A CLASS OF INFINITE CONVEX GEOMETRIES

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ABSTRACT. Various characterizations of finite convex geometries are well known. This note provides similar characterizations for possibly infinite convex geometries whose lattice of closed sets is strongly coatomic and lower continuous. Some classes of examples of such convex geometries are given.

1. INTRODUCTION

There are various ways to characterize finite convex geometries; see Chapter 3 of our [4], which combines results from Dilworth [9], Edelman and Jamison [11], Duquenne [10], and Monjardet [15]. These characterizations can be either combinatorial or lattice theoretical in nature.

Infinite versions of convex geometries occur in several sources, each of which is equivalent to a closure operator with the anti-exchange property, plus some finiteness conditions to provide structure. Crawley and Dilworth [7] consider dually algebraic, dually coatomic, locally distributive lattices. Adaricheva, Gorbunov and Tumanov [3] discuss closure operators with the anti-exchange property whose closure lattices are weakly atomic, dually spatial and atomistic. Adaricheva and Nation [4] are concerned with algebraic closure operators with the anti-exchange property. See also Nakamura and Sakaki [16], Adaricheva and Pouzet [5], and Adaricheva [2].

Here we consider a fourth type, inspired by Gorbunov's classic result [12]: *Every element of a complete lattice L has a canonical join decomposition if and only if L is strongly coatomic, lower continuous, and join semidistributive.* Adding lower semidmodularity to the list will give us an infinite class of convex geometries; see Theorem 7.

2. THE ANTI-EXCHANGE PROPERTY

Definition 1. *A closure system (X, γ) satisfies the* anti-exchange property *if for all $x \neq y$ and all closed sets $A \subseteq X$,*

(AEP) $x \in \gamma(A \cup \{y\})$ *and* $x \notin A$ *imply that* $y \notin \gamma(A \cup \{x\})$.

Equivalently, a closure operator satisfies the anti-exchange property if for all closed sets $A \subseteq X$ and elements x, $y \notin A$, if $\gamma(A \cup \{x\}) = \gamma(A \cup \{y\})$ then $x = y$.

Examples of closure operators with the anti-exchange property include

- the convex hull operator on Euclidean space $\mathbb{E}^n$,
- the convex hull operator on an ordered set,
- the subalgebra-generated-by operator on a semilattice,
- the algebraic-subset-generated-by operator on a complete lattice.

For a closure system (X, γ), we will let $\mathrm{Cld}(X, \gamma)$ denote the lattice of γ-closed subsets of X. A closure system is *zero-closure* if $\gamma(\emptyset) = \emptyset$.

Definition 2. *A zero-closure system that satisfies the anti-exchange property is called a* convex geometry.

(This common convention is a bit awkward, as some useful closure operators with the anti-exchange property have a non-empty closure of the empty set. Nonetheless, we shall retain it.)

A lattice is *strongly coatomic* if $a < c$ in L implies that there exists b such that $a \leq b \prec c$. A closure system is strongly coatomic if its lattice of closed sets is so.

Theorem 3. *For a strongly coatomic closure system (X, γ), the following are equivalent.*

(1) *(X, γ) has the anti-exchange property.*
(2) *If $A \prec B$ in $\mathrm{Cld}(X, \gamma)$, then $|B \setminus A| = 1$.*

Proof. Assume that (X, γ) has the AEP. If $A \prec B$ in $\mathrm{Cld}(X, \gamma)$ and x, $y \in B \setminus A$, then $\gamma(A \cup \{x\}) = B = \gamma(A \cup \{y\})$, whence $x = y$ by the AEP.

Suppose that (X, γ) satisfies (2). Assume that $B = \gamma(A \cup \{x\}) = \gamma(A \cup \{y\}) > A = \gamma(A)$. As $\mathrm{Cld}(X, \gamma)$ is strongly coatomic, there is a closed set A' such that $A \leq A' \prec B$. Then $B = \gamma(A' \cup \{x\}) = \gamma(A' \cup \{y\})$, so x, $y \in B \setminus A'$. By (2) we have $x = y$, as desired. $\square$

The equivalence of the preceding theorem is also valid for algebraic closure systems [4] and [2].

3. SCLCC LATTICES

A complete lattice is *lower continuous* if for every down-directed set $D \subseteq L$, $a \vee \bigwedge D = \bigwedge_{d \in D}(a \vee d)$. We denote a strongly coatomic, lower continuous, complete lattice as SCLCC, and a closure system is SCLCC if its lattice of closed sets is so.

Some basic properties of SCLCC lattices follow.

Lemma 4. *Let L be an SCLCC lattice.*

(1) *Every nonzero join irreducible element is completely join irreducible.*
(2) *If $w \succ c$ in L, then there exists an element k that is minimal w.r.t. $k \leq w$, $k \nleq c$, and each such element is join irreducible.*
(3) *Every element of L is a (perhaps infinite) join of join irreducible elements.*

A complete lattice in which every element is a join of completely join irreducible elements is said to be *spatial.* Clearly, this is a desirable property for any sort of "geometry." In [5], it was shown that every weakly atomic convex geometries is spatial. The preceding lemma says that SCLCC lattices have an even better property: a complete lattice is *strongly spatial* if it is spatial and every nonzero join irreducible element is completely join irreducible. The set of nonzero join irreducible elements of a lattice L will be denoted by $\mathrm{Ji}(L)$.

The next lemma is useful in connection with local distributivity. A complete lattice is *coatomistic* if every element is a meet of coatoms.

Lemma 5. *A distributive, coatomistic SCLCC lattice is isomorphic to the boolean algebra of all subsets of its coatoms.*

4. Join semidistributive SCLCC lattices

Next we generalize some equivalences of join semidistributivity which are well-known for finite lattices.

The implication

$$(\mathrm{SD}_\vee) \qquad w = x \vee y = x \vee z \quad \text{implies} \quad w = x \vee (y \wedge z)$$

is known as the *join semidistributive law.* In view of Lemma 4 and the results in Jónsson and Kiefer [13], we consider two potentially infinite versions of the law:

$$(\mathrm{SD}'_\vee) \qquad w = x \vee y \ \text{ for all } y \in Y \quad \text{implies} \quad w = x \vee \bigwedge Y$$

$$(\mathrm{SD}^*_\vee) \qquad w = \bigvee Y = \bigvee Z \quad \text{implies} \quad w = \bigvee (y \wedge z)$$

It is an elementary exercise that join semidistributive, lower continuous, complete lattices satisfy $\mathrm{SD}'_\vee$, and we shall use that at will.

For subsets A, $B \subseteq L$ we say that A *refines* B, denoted $A \ll B$, if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. Note that $A \ll B$ implies $\bigvee A \leq \bigvee B$.

We say that $w = \bigvee A$ is a *canonical join decomposition* if the join is irredundant, and $w = \bigvee B$ implies $A \ll B$.

Theorem 6. *The following are equivalent for an SCLLC lattice L.*

(1) *L satisfies $\mathrm{SD}_\vee$.*
(2) *L satisfies $\mathrm{SD}_\vee^*$.*
(3) *Every element of L has a canonical join decomposition.*
(4) *If $w \succ c$ in L, then there exists a join irreducible k which is the unique minimal element such that $k \leq w$ but $k \not\leq c$.*

Proof. $(1) \Rightarrow (4)$ is an immediate consequence of $\mathrm{SD}_\vee'$.

$(4) \Rightarrow (3)$. Assume that property (4) holds, and fix an element $w \in L$. Let $C = \{c \in L : w \succ c\}$. For each $c \in C$, we can find an element k_c such that $k_c \leq w$ but $k_c \not\leq c$. We claim that $w = \bigvee_{c \in C} k_c$ canonically. Clearly $\bigvee_{c \in C} k_c = w$, since each k_c is below w, while the join is below no lower cover of w.

If $c \neq d \in C$, then $c \vee d = w$. Hence by $\mathrm{SD}_\vee'$ we have $c \vee \overline{c} = w$, where $\overline{c} = \bigwedge (C \setminus \{c\})$. By uniqueness, this implies $k_c \leq \overline{c}$, i.e., $k_c \leq d$ whenever $c \neq d \in C$. It follows that the join is irredundant.

Suppose $w = \bigvee A$ and consider $c \prec w$. There exists some $a_0 \in A$ such that $a_0 \not\leq c$, though $a_0 \leq w$, whereupon $k_c \leq a_0$. Since this holds for all $c \in C$, we have $\{k_c : c \in C\} \ll A$, as desired.

$(3) \Rightarrow (2)$. Suppose that $w = \bigvee S = \bigvee T$. If there is a canonical join decomposition $w = \bigvee U$ in L, then U refines both S and T, so that for each $u \in U$ there exist $s \in S$ with $u \leq s$, and $t \in T$ with $u \leq t$. Hence each $u \leq s \wedge t$ for some pair, and it follows that $w = \bigvee (s \wedge t)$.

$(2) \Rightarrow (1)$ clearly, as (1) is a special case of (2). $\qquad \square$

5. Characterizations of LCSCC convex geometries

First, we introduce a few more terms.

A strongly coatomic complete lattice is said to be *locally distributive* (or *lower locally distributive* or *meet distributive*) if for any $x \in L$ the interval $[\mu(x), x]$ where $\mu(x) = \bigwedge \{y : y \prec x\}$ is a distributive lattice (and hence for SCLCC lattices a boolean algebra).

For any $A \subseteq X$, $x \in A$ is called an *extreme point* of A if $x \notin \gamma(A \setminus \{x\})$. The set of extreme points of A is denoted $\mathrm{Ex}(A)$. In lattice terms, for a strongly spatial lattice L, we identify the element a with the set $\mathrm{Ji}(a) = \{p \in \mathrm{Ji}(L) : p \leq a\}$. Then $x \in \mathrm{Ji}(a)$ is an extreme point of a if $a > \bigvee (\mathrm{Ji}(a) \setminus \{x\})$. This means that (i) x is join prime in the ideal $\mathrm{id}(a)$, and (ii) there is no other join irreducible y with $x < y \leq a$.

Recall that a lattice L is called *lower semimodular* if $a \prec b$ implies $a \wedge c \preceq b \wedge c$ for all a, b, $c \in L$. Equivalently, a lattice is lower semimodular if $a \prec a \vee c$ implies $a \wedge c \prec c$.

We now extend some characterizations of finite convex geometries to SCLCC geometries. In this setting we want to think of a lattice

in terms of its standard representation as a closure system on its set of join irreducibles. Note that properties (1)–(2) of the next theorem are about closure systems, while (3)–(6) are lattice properties. For the finite case, various parts of the theorem are due to R.P. Dilworth, P. Edelman and R. Jamison, and V. Duquenne.

Theorem 7. *Let L be an SCLCC lattice. Then the following are equivalent.*

(1) *L is the closure lattice $\mathrm{Cld}(X, \gamma)$ of a closure space (X, γ) with the AEP.*
(2) *If $A \prec B$ in $\mathrm{Cld}(X, \gamma)$, then $|B \setminus A| = 1$.*
(3) *L is locally distributive and lower semimodular.*
(4) *L is join semidistributive and lower semimodular.*
(5) *Every element $a \in L$ is the join of $\mathrm{Ex}(a)$.*
(6) *Every element of L has a unique irredundant join decomposition.*

Proof. The equivalence of (1) and (2) is Theorem 3.

To see that (2) $\to$ (3), let $T = \mathrm{Ji}(x)$ and $M = \mathrm{Ji}(\mu(x))$. For every element z in the interval $[\mu(x), x]$, we have $M \subseteq \mathrm{Ji}(z) \subseteq T$. Moreover, for each $c \prec x$ there is a unique join irreducible $x_c \in T \setminus M$ such that $\mathrm{Ji}(c) = T \setminus \{x_c\}$. Taking intersections, every set Z with $M \subseteq Z \subseteq T$ is a closed set, representing $\mathrm{Ji}(z)$ for some element z in $[\mu(x), x]$. Thus the entire boolean algebra of subsets $[M, T]$ occurs, which exhausts the possibilities, leaving room for no other elements in the interval. Thus (2) implies local distributivity, and it is clear that (2) implies lower semimodularity.

Now assume (3), that L is locally distributive and lower semimodular, and we want to show that L is join semidistributive by means of property (4) of Theorem 6, which yields (4) of this theorem.

So let $w \succ c$, and let a be minimal w.r.t. $a \leq w$, $a \nleq c$. Suppose by way of contradiction that there exists another such element b. Choose an element b' with $b \leq b' \prec a \vee b$. Using strong coatomicity, form a (possibly transfinite) sequence a_α for $\alpha \leq \kappa$ such that

(i) $a_0 = a \vee b$ and $a_\kappa = a$,
(ii) $a_{\alpha+1} \prec a_\alpha$,
(iii) $a_\lambda = \bigwedge_{\alpha < \lambda} a_\alpha$ for limit ordinals λ.

We show by induction that $(a_\alpha \wedge b') \vee c = w$ for all $\alpha \leq \kappa$, so that $a_\alpha \wedge b' \nleq c$.

If the statement holds for a_α, then by lower semimodularity, $a_{\alpha+1}$, $a_\alpha \wedge b'$ and $a_\alpha \wedge c$ are three distinct lower covers of a_α. Hence by local distributivity $a_{\alpha+1} \wedge (a_\alpha \wedge b') \nleq a_\alpha \wedge c$, whereupon $a_{\alpha+1} \wedge b' \nleq c$

and $(a_{\alpha+1} \wedge b') \vee c = w$. At limit ordinals, the claim follows by lower continuity.

For $\alpha = \kappa$, this contradicts the minimality of a. Thus L is join semidistributive.

Before proving that (4) implies (5), let us interpret (5) in a more traditional way.

Sublemma 8. *In an SCLCC lattice, $w = \bigvee \mathrm{Ex}(w)$ iff for each element c with $c \prec w$, there exists a unique j_c in $\mathrm{Ji}\, L$ such that $j_c \leq w$ and $j_c \not\leq c$.*

Proof. Assume $w = \bigvee \mathrm{Ex}(w)$, and let $c \prec w$. There exists $j \in \mathrm{Ex}(w)$ such that $j \not\leq c$, and $j \not\leq \bigvee(\mathrm{Ji}(w) \setminus \{j\})$ because j is extreme. In that case, the latter join must in fact be c, since c is a join of join irreducibles below w. Hence j is unique.

Conversely, suppose that for each $c \prec w$ such a j_c exists. Then $\bigvee\{j_c : c \prec w\} = w$, since the join is below w but not below any of its lower covers. On the other hand, each such j_c is extreme, as $c = \bigvee(\mathrm{Ji}(w) \setminus \{j_c\})$, since j_c is the only join irreducible below w that not below c. □

Now assume (4), that L is join semidistributive and lower semimodular. Let $w \succ c$ in L. Then $\mathrm{SD}_\vee$ and lower continuity give us the existence of a join irreducible element j that is the unique minimal element w.r.t. $j \leq w$, $j \not\leq c$. If $j < y \leq w$, then by lower semimodularity $y \succ y \wedge c$, whence $y = j \vee (y \wedge c)$ is join reducible. Thus j is an extreme element for w.

The equivalence of (5) and (6), and that (6) implies (2) for the standard representation of a spatial lattice L as a closure operator on its join irreducibles, are both easy. □

A lattice L is called *atomistic* if every nonzero $a \in L$ is a join of atoms. Atomistic convex geometries were characterized in Proposition 3.1 of Adaricheva, Gorbunov and Tumanov [3]. For the SCLCC case, the proof is particularly easy.

Corollary 9. *Any SCLCC atomistic join semidistributive lattice is the closure lattice of some convex geometry.*

Proof. It is enough to show that a SCLCC atomistic join semidistributive lattice is lower semimodular. Indeed, it follows from join semidistributivity that if $a \prec b$, then there exists a unique atom t such that $t \leq b$, $t \not\leq a$. If $c \leq b$, then either $t \leq c$ and thus c and $c \wedge a$ differ by the single atom t, so that $c \wedge a \prec c$, or else $t \not\leq c$ in which case every atom below c is below a, whence $c \leq a$. □

6. Examples of SCLCC convex geometries

There are natural examples of the kind of geometries described in Corollary 9, obtained by taking standard convex geometries and adding finiteness conditions to ensure strong coatomicity and lower continuity.

- If P is an ordered set such that both the ideal $\mathrm{id}(x)$ and the filter $\mathrm{fil}(x)$ are finite for every $x \in P$, then the lattice of convex subsets $\mathrm{Co}(P)$ is an atomistic SCLCC convex geometry.
- If S is a meet semilatiice such that the filter $\mathrm{fil}(x)$ is finite for every $x \in S$, then the lattice of subsemilattices $\mathrm{Sub}_\wedge(S)$ is an atomistic SCLCC convex geometry.

(We should note that in each example, the closure operator is algebraic, and an algebraic lattice is dually algebraic iff it is lower continuous.)

For the first type, we know that the convex hull operator on an ordered set satisfies the AEP; it remains to show that if P has the property that every principal ideal and filter is finite, then $\mathrm{Co}(P)$ is strongly coatomic and lower continuous.

Suppose that P has that property, and that $A < B$ in $\mathrm{Co}(P)$. Let $b_0 \in B \setminus A$. Then either $\mathrm{id}(b) \cap A = \emptyset$ or $\mathrm{fil}(b) \cap A = \emptyset$, w.l.o.g. the former. Choose b_1 minimal in $B \cap \mathrm{id}(b_0)$. Then $A \subseteq B \setminus \{b_1\} \prec B$ in $\mathrm{Co}(P)$. Thus $\mathrm{Co}(P)$ is strongly coatomic.

For lower continuity, it is convenient to use the equivalent formulation in terms of chains; see e.g. Theorem 3.8 of [17]. Let X be convex, and let $\mathcal{C} = \{C_i : i \in I\}$ be a chain of convex subsets of P. We want to show that $\bigwedge_i (X \vee C_i) \subseteq X \vee \bigwedge_i C_i$. Let w be in the left-hand side, and w.l.o.g. $w \notin X$. Either there exists a co-initial subset $\mathcal{D} = \{C_j : j \in J\}$ for some $J \subseteq I$, and elements $x_j \in X$ and $y_j \in C_j$, such that $x_j \leq w \leq y_j$ for all $j \in J$, or dually; assume the first. Because $\mathrm{fil}(w)$ is finite, there exists a co-initial subset $\mathcal{E} = \{C_k : k \in K\}$ with $K \subseteq J$ such that $y_k = y_{k'}$ for all k, $k' \in K$. Then $y_k \in \bigcap_{k \in K} C_k = \bigcap_{i \in I} C_i$, and fixing any $k_0 \in K$ we have $x_{k_0} \leq w \leq y_{k_0}$. Thus w is in the right-hand side, and $\mathrm{Co}(P)$ is lower continuous.

Now consider $\mathrm{Sub}_\wedge(S)$ for a meet semilattice S. Again, we know that the subsemilattice operator satisfies the AEP, and it remains to show that if S has the property that every principal filter is finite, then $\mathrm{Sub}_\wedge(S)$ is strongly coatomic and lower continuous.

If $A < B$ in $\mathrm{Sub}_\wedge(S)$, then we can choose b_0 maximal in $B \setminus A$ to obtain $A \leq B \setminus \{b_0\} \prec B$.

To show lower continuity, let X be a subsemilattice and let $\mathcal{C} = \{C_i : i \in I\}$ be a chain of subsemilattices of S. Again, we want to show that $\bigwedge_i (X \vee C_i) \subseteq X \vee \bigwedge_i C_i$, so let w be in the left-hand side. That

means that for each $i \in I$ there exist $x_i \in X$ and $y_i \in C_i$ such that $w = x_i \wedge y_i$. Let $x = \bigwedge_i x_i$, which really is a finite meet. If $x = w$, we are done. Otherwise, we continue: because $\mathrm{fil}(w)$ is finite, there exists a co-initial subset $\mathcal{E} = \{C_k : k \in K\}$ for some $K \subseteq I$ such that $y_k = y_{k'}$ for all k, $k' \in K$. Then $w = x \wedge y_k$, and $y_k \in \bigcap_{k \in K} C_k = \bigcap_{i \in I} C_i$, so $w \in X \vee \bigwedge_i C_i$, as desired.

We can even combine these examples: if S is a meet semilattice in which every principal filter $\mathrm{fil}(x)$ is a finite tree, then the lattice of convex subsemilattices of S is an SCLCC convex geometry. See Adaricheva [1] and Cheong and Jones [6].

In a similar vein, if $(P, \leq)$ is an ordered set in which every chain is finite and every interval is finite, then the lattice of suborders of $\leq$ on P is an SCLCC convex geometry. See Semenova [18].

Another construction yields SCLCC convex geometries that need not be atomistic. Our inspriration is the fact that a geometric lattice is isomorphic to the ideal lattice of its finite dimensional elements. (There is no chance for a similar characterization here, since for any non-limit ordinal α, the dual α^d is an SCLCC convex geometry.) Our construction uses Jónsson and Rival's characterization of join semidistributive varieties [14].

Define certain lattice terms recursively: for $k \geq 0$,

$$y_0 = y \qquad\qquad z_0 = z$$
$$y_{k+1} = y \wedge (x \vee z_k) \qquad z_{k+1} = z \wedge (x \vee y_k).$$

Then consider the lattice inclusions

$$\mathrm{SD}_\vee(k) \qquad y_k \leq x \vee (y \wedge z) \ .$$

These are equivalent to the corresponding identities $x \vee y_k \approx x \vee (y \wedge z)$. For example, $\mathrm{SD}_\vee(1)$ is equivalent to the distributive law.

Lemma 10. *Let $\mathcal{V}$ be a lattice variety. Then every lattice in $\mathcal{V}$ is join semidistributive if and only if $\mathcal{V}$ satisfies $\mathrm{SD}_\vee(n)$ for some $n < \omega$.*

Let $\mathbf{SD}_\vee(n)$ be the variety of all lattices satisfying $\mathrm{SD}_\vee(n)$.

Theorem 11. *Let L_0 be a lattice with the following properties.*

- *$\mathrm{fil}(x)$ is finite for each $x \in L_0$.*
- *$L_0 \in \mathbf{SD}_\vee(n)$ for some $n < \omega$.*
- *L_0 is lower semimodular.*

Then the filter lattice $L = \mathrm{Fil}(L_0)$ is an SCLCC convex geometry.

Proof. As usual, we order the filter lattice by reverse set inclusion: $F \leq G$ iff $F \supseteq G$. The filter lattice of any lattice is lower continuous and

satisfies the equations of the original, in particular $\mathrm{SD}_\vee(n)$ in this case. It remains to show that L is strongly coatomic and lower semimodular.

Suppose $F < G$ in L, i.e., $F \supset G$. Let k be an element maximal in $F \setminus G$, and note that k is meet irreducible. We claim that the filter generated by $G \cup \{k\}$, say $H = \mathrm{fil}(G, k)$, satisfies $F \leq H \prec G$. Let ℓ be any element of $H \setminus G$. Then $\ell \geq g \wedge k$ for some $g \in G$, and we may take $g \leq k^*$, where k^* denotes the unique upper cover of k in L_0. In that case, by lower semimodularity, $g \succ g \wedge k$, whence also $g \wedge k = g \wedge \ell$. It follows that $H = \mathrm{fil}(G, \ell)$, and since ℓ is arbitrary, $H \prec G$. Thus L is strongly coatomic.

The proof that L is lower semimodular is an adaptation of that for the corresponding dual claim in Theorem 11.1 of [17]. Assume that L is lower semimodular, and suppose that $F \prec F \vee G = F \cap G$ in $\mathrm{Fil}(L_0)$. Choose an element a maximal in $F \setminus G$, and note that a is meet irreducible, thus by the finiteness of $\mathrm{fil}(a)$ completely meet irreducible. Then $F = (F \vee G) \wedge \mathrm{fil}(a)$, and hence $F \wedge G = \mathrm{fil}(a) \wedge G$. Let x be any element in $(F \wedge G) \setminus G$. Since $x \in F \wedge G$, there exists $g \in G$ such that $x \geq a \wedge g$. Because L is lower semimodular, $a \wedge g \prec a^* \wedge g$. On the other hand, every element of L is a meet of finitely many meet irreducibles, so $x \notin G$ implies there exists a meet irreducible element $b \geq x$ with $b \notin G$. Now $b \geq a \wedge g$ and $b \not\geq g$, so $b \wedge g = a \wedge g$, whence $a \geq b \wedge g$. Thus $\mathrm{fil}(b) \wedge G = \mathrm{fil}(a) \wedge G = F \wedge G$; if follows *a fortiori* that $\mathrm{fil}(x) \wedge G = F \wedge G$. As this holds for every $x \in (F \wedge G) \setminus G$, we have $F \wedge G \prec G$, as desired. $\square$

So in particular, we could take L_0 to be the elements of finite depth in a direct product of finite convex geometries that satisfy $\mathrm{SD}_\vee(n)$ for some fixed n.

7. Discussion

In some sense, algebraic closure operators are the natural settings for any type of geometry. On the other hand, Crawley and Dilworth's setting of dually algebraic and strongly coatomic gives the nice equivalence of local distributivity and unique representations. Our hypothesis of SCLCC is slightly weaker, but means that we must assume lower semimodulariy along with local distributivity. The lattice $(\omega + 1)^d \times \mathbf{2}$, with its atom doubled, shows that local distributivity by itself is not sufficient.

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