduce and discuss several useful tools that are key ingredients in the proofs presented in subsequent chapters. These tools serve as the foundation for the rigorous analysis of the polyharmonic Green's function and its properties. We discuss then the proofs of the pointwise estimates of Green's function presented by Gazzola et al. [12], Pulst [30], and the estimates of Green's function derivatives by Gazzola et al. [12], Krasovskii [25], Dallacqua and Sweers [7]. Moving on to Chapter 3, we proceed into the proof of an inequality from Mazya's list of "Seventy Five (thousand) unsolved problems in analysis and partial differential equations" from [27]. Finally, the thesis concludes with a summary of the main findings.

Chapter 2

Preliminaries

2.1 Useful tools

In this section we will provide useful tools that will help to obtain Green's function's and its derivatives' estimates.
The following lemma gives a representation of the Green function:

Lemma 2.1.1. *[20] If $n > 2m$, then Green's function of the polyharmonic Dirichlet problem* (1.1.2)- (1.1.3) *can be represented in the form*

$$G_{m,n,r}(x,y) = d_{m,n}\Big[X^{2m-n} - Y_r^{2m-n} - \sum_{k=1}^{m-1} \frac{(-1)^k}{k!}\Big(m-\frac{n}{2}\Big)\Big(m-\frac{n}{2}-1\Big)...\Big(m-\frac{n}{2}-k+1\Big)Y_r^{2m-n-2k}Z_r^{2k}\Big],$$

where

$$d_{m,n} = 2\Gamma(n/2-m)\Gamma(1+n/2)/(n\pi^{n/2}\times 4^m\Gamma(n/2)(m-1)!). \tag{2.1.1}$$

Here $X(x,y), Y_r(x,y)$ and $Z_r(X,Y)$ are the functions such that:

$$X^2 = |x-y|^2, Y_r^2 = \left|\frac{y}{r}\right|^2\left|x-\frac{y}{|y|^2}r^2\right|^2 \text{ and } Z_r = r^2\left(1-\left|\frac{y}{r}\right|^2\right)\left(1-\left|\frac{x}{r}\right|^2\right),$$

respectively.

Proof. Clearly, for all $x, y \in B_r$ the relation $X^2 = Y_r^2 - Z_r^2$ holds true. Then we can use the relation in the following way:

$$|x-y|^{2m-n} = \left(X^2\right)^{m-n/2} = \left(Y_r^2\right)^{m-n/2} \left(1 - \frac{Z_r^2}{Y^2}\right)^{m-n/2}.$$

Also, we can represent $(1 - Z_r^2/Y_r^{-2})^{m-n/2}$ by the series. Then $|x-y|^{2m-n}$ can be expressed as the following:

$$|x-y|^{2m-n} = \left(X^2\right)^{m-n/2} = \left(Y_r^2\right)^{m-n/2} \left(1 - \frac{Z_r^2}{Y_r^2}\right)^{m-n/2} =$$
$$= \left(Y_r^2\right)^{m-n/2} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(m - \frac{n}{2}\right)\left(m - \frac{n}{2} - 1\right) \dots \left(m - \frac{n}{2} - k + 1\right) \left(\frac{Z_r^2}{Y_r^2}\right)^k\right]$$
$$= Y_r^{2m-n} + \sum_{k=1}^{m-1} \frac{(-1)^k}{k!} \left(m - \frac{n}{2}\right)\left(m - \frac{n}{2} - 1\right) \dots \left(m - \frac{n}{2} - k + 1\right) (Y_r^2)^{m-n/2-k} (Z_r^2)^k$$
$$+ \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left(m - \frac{n}{2}\right)\left(m - \frac{n}{2} - 2\right) \dots \left(m - \frac{n}{2} - k - 1\right) (Y_r^2)^{m-n/2-k} (Z_r^2)^k.$$

The last relation allows us to define Green's function as follows:

$$G_{m,n,r}(x,y) = d_{m,n}\Big[X^{2m-n} - Y_r^{2m-n} - \sum_{k=1}^{m-1} \frac{(-1)^k}{k!} \left(m - \frac{n}{2}\right)\left(m - \frac{n}{2} - 1\right) \dots \left(m - \frac{n}{2} - k + 1\right) Y_r^{2m-n-2k} Z_r^{2k}\Big],$$

$\square$

The following lemma shows the positivity of Green's function in a ball with arbitrary radius:

Lemma 2.1.2. *[15] Suppose $n > 2m$ and B_r is a ball with radius r. Green's function can be represented as a function in a ball B_1 with radius 1 with re-scaling factor and has the following form:*

$$G_{m,n,r}(x,y) = r^{2m-n} G_{m,n,B_1}\left(\frac{1}{r}x, \frac{1}{r}y\right).$$

Proof. We need to check that $r^{2m-n}G_{m,n,B_1}\Big(\frac{1}{r}x,\frac{1}{r}y\Big)$ satisfies equations (1.1.2)-(1.1.3). Let's first check that $r^{2m-n}G_{m,n,B_1}\Big(\frac{1}{r}x,\frac{1}{r}y\Big)$ satisfies equation (1.1.3). Since $G_{m,n,B_1}\Big(x,y\Big)$ is the solution for Dirichlet problem in unit ball, then it satisfies the boundary condition

$$\frac{\partial^i}{\partial n_x^i}G_{m,n}=0 \text{ on } \partial B_1.$$

It is clear that for any $x\in\partial G_{m,n,r}$, $\frac{x}{r}$ will be in ∂B_1, then $\frac{\partial^i}{\partial n_x^i}G_{m,n,B_1}\Big(\frac{1}{r}x,\frac{1}{r}y\Big)$ is also 0 on ∂B_1. Thus, it satisfies boundary conditions of the problem and $\frac{\partial^i}{\partial n_x^i}\Big(r^{2m-n}G_{1,n,B_1}(\frac{1}{r}x,\frac{1}{r}y)\Big)=0$ on ∂B_1.

Now, we need to check that $r^{2m-n}G_{m,n,B_1}\Big(\frac{1}{r}x,\frac{1}{r}y\Big)$ satisfies (1.1.2). For arbitrary m, the equation becomes:

$$(-\Delta)^m G_{m,n}(x,y)=\delta(x-y) \text{ in } B_1.$$

Now, let's check for $r^{2m-n}G_{m,n,B_1}\Big(\frac{1}{r}x,\frac{1}{r}y\Big)$ for $x,y\in B_r$.

$$\begin{aligned}(-\Delta)^m\Big(r^{2m-n}G_{m,n,B_1}\Big(\frac{1}{r}x,\frac{1}{r}y\Big)\Big)&=r^{2m-n}\frac{1}{r^{2m}}(-\Delta)^m G_{m,n,B_1}\Big(\frac{1}{r}x,\frac{1}{r}y\Big)\\&=\frac{1}{r^n}(-\Delta)^m G_{m,n,B_1}\Big(\frac{1}{r}x,\frac{1}{r}y\Big).\end{aligned}$$

For all $(x,y)\in G_{m,n,r}$, $\Big(\frac{1}{r}x,\frac{1}{r}y\Big)\in B_1$.

Thus, $(-\Delta)^m G_{m,n,B_1}\Big(\frac{1}{r}x,\frac{1}{r}y\Big)=0$ and $r^{2m-n}G_{m,n,B_1}\Big(\frac{1}{r}x,\frac{1}{r}y\Big)=0$ for all $x\neq y$.

For all $(x,y)\in G_{m,n,r}$, such that $x=y$, $\Big(\frac{1}{r}x,\frac{1}{r}y\Big)\in B_1$ and $\frac{1}{r}x=\frac{1}{r}y$. Thus, $(-\Delta)^m G_{m,n,B_1}\Big(\frac{1}{r}x,\frac{1}{r}y\Big)=\delta(x-y)$ for any $x,y\in B_r$.

The statement is proven. □

The following proposition presents Green's second identity for polyharmonic case:

Proposition 2.1.3. *[30] Let Ω be a domain for which the divergence theorem holds* *and let $u, v \in C^{2m}(\bar{\Omega})$. Then it holds*

$$\int_\Omega (-\Delta)^m uv - u(-\Delta)^m v dx = \sum_{\ell=0}^{m-1} \int_{\partial\Omega} \frac{\partial}{\partial n}(-\Delta)^{m-1-\ell} v(-\Delta)^\ell u - \frac{\partial}{\partial n}(-\Delta)^\ell u(-\Delta)^{m-1-\ell} v dS$$

,

where n is the exterior unit normal on $\partial\Omega$.

Proof. For the case when $m = 1$:

$$\int_\Omega (-\Delta u)v dx = \int_\Omega u(-\Delta v)dx + \int_{\partial\Omega} (\frac{\partial u}{\partial n} v - u\frac{\partial v}{\partial n})dS.$$

Using integration by parts for LHS when $m = 2$:

$$\int_\Omega ((-\Delta)^2 uv - u(-\Delta)^2 v)dx =$$

$$= \int_\Omega (-\Delta)u(-\Delta)v - \int_{\partial\Omega} (\frac{\partial(-\Delta)u}{\partial n} v - (-\Delta)u\frac{\partial v}{\partial n})dS$$
$$- \int_\Omega (-\Delta)u(-\Delta)v + \int_{\partial\Omega} (\frac{\partial(-\Delta)v}{\partial n} u - (-\Delta)v\frac{\partial u}{\partial n})dS$$

Solving for RHS when $m = 2$:

$$\int_{\partial\Omega} \frac{\partial(-\Delta)v}{\partial n} u - \frac{\partial u}{\partial n}(-\Delta)v + \frac{\partial v}{\partial n}(-\Delta)u - \frac{\partial(-\Delta)u}{\partial n} vdS$$

The RHS and LHS are identical. Inserting $-\Delta^l u$ and $-\Delta^{m-1-l} v$ into Green's second identity will sum to the desired expression. □

As a consequence of the previous proposition we have the next corollary:

Corollary 1. *[30] Let Ω a domain for which the divergence theorem holds and $u, v \in C^{2m}(\bar{\Omega})$. Let $\frac{\partial^i}{\partial n^i} v = 0$ on $\partial\Omega$ for $i = 0, .., m-1$. Then for $k \in \mathrm{N}$ the following is true:*

1. If $m = 2k$

$$\sum_{l=1}^{k} \int_{\partial\Omega} \frac{\partial}{\partial n}(-\Delta)^{l-1}u(-\Delta)^{m-l}v - \frac{\partial}{\partial n}(-\Delta)^{m-l}v(-\Delta)^{l-1}u d\sigma$$
$$+ \int_{\Omega} v(-\Delta)^m u - u(-\Delta)^m v dx = 0. \quad (111)$$

2. If $m = 2k + 1$

$$\sum_{l=1}^{k} \int_{\partial\Omega} \frac{\partial}{\partial n}(-\Delta)^{l-1}u(-\Delta)^{m-l}v - \frac{\partial}{\partial n}(-\Delta)^{m-l}v(-\Delta)^{l-1}u d\sigma$$
$$- \int_{\partial\Omega} \frac{\partial}{\partial n}(-\Delta)^k v(-\Delta)^k u d\sigma + \int_{\Omega} v(-\Delta)^m u - u(-\Delta)^m v dx = 0. \quad (111)$$

The following lemma presents the difference between cases when x and y are closer to the boundary $\partial\Omega$ than to each other and visa versa.

Lemma 2.1.4. *[12] Let $x, y \in \bar{B}$.*
If $|x - y| \geq \frac{1}{2}[XY]$, then

$$d(x)d(y) \leq 3|x - y|^2 \tag{2.1.2}$$

$$\max\{d(x), d(y)\} \leq 3|x - y| \tag{2.1.3}$$

If $|x - y| \leq \frac{1}{2}[XY]$, then

$$\frac{3}{4}|x - y|^2 \leq \frac{3}{16}[XY] \leq d(x)d(y) \tag{2.1.4}$$

$$\frac{1}{4}d(x) \leq d(y) \leq 4d(x) \tag{2.1.5}$$

$$|x - y| \leq 3\min\{d(x)d(y)\} \tag{2.1.6}$$

$$[XY] \leq 5\min\{d(x), d(y)\}. \tag{2.1.7}$$

Moreover, for all $x, y \in \bar{B}$ we have

$$dx \leqslant [XY], \quad d(y) \leq [XY],$$

$$[XY] \approx d(x) + d(y) + |x - y|$$

Proof. In the subsequent chapters, we will use the estimates 2.1.3, 2.1.4. The proof of two estimates provided below.

Let $|x-y| \geq \frac{1}{2}[XY]$.

$$d(x)d(y) = (1-|x|)(1-|y|) \leq (1-|x|^2)(1-|y|^2) = [XY]^2 - |x-y|^2,$$

but since $|x-y|^2 \geq \frac{1}{4}[XY]$, then

$$d(x)d(y) \leq [XY]^2 - |x-y|^2 \leq 3|x-y|^2.$$

The proof for (2.1.4) is complete. Now, the estimate (2.1.3) follows from the following:

$$d(x)^2 \leq d(x)(d(y) + |x-y|) \leq 3|x-y|^2 + d(x)|x-y| \leq 4|x-y|^2 + \frac{1}{4}d(x)^2,$$

thus $d(x)^2 \leq \frac{16}{3}|x-y|^2$.

Let $|x-y| \leq \frac{1}{2}[XY]$. Since $(1+|x|)(1+|y|) \leq 4$ in unit ball, then

$$(1-|x|)(1-|y|) = d(x)d(y) \geq \frac{1}{4}(1-|x|)(1-|y|)(1+|x|)(1+|y|).$$

Then

$$d(x)d(y) \geq \frac{1}{4}([XY] - |x-y|^2) \geq \frac{3}{4}|x-y|^2.$$

The proof of (2.1.4) is done. □

The following lemma is the consequence of the previous lemma in general domain.

Lemma 2.1.5. *[12] Let $\Omega \subset \mathbb{R}''$ be a bounded domain and let $p, q, \geq 0$ be fixed. For $(x, y) \in \bar{\Omega}^2$ we have:*

$$\min\left\{1, \frac{d(y)}{|x-y|}\right\} \simeq \min\left\{1, \frac{d(y)}{d(x)}, \frac{d(y)}{|x-y|}\right\}$$
$$\min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} \simeq \min\left\{\frac{d(y)}{d(x)^2}, \frac{d(x)}{d(y)}, \frac{d(x)d(y)}{|x-y|^2}\right\}$$
$$\min\left\{1, \frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right\} \simeq \min\left\{1, \frac{d(x)^p}{|x-y|^p}, \frac{d(y)^q}{|x-y|^q}, \frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right\}$$
$$\min\left\{1, \frac{d(x)^p d(y)^p}{|x-y|^{p+q}}\right\} \simeq \min\left\{1, \frac{d(x)}{|x-y|}\right\}^p \min\left\{1, \frac{d(y)}{|x-y|}\right\}^q$$

and assuming moreover that $p+q>0$, we also have

$$\log\left(1+\frac{d(x)^p d(y)^p}{|x-y|^{p+q}}\right) \simeq \log\left(2+\frac{d(y)}{|x-y|}\right)\min\left\{1,\frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right\}$$

2.2 Known estimates of Green's function

In this section we focus on polyharmonic clamped plate boundary value problem.

$$\begin{cases} (-\Delta^m)u = f & \text{in } \Omega, \\ \frac{\partial^j}{\partial n^j}u = 0 & \text{on } \partial\Omega \text{ with } \quad |j| \le m-1, \end{cases} \tag{2.2.1}$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain, f is in suitable functional space. For a smooth domain, Green's function $G_{-\Delta^m,\Omega}$ exists and the solution of the problem (2.2.1) is of the following form:

$$u = \int_\Omega G_{-\Delta^m,\Omega}(x,y)f(y)dy, \quad x \in \Omega.$$

The following theorem provides two-sided estimate of Green's function:

Theorem 2. *[12]*
In $\bar{B}\times\bar{B}$ we have

$$G_{m,n}(x,y) \simeq \begin{cases} |x-y|^{2m-n}\min\left\{1,\frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\} & \text{if } n>2m; \\ \log\left(1+\frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) & \text{if } n=2m; \\ d(x)^{m-\frac{n}{2}}d(y)^{m-\frac{n}{2}}\min\left\{1,\frac{d(x)^{\frac{n}{2}}d(y)^{\frac{n}{2}}}{|x-y|^n}\right\} & \text{if } n<2m. \end{cases}$$

Proof. Using the results of Lemma 2.1.4, let's discuss two cases: $|x-y| \ge \frac{1}{2}[XY]$, and $|x-y| \le \frac{1}{2}[XY]$.

Case 1: $|x-y| \le \frac{1}{2}[XY]$.
We are interested in case when $n>2m$. For $n>2m$ it is enough to show that $G_{m,n}(x,y) \simeq |x-y|^{2m-n}$, since (2.1.4) applies here.

According to [3], the Green function in a ball has the following form:

$$G_{m,n,B_1}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{[XY]/|x-y|} (v^2-1)^{m-1} v^{1-n} dv.$$

Since $|x-y| \leq \frac{1}{2}[XY]$, for any $a \in [2,\infty)$

$$\int_1^a \left(v^2-1\right)^{m-1} v^{1-n} dv \simeq \int_1^a v^{2m-n-1} dv$$

holds true. According to our assumption, the Green function is then

$$\begin{aligned} G_{m,n}(x,y) &\simeq |x-y|^{2m-n} \int_1^{[XY]/|x-y|} \left(v^2-1\right)^{m-1} v^{1-n} dv \\ &\simeq |x-y|^{2m-n} \int_1^{[XY]/|x-y|} v^{2m-n-1} dv. \end{aligned}$$

$$G_{m,n}(x,y) \simeq |x-y|^{2m-n} \int_1^{[XY]/|x-y|} v^{2m-n-1} dv \simeq |x-y|^{2m-n}.$$

Case 2: $|x-y| \geq \frac{1}{2}[XY]$.
According to (2.1.3), $\frac{d(x)}{|x-y|} \leq 3$, $\frac{d(y)}{|x-y|} \leq 3$. Thus, we have to show that

$$G_{m,n}(x,y) \simeq |x-y|^{-n} d(x)^m d(y)^m.$$

Again, using Boggio's formula [3] with the upper integration bound $[XY]/|x-y|$ is in $[1,2]$, since $|x-y| \geq \frac{1}{2}[XY]$. On $[1,2]$ we have $v^{-n} \approx 1$ and may conclude the following:

$$\begin{aligned} G_{m,n}(x,y) &\simeq |x-y|^{2m-n} \int_1^{[XY]/|x-y|} \left(v^2-1\right)^{m-1} v dv \\ &\simeq |x-y|^{2m-n} \left(\frac{[XY]^2}{|x-y|^2} - 1 \right)^m = |x-y|^{-n} \left([XY]^2 - |x-y|^2\right)^m \\ &= |x-y|^{-n} \left(\left(1-|x|^2\right)\left(1-|y|^2\right)\right)^m \simeq |x-y|^{-n} d(x)^m d(y)^m. \end{aligned}$$

We completed the proof for $n > 2m$. □

The next theorem presents estimates of Green's function from two sides:

Theorem 3. *[30] Let $\Omega \subset \mathbb{R}^n$ be a bounded and $C^{2m,\gamma}$ smooth domain with $m \geq 2$ and $n \geq 2$. Then there exist constants $c_1 \geq 0, c_2 > 0, c_3 > 0$ that depend on the domain and m, such that the following estimate for the polyharmonic Green's function $G_{(-\Delta)^m,\Omega}$ is true:*

$$c_2^{-1} H_\Omega(x,y) \leq G_{-\Delta^m,\Omega}(x,y) + c_1 \mathbb{1}_{|x-y|\geq c_3}(x,y) d(x)^m d(y)^m \leq c_2 H_\Omega(x,y),$$

where

$$H_\Omega := \begin{cases} |x-y|^{2m-n} \min\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\}, & \text{if } n > 2m, \\ \log(1 + \frac{d(x)^m d(y)^m}{|x-y|^{2m}}), & \text{if } n = 2m \\ d(x)^{m-n/2} d(y)^{m-n/2} \min\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\}, & \text{if } n < 2m \end{cases}$$

and indicator function

$$\mathbb{1}_{\{|x-y|\geq c_3\}} = \begin{cases} 1 & \text{if } |x-y| \geq c_3 \\ 0 & \text{if } |x-y| < c_3 \end{cases}$$

Proof. Since we know the estimate of the Green function from Theorem 2, it is necessary to show that on $\Omega \times \Omega$:

$$c_1 d(x)^m d(y)^m \leq c_2 H_\Omega(x,y)$$

to complete the proof for the estimate from above. We are interested in case when $n > 2m$.

Let $diam(\Omega) = sup\{|x-y| : x, y \in \Omega\}$.

Case 1: $d(x)d(y) < |x-y|^2$. We know that $|x-y| < diam(\Omega)$, then it follows

$$|x-y|^{2m-n} \min\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\} = \frac{d(x)^m d(y)^m}{|x-y|^n} \geq \frac{d(x)^m d(y)^m}{(diam(\Omega))^n}$$

Case 2: $d(x)d(y) \geq |x-y|^2$.

$$|x-y|^{2m-n} \min\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\} = |x-y|^{2m-n} \geq (diam(\Omega))^{2m-n} \frac{d(x)^m d(y)^m}{(diam(\Omega))^{2m}}$$
$$\geq \frac{d(x)^m d(y)^m}{(diam(\Omega))^n}$$

□

The following theorem is an example of pointwise estimate for biharmonic Green function.

Theorem 4. *[12] Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain, then the following estimate for the Green function is true:*

$$G_{\Delta^2,n}(x,y) \leq C(\Omega) \begin{cases} |x-y|^{4-n} + max\{d(x), d(y)\}^{4-n} & \text{if } n > 4; \\ \log(1 + |x-y|^{-1} + max\{d(x), d(y)\}^{-1}) & \text{if } n = 4; \\ 1 & \text{if } n = 2, 3, \end{cases}$$

Proof. For brevity, let $G(x,y) = G_{\Delta^2,\Omega}(x,y)$. The Green function $G(x,y) = F_n(|x-y|) + h(x,y)$, where $F_n(x)$ is the fundamental solution and represented by

$$F_n(x) = \begin{cases} c_n |x|^{4-n} & \text{if } n > 4; \\ -2c_4 \log|x| & \text{if } n = 4; \\ 2c_2 |x|^2 \log|x| & \text{if } n = 2, 3, \end{cases}$$

where c_n is defined by :

$$c_n = \begin{cases} \frac{1}{2(n-4)(n-2)ne_n} & \text{if } n \notin \{2,4\} \\ \frac{1}{8ne_n} & \text{if } n \in \{2,4\}, \end{cases}$$

and $h(x,y)$ solves the next Dirichlet problem:

$$\begin{cases} \Delta_y^2 h(x,y) = 0 \\ h(x,y) = -F_n(|x-y|) \\ \frac{\partial}{\partial n_y} h(x,y) = -\frac{\partial}{\partial n_y} F_n(|x-y|), \end{cases}$$

Case $n > 4$:

$$\|h(x,.)\|_{C^{1,\gamma}(\bar{\Omega})} \leq C(\Omega) d(x)^{3-n-\gamma}.$$

If $d(x) \geq d(y)$, then $h(x,y) \leq C(\partial\Omega) d(x)^{4-n}$, hence,

$$|G(x,y)| \leq C(\partial\Omega)(|x-y|^{4-n} + d(x)^{4-n}).$$

For $d(y) \geq d(x)$, we use the fact of symmetry of the Green function:

$$|G(x,y)| = |G(y,x)| \leq C(\partial\Omega)(|x-y|^{4-n} + d(y)^{4-n}).$$

□

Gazzola et al. also provided estimates for derivatives of the Green function:

Theorem 5. *[12] Let $\alpha \in \mathbb{N}^n$ be a multiindex. Then in $\bar{B} \times \bar{B}$ we have*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq (*)$$

with $(*)$ as follows:
1. if $|\alpha| \geq 2m-n$ and n odd, or if $|\alpha| > 2m-n$ and n even

$$(*) = \begin{cases} |x-y|^{2m-n-|\alpha|} \min\left\{1, \frac{d(x)^{m-|\alpha|}|d(y)^m}{|x-y|^{2m-|\alpha|}}\right\} & \text{for } |\alpha| < m. \\ |x-y|^{2m-n-|\alpha|} \min\left\{1, \frac{d(y)^m}{|x-y|^m}\right\} & \text{for } |\alpha| \geq m; \end{cases}$$

2. if $|\alpha| = 2m-n$ and n even

$$(*) = \begin{cases} \log\left(2 + \frac{d(y)}{|x-y|}\right) \min\left\{1, \frac{d(x)^{m-|\alpha|} d(y)^m}{|x-y|^{2m-|\alpha|}}\right\} & \text{for } |\alpha| < m, \\ \log\left(2 + \frac{d(y)}{|x-y|}\right) \min\left\{1, \frac{d(y)^m}{|x-y|^m}\right\} & \text{for } |\alpha| \geq m \end{cases}$$

3. if $|\alpha| \leq 2m-n$ and n odd, or if $|\alpha| < 2m-n$ and n even

$$(*) = \begin{cases} d(x)^{m-\frac{n}{2}-|\alpha|} d(y)^{m-\frac{n}{2}} \min\left\{1, \frac{d(x)^{\frac{n}{2}} d(y)^{\frac{n}{2}}}{|x-y|^n}\right\} & \text{for } |\alpha| < m - \frac{n}{2}, \\ d(y)^{2m-n-|\alpha|} \min\left\{1, \frac{d(x)^{m-|\alpha|} d(y)^{n-m+|\alpha|}}{|x-y|^n}\right\} & \text{for } m - \frac{n}{2} \leq |\alpha| < m, \\ d(y)^{2m-n-|\alpha|} \min\left\{1, \frac{d(y)^{n-m+|\alpha|}}{|x-y|^{n-m+|\alpha|}}\right\} & \text{for } |\alpha| \geq m. \end{cases}$$

Proof. We want to show that:

$$|D_x^\alpha G_{m,n}(x,y)| \preceq |x-y|^{2m-n-|\alpha|} \left(\frac{d(x)}{|x-y|}\right)^{\max\{m-|\alpha|,0\}} \left(\frac{d(y)}{|x-y|}\right)^{m}.$$

Recall,

$$G_{m,n,B_1}(x,y) = k_{m,n}|x-y|^{2m-n}\int_1^{[XY]/|x-y|}(v^2-1)^{m-1}v^{1-n}dv.$$

Let's use transformation $s = 1 - \frac{1}{v^2}$ in the formula above. We have

$$G_{m,n}(x,y) = \frac{k_{m,n}}{2}|x-y|^{2m-n} f_{m,n}\left(A_{x,y}\right),$$

where

$$f_{m,n}(t) := \int_0^t s^{m-1}(1-s)^{n/2-m-1}ds$$
$$A_{x,y} := \frac{[XY]^2-|x-y|^2}{[XY]^2} = \frac{\left(1-|x|^2\right)\left(1-|y|^2\right)}{[XY]^2} \simeq \frac{d(x)d(y)}{|XY|^2}.$$

Now, since Green's function is the product of $\frac{k_{m,n}}{2}|x-y|^{2m-n}$ and $f_{m,n}\left(A_{x,y}\right)$, let's use product rule to find the derivative:

$$|D_x^\alpha G_{m,n}(x,y)| \preceq \sum_{\beta\le\alpha}\left|D_x^{\alpha-\beta}\right| x-y\left|^{2m-n}\right| \cdot |D_x^\beta f_{m,n}(A_{x,y})|$$

Pulling out the case when $\beta = 0$ and applying chain rule for $f_{m,n}(A_{x,y})$, we get:

$$|D_x^\alpha G_{m,n}(x,y)| \preceq |x-y|^{2m-n-|\alpha|} \cdot |f_{m,n}(A_{x,y})|$$
$$+\sum_{\substack{\beta\le\alpha\\ \beta\ne 0}}|x-y|^{2m-n-|\alpha|+|\beta|}\cdot\sum_{j=1}^{|\beta|}\left\{\left|f_{m,n}^{(j)}\left(A_{x,y}\right)\right|\cdot\sum_{\substack{\sum_{i=1}^{j}\beta^{(i)}=\beta\\ \left|\beta^{(i)}\right|\ge 1}}\prod_{i=1}^{j}\left|D_x^{\beta(i)}A_{x,y}\right|\right\}$$

Using the results (2.1.4), we have

$$A_{x,y} \simeq \frac{d(x)d(y)}{[XY]^2} \le \frac{3}{4}.$$

Here, i.e for $t \in \left[0, \frac{3}{4}\right]$, the following holds true:

$$\left|f_{m,n}^{(j)}(t)\right| \preceq t^{\max\{m-j,0\}}$$

Since $d(x) \leq [XY]$, for every multiindex $\beta \in \mathbb{N}^n$ one has

$$\left|D_x^\beta A_{x,y}\right| \preceq d(y)[XY]^{-1-|\beta|}.$$

Using facts above:

$$\begin{aligned}
|D_x^\alpha G_{m,n}(x,y)| &\preceq |x-y|^{2m-n-|\alpha|}\frac{d(x)^m d(y)^m}{[XY]^{2m}} \\
&\quad + \sum_{\substack{\beta<a \\ \beta\neq 0}} |x-y|^{2m-n-|\alpha|+|\beta|} \cdot \sum_{j=1}^{|\beta|}\left\{\left(\frac{d(x)d(y)}{[XY]^2}\right)^{\max\{m-j,0\}} \cdot \frac{d(y)^j}{[XY]^{j+|\beta|}\}}\right\} \\
&\preceq \sum_{\beta\leq\alpha} |x-y|^{2m-n-|\alpha|}\left(\frac{|x-y|}{|XY|}\right)^{|\beta|}\left(\frac{d(x)}{[XY]}\right)^{\max\{m-|\beta|,0\}}\left(\frac{d(y)}{[XY\ |}\right)^m \\
&\preceq |x-y|^{2m-n-|\alpha|}\left(\frac{d(x)}{[XY]}\right)^{max\{m-|\alpha|,0\}}\left(\frac{d(y)}{[XY]}\right)^m
\end{aligned}$$

The estimate is proven. □

Krasovskii provided estimates in a general contest in [25]. The next theorem provides estimates without boundary terms.

Theorem 6. *[12] Let Ω be a bounded $C^{2m,\gamma}$-smooth domain in R^n with $n \geq 2$ and G be a Green's function in Ω for the Dirichlet boundary value problem. Then there exists a constant $C = C(\Omega)$, such that for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \leq 2m$:*
- If $|\alpha| + |\beta| + n > 2m$:

$$\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq C(\Omega)|x-y|^{2m-n-|\alpha|-|\beta|} \text{ for all } x, y \in \Omega$$

- If $|\alpha| + |\beta| + n = 2m$ and n is even:

$$\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq C(\Omega)\log\left(1 + |x-y|^{-1}\right) \text{ for all } x, y \in \Omega.$$

- *If* $|\alpha| + |\beta| + n = 2m$ *and* n *is odd, or if* $|\alpha| + |\beta| + n < 2m$ *:*

$$\left|D_x^\alpha D_y^\beta C(x,y)\right| \leq C(\Omega) \text{ for all } x, y \in \Omega.$$

Proof. We are interested in case when $n > 2m$.
In Theorem 2 we showed that

$$|G(x,y)| \leq C|x-y|^{2m-n}.$$

Let $x \in \Omega, y \in \Omega\backslash\{x\}$ and $R = \frac{|x-y|}{4}$. First, let's consider the case when $|\alpha| = 0$. Then in $B_R(y) \subset B_{2R}(y)$ the following can be derived:

$$\left\|D_y^\beta G\left(x, \cdot\right)\right\|_{L^\infty(B_R(y)\cap\Omega)} \leq \frac{C}{|x-y|^{|\beta|}} \|G(x,.)\|_{L^\infty(B_{2R}(y)\cap\Omega)}$$

$$\leq \frac{C}{|x-y|^{|\beta|}} \||x - \cdot|^{2m-n}\|_{L^\infty(B_{2R}(y)\cap\Omega)}$$
$$\leq C|x-y|^{2m-n-|\beta|}$$

where we used for $z \in B_{2R}(y)$ that

$$|x-z| \geq |x-y| - |y-z| > \frac{1}{2}|x-y|.$$

Since the Green function is symmetric, the same can be proved for $|\beta| = 0$ and $|\alpha| > 0$ in a similar way. Moreover, since $y \mapsto D_x^\alpha G(x,y)$ solves the homogeneous Dirichlet boundary value problem, we can proceed as before for the mixed derivatives. □

Next, let us present the following estimates proved by Maz'ya and Mayboroda:

Theorem 7. *[28] For every* x, y *in a bounded domain* Ω*, there exist constants* C, C' *that depend only on* m, n*, such that the following estimates hold:*
- *if* $n \in \{3, 2m+1\}$ *is odd*

$$\left|\nabla_x^i \nabla_y^j G(x,y)\right| \leq C \min\left\{1, \left(\frac{d(x)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-i}, \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-j}\right\} \times \frac{1}{|x-y|^{n-2m+i+j}},$$

whenever $0 \leq i, j \leq m - \frac{n}{2} + \frac{1}{2}$ *are such that* $i + j \geq 2m - n$, *and*

$$| \nabla_2^i \nabla_y^j G(x,y) \leq C \min \left\{ 1, \left(\frac{d(x)}{|x-y|} \right)^{m - \frac{5}{2} + \frac{1}{2} - i}, \left(\frac{d(y)}{|x-y|} \right)^{m - \frac{4}{2} + \frac{1}{2} - j} \right\} \times$$

$$\times \frac{1}{|x-y|^{n-2m+i+j}} \min \left\{ \frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1 \right\}^{n-2m+i+j}$$

if $0 \leq i, j \leq m - \frac{n}{2} + \frac{1}{2}$ *are such that* $i + j \leq 2m - n$.

-If $n \in [2, 2m] \cap \mathbb{N}$ *is even, then*

$$\left| \nabla_x^i \nabla_y^j G(x,y) \right| \leq C \min \left\{ 1, \left(\frac{d(x)}{|x-y|} \right)^{m - \frac{n}{2} - i}, \left(\frac{d(y)}{|x-y|} \right)^{m - \frac{n}{2} - j} \right\} \times$$
$$\times \frac{1}{|x-y|^{n-2m+i+j}} \min \left\{ \frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1 \right\}^{n-2m+i+j} \times$$
$$\times \log \left(1 + \frac{\min\{d(x), d(y)\}}{|x-y|} \right)$$

for all $0 \leq i, j \leq m - \frac{n}{2}$.

Dall'Acqua and Sweers [7] presented the next estimates:

Theorem 8. *Let* $G_m(x,y)$ *be the Green function. Let* $k \in \mathbb{N}^n$. *The following estimates hold for every* $x, y \in Q$:

(1) For $|k| \geqslant m$:
(a) if $2m - n - |k| < 0$, *then*

$$|D_x^k G_m(x,y)| \preceq |x-y|^{2m-n-|k|} \min \left\{ 1, \frac{d(y)}{|x-y|} \right\}^m,$$

(b) if $2m-n-|k|=0$*, then*

$$\left|D_x^k G_m(x,y)\right| \preceq \log\left(1+\frac{d(y)^m}{|x-y|^m}\right)$$
$$\sim \log\left(2+\frac{d(y)}{|x-y|}\right)\min\left\{1,\frac{d(y)}{|x-y|}\right\}^m,$$

(c) if $2m-n-|k|>0$*, then*

$$|D_x^k G_m(x,y)| \preceq d(y)^{2m-n-|k|}\min\left\{1,\frac{d(y)}{|x-y|}\right\}^{n+|k|-m},$$

(2) For $|k|<m$ *:*
(a) if $2m-n-|k|<0$*, then*

$$\left|D_x^k G_m(x,y)\right| \preceq |x-y|^{2m-n-|k|}\min\left\{1,\frac{d(x)}{|x-y|}\right\}^{m-|k|}\min\left\{1,\frac{d(y)}{|x-y|}\right\}^m,$$

(b) if $2m-n-|k|=0$*, then*

$$\left|D_x^k G_m(x,y)\right| \preceq \log\left(1+\frac{d(y)^m d(x)^{m-|k|}}{|x-y|^{2m-|k|}}\right)$$
$$\sim \log\left(2+\frac{d(y)}{|x-y|}\right)\min\left\{1,\frac{d(y)}{|x-y|}\right\}^m\min\left\{1,\frac{d(x)}{|x-y|}\right\}^{m-|k|}$$

(c) if $2m-n-|k|>0$*, and moreover*
(i) $m-\frac{1}{2}n\leqslant|k|$*, then*

$$|D_x^k G_m(x,y) \preceq d(y)^{2m-n-|k|}\min\left\{1,\frac{d(x)}{|x-y|}\right\}^{m-|k|}$$
$$\times\min\left\{1,\frac{d(y)}{|x-y|}\right\}^{n-m+|k|},$$

(ii) $|k|<m-\frac{1}{2}n$*, then*

$$|D_x^k G_m(x,y) \preceq d(y)^{m-n/2}d(x)^{m-n/2-|k|}\min\left\{1,\frac{d(x)d(y)}{|x-y|^2}\right\}^{\frac{n}{2}}$$

Proof. We refer to [8] and provide sketch of the proof.

Dall'Acqua and Sweers started from estimates of Krasovskii [25] for higher-order derivatives of Green's function. To obtain lower-ordered estimates for derivatives, authors integrated higher-ordered estimates along some path γ_x^y. For $\alpha, \beta \in \mathbb{N}^n$, and $\tilde{x} \in \partial\Omega$ they found:

$$D_x^\alpha D_y^\beta G_k(x,y) = D_x^\alpha D_y^\beta G_m(\tilde{x},y) + \int_{\gamma_x^y} \nabla_z D_z^\alpha D_y^\beta G_m(z,y)dz.$$

If $|\alpha| \leq m-1$, then

$$|D_x^\alpha D_y^\beta G_k(x,y)| = \int_0^l |\nabla_x D_x^\alpha D_y^\beta G_m(\tilde{\gamma}_x^y(s),y)|ds$$

If $|\beta| \leq m-1$,integrating with respect to y:

$$|D_x^\alpha D_y^\beta G_k(x,y)| = \int_0^l |\nabla_y D_y^\beta D_x^\alpha G_m(\tilde{\gamma}_x^y(s),y)|ds$$

After that the authors considered two cases when $r \geq m$ and $r < m$.
Case $r \geq m$: According to [25],$|D_x^\alpha D_y^\beta G_k(x,y)| \leq |x-y|^{m-n-r}$ with $\beta = m-1$.
Case 1 (a) of the Theorem was proven then by applying m times the estimate proven by the authors earlier :

$$|H(x,y)| \preceq |x-y|^{-k+1} \min\{1, \frac{dx}{|x-y|}\}^{\nu_1+1} \min\{1, \frac{dy}{|x-y|}\}^{\nu_2},$$

where $\nu_1, \nu_2, k \in \mathbb{N}$.
Case 1 (b) was proven by applying the same estimate as in 1(a), $m-1$ times. Case 1(c) was proven by first applying estimate as above $n+r-m-1$ times. And after that the authors suggested to use the following estimate:
$|H(x,y)| \preceq \log\left(2 + \frac{d(x)d(y)}{|x-y|^2}\right) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}$, once they found

$$|D_y^{\tilde{\beta}} D_x^k G_m(x,y)| \preceq \log(1 + \frac{d(y)^{n+r-m}}{|x-y|^{n+r-m}}),$$

where $\tilde{\beta} \in \mathbb{N}^n, \tilde{\beta} \le \beta$ and $|\tilde{\beta}| = 2m - n - r$. Then, the estimate $|H(x,y)| \preceq d(x) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}$ was applied once and $|H(x,y)| \preceq d(x)^{\alpha_1+1} d(y)^{\alpha_2} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}$ was applied $2m-n-r-1$ times for some $\alpha_1, \alpha_2 \in \mathbb{N}$.
Case 2: $r < m$, $|\alpha| = m - r, |\beta| = m$. Again, starting from Krasovskii's estimate for $|D_y^\beta D_x^\alpha D_x^k G_m(x,y)|$, the authors integrated the estimate m times with respect to y and $m - r$ times with respect to x.
Case 2(a) was proven by applying the estimate

$$|H(x,y)| \preceq |x-y|^{-k+1} \min\{1, \frac{dx}{|x-y|}\}^{\nu_1+1} \min\{1, \frac{dy}{|x-y|}\}^{\nu_2}$$

m times with respect to y and $m - r$ times with respect to x.
Case 2(b) was proven by applying the same estimate as in case 2(a) m times with respect to y and $m-r-1$ times with respect to x and applying the estimate $|H(x,y)| \preceq \log\left(2 + \frac{d(x)d(y)}{|x-y|^2}\right) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}$ with respect to x.
Case 2(c), for $m - r \le n - 1$ the statement was proven by applying the following estimates:

$$|H(x,y)| \preceq |x-y|^{-k+1} \min\{1, \frac{dx}{|x-y|}\}^{\nu_1+1} \min\{1, \frac{dy}{|x-y|}\}^{\nu_2}$$

$n - 1$ times,

$$|H(x,y)| \preceq \log\left(2 + \frac{d(x)d(y)}{|x-y|^2}\right) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}$$

one time,

$$|H(x,y)| \preceq d(x) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}$$

and

$$|H(x,y)| \preceq d(x)^{\alpha_1+1} d(y)^{\alpha_2} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}$$

$2m - n - r - 1$ times.

Case 2(c), when $m - r > n - 1$, was proven by applying the following estimates:

$$|H(x,y)| \preceq |x-y|^{-k+1} \min\{1, \frac{dx}{|x-y|}\}^{\nu_1+1} \min\{1, \frac{dy}{|x-y|}\}^{\nu_2}$$

$n - 1$ times,

$$|H(x,y)| \preceq \log\left(2 + \frac{d(x)d(y)}{|x-y|^2}\right) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}$$

one time, and

$$|H(x,y)| \preceq d(x)^{\alpha_1+1} d(y)^{\alpha_2} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}$$

m times with respect to y and $m - r - n$ with respect to x. □

Chapter 3

Main result

3.1 Mazya's problem in a ball

The next problem is taken from collection of open problems published in the article "Seventy Five (Thousand) Unsolved Problems in Analysis and Partial Differential Equations" by Vladimir Maz'ya. The statement of the problem is the following:
Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain. Green's function is a solution of

$$(-\Delta_x)^m G_{m,n}(x,y) = \delta(x-y), \quad x, y \in \Omega, \tag{3.1.1}$$

with the homogeneous Dirichlet boundary conditions:

$$\frac{\partial^i}{\partial n_x^i} G_{m,n}|_{x \in \partial\Omega} = 0, \quad i = 0, ..., m-1, \tag{3.1.2}$$

where δ is the Dirac delta function.

Let $n > 2m$. Prove or disprove that

$$|G_{m,n,r}(x,y)| \leq \frac{d(m,n)}{|x-y|^{n-2m}}, \quad x, y \in \Omega, x \neq y,$$

where $d(m,n)$ does not depend on Ω.

Theorem 9. *Let $n > 2m$ and B_r be a ball with radius r. Then polyharmonic Green's function satisfies the pointwise estimate*

$$|G_{m,n,r}(x,y)| \leq \frac{d(m,n)}{|x-y|^{n-2m}}, \quad x, y \in B_r, x \neq y,$$

where $d(m,n)$ does not depend on r and is a positive constant given by formula (2.1.1).

Proof. By using Lemma 2.1.2 and Lemma 2.1.1, we compute

$$|G_{m,n,r}(x,y)| = G_{m,n,r}(x,y) = d_{m,n}[X^{2m-n} - Y_r^{2m-n} -$$

$$\sum_{k=1}^{m-1} \frac{(-1)^k}{k!}\Big(m - \frac{n}{2}\Big)\Big(m - \frac{n}{2} - 1\Big)...\Big(m - \frac{n}{2} - k + 1\Big)Y_r^{2m-n-2k} Z_r^{2k}] =$$

$$= d_{m,n}[X^{2m-n} - Y_r^{2m-n} -$$

$$\sum_{k=1}^{m-1} \frac{(-1)^k(-1)^k}{k!}\Big(-m + \frac{n}{2}\Big)\Big(-m + \frac{n}{2} + 1\Big)...\Big(-m + \frac{n}{2} + k - 1\Big)Y_r^{2m-n-2k} Z_r^{2k}]$$

$$\leq d_{m,n} X^{2m-n},$$

where $X^{2m-n} = \frac{1}{|x-y|^{n-2m}}$. Here we have used the fact that $n > 2m$ and all the terms except the first term on the right hand side are negative. □

We have provided a simple and short proof of Problem 32 from [27], that is, $|G_{m,n,r}(x,y)| \leq \frac{d_{m,n}}{|x-y|^{n-2m}}$ in a ball with radius r, where the positive coefficient $d_{m,n}$ is given explicitly and does not depend on the radius r.

3.2 Conclusion

We provided a simple and short proof of a particular case of Problem 32 from [27], that is, $|G_{m,n,r}(x,y)| \leq \frac{d_{m,n}}{|x-y|^{n-2m}}$ in a ball with radius r, where the positive coefficient $d_{m,n}$ is given explicitly and does not depend on the radius r. The explicit representation of polyharmonic Green's function in the chosen domain, along with its positivity, proved to be crucial in our approach. Our result was presented at the International Conference of Mathematical Sciences in Turkey, 2022, and has been accepted for publication in the AIP Proceedings [29]. In the future, one can consider polyharmonic Green's function in other domains.

Bibliography

[1] A. Ancona, *Comparaison des mesures harmoniques et des fonctions de Green pour des opérateurs elliptiques sur un domaine lipschitzien*, Comptes rendus de l'Académie des Sciences, 294 (1982), pp. 505–508.

[2] N. Aroua and M. Bellasoued, *Stable determination of a second order perturbation of the polyharmonic operator by Boundary Measurements*, Journal of Mathematical Analysis and Applications, 522 (2023), p. 126–965.

[3] T. Boggio, *Sulle funzioni di Green d'ordinem*, Rendiconti del Circolo Matematico di Palermo, 20 (1905), pp. 97–135.

[4] L. Challis and F. Sheard, *The Green of Green Functions*, Physics Today, 56 (2003), p. 41.

[5] M. Cranston, E. Fabes, and Z. Zhao, *Conditional gauge and potential theory for the Schrödinger operator*, Transactions of American Mathematical Society, 307 (1988), pp. 171–194.

[6] A. Dall'Acqua, C. Meister, and G. Sweers, *Separating positivity and regularity for fourth order Dirichlet problems in 2d-domains*, Analysis (Munich), 25 (2005), pp. 205–261.

[7] A. Dall'Acqua and G. Sweers, *Estimates for Green function and Poisson kernels of higher-order Dirichlet boundary value problems*, Journal of Differential Equations, 205 (2004), pp. 466–487.

[8] ——, *The clamped-plate equation for the limaçon*, Annali di Matematica Pura ed Applicata, 184 (2005), pp. 361–374.

[9] R. Duffin, *Some problems of mathematics and science*, Bulletin of the American Mathematical Society,, 80 (1974), pp. 1053–1070.

[10] S. Eichmann and R. M. Schätzle, *Positivity for the clamped plate equation under high tension*, Annali di Matematica Pura ed Applicata (1923 -), 201 (2022), p. 2001–2020.

[11] P. Garabedian, *A partial differential equation arising in conformal mapping*, Pacific Journal of Mathematics, 1 (1951), pp. 485–524.

[12] F. Gazzola, H.-C. Grunau, and G. Sweers, *Polyharmonic boundary value problems positivity preserving and nonlinear higher order elliptic equations in bounded domains*, Springer, 2010.

[13] F. Gazzola and G. Sperone, *Remarks on radial symmetry and monotonicity for solutions of semilinear higher order elliptic equations*, Mathematics in Engineering, 4 (2021), p. 1–24.

[14] H.-C. Grunau and G. Sweers, *Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions*, Mathematische Nachrichten, 179 (1996), pp. 89–102.

[15] H.-C. Grunau and G. Sweers, *Regions of positivity for Polyharmonic Green's functions in arbitrary domains*, Proceedings of the American Mathematical Society, 135 (2007), pp. 3537–3547.

[16] H.-C. Grunau and G. Sweers, *A clamped plate with a uniform weight may change sign*, Discrete and Continuous Dynamical Systems - Series S, 7 (2014), pp. 761–766.

[17] M. Gruter and K.-O. Widman, *The Green function for uniformly elliptic equations.*, Manuscripta Mathematica, 37 (1982), pp. 303–342.

[18] H.-C. Grunau, and F. Robert, *Positivity and almost positivity of biharmonic Green's function under Dirichlet boundary conditions*, Archive for Rational Mechanics and Analysis, 195 (2010), pp. 865–898.

[19] H. Huber and M. Sieveking, *Uniform bounds for quotients of Green functions on C 1,1-domains*, Annales de l'institut Fourier, 32 (1982), pp. 105–117.

[20] T. S. Kal'menov and D. Suragan, *On a new method for constructing Green function of the Dirichlet problem*, Differential Equations, 48 (2012), pp. 441–445.

[21] V. Karachik, *Green's functions of some boundary value problems for the biharmonic equation*, Complex Variables and Elliptic Equation, 67 (2021), p. 1712–1736.

[22] ——, *Solution to the Dirichlet problem for the polyharmonic equation in the ball*, Siberian Advances in Mathematics, 32 (2022), p. 197–210.

[23] C. Kockritz, *Lokale positivität der polyharmonischen greenschen funktion.*

[24] V. Kozlov, V. Kondratiev, and V. Maz'ya, *On sign variation and the absence of strong zeros of solutions of elliptic equations*, Mathematics of the USSR - Izvestiya,, 34 (1990), pp. 337–353.

[25] J. P. Krasovski, *Isolation of singularities of the Green's function*, Mathematics of the USSR-Izvestiya, 1 (1967), pp. 935–966.

[26] Z. Liu and W. Dai, *A Liouville type theorem for poly-harmonic system with Dirichlet boundary conditions in a half space*, Advanced Nonlinear Studies, 15 (2015), p. 117–134.

[27] V. Maz'ya, *Seventy Five (thousand) unsolved problems in analysis and partial differential equations*, Integral Equations and Operator Theory, 90 (2018).

[28] V. Maz'ya and S. Mayboroda, *Pointwise estimates for the polyharmonic Green function in general domains*, Analysis, Partial Differential Equations and Applications, (2009).

[29] G. Myrzabayeva and D. Suragan, *Pointwise estimates for polyharmonic green's function.* To appear, 2023.

[30] L. Pulst, *Dominance of positivity of the green's function associated to a perturbed polyharmonic dirichlet boundary value problem by pointwise estimates*, Mathematics, (2015).

[31] K.-O. Widman, *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations*, Mathematica Scandinavica, 21 (1968), pp. 17–37.

[32] Z. Nehari, *On the biharmonic Green's function*, Studies in Mathematics and Mechanics, presented to Richard von Mise, (1954), pp. 111–117.

[33] Z. Zhao, *Green function for Schrödinger operator and conditioned Feynman-Kac gauge*, Journal of Mathematical Analysis and Application, 116 (1986), pp. 309–334.

[34] ——, *Green functions and conditioned gauge theorem for a 2-dimensional domain*, Seminar on Stochastic Processes, 15 (1988), pp. 283–294.