

Pricing Convertible Bonds by Finite Element Method

by

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Abstract

In this thesis, the proposed problem is to solve the system of two-coupled Black-Scholes equations, which is the so called TF (Tsiveriotis and Fernandes) model [16], for pricing the convertible bonds by the finite element method. Firstly, the derivation of the TF model is reviewed and introduced based on original work of Tsiveriotis and Fernandes [16]. Standard transformations are applied to the Black-Scholes equations to obtain a system of two parabolic equations. After the transformations, the finite element method is introduced and is applied to solve the new system. The well-posedness of the method is discussed and numerical implementation of the schemes are presented. Finally, the some numerical examples by the finite element method are obtained.

Thesis Supervisor: Dongming Wei

Title: Full Professor

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Nomenclature

Main variables

U Value of the Convertible Bond

V Value of the Cash-Only Convertible Bond

Parameters for pricing Convertible Bonds

σ Volatility

C Coupon payments

D Dividends

F Face value

k^* Conversion ratio

P Hazard rate

q Dividend yield

r Risk free rate

r_c Credit spread reflecting payoff default risk

r_g Growth rate of the stock

S Underlying stock price

T Time to maturity

V_c Clean call price

V_p Clean put price

Finite element method

\underline{u} Global nodal solution

$\underline{u}^{(e)}$ Local nodal solution

\vec{u} Exact solution

$\vec{u}_{h^*}^{(e)}$ Local finite element solution

\vec{u}_{h^*} Finite element solution

A, p Denoted coefficients after the first transformation

t Value of the time in first transformation

$\alpha, \beta, \gamma, \lambda$ Denoted coefficients after the second transformation

\mathbf{K} Global stiffness matrix

\mathbf{M} Global mass matrix

\mathbf{Q} Global load vector

τ Value of time after second transformation

h, g Values of CB and COCB in the reduced form

h^* Maximum value of $l^{(e)}$

$h^{(e)}(x, \tau), g^{(e)}(x, \tau)$ Values of CB and COCB in the reduced form

$K^{(e)}$ Local stiffness matrix

$l^{(e)}$ Length of eth subinterval

$M^{(e)}$ Local mass matrix

$N^{(e)}(x)$ Local shape function for FEM

$N_j(x)$ Global shape function for FEM

$Q^{(e)}$ Local load vector

Chapter 1

Introduction

1.1 Background

A convertible bond is a type of financial derivatives, which is used in financial risk management and trading by investors and issuer. Pricing financial derivatives for convertible bonds is a complicated problem that involves solving stochastic partial differential equations. There are many models for pricing convertible bonds. One of them is the TF model which is a well-known model, it was first developed by Tsiveriotis and Fernandes in the 1998 paper [16]. This model is an exceptionally prevalent choice among specialists for estimating convertible bonds due to its relative simplicity and its capacity to join the basic characteristics of convertible bonds that have constrained advertise information. The authors have developed numerical solutions based on their model by using a finite difference method.

The K. Milanov and O. Kounchev have made critical analysis [12] on the framework of the TF model by using the Binomial-Tree approximation, which is a well-known theory in the pricing of financial derivatives. They have first developed a binomial model for pricing convertible bonds with credit risk. The key idea of their paper is that the standard Binomial-Tree approach is not convenient for practitioners, but they have developed a model with better performance in real-life practice with Binomial-Tree approach with the explicit method which can be very useful in pricing convertible bonds. The Victor Gushchin and Erwan Curien have a good paper [17] based on the TF model.

They have tried to use the TF model for pricing convertible bonds with an exogenous credit spread. The main goal of their research was to make a universal technique that can be used in real life without additional complete information in the market. They have taken into account that the existence of continuous credit spread could be used to correctly pricing the convertible bonds. Moreover, to achieve persuasive results, there was incorporated historical volatility. Therefore, the empirical analysis or practical approach for pricing convertible bonds with moving average credit spread has shown very impressive results.

The Giovanni Barone-Adesi, Ana Bermudez, and John Hatgioannides have presented a paper in 2003 [2], where they have investigated two-factor convertible bonds model. The key idea of this paper is that they have calibrated particularly each parameter such as interest rate and volatility by using market data and then applied characteristics methods together with finite element methods. Also, they have enhanced the efficiency of the numerical methods by improving the iterative algorithm which includes variational inequalities taking into account that it has a sequence of variational equalities.

The master thesis on the topic pricing convertible bonds using finite element methods [1]. He has considered a one-factor model for pricing operations where stock price is stochastic and value of convertible bond which satisfies linear complementarity problem, consequently, it goes to the two-factor variational inequality model with stochastic parameters. The finite element method was applied to the parabolic variational inequalities, which has no closed-form solution.

The book numerical methods in finance has lots of different approaches in pricing convertible bonds [4], stating concretely, in chapter 5 they have made a deep research in the two-factor model to price a convertible bonds with variational inequalities.

There is a very interesting master thesis by Qingkai Mo [13] which was engaged in pricing convertible bonds with dividend protection subject to credit risk by using a numerical approach. The models presented in this work are TF and AFV, which are well-known in the area of financial derivatives. Both of them based on the Black-Scholes [15] equations, which means for us that there are pretty good opportunities to use numerical approaches. These models were compared by different criteria such as

convergence rate, number of iterations and computational time. The key idea of this paper is to identify which model is the best for pricing convertible bonds with dividend protection.

From this literature survey, it is found that the various methods for numerical analysis of convertible bonds such as FDM, Binomial-Tree, Critical analysis, and finite element method (FEM) have been considered for numerical solutions of the TF model. The work of Russell and Wheeler [14] show solution equivalence between finite difference and finite element methods. Therefore, FEM is considered for numerical solutions of the model. The linear and quadratic elements will be applied to the system of parabolic equations obtained from the TF model by the change of variables. The advantage of the FEM is the ability to combine linear and quadratic polynomial approximations of the solutions in the same interval.

The value of a convertible bond [16], U is represented by the Black-Scholes condition, where r_g is the growth rate, r is the risk free rate, r_c is the credit spread reflecting payoff default risk, and σ is Volatility.

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + r_g S \frac{\partial U}{\partial S} - (r + r_c)U = 0 \quad (1.1)$$

We characterize the ‘cash only part of the convertible bond’ (COCB) as it were money streams that an optimum carrying on the holder of the comparing convertible would get. By definition, this value indicated as V , is defined by the conduct of U , S and t . Hence, like U , the COCB [2] cost V must satisfy the Black-Scholes condition and ought to include the issuer’s credit spread in a few cases as well as we are working with ‘risky’ in cash. $(U - V)$ shows to the value parcel of the convertible and may be marked down utilizing the risk-free rate. This led us to a novel definition of convertible bond valuation as a framework of two coupled Black-Scholes equations:

$$\begin{aligned} \frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + (r - q)S \frac{\partial U}{\partial S} - r(U - V) - (r + r_c)V &= 0 \\ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - (r + r_c)V &= 0 \end{aligned} \quad (1.2)$$

At the beginning in the original paper of the Black and Scholes mainly they have used for derivation such techniques as Geometric Brownian motion Ito's Lemma.

1.2 Convertible bonds

A convertible bond is security by which an investor can convert bonds for the number of shares that was predetermined in advance when those securities were issued [9]. It is worth noting that the investor is not obliged to traditionally change bonds, this is just an option that the investor has when buying securities. Convertible bonds are a very profitable option when providing finance to companies, it should be noted that the strength of these securities is that the holder has hybrid capabilities compared to other securities, stating concretely such as interest payments on shareholdings. There is a crucial point as a conversion rate [3] for holding convertible bonds, which means how many shares we can exchange bonds for. This coefficient is predetermined when a bond is issued on the market. Indeed on the off chance that the financial specialist claims, if the bond does not change over, he will get intermittent installments at the shown coupon level. The valuation of a convertible bond is complicated due to its many parameters and characteristics. When estimating, the fundamental data related to the bonds and stocks should be considered carefully. For example, stock price, maturity, coupon, volatility, and spread should be taken into account.

1.3 Geometric Brownian motion and Ito lemma

Geometric Brownian motion (GBM)[8] is a stochastic process which mostly is used for the price of assets, furthermore, in the origin of GBM lies the Wiener process. Also, GBM is used to present option prices with the Black–Scholes models. A stochastic

process S_t is said to be a GBM if it fulfills the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a Wiener process, μ is drift, and σ volatility.

Lemma 1.3.1 *Ito's Lemma.* Assume that the value of variable S follows Ito process [10]

$$dS = \mu(S, t)Sdt + \sigma(S, t)Sdz$$

where dz is a Wiener process and μ, σ are functions depend on S, t . Ito lemma [8] presents that a function G ($G = U = V$) can be written as following

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz$$

Mention that both S and G are impacted by a Wiener process. Therefore, according to Ito lemma it can concluded that it plays important role in derivation of Black-Scholes equation. G also follows Ito process, with dritt (μ) and variance (σ)

$$\frac{\partial G}{\partial S} \mu + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 \quad \text{and} \quad \left(\frac{\partial G}{\partial S} \right)^2 \sigma^2$$

these expressions above can be used in the derivation of the TF model in next section.

1.3.1 Replicating portfolio for option pricing

1.3.1.1 Ito lemma for pricing the convertible bonds

Our first step is to derive pricing financial derivatives with the Black-Scholes model as presented below:

- Let us assume that stock price satisfies the stochastic differential equation $dS = \mu S dt + \sigma S dW$
- Price of option V depends on time t and on the underlying stock price S , hence $V = V(S, t)$

- Pricing derivatives, especially stochastic differential equation is derived by means of Ito's Lemma

1.4 Derivation of the TF model

The Tsiveriotis and Fernandes framework is a well-known choice of model amongst practitioners for pricing convertible bonds due to its ability to incorporate the fundamental traits of convertible bonds that have limited market data. The TF model known for its free boundaries. Therefore, the main idea of the TF model is that convertible bond price U is stated as a sum of two elements [12] which is represented below:

$$U = V + E$$

where U is the value of the convertible bond, V is the value of COCB(cash-only Convertible Bond) and E is the equity component.

The value $\Pi = V + \delta S$ of the portfolio consisting of one option in a long position at the price V and δ underlying stock at the price S . Π changes over the time interval $[t, t + \Delta t]$ by selling $\Delta\delta < 0$ or buying $\Delta\delta > 0$ short positioned stocks by $\Delta\Pi = \Delta\Pi_{t+\delta t} - \Pi_t$.

$$\Delta\Pi = \Delta(V + \delta S)$$

Since the underlying asset follows the geometric Brownian motion we have

$$\Delta S = \rho S \Delta t + \sigma S \Delta W, \tag{1.3}$$

where $\Delta W = W_{t+\Delta t} - W_t$ is the increment of the Wiener process. Now, assuming the change $\Delta\Pi$ in the portfolio is balanced by a bond with the risk-free rate $r \geq 0$, i.e. $\Delta\Pi = r\Pi\Delta$, using Ito's lemma for ΔV and applying the delta hedging strategy $\delta = -V_S$, we obtain generalization of the Black-Scholes equation:

applying G in Ito lemma to U and V we get the below expression and ignoring higher

order terms in the equation of portfolio value differential equation

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} S \right) dt$$

the above expression is similar to Ito's Lemma. From the point of default situation, we can make an assumption that the bondholder losses all future cash flows, which is

$$d\Pi = -V.$$

The expected value is

$$E(d\Pi) = \left(\frac{\partial V}{\partial t} + \frac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} S - r_c V \right) dt$$

Eventually, non-arbitrage arguments give us

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - (r + r_c)V = 0. \quad (1.4)$$

Equity component value $E = U - V$ shows that value of CB is connected with payments in equity, therefore, E satisfies the Black-Scholes equation

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \left(\frac{\partial^2 U}{\partial S^2} - \frac{\partial^2 V}{\partial S^2} \right) + rS \left(\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S} \right) - r(U - V) = 0. \quad (1.5)$$

Furthermore, substituting the equation (1.4) into (1.5) we obtain the following equation

$$\frac{\partial U}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS \frac{\partial U}{\partial S} - rU - r_c V = 0 \quad (1.6)$$

Finally, we have derived the following system (1.7) of equations for pricing convertible bonds, TF model. The model for pricing convertible bonds is presented below and first type of boundary and initial conditions are proposed to pricing convertible bonds with

dividend protection subject to credit risk:

$$\begin{cases} \frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + r_g S \frac{\partial U}{\partial S} - r(U - V) - (r + r_c)V = 0 \\ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_g S \frac{\partial V}{\partial S} - (r + r_c)V = 0 \end{cases} \quad (1.7)$$

where U is the value of the convertible bond, V is the value of COCB, S is the underlying stock price, r is the risk free rate, r_g is the growth rate which can be counted as risk free rate r by referring to [13] and r_c is the credit spread reflecting payoff default risk. σ is volatility. For numerical solution, we have the dataset from the Master thesis of Computer Science by Qingkai Mo, University of Toronto [13].

1.5 The boundary and terminal conditions

1.5.1 Main type of boundary and terminal conditions

In the following, only this type of boundary and terminal conditions [13] was considered to solve the problem numerically.

$$\begin{cases} U(S, T) = k^* S, & \text{for } k^* S \geq F \\ U(S, T) = F + C, & \text{otherwise} \end{cases}$$

$$\begin{cases} V(S, T) = 0, & \text{for } k^* S \geq F \\ V(S, T) = F + C, & \text{otherwise} \end{cases}$$

Upside constraint due to conversion of bonds:

$$U(S, T) \geq k^* S, \quad V(S, T) = 0 \quad \text{if } U(S, T) \leq k^* S.$$

Upside constraint due to callability (at call price V_c)

$$U(S, T) \leq \max(V_c, k^* S), \quad V(S, T) = 0 \quad \text{if } U(S, T) \geq B_c.$$

Downside constraint due to putability (at put price V_p)

$$U(S, t) \geq V_p, \quad V(S, t) = V_p \quad \text{if} \quad U(S, t) \leq V_p.$$

At $S = 0$, the boundary condition is presented as system of ordinary differential equations.

$$\begin{cases} \frac{\partial U(S, t)}{\partial t} - rU(S, t) - r_c V(S, t) = 0 \\ \frac{\partial V(S, t)}{\partial t} - (r + r_c)V(S, t) = 0 \end{cases}$$

As $S \rightarrow \infty$, the convertible bond is going to be converted into stock.

$$\begin{cases} U(S, t) = k^* S \\ V(S, t) = 0 \end{cases}$$

where k is conversion ratio, F is the redemption value of the bond and C is the coupon payment at maturity.

1.5.2 Additional type of boundary and expiry conditions

In the framework of Tsiveriotis and Fernandes model [16], authors have presented the following terminal conditions, upside and downside constraints. The key idea of using these conditions is that cash-only part of convertible bond independent of credit risk while the value of a convertible bond depends on the credit risk. Hence, framework leads to application of free boundary conditions, and conversion function has a specific role, such as examining the credit risk. Therefore, authors decided that boundary and initial conditions depend on the conversion function. This kind of framework with corresponding boundary and expiry conditions was solved by using the binomial-tree method [12] as well as finite difference method numerically.

We have the conversion function:

$$cnv(S, t) = \begin{cases} k^* S, & \text{if } t \text{ belongs to Conversion periods of the Contract,} \\ 0, & \text{otherwise.} \end{cases}$$

The Put Back function

$$V^{Put}(t) = \begin{cases} b(t), & \text{for } t \text{ belongs to Put periods of the Contract,} \\ 0, & \text{otherwise.} \end{cases}$$

The Call back function

$$V^{Call}(t) = \begin{cases} c(t), & \text{for } t \text{ belongs to Call periods of the Contract,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Expiry Conditions for $t=T$

$$U(S, T) = \max(cnv(S, T), N)$$

$$B(S, T) = \begin{cases} N, & cnv(S, T) \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

Boundary Conditions: for $S = 0$, $S = \infty$, when $S = 0$, $0 \leq t \leq T$

$$U(0, T) = \max(V^{Put}(t), U(t))$$

$$V(0, t) = \max(V^{Put}(t), V(t))$$

where $U(t) = V(t) = Ne^{-(r+r_c)(T-t)}$ and when $S \rightarrow \infty$: for $0 \leq t \leq T$

$$U(S, t) = cnv(S, T), \quad V(S, t) = 0$$

The payoff constraints:

$$U(S, t) = \max(V^{Put}(t), cnv(S, t), \min(V^{Call}(t), U_{held}(S, t)))$$

for $0 \leq S \leq +\infty$, $0 \leq t \leq T$, also we should notice that all conditions and constraints for zero coupon Convertible Bonds which is good enough for our present considerations.

1.6 Parameters for pricing the CB

As an numerical example, we adapt the data from [13] listed in Table 1.

Dataset of the Model for the CBs with dividends	
Parameters	Value
Time to maturity T	5 years
Conversion	0 to 5 years into k^* shares
Conversion ratio k^*	1.0
Face Value	100
Clean call price V_c	110 from 3 to 5 year
Clean put price V_p	105 during 3 year
Coupon payments C	\$4.0
Coupon dates	.5, 1.0, 1.5, ... ,5.0
Dividends D_i	2,3,4,4,4
Dividend dates	0, 1.0, 2.0, 3.0,4.0
Risk-free interest rate r	5% or 0.05
Credit risk r_c	2% or 0.02
Hazard rate P	2% or 0.02
Volatility σ	20% or 0.2
Underlying stock price at $t = 0(S_{int})$	100
Tolerance for penalty iteration tol	1.0e-06

Table 1.1: Parameters for the Pricing of Convertible Bonds

1.7 Research methods and expected results

Usually convertible bonds consists call option that give the issuer with the right to buy back the bond at a specified price before its maturity time. Also, the bonds may include put options that allow the holder to sell the bond back to issuer at a predetermined price in advance. The conversion ratio and the call (or put) price are often a function of time. The convertible may also have soft call features and the conversion value may

be restricted. However, we are planning these options for consideration in the future relevant works.

It is already known that $U(S, t)$ and $V(S, t)$ fully satisfy our equation (1.6), so it is confident decision in derivation that can make nice use of the equation for new functions $u(x, \tau)$ and $v(x, \tau)$ defined in terms of the $U(S, t)$ and $V(S, t)$. By using the Chain rule the partial derivatives $V_t, V_S, V_{SS}, U_t, U_S,$ and U_{SS} can be found. After the transformation, the next step is to apply the FEM to the derived system of parabolic equations (1.6) and obtain the numerical solution for our proposed TF model with dividend protection.

Chapter 2

Transformation of the TF system to a parabolic system

2.1 Reduced form of the TF model

Firstly, we use the change of variables to our system of equations (1.7) to obtain the system of parabolic equations. Furthermore, next step concentrates on doing numerical approach on the system of parabolic equations. Therefore, numerical results obtained by using finite element approach are going to be a solution for the main system of Black-Scholes equations. As mentioned before, change of variables must be applied to the system of equations (1.7). Hence, boundary, initial conditions, and notations for applying the change of variables are presented below.

2.2 Change of variables:

$$t = T - \frac{2\tau}{\sigma^2} \quad \text{or} \quad \tau = \frac{\sigma^2}{2}(T - t); \quad x = \log(S); \quad U(S, t) = u(x, \tau); \quad V(S, t) = v(x, \tau);$$

It can be seen that the parabolic equation on an infinite interval has the first derivative with respect to time t and has a second derivative with respect to S . Each variable is transformed via a corresponding change of variables. The target is to reduce the system of Black-Scholes equations and its conditions to the system of parabolic equations.

The First and Second derivatives are:

$$\frac{\partial U}{\partial t} = \frac{\partial u}{\partial t} \frac{\partial \tau}{\partial t} = \frac{\partial u}{\partial \tau} \frac{(-\sigma^2)}{2} \quad (2.1)$$

$$\frac{\partial U}{\partial S} = \frac{\partial u}{\partial x} \frac{dx}{dS} = \frac{\partial u}{\partial x} \frac{1}{S} \quad (2.2)$$

$$\begin{aligned} \frac{\partial^2 U}{\partial S^2} &= \frac{\partial}{\partial S} \left(\frac{\partial u}{\partial S} \right) = \frac{\partial}{\partial S} \left(\frac{\partial u}{\partial x} \frac{1}{S} \right) = \frac{\partial u}{\partial x} \frac{(-1)}{S^2} + \frac{\partial}{\partial S} \left(\frac{\partial u}{\partial x} \right) \frac{1}{S} = \\ &= \frac{\partial u}{\partial x} \frac{(-1)}{S^2} + \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) \frac{dx}{dS} \frac{1}{S} = \frac{\partial u}{\partial x} \frac{(-1)}{S^2} + \frac{\partial^2 u}{\partial x^2} \frac{1}{S^2} \end{aligned} \quad (2.3)$$

2.2.1 First transformation

The transformed initial and boundary conditions are:

$$\begin{cases} u(x, 0) = k^* e^x, & \text{for } ke^x \geq F \\ u(x, 0) = F + C, & \text{otherwise} \end{cases}$$

$$\begin{cases} v(x, 0) = 0, & \text{for } ke^x \geq F \\ v(x, 0) = F + C, & \text{otherwise} \end{cases}$$

At $x = x_{min}$,

$$\begin{cases} \frac{(-\sigma^2)}{2} \frac{\partial u(x, \tau)}{\partial \tau} - ru(x, \tau) - r_c v(x, \tau) = 0 \\ \frac{(-\sigma^2)}{2} \frac{\partial v(x, \tau)}{\partial \tau} - (r + r_c)v(x, \tau) = 0 \end{cases}$$

As $x = x_{max}$

$$\begin{cases} u(x, \tau) = k^* e^x \\ v(x, \tau) = 0 \end{cases}$$

$\gamma = -\frac{(A-1)^2}{4} - A - p$, as well as for initial and boundary conditions we have applied change of variables, thus, we obtained the following system of equations.

2.2.2 Second transformation

After second transformation equation (2.7) has changed to the (2.8), moreover, the boundary and initial conditions also get transformed.

$$\begin{cases} h_\tau = h_{xx} - pg \\ g_\tau = g_{xx} \end{cases} \quad (2.8)$$

Boundary and Initial conditions:

$$\begin{cases} h(x, 0) = k^* e^{x(\frac{A+1}{2})}, & \text{for } k^* e^x \geq F \\ h(x, 0) = (F + C) e^{x(\frac{A-1}{2})}, & \text{otherwise} \end{cases}$$

$$\begin{cases} g(x, 0) = 0, & \text{for } ke^x \geq F \\ g(x, 0) = (F + C) e^{x(\frac{A-1}{2})}, & \text{otherwise} \end{cases}$$

At $x = x_{min}$

$$\begin{cases} \left(\frac{\sigma^2}{2} \left(\frac{(A-1)^2}{4} + A \right) - r \right) h(x, \tau) - \frac{\sigma^2}{2} h_\tau(x, \tau) - r_c g(x, \tau) = 0 \\ \left(\frac{\sigma^2}{2} \left(\frac{(A-1)^2}{4} + A + p \right) - r - r_c \right) g(x, \tau) - \frac{\sigma^2}{2} g_\tau(x, \tau) = 0 \end{cases}$$

As $x = x_{max}$

$$\begin{cases} h(x, \tau) = k^* e^{\left(x \frac{(A+1)}{2} + \frac{(A-1)^2}{4} \tau \right)} \\ g(x, \tau) = 0 \end{cases}$$

Let us combine same elements of boundary conditions, so it has the following form, and there are coefficients such as

$$J_1 = \frac{\sigma^2}{2} \left(\frac{(A-1)^2}{4} + A \right) - r \quad \text{and} \quad J_2 = \frac{\sigma^2}{2} \left(\frac{(A-1)^2}{4} + A + p \right) - r - r_c$$

At $x = x_{min}$,

$$\begin{cases} J_1 h(x, \tau) - \frac{\sigma^2}{2} h_\tau(x, \tau) - r_c g(x, \tau) = 0 \\ J_2 g(x, \tau) - \frac{\sigma^2}{2} g_\tau(x, \tau) = 0 \end{cases}$$

As $x = x_{max}$

$$\begin{cases} h(x, \tau) = k^* e^{\left(x \frac{(A+1)}{2} + \frac{(A-1)^2}{4} \tau\right)} \\ g(x, \tau) = 0 \end{cases}$$

The boundary condition at $x = x_{min}$ is initial value problem and to solve that we used initial conditions at $x = x_{min}$ which has two sets of solution when initial condition $k^* e^{x_{min}} \geq 0$ and $k^* e^{x_{min}} < 0$.

when $k^* e^{x_{min}} = k^* e^{\ln(S_{min})} \geq 0$ the boundary conditions as following

$$\begin{cases} h(x_{min}, \tau) = 0, \\ g(x_{min}, \tau) = k^* e^{x_{min} \left(\frac{A+1}{2}\right)} e^{\frac{\tau}{2}}, \\ h(x_{min}, 0) = k^* e^{x_{min} \left(\frac{A+1}{2}\right)}, \\ g(x_{min}, 0) = 0 \end{cases}$$

when $k^* e^{x_{min}} = k^* e^{\ln(S_{min})} < 0$ the boundary conditions as following

$$\begin{cases} h(x_{min}, \tau) = (F + C) e^{x_{min} \left(\frac{A-1}{2}\right)} e^{\frac{\tau}{2}}, \\ g(x_{min}, \tau) = (F + C) e^{x_{min} \left(\frac{A-1}{2}\right)} e^{\frac{\tau}{2}} - \frac{T}{2} (F + C) e^{x_{min} \left(\frac{A-1}{2}\right)} e^{\frac{\tau}{2}}, \\ h(x_{min}, 0) = (F + C) e^{x_{min} \left(\frac{A-1}{2}\right)}, \\ g(x_{min}, 0) = (F + C) e^{x_{min} \left(\frac{A-1}{2}\right)} \end{cases}$$

$$\begin{cases} h(x_{max}, \tau) = k^* e^{\left(x_{max} \frac{(A+1)}{2} + \frac{(A-1)^2}{4} \tau\right)} \\ g(x_{max}, \tau) = 0 \end{cases}$$

Final form of TF model

After applying the change of variables to the system of BS equations, as a result, the following system of parabolic equations was obtained and boundary and initial conditions are in the transformed form too. The final version of the problem before we start the application in Finite Element Analysis is presented below.

$$\begin{cases} h_\tau = h_{xx} - pg \\ g_\tau = g_{xx} \end{cases} \quad (2.9)$$

$$\begin{cases} h(x, 0) = k^* e^{x(\frac{A+1}{2})}, & \text{for } k^* e^x \geq F \\ h(x, 0) = (F + C) e^{x(\frac{A-1}{2})}, & \text{otherwise} \end{cases}$$

$$\begin{cases} g(x, 0) = 0, & \text{for } k e^x \geq F \\ g(x, 0) = (F + C) e^{x(\frac{A-1}{2})}, & \text{otherwise} \end{cases}$$

Boundary conditions:

As mentioned before, the boundary conditions at the point x_{min} were implemented (Figure 2-1) in the MATLAB and the values of boundary conditions are obtained in order to obtain the numerical solution of the system of parabolic equations (2.9).

when $k^* e^{x_{min}} = k^* e^{\ln(S_{min})} \geq 0$ the boundary conditions as following

$$\begin{cases} h(x_{min}, \tau) = 0, \\ g(x_{min}, \tau) = k^* e^{x_{min}(\frac{A+1}{2})} e^{\frac{\tau}{2}}, \\ h(x_{min}, 0) = k^* e^{x_{min}(\frac{A+1}{2})}, \\ g(x_{min}, 0) = 0 \end{cases}$$

when $k^* e^{x_{min}} = k^* e^{\ln(S_{min})} < 0$ the boundary conditions as following

$$\begin{cases} h(x_{min}, \tau) = (F + C)e^{x_{min}(\frac{A-1}{2})} e^{\frac{\tau}{2}}, \\ g(x_{min}, \tau) = (F + C)e^{x_{min}(\frac{A-1}{2})} e^{\frac{\tau}{2}} - \frac{\tau}{2}(F + C)e^{x_{min}(\frac{A-1}{2})} e^{\frac{\tau}{2}}, \\ h(x_{min}, 0) = (F + C)e^{x_{min}(\frac{A-1}{2})}, \\ g(x_{min}, 0) = (F + C)e^{x_{min}(\frac{A-1}{2})} \end{cases}$$

$$\begin{cases} h(x_{max}, \tau) = k^* e^{\left(x_{max} \frac{(A+1)}{2} + \frac{(A-1)^2}{4} \tau\right)} \\ g(x_{max}, \tau) = 0 \end{cases}$$

Chapter 3

Well-posedness of the TF model

Let us consider the parabolic equation to illustrate the general steps of the proof of the well-posedness. The standard proof of the well-posedness of the standard parabolic equation was reviewed in this section [6].

3.1 Well-posedness of the standard parabolic equation

In order to prove well-posedness of the standard parabolic equations, the parabolic equation was changed from strong to the weak formulation and then make an analysis to identify the well-posedness.

$$u_t = u_{xx} + f(x, t)$$

with boundary and initial conditions

$$\begin{cases} u(x, 0) = A(x) \\ u(0, t) = B \\ u(l, t) = C \end{cases}$$

The function w is multiplied by both sides to obtain the weak formulation. Assume u_1 and u_2 are different solutions of the parabolic equation and by taking their subtraction, then weak formulation has the following form:

$$w(x, t) = u_1(x, t) - u_2(x, t)$$

$$w_t = w_{xx}$$

Weak formulation is presented below:

$$\int_0^l w_t w dx = \int_0^l w_{xx} w dx \quad (3.1)$$

Applying integration by parts to the RHS of (3.1) we get:

$$\frac{1}{2} \frac{d}{dt} \int_0^l w^2 dx = -\frac{1}{2} \int_0^l w_x^2 + w_x w|_0^l$$

$$\frac{1}{2} \frac{d}{dt} \int_0^l w^2 dx + \frac{1}{2} \int_0^l w_x^2 = 0$$

Let us assume that

$$\int_0^l w^2 dx = H(t), \quad H(t) \rightarrow 0, \quad \frac{1}{2} \frac{d}{dt} \int_0^l w^2 dx \leq 0 \quad \text{and} \quad \frac{1}{2} \int_0^l w_x^2 \geq 0$$

Finally, by making an analysis of the above assumptions it can be stated that the well-posedness of the solution is proved.

$$\frac{1}{2} \frac{dH(t)}{dt} \leq 0 \quad \text{so that} \quad H(t) \leq 0 \implies H(0) = 0 \implies H(t) = 0 \quad \text{for all } t.$$

3.2 The well-posedness of solution of the system of parabolic equations

In the original paper "Valuing Convertible Bonds "[16] authors stated that their model has a solution and it is well-posed referred to the book Pearson and Carrier [5], further-

more, in order to add details of the proof given by this paper we are going to do it as following. Now, we prove the existence of weak formulation of the system of equations in the same manner as for the standard parabolic equation

$$\begin{cases} h_\tau = h_{xx} - pg \\ g_\tau = g_{xx} \end{cases} \quad (3.2)$$

with boundary conditions:

At $x = x_{min}$,

$$\begin{cases} J_1 h(x, \tau) - \frac{\sigma^2}{2} h_\tau(x, \tau) - r_c g(x, \tau) = 0 \\ J_2 g(x, \tau) - \frac{\sigma^2}{2} g_\tau(x, \tau) = 0 \end{cases} \quad (3.3)$$

As $x = x_{max}$

$$\begin{cases} h(x, \tau) = ke^{\left(x \frac{(A+1)}{2} + \frac{(A-1)^2}{4} \tau\right)} \\ g(x, \tau) = 0 \end{cases} \quad (3.4)$$

our system can be written in the following vector form [18]

$$\vec{u}_\tau = L\vec{u} + \vec{f},$$

where

$$L\vec{u} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ g \end{bmatrix}_{xx} \text{ is differential operator, } \vec{u} = \begin{bmatrix} h \\ g \end{bmatrix}, \text{ and } \vec{f} = \begin{bmatrix} pg \\ 0 \end{bmatrix}$$

for given coefficients

$$a^{ij} = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

For considering \vec{w} , it is necessary to define \vec{u}_1 and \vec{u}_2

$$\vec{u}_1 = \begin{bmatrix} h_1 \\ g_1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} h_2 \\ g_2 \end{bmatrix}$$

Let us consider our solution in the following form:

$$\vec{w}_\tau = \vec{w}_{xx} \implies \vec{w} = \vec{u}_1 - \vec{u}_2$$

Before we prove existence of weak formulation of the system, let us present small part of theory according to this proof, let us denote $x_{min} = a$ and $x_{max} = b$:

$$\vec{u} \in L^2(0, \infty; H^1(a, b))$$

where H is Hilbert space, so

$$\vec{u} \in H^1(a, b) \times H^1(a, b)$$

where

$$H^1(a, b) = \{u | u, u_x \in L^2(a, b)\}$$

we now define two subspaces of $H^1(a, b)$ satisfying certain boundary conditions

$$H_b^1(a, b) = \{\vec{u} | \vec{u}, \vec{u}_x \in L^2(a, b), u \text{ satisfies the boundary conditions (3.3) and (3.4)}\}$$

$$H_0^1(a, b) = \{\vec{u} | \vec{u}, \vec{u}_x \in L^2(a, b), u \text{ satisfies the zero corresponding B.Cs (3.3) and (3.4)}\}$$

Let us now define the time dependent bilinear form and definition of weak solution of the system (2.9) by referring to the book [6]

$$B[\vec{u}, \vec{v}; \tau] := \int_a^b [L\vec{u}\vec{v}] dx$$

where

$$\vec{v} = \int_a^b (h_{xx}v_1 + g_{xx}v_2)dx = \int_a^b (h_xv_{1_x} + g_xv_{2_x})dx$$

We say a function

$$\vec{u} \in L^2(0, T; H_b^1(a, b) \times H_b^1(a, b)), \vec{v} \in H_0^1(a, b) \times H_0^1(a, b), \vec{u}_\tau \in L^2(0, T; H_b^{-1}(a, b) \times H_b^{-1}(a, b))$$

is a weak solution of the parabolic initial/boundary value problem (2.9) provided

$$(i) \int_a^b \vec{u}_\tau \vec{v} = \int_a^b (L\vec{u}\vec{v} + \vec{f}\vec{v}) \equiv -B[\vec{u}, \vec{v}; \tau] + \int_a^b \vec{f}\vec{v} \rightarrow \langle \vec{u}_\tau, \vec{v} \rangle + B[\vec{u}, \vec{v}; \tau] = \langle \vec{f}, \vec{v} \rangle$$

for each $\vec{v} \in H_0^1(a, b) \times H_0^1(a, b)$ and a.e time $0 \leq \tau \leq T$ and

(ii) $u(\vec{0}) = \vec{g}$. The weak formulation of (2.9) is to solve for $\vec{u} \in H_b^1(a, b) \times H_b^1(a, b)$, such that

$$\int_a^b \vec{u}_\tau \vec{v} dx = \int_a^b \vec{u}_{xx} \vec{v} dx + \int_a^b \vec{f} \vec{v} dx$$

where $\vec{v} \in H_0^1(a, b) \times H_0^1(a, b)$. Weak formulation is performed as following:

$$\int_{x_{min}}^{x_{max}} \vec{w}_\tau \cdot \vec{w} dx = \int_{x_{min}}^{x_{max}} \vec{w}_{xx} \cdot \vec{w} dx$$

$$\frac{1}{2} \frac{d}{d\tau} \int_{x_{min}}^{x_{max}} \vec{w} \cdot \vec{w} dx = - \int_{x_{min}}^{x_{max}} \vec{w}_x \cdot \vec{w}_x dx + \vec{w}_x \cdot \vec{w} \Big|_{x_{min}}^{x_{max}}$$

$$\vec{w} = \begin{bmatrix} h_1 \\ g_1 \end{bmatrix} - \begin{bmatrix} h_2 \\ g_2 \end{bmatrix} = \vec{u}_1 - \vec{u}_2$$

$$\frac{1}{2} \frac{d}{d\tau} \int_{x_{min}}^{x_{max}} \|\vec{w}\|^2 dx + \int_{x_{min}}^{x_{max}} \|\vec{w}_x\|^2 dx = \vec{w}_x \cdot \vec{w} \Big|_{x_{min}}^{x_{max}}$$

Applying the boundary conditions to the RHS of the above expression, we have to check the value of the expression at the point x_{max} , in order to prove the well-posedness of the system of Parabolic equations:

$$\vec{w}_x \cdot \vec{w} \Big|_{x_{max}} = \left[(h_1(x_{min}, \tau) - h_2(x_{min}, \tau))_x (h_1(x_{min}, \tau) - h_2(x_{min}, \tau)) + (g_1(x_{min}, \tau) - g_2(x_{min}, \tau))_x (g_1(x_{min}, \tau) - g_2(x_{min}, \tau)) \right] = 0$$

We can easily justify that if we apply boundary conditions at x_{max} above expression is going to be zero. So next step is checking at x_{min} :

$$\frac{1}{2} \frac{d}{d\tau} \int_{x_{min}}^{x_{max}} \|\vec{w}\|^2 dx + \int_{x_{min}}^{x_{max}} \|\vec{w}_x\|^2 dx = -\vec{w}_x \cdot \vec{w} \Big|_{x_{min}} \quad (3.5)$$

where

$$\begin{aligned} \vec{w}_x \cdot \vec{w} \Big|_{x_{min}} = & - \left[(h_1(x_{min}, \tau) - h_2(x_{min}, \tau))_x (h_1(x_{min}, \tau) - h_2(x_{min}, \tau)) + \right. \\ & \left. + (g_1(x_{min}, \tau) - g_2(x_{min}, \tau))_x (g_1(x_{min}, \tau) - g_2(x_{min}, \tau)) \right] \end{aligned}$$

Let us turn back to the boundary conditions in order to prove the above expressions, however, before we need to prove the well-posedness for the system of Boundary Conditions at the point x_{min} , so that we could finish the proof.

At $x = x_{min}$:

$$\begin{cases} \dot{h} = -\frac{2J_1}{\sigma^2} h - \frac{2r_c}{\sigma^2} g \\ \dot{g} = -\frac{2J_2}{\sigma^2} g \end{cases}$$

The system of equation can be expressed in the matrix form as following:

$$\begin{bmatrix} \dot{h} \\ \dot{g} \end{bmatrix} = \begin{bmatrix} -\frac{2J_1}{\sigma^2} & -\frac{2r_c}{\sigma^2} \\ -\frac{2J_2}{\sigma^2} & 0 \end{bmatrix} \begin{bmatrix} h \\ g \end{bmatrix}$$

then,

$$\dot{\mathbf{u}} = A\mathbf{u} \quad (3.6)$$

where,

$$\dot{\mathbf{u}} = \begin{bmatrix} \dot{h} \\ \dot{g} \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{2J_1}{\sigma^2} & -\frac{2r_c}{\sigma^2} \\ -\frac{2J_2}{\sigma^2} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} h \\ g \end{bmatrix}$$

Note that the coefficient matrix A is constant. So

$$\det(A) = \det \left(\begin{bmatrix} -\frac{2J_1}{\sigma^2} & -\frac{2r_c}{\sigma^2} \\ -\frac{2J_2}{\sigma^2} & 0 \end{bmatrix} \right) = -\frac{2J_1}{\sigma^2} * 0 - \left(\frac{2r_c}{\sigma^2} * \frac{2J_2}{\sigma^2} \right) = \frac{4J_2 r_c}{\sigma^4} \neq 0$$

It can be easily concluded that the matrix A is *non-singular* because its determinant is not equal to zero. Therefore, we can apply the following theory.

Theorem 3.2.1 (*Existence-Uniqueness for Constant Linear Systems*) [7]

Let A be an $m \times n$ non-singular matrix with constant entries and let τ_0 be any m -vector. Then the initial value problem

$$\dot{\mathbf{u}} = A\mathbf{u}, \quad \mathbf{u}(0) = \tau_0$$

has a unique solution $U(\tau)$ defined for all values of τ .

This can be completely confirmed from Theorem 1, by referring to the following book [7], that the system (3.3) has a solution and it is the unique solution, in this way, we can continue to prove the well-posedness for the system (2.9). Once we proved the existence and uniqueness for the (3.3), consequently, RHS of the (3.5) is going to be zero, because all solutions are unique, that's why their differences at the point x_{min} has to be zero. Which means:

$$\begin{aligned} \vec{w}_x \cdot \vec{w} \Big|_{x_{min}} &= - \left[(h_1(x_{min}, \tau) - h_2(x_{min}, \tau))_x (h_1(x_{min}, \tau) - h_2(x_{min}, \tau)) + \right. \\ &\left. + (g_1(x_{min}, \tau) - g_2(x_{min}, \tau))_x (g_1(x_{min}, \tau) - g_2(x_{min}, \tau)) \right] = 0 \end{aligned}$$

then equation (3.2) has taken the following form:

$$\frac{1}{2} \frac{d}{d\tau} \int_{x_{min}}^{x_{max}} \|\vec{w}\|^2 dx + \int_{x_{min}}^{x_{max}} \|\vec{w}_x\|^2 dx = 0 \quad (3.7)$$

$L^2(0, \infty; S_0^{h^*}(a, b) \times S_0^{h^*}(a, b))$ is semi-discrete subspace of $L^2(0, \infty; H^1(a, b) \times H^1(a, b))$, so

$$\vec{u}_h \in S_b^{h^*}(a, b) \times S_b^{h^*}(a, b) \subset H^1(a, b) \times H^1(a, b)$$

$$H_0^1(a, b) \times H_b^1(a, b) \subset H^1(a, b) \times H^1(a, b)$$

where h^* is the maximum of the length of the element interval, when $h^* \rightarrow 0$, then

$\vec{u}_{h^*} \rightarrow \vec{u}$, where $\vec{u} = \begin{bmatrix} h(x, \tau) \\ g(x, \tau) \end{bmatrix}$, is true solution and $\underline{u} = \begin{bmatrix} h_1(\tau) \\ \dots \\ g(\tau) \end{bmatrix}$ is nodal solution in

the finite element method.

$$\|\vec{u} - \vec{u}_{h^*}\| \leq ch^{*2}$$

Global equation has the following form:

$$\mathbf{M}\dot{\underline{u}} + \mathbf{K}\underline{u} - \mathbf{Q} = 0$$

$$\vec{u}_{h^*} = N^T \vec{u}_{h^*}^{(e)}$$

where N is the basis of $S_{h^*}(a, b)$ and $\vec{u}_{h^*}^{(e)}$ is local finite element solution.

Let us define that

$$\int_{x_{min}}^{x_{max}} \|\vec{w}\|^2 dx = H(\tau), \quad H(\tau) \rightarrow x_{min},$$

$$\frac{1}{2} \frac{d}{d\tau} \int_{x_{min}}^{x_{max}} \|\vec{w}\|^2 dx \leq 0 \quad \text{and} \quad \int_{x_{min}}^{x_{max}} \|\vec{w}_x\|^2 dx \geq 0$$

$$\frac{1}{2} \frac{dH(\tau)}{dt} \leq 0 \quad \text{so that} \quad H(\tau) \leq 0 \implies H(\tau) = 0 \implies H(\tau) = 0 \quad \text{for all } \tau.$$

Finally, we prove the well-posedness of the solution.

Chapter 4

Finite element method

In this chapter, we will develop the numerical solution of (2.9) with initial and boundary conditions described in Chapter 2 by using the Galerkin finite element method. We first present a weak formulation of the problem (2.9) and then we have developed finite element linear and quadratic local and global shape functions, Galerkin finite element formulation and local finite element matrix equations, connectivity, boundary, and initial conditions and assembling of the global system of ODEs. Finally, we apply the finite difference Crank-Nicolson method to solve the global system.

4.1 Introduction to local and global shape functions

Before looking into the Galerkin finite element method, we introduce some basic Lagrange finite element interpolation shape functions [11], which are used in solving the second-order equations. We discretize a finite interval $[a, b]$ by dividing into NE subintervals by a partition

$$a = x_1 < x_2 < \cdots < x_{NE} < x_{NE+1} = b$$

and denote a typical subinterval by $[x_1^{(e)}, x_2^{(e)}]$, where $e = 1, \dots, NE$. This subinterval denotes a linear finite element $\Omega^{(e)}$ of length $l^{(e)} = x_2^{(e)} - x_1^{(e)}$. The end points of this

subinterval are called the *global nodes*(NG) of the element. In order to achieve the general illustration, the linear and quadratic shape functions are chosen to solve the system (2.9) by using Galerkin finite element.

Linear element shape functions:

$$[a, b] = \bigcup_{e=1}^{NE} [x_1^{(e)}, x_2^{(e)}], \quad l^{(e)} = x_2^{(e)} - x_1^{(e)}, \quad h^* = \max_{e=1, \dots, NE} (l^{(e)})$$

$$x = \frac{1 - \xi}{2} x_1^{(e)} + \frac{1 + \xi}{2} x_2^{(e)} \leftrightarrow \xi = \frac{2x - (x_1^{(e)} + x_2^{(e)})}{l^{(e)}}$$

$$N_j^{(e)} \equiv N_j(\xi(x)), \quad x_1^{(e)} \leq x \leq x_2^{(e)}, \quad j = 1, 2$$

we use the following finite element approximation for linear elements

$$h_1^{(e)} = h_1(\tau), \quad h_2^{(e)} = g_1(\tau), \quad h_3^{(e)} = h_2(\tau), \quad h_4^{(e)} = g_2(\tau)$$

$$\begin{bmatrix} h^{(e)}(x, \tau) \\ g^{(e)}(x, \tau) \end{bmatrix} \cong \begin{bmatrix} N_1^{(e)} & 0 & N_2^{(e)} & 0 \\ 0 & N_1^{(e)} & 0 & N_2^{(e)} \end{bmatrix} \begin{bmatrix} h_1^{(e)} \\ h_2^{(e)} \\ h_3^{(e)} \\ h_4^{(e)} \end{bmatrix}, \quad x_1^{(e)} \leq x \leq x_2^{(e)}$$

The global piecewise linear interpolation functions associated with the below partition

$$0 = x_1 < x_2 < \dots < x_{NG}, \quad NE + 1 = NG$$

$$[x_1^{(1)}, x_2^{(1)}] = [x_1, x_2], \dots, [x_1^{(e)}, x_2^{(e)}] = [x_e, x_{e+1}], \dots, [x_1^{NE}, x_2^{NE}] = [x_{NE}, x_{NE+1}]$$

is defined by

$$h(x) = \sum_{e=1}^{NE} \chi^{(e)}(x) h^{(e)}(x, \tau) = \sum_{e=1}^{NE} \chi^{(e)}(x) [N_1^{(e)}(x) h_1^{(e)} + N_2^{(e)}(x) h_2^{(e)}]$$

For $e = 1, \dots, NE$. Denote $\mathbf{h} = [h_1, h_2, \dots, h_{2NE+2}]^T$ and $\mathbf{g} = [g_1, g_2, \dots, g_{2NE+2}]^T$ as the global nodal values, then the connection between the global nodal values and local

nodal values are

$$\begin{aligned} h_1 &= h_1^{(1)}, & h_3^{(1)} &= h_2 = h_1^{(2)}, \dots, h_3^{(j-1)} = h_j = h_1^{(j)}, \dots, h_3^{(NE)} = h_{NG}; \\ g_1 &= h_2^{(1)}, & h_4^{(1)} &= g_2 = h_2^{(2)}, \dots, h_4^{(j-1)} = g_j = h_2^{(j)}, \dots, h_4^{(NE)} = g_{NG}; \\ && & \text{for } e = 1, \dots, NE, \quad \text{where } NG = NE + 1 \end{aligned}$$

So the e th row of the connectivity matrix C for the linear elements can be defined by

$$C(e, NL) = [e, e + 1], \quad e = 1, 2, \dots, NE$$

Then the global linear shape functions are defined by

$$h(x, \tau) = \sum_{j=1}^{NG} N_j(x) h_{2j-1}(\tau) \quad (4.1)$$

$$g(x, \tau) = \sum_{j=1}^{NG} N_j(x) g_{2j}(\tau) \quad (4.2)$$

where $N_j(x) = N_i^{(e)}(x)$ for x in the e th element interval where $j = C(e, i)$ for some $i = 1, 2$ for linear elements. Let us consider local and global quadratic shape functions. Suppose a finite element $\Omega^{(e)}$ that consists of the end points $x_1^{(e)}$ and $x_2^{(e)}$ and the midpoint with $x_3^{(e)}$ such that $x_3^{(e)} = x_1^{(e)} + \frac{l^{(e)}}{2}$, where $l^{(e)} = x_2^{(e)} - x_1^{(e)}$ denotes the length of the element.

Quadratic element shape functions:

$$[a, b] = \bigcup_{e=1}^{NE} [x_1^{(e)}, x_3^{(e)}], \quad l^e = x_3^{(e)} - x_1^{(e)} \quad x_2^{(e)} = \frac{x_1^{(e)} + x_3^{(e)}}{2}$$

$$x = \frac{1 - \xi}{2} x_1^{(e)} + \frac{1 + \xi}{2} x_3^{(e)} \leftrightarrow \xi = \frac{2x - (x_1^{(e)} + x_3^{(e)})}{l^{(e)}}$$

$$N_j^{(e)} \equiv N_j(\xi(x)), \quad x_1^{(e)} \leq x \leq x_3^{(e)}, \quad j = 1, 2, 3$$

we use the following finite element approximation for quadratic elements

$$h_1^{(e)} = h_1(\tau), h_2^{(e)} = g_1(\tau), h_3^{(e)} = h_2(\tau), h_4^{(e)} = g_2(\tau), h_5^{(e)} = h_3(\tau), h_6^{(e)} = g_3(\tau)$$

$$\begin{bmatrix} h^{(e)}(x, \tau) \\ g^{(e)}(x, \tau) \end{bmatrix} \cong \begin{bmatrix} N_1^{(e)} & 0 & N_2^{(e)} & 0 & N_3^{(e)} & 0 \\ 0 & N_1^{(e)} & 0 & N_2^{(e)} & 0 & N_3^{(e)} \end{bmatrix} \begin{bmatrix} h_1^{(e)} \\ h_2^{(e)} \\ h_3^{(e)} \\ h_4^{(e)} \\ h_5^{(e)} \\ h_6^{(e)} \end{bmatrix}, \quad x_1^{(e)} \leq x \leq x_3^{(e)}$$

where $l^{(e)}$ is the length of e-th subinterval.

Global piecewise quadratic interpolation function is

$$0 = x_1 < x_2 < \cdots < x_{NG}, \quad NE + 1 = NG$$

$$[x_1^{(1)}, x_2^{(1)}] = [x_1, x_2], \dots, [x_1^{(e)}, x_2^{(e)}] = [x_e, x_{e+1}], \dots, [x_1^{NE}, x_2^{NE}] = [x_{NE}, x_{NE+1}]$$

is defined by

$$h(x) = \sum_{e=1}^{NE} \chi^{(e)}(x) h^{(e)}(x, \tau) = \sum_{e=1}^{NE} \chi^{(e)}(x) [N_1^{(e)}(x) h_1^{(e)} + N_2^{(e)}(x) h_2^{(e)} + N_3^{(e)}(x) h_3^{(e)}]$$

For $e = 1, \dots, NE$. Denote $\mathbf{h} = [h_1, h_2, \dots, h_{2NE+2}]^T$ and $\mathbf{g} = [g_1, g_2, \dots, g_{2NE+2}]^T$ as the global nodal values, then the connection between the global nodal values and local nodal values are

$$h_1 = h_1^{(1)}, \quad h_3^{(1)} = h_2 = h_1^{(2)}, \quad h_5^{(1)} = h_3 = h_2^{(2)}, \dots, h_5^{(j-1)} = h_j = h_2^{(j)}, \dots, h_5^{(NE)} = h_{NG};$$

$$g_1 = h_2^{(1)}, \quad h_4^{(1)} = g_2 = h_2^{(2)}, \quad h_6^{(1)} = g_3 = h_3^{(2)}, \dots, h_6^{(j-1)} = g_j = h_3^{(j)}, \dots, h_6^{(NE)} = g_{NG};$$

for $e = 1, \dots, NE$, where $NG = NE + 1$

So the eth row of the connectivity matrix C for the quadratic elements can be defined by

$$C(e, 3) = [e, e + 1, e + 2], \quad e = 1, \dots, NE$$

Then the global quadratic shape functions are defined by

$$h(x, \tau) = \sum_{j=1}^{NG} N_j(x) h_{2j-1}(\tau) \quad (4.3)$$

$$g(x, \tau) = \sum_{j=1}^{NG} N_j(x) h_{2j}(\tau) \quad (4.4)$$

where $N_j(x) = N_i^{(e)}(x)$ for x in the e th element interval where $j = C(e, i)$ for some $i = 1, 2, 3$ for quadratic elements, otherwise $N_j(x) = 0$

4.2 Weak formulation

To construct the finite element approximation of the solution of the weak formulation, a subspace of the solution space V is constructed by piecewise polynomial functions over the interval $[x_{min}, x_{max}]$. Linear and quadratic interpolation functions are considered. In the following the finite element interpolation subspace constructed is denoted by $S^{h^*}(a, b)$. Corresponding to V , we have $S_0^{h^*}(a, b)$. The weak formulation can be obtained by multiplying both sides of the differential equation by test functions and integrating by parts over the interval $[a, b]$. The finite element approximation of the weak formulation is to solve for $\vec{u}_{h^*} \in S_{b_h}^{h^*}(a, b) \times S_{b_g}^{h^*}(a, b)$, such that

$$\int_a^b \vec{u}_\tau \cdot \vec{v} dx = \int_a^b \vec{u}_{xx} \cdot \vec{v} dx + \int_a^b \vec{f} \cdot \vec{v} dx$$

where $\vec{v} \in S_{0_h}^{h^*}(a, b) \times S_{0_g}^{h^*}(a, b)$. Now let us define finite element subspaces for h and g respectively:

$$S_{b_h}^{h^*}(a, b) = \{h | h(x, \tau) = \sum_{e=1}^{NE} \chi^{(e)}(x) h^{(e)}(x, \tau), h \text{ satisfies the B.Cs (3.3) and (3.4)}\} \subset H_{b_h}^1(a, b)$$

$$S_{b_g}^{h^*}(a, b) = \{g | g(x, \tau) = \sum_{e=1}^{NE} \chi^{(e)}(x) g^{(e)}(x, \tau), g \text{ satisfies the B.Cs (3.3) and (3.4)}\} \subset H_{b_g}^1(a, b)$$

$$S_{0_h}^{h^*}(a, b) = \{h|h(x, \tau) = \sum_{e=1}^{NE} \chi^{(e)}(x)h^{(e)}(x, \tau), h \text{ satisfies the zero B.Cs (3.3) and (3.4)}\} \subset H_{0_h}^1(a, b)$$

$$S_{0_g}^{g^*}(a, b) = \{g|g(x, \tau) = \sum_{e=1}^{NE} \chi^{(e)}(x)g^{(e)}(x, \tau), g \text{ satisfies the zero B.Cs (3.3) and (3.4)}\} \subset H_{0_g}^1(a, b)$$

where local true values of h and g with the linear shape functions presented below

$$h^{(e)}(x, \tau) = N_1^{(e)}(x)h_1^{(e)}(\tau) + N_2^{(e)}(x)h_2^{(e)}(\tau), \quad x \in [x_1^{(e)}, x_2^{(e)}] \quad (4.5)$$

$$g^{(e)}(x, \tau) = N_1^{(e)}(x)g_1^{(e)}(\tau) + N_2^{(e)}(x)g_2^{(e)}(\tau), \quad x \in [x_1^{(e)}, x_2^{(e)}] \quad (4.6)$$

also, local true values of h, g with the quadratic shape functions presented below

$$h^{(e)}(x, \tau) = N_1^{(e)}(x)h_1^{(e)}(\tau) + N_2^{(e)}(x)h_2^{(e)}(\tau) + N_3^{(e)}(x)h_3^{(e)}(\tau), \quad x \in [x_1^{(e)}, x_3^{(e)}] \quad (4.7)$$

$$g^{(e)}(x, \tau) = N_1^{(e)}(x)g_1^{(e)}(\tau) + N_2^{(e)}(x)g_2^{(e)}(\tau) + N_3^{(e)}(x)g_3^{(e)}(\tau) \quad x \in [x_1^{(e)}, x_3^{(e)}] \quad (4.8)$$

and characteristic function

$$\chi^{(e)}(x) = \begin{cases} 1, & \text{if } x \in [x_1^{(e)}, x_2^{(e)}] \\ 0, & \text{if } x \notin [x_1^{(e)}, x_2^{(e)}] \end{cases}$$

Let us discretize an interval from a to b by NG intervals ($NG \geq 1$) period $N_i^{(e)}$ is the shape function which is defined in the e -th interval: $x_1^{(e)} < x < x_2^{(e)}$; $h_i^{(e)}$ and $g_i^{(e)}$ are the nodal values at τ .

$$S^{h^*}(a, b) = \text{span}\{N_j(x) : j = 2, \dots, NG - 1\} \subset H^1(a, b)$$

where N_i is the global shape functions. For system of equations

$$S^{h^*}(a, b) \times S^{g^*}(a, b) \subset H^1(a, b) \times H^1(a, b)$$

In the Galerkin finite element method, we multiply both sides of system of equations(2.9) by test functions $N_j(x)$ and integrate over the interval $[x_2^{(e)}, x_1^{(e)}]$, then it

has the following form:

$$-\int_{x_1^{(e)}}^{x_2^{(e)}} h_x^{(e)} N_{j,x}^{(e)}(x) dx = \int_{x_1^{(e)}}^{x_2^{(e)}} (h_\tau^{(e)} + pg^{(e)}) N_j^{(e)}(x) dx - h_x N_j^{(e)}(x) \Big|_{x_1^{(e)}}^{x_2^{(e)}} \quad (4.9)$$

$$-\int_{x_1^{(e)}}^{x_2^{(e)}} g_x^{(e)} N_{j,x}^{(e)}(x) dx = \int_{x_1^{(e)}}^{x_2^{(e)}} (g_\tau^{(e)}) N_j^{(e)}(x) dx - g_x N_j^{(e)}(x) \Big|_{x_1^{(e)}}^{x_2^{(e)}} \quad (4.10)$$

$$\Rightarrow \sum_{i=1}^{NG} K_{ij} h_i + \sum_{i=1}^{NG} M_{ij} \dot{h}_i - Q_j = 0, \quad i, j = 1, \dots, NG \quad (4.11)$$

$$\Rightarrow \sum_{i=1}^{NG} K_{ij} g_i + \sum_{i=1}^{NG} M_{ij} \dot{g}_i - Q_j = 0, \quad i, j = 1, \dots, NG \quad (4.12)$$

The local mass and stiffness matrices written below corresponding to (4.9),(4.10). Load vectors are presented as well.

$$K_{ij}^{(e)} = \int_{x_1^{(e)}}^{x_2^{(e)}} N_{i,x}^{(e)} N_{j,x}^{(e)} dx, \quad M_{ij}^{(e)} = \int_{x_1^{(e)}}^{x_2^{(e)}} N_i^{(e)} N_j^{(e)} dx, \quad (4.13)$$

$$Q_j^{(e)} = h_x N_j^{(e)} \Big|_{x_1^{(e)}}^{x_2^{(e)}}$$

$i, j = 1, 2$ for the two-node linear element and $i, j = 1, 2, 3$ for the three-node quadratic element, for $K^{(e)}, M^{(e)}$ respectively. We know that $j = 1, \dots, NG$ and the number unknown values is $NG + 4$, because we have unknown load vectors from each equation. The local finite element matrix equation for the system of equations is

$$K^{(e)} h^{(e)} + M^{(e)} \dot{h}^{(e)} - Q^{(e)} = 0 \quad (4.14)$$

The following diagram is a simple example of how elements of local mass and stiffness matrices map to the global ones.

For stiffness matrix

$$\begin{aligned}
k_{11}^{(e)} &\rightarrow k_{2e-1,2e-1}, & k_{12}^{(e)} &\rightarrow k_{2e-1,2e}, & k_{13}^{(e)} &\rightarrow k_{2e-1,2e+1}, & k_{14}^{(e)} &\rightarrow k_{2e-1,2e+2} \\
k_{21}^{(e)} &\rightarrow k_{2e,2e-1}, & k_{22}^{(e)} &\rightarrow k_{2e,2e}, & k_{23}^{(e)} &\rightarrow k_{2e,2e+1}, & k_{24}^{(e)} &\rightarrow k_{2e,2e+2} \\
k_{31}^{(e)} &\rightarrow k_{2e+1,2e-1}, & k_{32}^{(e)} &\rightarrow k_{2e+1,2e}, & k_{33}^{(e)} &\rightarrow k_{2e+1,2e+1}, & k_{34}^{(e)} &\rightarrow k_{2e+1,2e+2} \\
k_{41}^{(e)} &\rightarrow k_{2e+2,2e-1}, & k_{42}^{(e)} &\rightarrow k_{2e+2,2e}, & k_{43}^{(e)} &\rightarrow k_{2e+2,2e+1}, & k_{44}^{(e)} &\rightarrow k_{2e+2,2e+2}
\end{aligned}$$

For mass matrix

$$\begin{aligned}
m_{11}^{(e)} &\rightarrow m_{2e-1,2e-1}, & m_{12}^{(e)} &\rightarrow m_{2e-1,2e}, & m_{13}^{(e)} &\rightarrow m_{2e-1,2e+1}, & m_{14}^{(e)} &\rightarrow m_{2e-1,2e+2} \\
m_{21}^{(e)} &\rightarrow m_{2e,2e-1}, & m_{22}^{(e)} &\rightarrow m_{2e,2e}, & m_{23}^{(e)} &\rightarrow m_{2e,2e+1}, & m_{24}^{(e)} &\rightarrow m_{2e,2e+2} \\
m_{31}^{(e)} &\rightarrow m_{2e+1,2e-1}, & m_{32}^{(e)} &\rightarrow m_{2e+1,2e}, & m_{33}^{(e)} &\rightarrow m_{2e+1,2e+1}, & m_{34}^{(e)} &\rightarrow m_{2e+1,2e+2} \\
m_{41}^{(e)} &\rightarrow m_{2e+2,2e-1}, & m_{42}^{(e)} &\rightarrow m_{2e+2,2e}, & m_{43}^{(e)} &\rightarrow m_{2e+2,2e+1}, & m_{44}^{(e)} &\rightarrow m_{2e+2,2e+2}
\end{aligned}$$

Using the above connectivity diagram we generate the system of global finite element equations as in the following form:

$$\mathbf{K}h + \mathbf{M}\dot{h}_\tau - \mathbf{Q} = 0 \tag{4.15}$$

where

$$\begin{aligned}
\mathbf{K}_{ij} &= \int_a^b N_{i,x}N_{j,x} dx, & \mathbf{M}_{ij} &= \int_a^b N_iN_j dx, \\
\mathbf{Q}_j &= h_x N_j \Big|_a^b
\end{aligned} \tag{4.16}$$

for $i, j = 1, \dots, NG$, where \mathbf{M} is global mass matrix, \mathbf{K} is global stiffness matrix, and \mathbf{Q} is global load vector.

Local finite element matrix equations for linear and quadratic element

$$\begin{aligned}
 & \frac{1}{l^{(e)}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_1^{(e)}(\tau) \\ h_2^{(e)}(\tau) \\ h_3^{(e)}(\tau) \\ h_4^{(e)}(\tau) \end{bmatrix} + \frac{pl^{(e)}}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} h_1^{(e)}(\tau) \\ h_2^{(e)}(\tau) \\ h_3^{(e)}(\tau) \\ h_4^{(e)}(\tau) \end{bmatrix} + \\
 & + \frac{l^{(e)}}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} \dot{h}_1^{(e)}(\tau) \\ \dot{h}_2^{(e)}(\tau) \\ \dot{h}_3^{(e)}(\tau) \\ \dot{h}_4^{(e)}(\tau) \end{bmatrix} - \begin{bmatrix} -h_x(x_1^{(e)}, \tau) \\ -g_x(x_1^{(e)}, \tau) \\ h_x(x_2^{(e)}, \tau) \\ g_x(x_2^{(e)}, \tau) \end{bmatrix} = 0
 \end{aligned} \tag{4.17}$$

$$\begin{aligned}
 & \frac{1}{3l^{(e)}} \begin{bmatrix} 7 & 0 & -8 & 0 & 1 & 0 \\ 0 & 7 & 0 & -8 & 0 & 1 \\ -8 & 0 & 16 & 0 & -8 & 0 \\ 0 & -8 & 0 & 16 & 0 & -8 \\ 1 & 0 & -8 & 0 & 7 & 0 \\ 0 & 1 & 0 & -8 & 0 & 7 \end{bmatrix} \begin{bmatrix} h_1^{(e)}(\tau) \\ h_2^{(e)}(\tau) \\ h_3^{(e)}(\tau) \\ h_4^{(e)}(\tau) \\ h_5^{(e)}(\tau) \\ h_6^{(e)}(\tau) \end{bmatrix} + \frac{pl^{(e)}}{60} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 16 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} h_1^{(e)}(\tau) \\ h_2^{(e)}(\tau) \\ h_3^{(e)}(\tau) \\ h_4^{(e)}(\tau) \\ h_5^{(e)}(\tau) \\ h_6^{(e)}(\tau) \end{bmatrix} +
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{l^{(e)}}{30} \begin{bmatrix} 4 & 0 & 2 & 0 & -1 & 0 \\ 0 & 4 & 0 & 2 & 0 & -1 \\ 2 & 0 & 16 & 0 & 2 & 0 \\ 0 & 2 & 0 & 16 & 0 & 2 \\ -1 & 0 & 2 & 0 & 4 & 0 \\ 0 & -1 & 0 & 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} \dot{h}_1^{(e)}(\tau) \\ \dot{h}_2^{(e)}(\tau) \\ \dot{h}_3^{(e)}(\tau) \\ \dot{h}_4^{(e)}(\tau) \\ \dot{h}_5^{(e)}(\tau) \\ \dot{h}_6^{(e)}(\tau) \end{bmatrix} - \begin{bmatrix} -h_x(x_{min}, \tau) \\ -g_x(x_{min}, \tau) \\ 0 \\ 0 \\ h_x(x_{max}, \tau) \\ g_x(x_{max}, \tau) \end{bmatrix} = 0
 \end{aligned} \tag{4.18}$$

4.3 Assembling of global finite element system

At the end of the sections 4.4 and 4.5 it was derived local linear (4.17) and quadratic (4.18) finite element systems respectively, therefore, these finite element systems is going to be combined and as a result there will be obtained global finite element system:

$$\begin{aligned}
& \frac{1}{3} \begin{bmatrix} \frac{1}{3l^{(1)}} & 0 & \frac{-1}{3l^{(1)}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3l^{(1)}} & 0 & \frac{-1}{3l^{(1)}} & 0 & 0 & 0 & 0 \\ \frac{-1}{3l^{(1)}} & 0 & \frac{1}{3l^{(1)}} + \frac{7}{l^{(2)}} & 0 & \frac{-8}{l^{(2)}} & 0 & \frac{1}{l^{(2)}} & 0 \\ 0 & \frac{-1}{3l^{(1)}} & 0 & \frac{1}{3l^{(1)}} + \frac{7}{l^{(2)}} & 0 & \frac{-8}{l^{(2)}} & 0 & \frac{1}{l^{(2)}} \\ 0 & 0 & \frac{-8}{l^{(2)}} & 0 & \frac{16}{l^{(2)}} & 0 & \frac{-8}{l^{(2)}} & 0 \\ 0 & 0 & 0 & \frac{-8}{l^{(2)}} & 0 & \frac{16}{l^{(2)}} & 0 & \frac{-8}{l^{(2)}} \\ 0 & 0 & \frac{1}{l^{(2)}} & 0 & \frac{-8}{l^{(2)}} & 0 & \frac{7}{l^{(2)}} & 0 \\ 0 & 0 & 0 & \frac{1}{l^{(2)}} & 0 & \frac{-8}{l^{(2)}} & 0 & \frac{7}{l^{(2)}} \end{bmatrix} \begin{bmatrix} h_1(\tau) \\ h_2(\tau) \\ h_3(\tau) \\ h_4(\tau) \\ h_5(\tau) \\ h_6(\tau) \\ h_7(\tau) \\ h_8(\tau) \end{bmatrix} + \\
& + \frac{p}{60} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2l^{(1)}}{10} & 0 & \frac{l^{(1)}}{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{l^{(1)}}{10} & 0 & \frac{2l^{(1)}}{10} + 4l^{(2)} & 0 & 2l^{(2)} & 0 & -l^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2l^{(2)} & 0 & 16l^{(2)} & 0 & 2l^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -l^{(2)} & 0 & 2l^{(2)} & 0 & 4l^{(2)} \end{bmatrix} \begin{bmatrix} h_1(\tau) \\ h_2(\tau) \\ h_3(\tau) \\ h_4(\tau) \\ h_5(\tau) \\ h_6(\tau) \\ h_7(\tau) \\ h_8(\tau) \end{bmatrix} + \\
& + \frac{1}{30} \begin{bmatrix} \frac{2l^{(1)}}{5} & 0 & \frac{l^{(1)}}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2l^{(1)}}{5} & 0 & \frac{l^{(1)}}{5} & 0 & 0 & 0 & 0 \\ \frac{l^{(1)}}{5} & 0 & \frac{2l^{(1)}}{5} + 4l^{(2)} & 0 & 2l^{(2)} & 0 & -l^{(2)} & 0 \\ 0 & \frac{l^{(1)}}{5} & 0 & \frac{2l^{(1)}}{5} + 4l^{(2)} & 0 & 2l^{(2)} & 0 & -l^{(2)} \\ 0 & 0 & 2l^{(2)} & 0 & 16l^{(2)} & 0 & 2l^{(2)} & 0 \\ 0 & 0 & 0 & 2l^{(2)} & 0 & 16l^{(2)} & 0 & 2l^{(2)} \\ 0 & 0 & -l^{(2)} & 0 & 2l^{(2)} & 0 & 4l^{(2)} & 0 \\ 0 & 0 & 0 & -l^{(2)} & 0 & 2l^{(2)} & 0 & 4l^{(2)} \end{bmatrix} \begin{bmatrix} h_1(\dot{\tau}) \\ h_2(\dot{\tau}) \\ h_3(\dot{\tau}) \\ h_4(\dot{\tau}) \\ h_5(\dot{\tau}) \\ h_6(\dot{\tau}) \\ h_7(\dot{\tau}) \\ h_8(\dot{\tau}) \end{bmatrix} - \begin{bmatrix} -h_x(x_{min}, \tau) \\ -g_x(x_{min}, \tau) \\ 0 \\ 0 \\ 0 \\ 0 \\ h_x(x_{max}, \tau) \\ g_x(x_{max}, \tau) \end{bmatrix} = 0
\end{aligned} \tag{4.19}$$

The global finite element system for 1 linear and 1 quadratic element as following

$$\begin{aligned}
& \begin{bmatrix} \frac{1}{9l^{(1)}} + \frac{7}{3l^{(2)}} & 0 & \frac{-8}{3l^{(2)}} & 0 \\ 0 & \frac{1}{9l^{(1)}} + \frac{7}{3l^{(2)}} + \frac{2pl^{(1)}}{600} + \frac{4pl^{(2)}}{60} & 0 & \frac{-8}{3l^{(2)}} + \frac{l^{(2)}}{30} \\ \frac{-8}{3l^{(2)}} & 0 & \frac{16}{3l^{(2)}} & 0 \\ 0 & \frac{-8}{3l^{(2)}} + \frac{l^{(2)}}{30} & 0 & \frac{16}{3l^{(2)}} + \frac{8pl^{(2)}}{30} \end{bmatrix} \begin{bmatrix} h_3(\tau) \\ h_4(\tau) \\ h_5(\tau) \\ h_6(\tau) \end{bmatrix} \\
& + \frac{1}{30} \begin{bmatrix} \frac{2l^{(1)}}{5} + 4l^{(2)} & 0 & 2l^{(2)} & 0 \\ 0 & \frac{2l^{(1)}}{5} + 4l^{(2)} & 0 & 2l^{(2)} \\ 2l^{(2)} & 0 & 16l^{(2)} & 0 \\ 0 & 2l^{(2)} & 0 & 16l^{(2)} \end{bmatrix} \begin{bmatrix} \dot{h}_3(\tau) \\ \dot{h}_4(\tau) \\ \dot{h}_5(\tau) \\ \dot{h}_6(\tau) \end{bmatrix} = \tag{4.20} \\
& = \begin{bmatrix} \frac{-1}{9l^{(1)}} h_1(\tau) \\ (\frac{-1}{9l^{(1)}} + \frac{pl^{(1)}}{600}) h_2(\tau) \\ \frac{-8}{3l^{(2)}} h_7(\tau) \\ (\frac{-8}{3l^{(2)}} + \frac{pl^{(2)}}{30}) h_8(\tau) \end{bmatrix} + \frac{1}{30} \begin{bmatrix} \frac{l^{(1)}}{5} \dot{h}_1(\tau) \\ \frac{l^{(1)}}{5} \dot{h}_2(\tau) \\ l^{(2)} \dot{h}_7(\tau) \\ 2l^{(2)} \dot{h}_8(\tau) \end{bmatrix}
\end{aligned}$$

Initial conditions for each x_2, x_3 and some coefficients are presented below. Boundary conditions are here to substitute the corresponding nodal values in global system.

$$\begin{cases} h_3(\tau) = h(x_2, 0) = k^* e^{x_2(\frac{A+1}{2})}, & \text{for } k^* e^{x_2} \geq F \\ h_3(\tau) = h(x_2, 0) = (F + C) e^{x_2(\frac{A-1}{2})}, & \text{otherwise} \end{cases}$$

$$\begin{cases} h_4(\tau) = g(x_2, 0) = 0, & \text{for } ke^{x_2} \geq F \\ h_4(\tau) = g(x_2, 0) = (F + C) e^{x_2(\frac{A-1}{2})}, & \text{otherwise} \end{cases}$$

$$\begin{cases} h_5(\tau) = h(x_3, 0) = k^* e^{x_3(\frac{A+1}{2})}, & \text{for } k^* e^{x_3} \geq F \\ h_5(\tau) = h(x_3, 0) = (F + C) e^{x_3(\frac{A-1}{2})}, & \text{otherwise} \end{cases}$$

$$\begin{cases} h_6(\tau) = g(x_3, 0) = 0, & \text{for } ke^{x_3} \geq F \\ h_6(\tau) = g(x_3, 0) = (F + C) e^{x_3(\frac{A-1}{2})}, & \text{otherwise} \end{cases}$$

Boundary conditions to substitute the corresponding nodal values in global system

$$\left\{ \begin{array}{l} h_1(\tau) = h(x_{min}, \tau) = \frac{e^{\tau/2}}{50} \\ h_2(\tau) = g(x_{min}, \tau) = \frac{e^{\tau/2}}{50} - (te^{\tau/2})/100 \\ h_7(\tau) = h(x_{max}, \tau) = e^{(\ln(140)(\frac{A+1}{2}) + \frac{(A-1)^2}{4}\tau)} \\ h_8(\tau) = g(x_{max}, \tau) = 0 \\ \dot{h}_2(\tau) = \dot{g}(x_{min}, \tau) = -\frac{\tau e^{\tau/2}}{200} \\ \dot{h}_1(\tau) = \dot{h}(x_{min}, \tau) = \frac{e^{\tau/2}}{100} \\ \dot{h}_7(\tau) = \dot{h}(x_{max}, \tau) = \frac{(A-1)^2}{4} e^{(\ln(140)(\frac{A+1}{2}) + \frac{(A-1)^2}{4}\tau)} \\ \dot{h}_8(\tau) = \dot{g}(x_{max}, \tau) = 0 \end{array} \right.$$

Finite element parameters and denoted coefficients

$$l^{(1)} = l^{(2)} = (x_{max} - x_{min})/2 = 0.62, \quad x_{min} = \ln(S_{min}) = \ln(40), \quad A = 2.5$$

$$x_{max} = \ln(S_{max}) = \ln(140), \quad k^* = 1, \quad p = 1$$

The derived global finite element system (4.20) can be solved by using MATLAB code by implementing Crank-Nicolson finite difference scheme. In Chapter 5 the numerical results presented in details.

Chapter 5

Numerical results

5.1 Numerical solution of the TF model

As described in the previous chapters, the main goal of this thesis was to make finite element analysis to the system of two-coupled Black-Scholes equations which was used for pricing convertible bonds with dividends. For obtaining the numerical results on the problem of convertible bonds, it is necessary to identify our boundary and initial conditions. The key idea of the thesis is that all previous scientific works which are engaged in this problem of convertible bonds were concentrated in obtaining the numerical results by using finite difference method while our vector of movement is using the finite element analysis in space for this task. In chapter 2, the initial conditions were calculated by using the given parameters (Table 1) and boundary conditions at x_{min} was solved in order to use them for solving the finite element system. The whole process of implementation and numerical calculations were done in MATLAB.

The value of convertible and cash-only convertible bonds by pricing under the TF model with dividends was produced using the finite element method. Some numerical examples was done by a combination of linear and quadratic elements which provides the smooth curvature in the illustration part and accurate results at the end.

Let us present some numerical examples of pricing convertible bonds by using 2 quadratic and 1 linear elements:

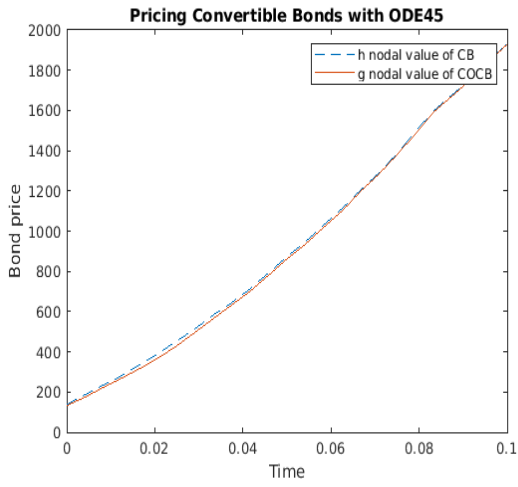


Figure 5-1: Nodal value of CB and COCB when $x_1 = \log(60\$)$, $0 < \tau < 0.1$

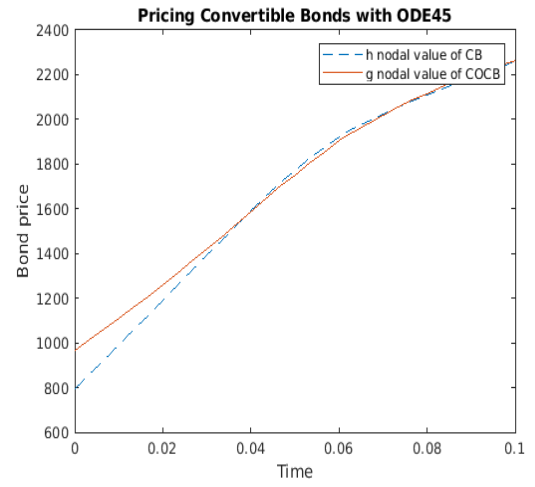


Figure 5-2: Nodal value of CB and COCB when $x_2 = \log(80\$)$, $0 < \tau < 0.1$

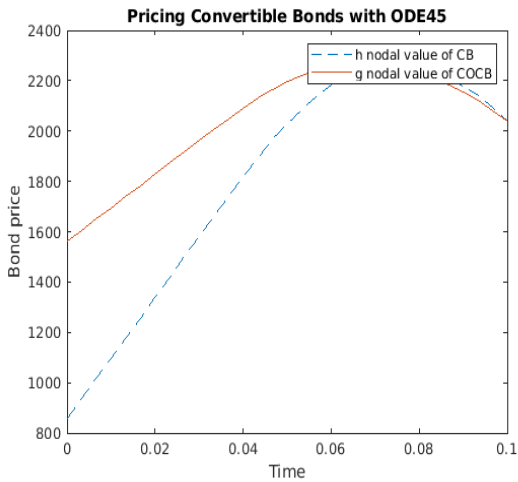


Figure 5-3: Nodal value of CB and COCB when $x_3 = \log(100\$)$, $0 < \tau < 0.1$

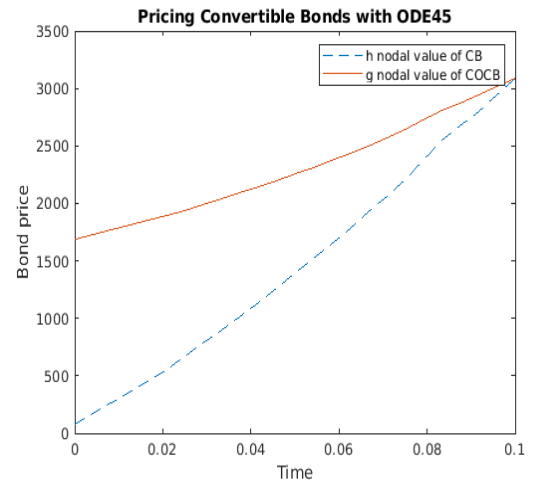


Figure 5-4: Nodal value of CB and COCB when $x_4 = \log(120\$)$, $0 < \tau < 0.1$

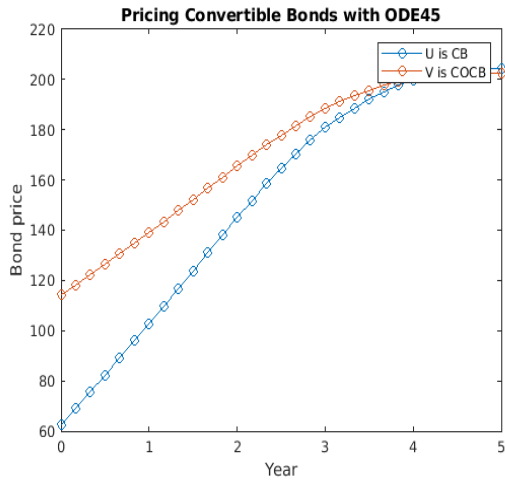


Figure 5-5: Value of CB and COCB when $S_1 = 60\$, 0 < t < 5$

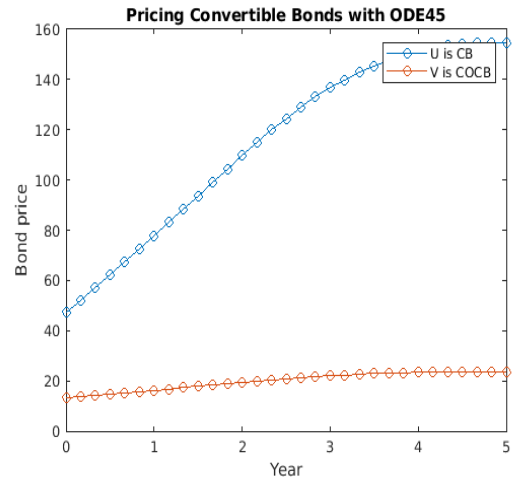


Figure 5-6: Value of CB and COCB when $S_2 = 80\$, 0 < t < 5$

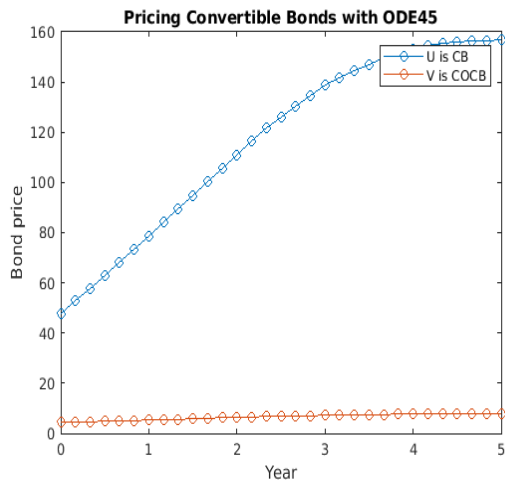


Figure 5-7: Value of CB and COCB when $S_3 = 100\$, 0 < t < 5$

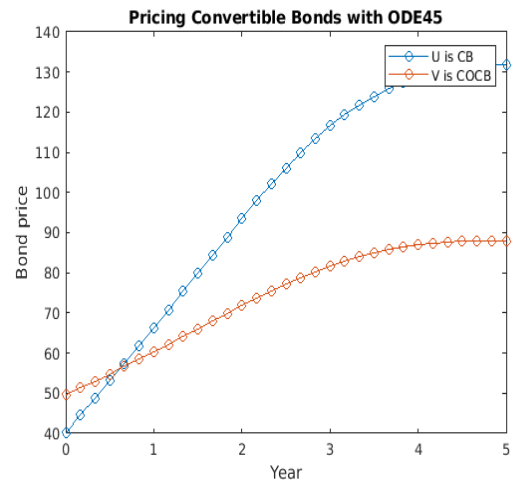


Figure 5-8: Value of CB and COCB when $S_4 = 120\$, 0 < t < 5$

The produced numerical examples were compared with the results of the paper [16] in a similar topic and the only difference is that they have used finite difference approximation for pricing convertible bonds. The numerical examples under the same problem correspondingly converges with each other.

Chapter 6

Conclusion

In conclusion, the system for pricing Convertible bonds was solved by using of Finite Element method. The system of Black-Scholes equations was transformed to the system of parabolic equations by using a change of variables. Moreover, the well-posedness of the system of parabolic equations was studied. Galerkin finite element solution of the system of parabolic equations was applied to the transformed TF model, and some examples of numerical solution were presented. In the future, we are planning to write a general finite element code of the TF model for pricing convertible bonds. Also, one of the major applications of convertible bonds by the TF model is to find an efficient pricing method that is consistent with observed market prices which can be done by considering an extension of the problem to a nonlinear system by considering transaction costs.

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