A Finite Element Approach to Solving Leland Model for Options Pricing

by

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Abstract

In this thesis work, the Leland model for pricing of European options is studied. Firstly, the derivation of the Leland model is introduced by using Ito's lemma and synthesized replicate portfolio methodology. Then the model is transformed to a system of equations by change of variables to which the Galerkin finite element model can be applied. Crank-Nicolson finite difference method is adopted to solve the resulting differential differential algebraic finite element system with data from literature. Some numerical solutions are presented by example.

Thesis Supervisor: Dongming Wei
Title: Full Professor

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Chapter 1

Introduction

An option is an important financial security, giving the holder the right, but not an obligation, to purchase (call) or sell (put) some assets or other securities at an agreed-upon price (the strike price K) on a specified date (expiration date T) [13]. An "American option" is one that can be exercised at any time before the expiration date, whereas a "European option" can be used only at the expiration date. In this thesis work, the pricing of European options is considered only, however, our method can be applied and adapted for Leland model of other options. In 1973, Black and Scholes developed an option pricing theory to value a fair price of various financial securities [6], and earlier some research was done by Merton who made the conclusion that the model for option pricing has a form of a parabolic partial differential equation [15]. Until now, the Leland model has been solved numerically by binary method [13, 17] and finite difference method (FDM) [3, 10] in literature . There are not many papers in literature on the application of finite element method (FEM) to solve the Leland model [8, 16, 19]. In most cases, numerical solutions for the Leland model were obtained using finite difference methods and other discrete methods. In the paper of M. Ehrhardt and J. Ankudinova [10], Crank-Nicolson method was used to solve numerically nonlinear Black-Scholes models for different volatility functions for European options including Leland model. Their results of the Black-Scholes models were compared using some parameters. In [20], Liao and Khaliq proposed high-order compact scheme for solving a nonlinear Black-Scholes equation with different transaction costs. This paper presents a new algorithm, solving a system of two reaction-diffusion equations instead of solving a single convection-diffusion-reaction equation. Paper [4] presents an approach based on a solution to the nonlinear Gamma equation of the Leland model with variable transaction costs, whereas the other finite difference approximation schemes are based on discretization of the original fully nonlinear Black-Scholes equation. The numerical approximation scheme is used in [2] is semi-implicit in time.

The structure of the thesis is outlined as follows. For completeness, the celebrated Black-Scholes model for European options with some basic assumptions is introduced and its solution is derived in Chapter 1. Removing some of the assumptions leads to nonlinear Black-Scholes models including the nonlinear Leland model for European options. Leland model is derived in Chapter 2. In addition, transformation of the Leland model into a quasilinear parabolic Gamma equation is presented and reduction of the partial differential equation of the model with transformed initial and boundary conditions are derived. Chapter 3 devotes to the application of Galerkin finite method to Leland model. The finite element linear and quadratic shape functions and finite element matrices are introduced. The finite element method transforms the partial differential equations into a differential algebraic system of ordinary differential equations. In Chapter 4, Crank-Nicolson method is modified and applied to the differential algebraic system. A method code is developed to implement a numerical example with parameters from [10] are used to produce numerical solutions for the Leland model.

1.1 Linear Black-Scholes model for European options

The following Black-Scholes equation was derived by Black and Scholes and Merton. This equation defines the option value $V(S,t)$ as a solution of the equation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0 \tag{1.1}$$

with the terminal and boundary conditions

$$\begin{aligned} &V(S,T) = \max(S-K,0), \quad S \geq 0 \\ &V(0,t) = 0, \quad t \geq 0 \\ &V(S,t) = S, \quad S \to \infty \end{aligned} \tag{1.2}$$

where V is the option price, S is the value of the underlying asset, $r \geq 0$ and σ are constant interest rate and volatility, correspondingly, t is the time between 0 and the expiry date T. In the original paper of Black and Scholes [6], the following conditions are assumed in the market for modeling of option pricing:

- The price of the underlying asset $S = S(t)$ follows a Geometric Brownian motion $W(t)$, which means that S satisfies the following stochastic differential equation:

$$dS = \mu S dt + \sigma S dW \tag{1.3}$$

  where the trend or drift μ, the volatility σ and the interest rate r are constants for the time interval $0 \leq t \leq T$.

- The market is frictionless, therefore there are no transaction costs in the market and one can borrow and lend money with the same interest rates, also all information can be accessed by all. In addition, personal trading does not have so much influence on the option price.

- The distribution of the stock price is lognormal, with the constant rate of variance that is proportional to the square of the asset price.

- There are no arbitrage opportunities [6].

Under these assumptions, the market is complete, means that any underlying asset may be hedged or replicated with a portfolio of other assets. For completeness, we state Ito's lemma that was used by Black and Scholes and Merton [12]:

$$\triangle V = V_S \triangle S + (V_t + \frac{\sigma^2}{2} S^2 V_{SS}) \triangle t$$

For completeness, the exact solution of the classical Black-Scholes equation is introduced as following. We transform the partial differential equation into the heat equation making the following change of variables, expressing the option price V through a new function v, and also the new τ and x in terms of the t and S:

$$\begin{aligned} \tau &= \frac{\sigma^2}{2}(T-t) \\ x &= \log(\frac{S}{K}) \\ V(S,t) &= Kv(x,\tau) \end{aligned} \tag{1.4}$$

The first derivatives are

$$\begin{aligned} \frac{\partial V}{\partial t} &= K\frac{\partial v}{\partial \tau}\cdot\frac{\partial \tau}{\partial t} = -K\frac{\partial v}{\partial \tau}\cdot\frac{\sigma^2}{2} \\ \frac{\partial V}{\partial S} &= K\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial S} = K\frac{\partial v}{\partial x}\cdot\frac{1}{S} \end{aligned}$$

and the second derivative of the option price V is:

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\Big(K\frac{\partial v}{\partial x}\frac{1}{S}\Big) = -K\frac{\partial u}{\partial x}\cdot\frac{1}{S^2} + K\frac{\partial^2 v}{\partial x^2}\cdot\frac{1}{S^2}$$

Substituting all the above derivatives into the Black-Scholes equation (1.1) and making simplifications, we obtain constant coefficient equation:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv \tag{1.5}$$

where k is defined as a multiple of the volatility and a rescaled time to expiry, $k = \frac{\sigma^2}{2}T$. The initial and boundary conditions are need to be transformed too. Then the equation (1.2) becomes

$$\begin{aligned} &v(x,0) = \max(e^x - 1, 0) \\ &v(0,\tau) = 0 \\ &\lim_{x\to\infty}\frac{v(x,\tau)}{e^x} = 1 \end{aligned} \tag{1.6}$$

Again, we make the change of variables to simplify further

$$v = e^{\alpha x+\beta\tau}u(x,\tau) \tag{1.7}$$

where α and β are yet to be determined. Finding partial derivatives of v and putting these derivatives into the coefficient partial differential equation, then dividing by the common factor of $e^{+\beta\tau}$ throughout, we have:

$$u_\tau = u_{xx} + [2\alpha + (k-1)]u_x + [\alpha^2 + (k-1)\alpha - k - \beta]u \tag{1.8}$$

Choose $\alpha = -\frac{k-1}{2}$ and $\beta = \alpha^2 + (k-1)\alpha - k = -\frac{(k+1)^2}{4}$ so that the coefficient of u_x and u is zero. The equation (1.9) is reduced to

$$u_\tau = u_{xx} \tag{1.9}$$

The transformed initial and boundary conditions (1.6) become:

$$\begin{aligned} &u(x,0) = u_0(x) = \max\Big(e^{\left(\frac{(k+1)}{2}\right)x} - e^{\left(\frac{(k-1)}{2}\right)x}, 0\Big) \\ &u(0,\tau) = 0 \\ &\lim_{x\to\infty} \frac{u(x,\tau)}{e^{\frac{k+1}{2}x}} = 1 \end{aligned} \tag{1.10}$$

Now we can apply the heat-equation solution representation formula:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\in}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds \tag{1.11}$$

First, we make a change of variable in the integration, by assuming $z = \frac{(s-x)}{\sqrt{2\tau}}$, thus the integration (1.11) becomes

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0\big(z\sqrt{2\tau} + x\big) e^{-\frac{z^2}{2}} dz \tag{1.12}$$

The solution of the transformed heat equation (1.12) is

$$u(x,\tau) = e^{\frac{(k+1)x}{2}+\frac{\tau(k+1)^2}{4}}\Phi(d_1) - e^{\frac{(k-1)x}{2}+\frac{\tau(k-1)^2}{4}}\Phi(d_2)$$

where $d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)$ and $d_2 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k-1)$, and Φ is the cumulative distribution function of a normal random variable:

$$\Phi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}dy$$

Now we must systematically unwind each of the changes of variables, starting from u. First, $v(x,\tau) = e^{-\frac{(k-1)x}{2}-\frac{(k+1)^2\tau}{4}}u(x,\tau)$, next put $x = \log(S/K), \tau = (\frac{1}{2})\sigma^2(T-t)$ and $V(S,t) = Kv(x,\tau)$. The ultimate result is the Black-Scholes formula for the value of a European call option at time T with the strike price K, if the current time is t and the underlying security price is S, the risk-free interest rate is r and the volatility is σ:

$$\begin{aligned} V(S,t) =& S\Phi\left(\frac{\log(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & - Ke^{-r(T-t)}\Phi\left(\frac{\log(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned} \tag{1.13}$$

Specifically, let

$$\begin{aligned} d_1 &= \frac{\log(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\log(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

We now have the exact solution of the Black-Scholes equation :

$$V(S,t) = S\cdot\Phi(d_1) - Ke^{-r(T-t)}\cdot\Phi(d_2) \tag{1.14}$$

Example 1.1.1 *Suppose that the strike price of a call option* $K = 70$*. The risk-free rate per year, continuously compounded is* 14%*, the expiration time is* $T = 1$

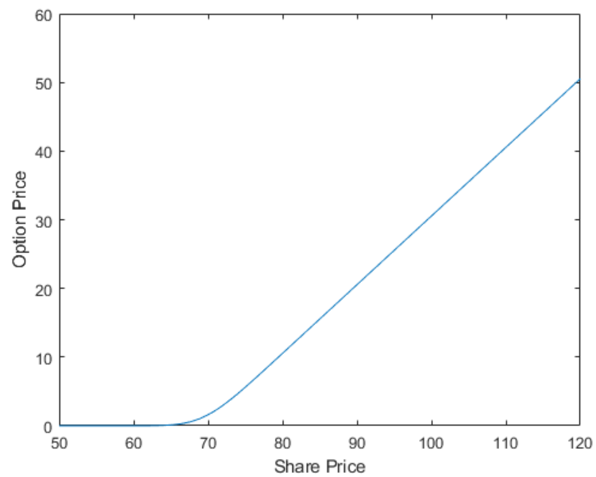

Figure 1-1: Value of the call at expiration

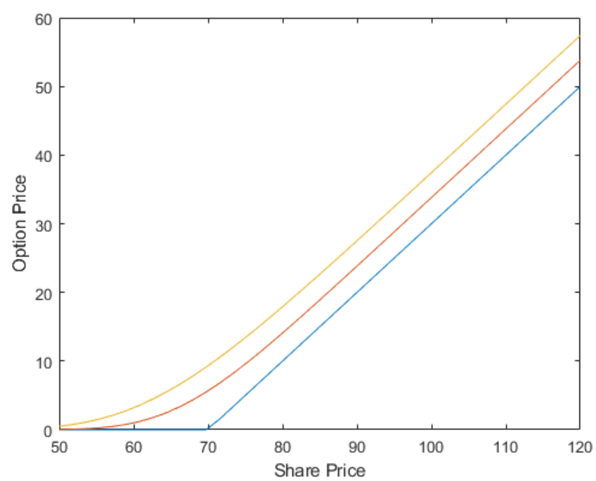

Figure 1-2: Value of the call option at various times

measured in years, and the standard deviation per year on the return of the stock, or the volatility is $\sigma = 0.2$*. Find the option price* V*.*

The Black-Scholes formula (1.14) is used to compute the value of the option before expiration. The value of the call option at maturity is plotted over a range of stock prices $50 \leq S \leq 120$ (Figure 1-1). With the same parameters as above the value of the call option is plotted over a range of stock prices $50 \leq S \leq 120$ at time remaining to expiration $t = 1$ (yellow), $t = 0.5$ (red), and at maturity $t = 0$ (blue), shown in the Figure 1-2.

Using this graph, two trends in the option value are observed:

- As the stock value increases for a fixed time, the option price increases too.
- For a fixed stock price, when the time to maturity decreases, the price of option decreases too to the price at the expiration.

Although the solution to the Black-Scholes (1.13)-(1.14) is very important and useful and it cannot be applied to options in which transaction cost are considered. In the following we will introduce several nonlinear Black-Scholes model which take into consideration transaction costs.

1.2 Nonlinear Black-Scholes models for European options

The "ideal conditions" of the standard Black-Scholes equation never occur in reality because of the presence of transaction costs, preference of investors, also the existence of incomplete markets [10]. By removing some of the assumptions in the method used to derive standard Black-Scholes equation leads to nonlinear Black-Scholes models. These nonlinear Black-Scholes models consider volatility as a function of time t, the underlying asset S and the partial derivatives of the option price $V(S,t)$ with respect to S in the following form

$$\tilde{\sigma}^2 = \tilde{\sigma}^2(t, S, V_S, V_{SS}) \tag{1.15}$$

Then the corresponding option pricing equation becomes a nonlinear Black-Scholes model:

$$V_t + \frac{1}{2}\tilde{\sigma}^2(t, S, V_S, V_{SS})S^2 V_{SS} + rSV_S - rV = 0 \tag{1.16}$$

There are a number of nonlinear volatility models that incorporate transaction costs with the risk from unprotected portfolios. In the Black-Scholes model, a continuous portfolio adjustment should be required to make a hedging with no risk. However, if there exist transaction costs it is possible that adjustments become more expensive because of numerous number of financial operations. A finding of the balance between the implied costs of errors in hedging and the transaction costs that are needed to balance the portfolio is the main for the hedger. As a result, the option price can be over- or under-prices [10].

1.2.1 Risk adjusted pricing methodology and nonlinear Black-Scholes models

In [14], M.Kratka derived the risk adjusted pricing model and then this model was developed by Jandačka and Ševčovič. The sum of the transaction costs rate and the rate of risk from an unprotected portfolio is minimized by the optimal time-lag δt between the operations.

$$\widetilde{\sigma}^2 = \sigma^2 \left(1 + 3 \left(\frac{C^2 M}{2\pi} S V_{SS} \right)^{\frac{1}{3}} \right) \tag{1.17}$$

where $M \geq 0$ is the measure value of the transaction cost and $C \geq 0$ is the risk premium measure. The models stated here are assumed to have nonzero transaction costs. Leland's idea states that the trading at discrete times can reduce risks from unprotected portfolios [9]. In this model, $\frac{\kappa}{2}|\triangle|S$ is the transaction cost, where c means the round trip of the transaction cost per one dollar and $\triangle$ is the number of assets that are sold ($\triangle < 0$) and assets are bought ($\triangle > 0$)

$$\widetilde{\sigma}^2 = \sigma^2 \left(1 + Le \, \mathrm{sgn}(V_{SS}) \right) \tag{1.18}$$

where σ stands for the historical volatility and Le the Leland number:

$$Le = \sqrt{\frac{2}{\pi}} \frac{c}{\sigma \sqrt{\delta t}} \tag{1.19}$$

where δt is the frequency of the transactions. Before Leland, Boyle and Vorst [6] showed if the transaction frequency equals the time step in the binomial model and the transaction cost c approaches zero applying the central limit theorem, then the option price with transaction costs converges to the classical Black-Scholes equation for a European call option with the following modified volatility function:

$$\widetilde{\sigma}^2 = \sigma^2 \left(1 + \frac{2c}{\sigma\sqrt{\delta t}} \operatorname{sgn}(V_{SS})\right) \tag{1.20}$$

In the above model, Leland factor $\sqrt{2/\pi}$ is approximated as 2. For both two above models, the parameters c and δt should be chosen so that $\tilde{\sigma}^2 > 0$. After two years, Hoggard proposed a new nonlinear volatility model for a call option [18]:

$$\widetilde{\sigma}^2 = \sigma^2 \left(1 - \sqrt{\frac{2}{\pi}} \frac{2c}{\sigma\sqrt{\delta t}} \operatorname{sgn}(V_{SS})\right) \tag{1.21}$$

which includes the negative of the product of the Boyle and Vorst factor 2 and the Leland factor $\sqrt{2/\pi}$. Both Boyle and Vorst and Hoggard assumed the opposite sign of the Leland factor for a call option in a short position. In [7], Barles and Soner proposed a mathematical model to price options based on the idea of Hodges and Neuberger which states that the behavior of the investor can be expressed by the utility function. The option price $V(S,t)$ is the only one solution of the following nonlinear volatility model:

$$V_t + \frac{1}{2}\tilde{\sigma}^2 S^2 V_{SS} + rSV_S - rV = 0, \tag{1.22}$$

where

$$\widetilde{\sigma}^2 = \sigma^2(1 + \Psi(e^{r(T-t)} a^2 S^2 V_{SS})) \tag{1.23}$$

where σ is the historical volatility, κ is the transaction cost, $a = \kappa/\sqrt{\epsilon}$ and $\Psi(x)$ is the solution of the following nonlinear ordinary differential equation:

$$\Psi'(x) = \frac{\Psi(x) + 1}{2\sqrt{x\Psi(x)} - x}, \ x \neq 0$$

with the initial condition $\Psi(0) = 0$. Then using the analysis by Barles and Soner in [7]:

$$\lim_{x\to\infty} \frac{\Psi(x)}{x} = 1 \ \lim_{x\to-\infty} \Psi(x) = -1 \tag{1.24}$$

We obtain the modified volatility function:

$$\widetilde{\sigma}^2 = \sigma^2(1 + e^{r(T-t)}a^2S^2V_{SS}). \tag{1.25}$$

The Leland model will be considered in the next chapter which is focus of the thesis.

Chapter 2

The Leland model

One of the model that takes into the consideration transaction costs is the Leland model [9]. This model was further developed by Hoggard, Whalley and Wilmott [18] for financial derivatives. The partial differential equation of the Leland model is

$$V_t + \frac{1}{2}\hat{\sigma}^2(SV_{SS})S^2V_{SS} + rSV_S - rV = 0, \tag{2.1}$$

with a volatility function $\hat{\sigma}^2$ is given by

$$\hat{\sigma}^2(SV_{SS}) = \sigma^2\Big(1 + \mathrm{Le}\,\mathrm{sgn}(SV_{SS})\Big)\begin{cases}\sigma^2(1+\mathrm{Le}), & \text{if}\, V_{SS} > 0\\ \sigma^2(1-\mathrm{Le}), & \text{if}\, V_{SS} < 0\end{cases} \tag{2.2}$$

where $\mathrm{Le} = \sqrt{\frac{2}{\pi}}\frac{c}{\sigma\sqrt{\delta t}}$ is the Leland number, σ is a constant historical volatility, $c > 0$ is a constant round trip transaction cost per unit dollar of transaction in the underlying asset market and δt is the time-lag between consecutive portfolio adjustments.

2.1 Derivation of the Leland model

In the option pricing theory of Black and Scholes, the assumption of continuous adjustment of a portfolio leads to an infinite number of transactions in the context of transaction costs because of diffusion processes have infinite variation and then continuous trading would be very expensive no matter how small transaction costs might be. In 1985, Leland developed the idea of the classical Black-Scholes model, integrating transaction costs and opportunity of rearranging the portfolio at discrete times [9]. Since the portfolio is maintained at regular intervals, it means that the total transaction costs are limited. The cost $\bar{c}$ per one transaction is assumed to be a constant and the underlying asset is bought at a higher *ask* price S_{ask} and it is sold for a lower *bid* price S_{bid}. The price of S is then computed as an average of ask and bid prices:

$$S = \frac{(S_{ask} + S_{bid})}{2} \tag{2.3}$$

Then $\bar{c} > 0$ represents a constant percentage of the cost of the sale and purchase of a share relative to the price S:

$$\bar{c} = \frac{S_{ask} - S_{bid}}{S} = 2\frac{S_{ask} - S_{bid}}{S_{ask} + S_{bid}} \tag{2.4}$$

Let $\Pi = V + \delta S$ be the value of the synthesized portfolio consisting of one option in a long position at the price V and δ underlying assets at the price S changes over the time interval $[t, t + \triangle t]$ by selling $\triangle\delta < 0$ or buying $\triangle\delta > 0$ short positioned assets. It means that the purchase or selling $\triangle\delta$ assets at a price of S yields the additional cost $\triangle TC$ for the option holder

$$\triangle TC = \frac{S}{2}\bar{c}|\triangle\delta| \tag{2.5}$$

Consequently, the value of the portfolio changes to:

$$\triangle\Pi = \triangle(V + \delta S) - \triangle TC \tag{2.6}$$

during the time interval $[t, t + \triangle t]$. The key step in derivation of the Leland model consists in approximation of the change $\triangle TC$ of transaction costs by its expected value $\mathbb{E}[TC]$ i.e. $\triangle TC \approx r_{TC} S \triangle t$, where the transaction costs measure r_{TC} is defined as the expected value of the change of the transaction costs per unit time interval $\triangle t$

and price S:

$$r_{TC} = \frac{\mathbb{E}[\triangle TC]}{S\triangle t} \tag{2.7}$$

Hence equation (2.6) describing the change in the portfolio has the form:

$$\triangle\Pi = \triangle(V + \delta S) - r_{TC} S \triangle t \tag{2.8}$$

We assumed the underlying asset follows the geometric Brownian motion (1.3) in the derivation of the Black-Scholes model. Now suppose the change $\triangle\Pi$ in the portfolio is balanced by a bond with the risk-free rate $r \geq 0$, i.e. $\triangle\Pi = r\Pi\triangle t$, using Ito's lemma for $\triangle V$, then applying the delta hedging strategy $\delta = -V_S$, we obtain generalization of the Black-Scholes equation:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV - r_{TC}S = 0 \tag{2.9}$$

Furthermore, applying Ito's formula for the function $\delta = -V_S$, we obtain

$$\triangle\delta = -\sigma S V_{SS} \triangle W = -\sigma S V_{SS}\ \Phi\sqrt{\triangle t} \tag{2.10}$$

plus higher order terms in $\sqrt{\triangle t}$. Here $\Phi \sim N(0,1)$ is a normally distributed random variable. Hence, in the lowest order $O(\sqrt{\triangle t})$ we have that

$$|\triangle\delta| = \alpha|\Phi|, \quad \text{where} \quad \alpha := \sigma S|V_{SS}|\sqrt{\triangle t} \tag{2.11}$$

For the case of constant transaction costs given by (2.5), using the fact that $\mathbb{E}[|\Phi|] = \sqrt{2/\pi}$, we obtain

$$r_{TC}S = \frac{\mathbb{E}[\triangle TC]}{\triangle t} = \frac{1}{2}cS\frac{\mathbb{E}[|\triangle\delta|]}{\triangle t} = \frac{1}{2}S^2\sqrt{\frac{2}{\pi}}\frac{c}{\sqrt{\triangle t}}\sigma|V_{SS}| = \frac{1}{2}\sigma^2 S^2 \mathrm{Le}|V_{SS}| \tag{2.12}$$

Inserting the term $r_{TC}S$ into (2.9) we obtain the Leland equation (2.1) with the volatility function given by (2.2):

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}\left[1 + \frac{c}{\sigma\sqrt{\delta t}}\sqrt{\frac{2}{\pi}}\text{sign}\Big(V_{SS}\Big)\right] + rSV_S - rV = 0 \tag{2.13}$$

Remark 1 *The Leland model (2.13) is in a form to which Galerkin method cannot be applied directly and we will perform transformations to this model for our numerical scheme, introduced in Chapter 4 later.*

2.2 Analysis of the Leland model

This section examines the transformation of the nonlinear Black-Scholes equation, the Leland model, into a quasilinear parabolic equation for the second derivative of the option price V, Γ, proposed by Jandačka and Ševčovič [14]. In addition, this section presents the transformation of the partial differential equation of the Leland model into a system of two equations using some change of variables so that weak formulation and Galerkin method can be applied to these equations.

2.2.1 Transformation of the Leland model into a quasilinear parabolic equation

For the Leland model (2.13), we introduce a new change of variables that is a product of Γ and the underlying asset:

$$H(x,\tau) = SV_{SS} = S\Gamma \tag{2.14}$$

and use the following notation

$$\beta(H) = \frac{1}{2}\hat{\sigma}^2(H)H \tag{2.15}$$

where, $\hat{\sigma}$ is the modified volatility function. We adapt the following proposition from the paper "Analysis of the nonlinear option pricing model under variable transaction costs" [4]. The following proposition states if $H = H(x, \tau)$ is a solution to a nonlinear parabolic equation (2.17) satisfying the initial and boundary conditions (2.18), then the nonlinear Black-Scholes equation has a solution.

Proposition 1 *Assume the function $V = V(S, t)$ is a solution to the nonlinear Black-Scholes equation*

$$V_t + S\beta(SV_{SS}) + rSV_S - rV = 0, \quad S > 0,\ t \in (0, T). \tag{2.16}$$

Then the transformed function $H = H(x, \tau) = SV_{SS}(S, t)$, where $x = ln(\frac{S}{K})$, $\tau = T - t$ is a solution to the quasilinear parabolic (Gamma) equation

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + r\partial_x H. \tag{2.17}$$

If H is a solution to (2.17) such that

$$H(-\infty, \tau) = \partial_x H(-\infty, \tau) = 0 \tag{2.18}$$

and $\beta'(0)$ is finite, then the function

$$V(S, t) = aS + be^{-r(T-t)} + \int_{-\infty}^{\infty} (S - Ke^x)^+ H(x, T - t)\ dx, \tag{2.19}$$

is a solution to the nonlinear Black-Scholes equation (2.16) for any $a, b \in \mathbb{R}$.

2.2.2 Reduction of the Leland model to a system of equations

We use the following change of variables to transform the nonlinear Black-Scholes model (2.13) into a diffusion partial differential equation:

$$\begin{aligned} &x = \ln S + k\tau,\ \tau = \frac{\sigma^2}{2}(T - t),\ k = \frac{2r}{\sigma^2} \\ &u(x, \tau) = e^{k\tau}\, V(e^{x - k\tau}, T - \frac{2}{\sigma^2}\tau) \end{aligned} \tag{2.20}$$

The partial derivatives of the option price V with respect to the asset price S and time t are expressed with partial derivatives of u in terms of x and τ:

$$
\begin{aligned}
V_t &= e^{-k\tau}(-k)(-\frac{\sigma^2}{2})u + e^{-k\tau}(-\frac{\sigma^2}{2})u_\tau = \frac{\sigma^2}{2}e^{-k\tau}(ku - u_\tau) \\
V_S &= \frac{1}{S}e^{-k\tau}u_x \\
V_{SS} &= -\frac{1}{S^2}e^{-k\tau}u_x + \frac{1}{S^2}e^{-k\tau}u_{xx}
\end{aligned} \tag{2.21}
$$

Substitution the above equation (2.21) into the equation (2.13) gives us

$$
u_{xx} + (k-1)u_x + \sqrt{\frac{2}{\pi}}\Big(\frac{2c}{\sigma\sqrt{\delta t}}\Big)\Big|u_{xx} - u_x\Big| = u_\tau \tag{2.22}
$$

Equation (2.22) can be transformed into the following system of two equations:

$$
\begin{cases}
u_{xx} - u_x = v \\
u_\tau = ku_x + v + \tilde{k}|v|
\end{cases} \tag{2.23}
$$

where $\tilde{k} = \sqrt{\frac{2}{\pi}}\Big(\frac{2c}{\sigma\sqrt{\delta t}}\Big)$. The above system will be used later for numerical solutions. The terminal and boundary conditions of (2.13) for $(S,t) \in (0,\infty) \times (0,T)$

$$
\begin{aligned}
&V(S,T) = \max(S-K,0) \quad \forall S \\
&V(0,t) = 0 \\
&V(S,t) = S \quad S \to \infty
\end{aligned} \tag{2.24}
$$

After applying the transformation (2.20), the equation (2.24) becomes

$$
\begin{aligned}
&u(x,0) = V(e^x,T) = \max(e^x - K, 0) \\
&u(x,\tau) = 0, \quad x \to -\infty \\
&u(x,\tau) = e^x, \quad x \to +\infty
\end{aligned} \tag{2.25}
$$

For computational reasons, the boundary conditions at $x = \pm\infty$ is replaced $x = \pm L$. Then we use the updated initial and boundary conditions for large number L:

$$\begin{aligned} &u(L, 0) = V(e^L, T) = \max(e^L - K, 0) \\ &u(-L, \tau) = 0 \\ &u(L, \tau) = e^L \end{aligned} \tag{2.26}$$

2.3 Well-posedness of the Leland model

A well-posedness of (2.22) with initial and boundary conditions (2.25) has been established in the paper "Solution to a nonlinear Black-Scholes equation" [2], in which the result a more general equation. For completeness, we introduce definitions and the existing results of this paper [2] in the following.

Let $\Omega \in \mathbb{R}^p$, $p \in \mathbb{N}$, be an open subset, let $C^k(\Omega; \mathbb{R}^n)$, $0 \leq k < \infty$ denote the set of all functions $f : \Omega \to \mathbb{R}^n$ that are continuously differentiable up to the order k. Further, $C^\infty(\Omega, \mathbb{R}^n)$ denotes the set of all arbitrary often differentiable functions. $C_0^\infty(\Omega, \mathbb{R}^n)$ is the set of test functions on Ω [5]. In our problem, Ω is the open interval $(-L, L)$.

Definition 2.3.1 (Locally integrable functions) *The set of locally integrable functions on* (Ω) *is denoted by*

$$L^1_{loc}(\Omega) := \{f : f \in L^1(K) \textit{ for all compact } K \subset \textit{int } (\Omega)\}$$

Definition 2.3.2 (Multi-index) *For a vector* $\alpha \in \mathbb{N}_0^d$ *used as a multi-index, its length is defined as*

$$|\alpha| := \sum_{j=1}^{d} \alpha_j$$

and for some $\phi \in C_0^\infty(\Omega)$*. We will write* $D^\alpha\phi$*. The order of the derivative is* $|\alpha|$*.*

Definition 2.3.3 *Suppose* u *and* $v \in L^1_{loc}(\Omega)$*, and alpha is a multi-index constant.* v *is said to be the* α^{th}*-weak partial derivative of* u*, denoted* $D^\alpha u = v$*, if*

$$\int_\Omega u D^\alpha \phi \; dx = (-1)^{|\alpha|} \int_\Omega v \phi \; dx$$

for any test function $\phi \in C_0^\infty(\Omega)$.

The Sobolev space

$$H^m(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), \text{for any multiindex } |\alpha| \text{with } |\alpha| \leq m\},$$

where the derivatives are taken in the weak sense, is a Hilbert space when endowed with the inner product

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}$$

Let $H_0^1(\Omega) = \{u \in H^1(\Omega)$ such that $u = 0$ on $\partial\Omega\}$ be the closure of C_0^∞ in $H^1(\Omega)$. Let the space $H^{-1}(\Omega)$ is the topological dual of $H_0^1(\Omega)$. Let X be a Banach space and let T^* be a nonnegative integer. The space $L^2(0, T^*; X)$ consists of all measurable functions $u : (0, T^*) \to X$ with

$$||u||_{L^2(0,T^*;X)} := \left(||u(\tau)||_X^2 d\tau\right)^{1/2} < \infty$$

The space $C([0, T^*; X])$ consists of all continuous functions $u := [0, T^*] \to X$ with

$$||u||_{C^2(0,T^*;X)} := \max_{0 \leq \tau \leq T^*} ||u(\tau)||_X < \infty$$

We take a derivative on both sides of (2.22) with respect x and set $w = u_x$:

$$w_{xx} + (k-1)w_x + \tilde{k}\Big|w_x - w\Big| = w_\tau \tag{2.27}$$

In paper [2], a similar problem was considered

$$-w_\tau + w_{xx} + \alpha\ w_x = -\beta F_x(w, w_x) \quad (x, \tau) \in \omega \times (0, T^*) \tag{2.28}$$

$$w(x, 0) = w_0(x) \quad x \in \Omega \tag{2.29}$$

$$w(x, \tau) = 0, \quad (x, \tau) \in \partial\Omega \times [0, T^*] \tag{2.30}$$

The equations (2.28)-(2.30) are reduced to our equation (2.27) when $\alpha = k-1,\ \beta = \tilde{k}$ and $F = |w_x - w|$.

Definition 2.3.4 *A function w is said to be a weak solution of (2.27)-(2.29) if $w \in L^2(0,T;H_0^1(\Omega)), w_\tau \in L^2(0,T;H^{-1}(\Omega))$ and*

$$\int_\Omega (w_\tau \phi + w_x \phi_x - \alpha w \phi_x) = -\beta \int_\Omega F_x(w, w_x)\phi_x \, dx \tag{2.31}$$

for all $\phi \in H_0^1(\Omega)$.

Theorem 1 *If F satisfies the following assumptions (A1)-(A5)*

$$F = |w_x - w|$$

(A1) $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ *is a continuous function*

(A2) $F(p,q) \leq |p| + |q|$

(A3) *For $u \in H^2_{loc}(\Omega)$, $(F(u,u_x))_x \in L^2(0,T;L^2_{loc}(\Omega))$. Then if $w_k \to w$ in $L^2(0,T;H_0^1(\Omega))$, then $F_x(w_k, w_{x_k}) \to F_x(w,w_x)$ in $L^2(0,T;L^2(\Omega))$*

(A4) $u_0 \in H^1_{loc}(\Omega)$

(A5) $\beta < 1$

then (2.28)-(2.30) has a weak solution $w \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$.

Remark 2 *Theorem 1 shows that $w = u_x \in L^2(0,T;H_0^1(\Omega))$ solves problem (2.28)-(2.30); so $u \in L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega))$ and is a strong solution of problem (2.22) in the bounded domain $\Omega \times [0,T]$ with zero Dirichlet condition on the lateral boundary of the domain.*

Remark 3 *It is shown in [2] the above theorem also holds $\Omega = R$ by taking limit.*

Remark 4 *The uniqueness of the solution is open.*

Since the nonlinear Leland model does not have a closed-form solution, then we will consider a finite element numerical method to solve (2.23) with the initial and boundary conditions (2.25) in the next chapter.

Chapter 3

Finite element formulation of the Leland model

This chapter presents the numerical solution of the transformed nonlinear Leland model (2.23) and (2.25) with the application of Galerkin finite element method. Finite element local and quadratic shape functions, finite element matrix equations, connectivity matrix and assembling of the global system are illustrated and then a weak formulation of the problem are presented. The finite element method converts the first equation in (2.23) into algebraic equations and the second equation into ordinary differential equations. The coupled algebraic equations with ordinary differential equations is called a differential algebraic system (DAE).

3.1 Introduction to finite element local and global shape functions

Before looking into the Galerkin finite element method, we introduce some basic Lagrange finite element interpolation shape functions, which are used in solving the second-order equations. We discretize a finite interval $[-L, L]$ by dividing into NE

subintervals by a partition

$$-L = x_1 < x_2 < \cdots < x_{NE} < x_{NE+1} = L$$

and denote a typical subinterval by $[x_1^{(e)}, x_2^{(e)}]$, where $e = 1, \ldots, NE$. This subinterval denotes a linear finite element $\Omega^{(e)}$ of length $l^{(e)} = x_2^{(e)} - x_1^{(e)}$. The end points of this subinterval are called the *global nodes(NG)* of the element. The point-slope form of the equation of the straight line passing through the two points $\left(x_1^{(e)}, u_1^{(e)}\right)$ and $\left(x_2^{(e)}, u_2^{(e)}\right)$ is

$$y - u_1^{(e)} = \frac{u_2^{(e)} - u_1^{(e)}}{x_2^{(e)} - x_1^{(e)}}\left(x - x_1^{(e)}\right)$$

which can be rewritten as

$$y = N_1^{(e)}(x)u_1^{(e)} + N_2^{(e)}(x)u_2^{(e)}$$

where

$$N_1^{(e)} = \frac{x_2^{(e)} - x}{x_2^{(e)} - x_1^{(e)}}, \quad N_2^{(e)} = \frac{x - x_1^{(e)}}{x_2^{(e)} - x_1^{(e)}} \tag{3.1}$$

are called the *local linear interpolation shape functions*, or the *local shape functions* [11]. For transient analysis of our system (2.23) with two unknowns, we use the following finite element approximation for linear elements

$$\begin{bmatrix} u(x,\tau) \\ v(x,\tau) \end{bmatrix} \equiv \begin{bmatrix} N_1^{(e)}(x) & 0 & N_2^{(e)}(x) & 0 \\ 0 & N_1^{(e)}(x) & 0 & N_2^{(e)}(x) \end{bmatrix} \begin{bmatrix} u_1^{(e)}(\tau) \\ u_2^{(e)}(\tau) \\ u_3^{(e)}(\tau) \\ u_4^{(e)}(\tau) \end{bmatrix} = \begin{bmatrix} u^{(e)}(x,\tau) \\ v^{(e)}(x,\tau) \end{bmatrix}$$

Let

$$\begin{aligned} u^{(e)}(x,\tau) &= N_1^{(e)}(x)u_1^{(e)}(\tau) + N_2^{(e)}(x)u_3^{(e)}(\tau), \quad x \in [x_1^{(e)}, x_2^{(e)}] \\ v^{(e)}(x,\tau) &= N_1^{(e)}(x)u_2^{(e)}(\tau) + N_2^{(e)}(x)u_4^{(e)}(\tau), \quad x \in [x_1^{(e)}, x_2^{(e)}] \end{aligned} \tag{3.2}$$

denotes the linear interpolation function for the interval $[x_1^{(e)}, x_2^{(e)}]$. The global piece-wise linear interpolation functions associated with the above partition
$-L = x_1 < x_2 < \cdots < x_{NE} < x_{NE+1} = L,\ \ NE + 1 = NG$ for the subintervals

$$[x_1^{(1)}, x_2^{(1)}] = [x_1, x_2], \ldots, [x_1^{(e)}, x_2^{(e)}] = [x_e, x_{e+1}], \ldots, [x_1^{(NE)}, x_2^{(NE)}] = [x_{NE}, x_{NE+1}]$$

is defined by

$$u(x) = \sum_{e=1}^{NE} \chi^{(e)}(x) u^{(e)}(x, \tau) = \sum_{e=1}^{NE} \chi^{(e)}(x)[N_1^{(e)}(x) u_1^{(e)}(\tau) + N_2^{(e)}(x) u_2^{(e)}(\tau)]$$

where $\chi^{(e)}$ is the set of characteristic function associated with the interval $[x_1^{(e)}, x_2^{(e)}]$:

$$\chi^{(e)}(x) = \begin{cases} 1, & x \in [x_1^{(e)}, x_2^{(e)}] \\ 0, & x \notin [x_1^{(e)}, x_2^{(e)}] \end{cases} \tag{3.3}$$

for $e = 1, \ldots, NE$. Denote $\mathbf{U} = [U_1, U_2, \ldots, U_{2NE+2}]^T$ as the global nodal values, then the connection between the global nodal values and local nodal values are

$$\begin{aligned} &U_{2e-1} = u_1^{(e)}, \quad U_{2e} = u_2^{(e)}, \\ &U_{2e+1} = u_3^{(e)}, \quad U_{2(e+1)} = u_4^{(e)}, \quad \text{for } e = 1, \ldots, NE \end{aligned} \tag{3.4}$$

So the eth row of the connectivity matrix C for the linear elements can be defined by

$$C(e, NL) = [e, e + 1], \quad e = 1, 2, \ldots NE \tag{3.5}$$

Then the global linear shape functions are defined by

$$u(x, \tau) = \sum_{j=1}^{NG} N_j(x) U_{2j-1}(\tau) \tag{3.6}$$

$$v(x, \tau) = \sum_{j=1}^{NG} N_j(x) U_{2j}(\tau) \tag{3.7}$$

where $N_j(x) = N_i^{(e)}(x)$ for x in the eth element interval where $j = C(e,i)$ for some $i = 1, 2$ for linear elements, otherwise $N_j(x) = 0$.

Let us consider local and global quadratic shape functions. Suppose a finite element $\Omega^{(e)}$ that consists of the end points $x_1^{(e)}$ and $x_2^{(e)}$ and the midpoint with $x_3^{(e)}$ such that $x_3^{(e)} = x_1^{(e)} + \frac{l^{(e)}}{2}$, where $l^{(e)} = x_2^{(e)} - x_1^{(e)}$ denotes the length of the element. Then, the quadratic interpolation function is

$$\begin{aligned} u^{(e)}(x,\tau) &= N_1^{(e)}(x)u_1^{(e)}(\tau) + N_2^{(e)}(x)u_3^{(e)}(\tau) + N_3^{(e)}(x)u_5^{(e)}(\tau) \\ v^{(e)}(x,\tau) &= N_1^{(e)}(x)u_2^{(e)}(\tau) + N_2^{(e)}(x)u_4^{(e)}(\tau) + N_3^{(e)}(x)u_6^{(e)}(\tau) \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} N_1^{(e)}(x) &= \frac{\left(x_2^{(e)} - x\right)\left(x_3^{(e)} - x\right)}{\left(x_2^{(e)} - x_1^{(e)}\right)\left(x_3^{(e)} - x_1^{(e)}\right)} \\ N_2^{(e)}(x) &= \frac{\left(x_1^{(e)} - x\right)\left(x_3^{(e)} - x\right)}{\left(x_1^{(e)} - x_2^{(e)}\right)\left(x_3^{(e)} - x_2^{(e)}\right)} \\ N_3^{(e)}(x) &= \frac{\left(x_1^{(e)} - x\right)\left(x_2^{(e)} - x\right)}{\left(x_1^{(e)} - x_2^{(e)}\right)\left(x_2^{(e)} - x_3^{(e)}\right)} \end{aligned} \tag{3.9}$$

These quadratic functions $N_1^{(e)}(x), N_2^{(e)}(x)$, and $N_3^{(e)}(x)$ are called the *local quadratic shape functions*. For transient analysis of our system (2.23) with two unknowns, we use the following finite element approximation for quadratic elements

$$\begin{bmatrix} u^{(e)}(x,\tau) \\ v^{(e)}(x,\tau) \end{bmatrix} \equiv \begin{bmatrix} N_1^{(e)}(x) & 0 & N_2^{(e)}(x) & 0 & N_3^{(e)}(x) & 0 \\ 0 & N_1^{(e)}(x) & 0 & N_2^{(e)}(x) & 0 & N_3^{(e)}(x) \end{bmatrix} \begin{bmatrix} u_1^{(e)}(\tau) \\ u_2^{(e)}(\tau) \\ u_3^{(e)}(\tau) \\ u_4^{(e)}(\tau) \\ u_5^{(e)}(\tau) \\ u_6^{(e)}(\tau) \end{bmatrix}$$

The global piecewise quadratic interpolation functions associated with the above partition $-L = x_1 < x_2 < \cdots < x_{NE} < x_{NE+1} = L,\ \ NE + 1 = NG$ for the subintervals

$$[x_1^{(e)}, x_2^{(e)}] = [x_e, x_{e+1}], \ldots, [x_1^{(NE)}, x_2^{(NE)}] = [x_{NE}, x_{NE+1}]$$

is defined by as following

$$u(x,\tau) = \sum_{e=1}^{NE} \chi^{(e)}(x)u^{(e)}(x,\tau) = \sum_{e=1}^{NE} \chi^{(e)}(x)[N_1^{(e)}(x)u_1^{(e)}(\tau) + N_2^{(e)}(x)u_2^{(e)}(\tau) + N_3^{(e)}(x)u_3^{(e)}(\tau)]$$
$$v(x,\tau) = \sum_{e=1}^{NE} \chi^{(e)}(x)v^{(e)}(x,\tau) = \sum_{e=1}^{NE} \chi^{(e)}(x)[N_1^{(e)}(x)u_2^{(e)}(\tau) + N_2^{(e)}(x)u_4^{(e)}(\tau) + N_3^{(e)}(x)u_6^{(e)}(\tau)]$$

where $\chi^{(e)}$ is the set of characteristic function associated with the interval $[x_1^{(e)}, x_3^{(e)}]$:

$$\chi^{(e)}(x) = \begin{cases} 1, & x \in [x_1^{(e)}, x_3^{(e)}] \\ 0, & x \notin [x_1^{(e)}, x_3^{(e)}] \end{cases} \tag{3.10}$$

for $e = 1, \ldots, NE$. Assume that $\mathbf{U} = [U_1, U_2, \ldots, U_{2NE+2}]^T$ are global nodal values, then the global nodal values and local nodal values are connected as following

$$\begin{aligned} &U_{2e-1} = u_1^{(e)}, \quad U_{2e} = u_2^{(e)}, \\ &U_{2e+1} = u_3^{(e)}, \quad U_{2(e+1)} = u_4^{(e)}, \quad \text{for } e = 1, \ldots, NE \end{aligned} \tag{3.11}$$

So the eth row of the connectivity matrix C for the quadratic elements can be defined by

$$C(e,3) = [e, e+1, e+2], \quad e = 1, \ldots NE$$

The global quadratic shape functions are defined by

$$u(x,\tau) = \sum_{j=1}^{NG} N_j(x)U_{2j-1}(\tau) \tag{3.12}$$

$$v(x,\tau) = \sum_{j=1}^{NG} N_j(x)U_{2j}(\tau) \tag{3.13}$$

where $N_j(x) = N_i^{(e)}(x)$ for x in the eth element interval where $j = C(e,i)$ for some $i = 1,2,3$ for quadratic elements, otherwise $N_j(x) = 0$

3.2 Weak formulation

Before applying the Galerkin method for finite element analysis, we discretize the interval $[-L, L]$ by subintervals. We define the finite element subspace for the solution of u and v of (2.23) and (2.25):

$$S_b^h(-L,L) = \{u | u(x,\tau) = \sum_{e=1}^{NE} = \chi^{(e)} u^{(e)}(x,\tau), u(L) = e^L, u(-L) = 0\} \subset H_b^1(-L,L)$$
$$S_0^h(-L,L) = \{u | u(x,\tau) = \sum_{e=1}^{NE} = \chi^{(e)} u^{(e)}(x,\tau), u(L) = 0, u(-L) = 0\} \subset H_0^1(-L,L)$$

where $\chi^{(e)}(x) = \begin{cases} 1, & \text{if} \quad x \in [x_1^{(e)}, x_2^{(e)}] \\ 0, & \text{if} \quad x \notin [x_1^{(e)}, x_2^{(e)}] \end{cases}$ is the characteristic function for linear elements and

$$u^{(e)}(x,\tau) = N_1^{(e)} u_1^{(e)}(\tau) + N_2^{(e)} u_2^{(e)}(\tau), \quad x \in [x_1^{(e)}, x_2^{(e)}]$$

a linear finite element interpolation function. The characteristic function for quadratic elements $\chi^{(e)}(x) = \begin{cases} 1, & \text{if} \quad x \in [x_1^{(e)}, x_3^{(e)}] \\ 0, & \text{if} \quad x \notin [x_1^{(e)}, x_3^{(e)}] \end{cases}$ and then the quadratic finite element interpolation function:

$$u^{(e)}(x,\tau) = N_1^{(e)} u_1^{(e)}(\tau) + N_2^{(e)} u_2^{(e)}(\tau) + N_3^{(e)} u_3^{(e)}, \quad x \in [x_1^{(e)}, x_3^{(e)}]$$

For the system of equations (2.23) we look for solutions $\vec{u} = \begin{Bmatrix} u \\ v \end{Bmatrix} \in H_b^1(-L,L) \times H^1(-L,L)$ for the finite element interpolation subspace is

$$\vec{u}_h = \begin{Bmatrix} u_h \\ v_h \end{Bmatrix} \in S_b^h(-L,L) \times S^h(-L,L) \subset H_b^1(-L,L) \times H^1(-L,L)$$

The finite element approximation of the weak formulation for the problem (2.23) with boundary conditions (2.25) is to solve for $\vec{u}_h \in S_b^h(-L,L) \times S^h(-L,L)$, such that

$$\int_{-L}^{L} u_h' \cdot w_1' \; dx + \int_{-L}^{L} u_h' \cdot w_1 = -\int_{-L}^{L} \; v_h \cdot w_1 \; dx + u_x \cdot w_1 \Big|_{-L}^{L} \tag{3.14}$$

$$\int_{-L}^{L} (u_h)_\tau \cdot \; w_2 \; dx = \int_{-L}^{L} v_h \cdot w_2 \; dx + \beta \int_{-L}^{L} |v_h| \cdot w_2 \; dx + \alpha \int_{-L}^{L} u_h' \cdot w_2 \; dx \tag{3.15}$$

where $\vec{w} = \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} \in S_0^h(-L,L) \times S_0^h(-L,L)$ and $w_1 = w_2 = N_j$ for $j = 1, 2, \ldots NG$. (3.14) and (3.15) are called the global system of finite element equation.

3.3 The local and global finite element system for the Leland model

We apply the Galerkin method for an approximation of solution u and v in the finite interval $[x_1^{(e)}, x_2^{(e)}]$, multiplying both sides of equation (3.14) by the global shape function, $N_j(x), j = 1, 2, \ldots NG$, and then we integrate the resulting equations over the interval $[x_1^{(e)}, x_2^{(e)}]$ to get

$$\begin{aligned} -\int_{x_1^{(e)}}^{x_2^{(e)}} u_x^{(e)} N_j'^{(e)} \; dx + u_x N_j^{(e)} \Big|_{x_1^{(e)}}^{x_2^{(e)}} - \int_{x_1^{(e)}}^{x_2^{(e)}} u_x^{(e)} N_j^{(e)} \; dx &= \int_{x_1^{(e)}}^{x_2^{(e)}} v^{(e)} N_j^{(e)} \; dx \\ \int_{x_1^{(e)}}^{x_2^{(e)}} u_\tau^{(e)} N_j^{(e)} \; dx = k \int_{x_1^{(e)}}^{x_2^{(e)}} u_x^{(e)} N_j^{(e)} \; dx + \int_{x_1^{(e)}}^{x_2^{(e)}} & v^{(e)} N_j^{(e)} \; dx + \tilde{k} \int_{x_1^{(e)}}^{x_2^{(e)}} |v^{(e)}| N_j^{(e)} \; dx \end{aligned} \tag{3.16}$$

where $u^{(e)}$ and $v^{(e)}$ are approximated by the equation (3.2), $|v^{(e)}|$ is approximated as following

$$|v^{(e)}| \approx \chi^{(e)}(x)[\; N_1^{(e)}(x)|v_1^{(e)}| + N_2^{(e)}(x)|v_2^{(e)}| \;]$$

And we integrate the system of two equations (2.23) over the interval $[x_1^{(e)}, x_3^{(e)}]$ and we obtain

$$\begin{aligned} -\int_{x_1^{(e)}}^{x_3^{(e)}} u_x^{(e)} N_j'^{(e)} \; dx + u_x N_j^{(e)} \Big|_{x_1^{(e)}}^{x_3^{(e)}} - \int_{x_1^{(e)}}^{x_3^{(e)}} u_x^{(e)} N_j^{(e)} \; dx &= \int_{x_1^{(e)}}^{x_3^{(e)}} v^{(e)} N_j^{(e)} \; dx \\ \int_{x_1^{(e)}}^{x_3^{(e)}} u_\tau^{(e)} N_j^{(e)} \; dx = k \int_{x_1^{(e)}}^{x_3^{(e)}} u_x^{(e)} N_j^{(e)} \; dx + \int_{x_1^{(e)}}^{x_3^{(e)}} & v^{(e)} N_j^{(e)} \; dx + \tilde{k} \int_{x_1^{(e)}}^{x_3^{(e)}} |v^{(e)}| N_j^{(e)} \; dx \end{aligned} \tag{3.17}$$

where $u^{(e)}$ and $v^{(e)}$ are approximated by the equation (3.8), $|v^{(e)}|$ is approximated as following

$$|v^{(e)}| \approx \ \chi^{(e)}(x)[N_1^{(e)}(x)|v_1^{(e)}| + N_2^{(e)}(x)|v_2^{(e)} + N_3^{(e)}(x)|v_3^{(e)}|]$$

The next step is the replacement of u by $u^{(e)}(x,\tau)$ and v by $v^{(e)}(x,\tau)$ in the equation (3.2) and take the sum over all over elements. As a result, we obtain

$$\sum_i^{NG}\left(\int_{x_1^{(e)}}^{x_2^{(e)}} N_i'^{(e)} N_j'^{(e)}\ dx + \int_{x_1^{(e)}}^{x_2^{(e)}} N_i'^{(e)} N_j^{(e)}\ dx\right)u_i + \sum_i^{NG}\left(\int_{x_1^{(e)}}^{x_2^{(e)}} N_i^{(e)} N_j^{(e)}\ dx\right)v_i = u_x N_1^{(1)}\Big|_{x_1^{(1)}}^{x_2^{(1)}} + u_x N_2^{(NE)}\Big|_{x_1^{(NE)}}^{x_2^{(NE)}}$$

$$\sum_i^{NG}\left(\int_{x_1^{(e)}}^{x_2^{(e)}} N_i^{(e)} N_j^{(e)}\ dx\right)\dot{u}_i = \sum_i^{NG}\left(k\int_{x_1^{(e)}}^{x_2^{(e)}} N_i'^{(e)} N_j^{(e)}\ dx\right)u_i + \sum_i^{NG}\left(\int_{x_1^{(e)}}^{x_2^{(e)}} N_i^{(e)} N_j^{(e)}\ dx\right)v_i + \\ + \sum_i^{NG}\left(\tilde{k}\int_{x_1^{(e)}}^{x_2^{(e)}} N_i^{(e)} N_j^{(e)}\ dx\right)|v_i|$$

The above local finite element matrix equation can be written in the following form

$$\begin{aligned} &\mathbf{K}^{(e)}\mathrm{U}^{(e)} + \mathbf{P}^{(e)}\mathrm{U}^{(e)} + \mathbf{N}^{(e)}\mathrm{V}^{(e)} = \mathbf{Q}^{(e)} \\ &\mathbf{M}^{(e)}\dot{\mathrm{U}}^{(e)} = k\mathbf{P}^{(e)}\mathrm{U}^{(e)} + \mathbf{N}^{(e)}\mathrm{V}^{(e)} + \tilde{k}\mathbf{N}^{(e)}|\mathrm{V}^{(e)}| \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} K_{ij}^{(e)} &= \int_{x_1^{(e)}}^{x_2^{(e)}} N_i'^{(e)} N_j'^{(e)}\ dx \\ P_{ij}^{(e)} &= \int_{x_1^{(e)}}^{x_2^{(e)}} N_i'^{(e)} N_j^{(e)}\ dx \\ M_{ij}^{(e)} &= N_{ij}^{(e)} = \int_{x_1^{(e)}}^{x_2^{(e)}} N_i^{(e)} N_j^{(e)}\ dx \\ Q_j^{(e)} &= u_x N_j^{(e)}\Big|_{x_1^{(e)}}^{x_2^{(e)}} \end{aligned} \tag{3.19}$$

$i,j = 1,2$ for the two-node linear element and $i,j = 1,2,3$ for the three-node quadratic element, for $\mathbf{K}^{(e)}, \mathbf{N}^{(e)}, \mathbf{M}^{(e)}$ respectively. Note that $Q = 0$ for $j = 2,\ldots,NG-1$, then we have NG equations and $NG+2$ unknowns. Then we assemble the global matrices adding the above local matrices and using the following connectivity mappings:

For the stiffness global matrix $\mathbf{K}$

$$
\begin{array}{llll}
K_{11}^{(e)} \to K_{2e-1,2e-1}, & K_{12}^{(e)} \to K_{2e-1,2e}, & K_{13}^{(e)} \to K_{2e-1,2e+1}, & K_{14}^{(e)} \to K_{2e-1,2e+2} \\
K_{21}^{(e)} \to K_{2e,2e-1}, & K_{22}^{(e)} \to K_{2e,2e}, & K_{23}^{(e)} \to K_{2e,2e+1}, & K_{24}^{(e)} \to K_{2e,2e+2} \\
K_{31}^{(e)} \to K_{2e+1,2e-1}, & K_{32}^{(e)} \to K_{2e+1,2e}, & K_{33}^{(e)} \to K_{2e+1,2e+1}, & K_{34}^{(e)} \to K_{2e+1,2e+2} \\
K_{41}^{(e)} \to K_{2e+2,2e-1}, & K_{42}^{(e)} \to K_{2e+2,2e}, & K_{43}^{(e)} \to K_{2e+2,2e+1}, & K_{44}^{(e)} \to K_{2e+2,2e+2}
\end{array}
$$

For the global mass matrix $\mathbf{M}$

$$
\begin{array}{llll}
M_{11}^{(e)} \to M_{2e-1,2e-1}, & M_{12}^{(e)} \to M_{2e-1,2e}, & M_{13}^{(e)} \to M_{2e-1,2e+1}, & M_{14}^{(e)} \to M_{2e-1,2e+2} \\
M_{21}^{(e)} \to M_{2e,2e-1}, & M_{22}^{(e)} \to M_{2e,2e}, & M_{23}^{(e)} \to M_{2e,2e+1}, & M_{24}^{(e)} \to M_{2e,2e+2} \\
M_{31}^{(e)} \to M_{2e+1,2e-1}, & M_{32}^{(e)} \to M_{2e+1,2e}, & M_{33}^{(e)} \to M_{2e+1,2e+1}, & M_{34}^{(e)} \to M_{2e+1,2e+2} \\
M_{41}^{(e)} \to M_{2e+2,2e-1}, & M_{42}^{(e)} \to M_{2e+2,2e}, & M_{43}^{(e)} \to M_{2e+2,2e+1}, & M_{44}^{(e)} \to M_{2e+2,2e+2}
\end{array}
$$

Similarly, for the global matrix $\mathbf{P}$

$$
\begin{array}{llll}
P_{11}^{(e)} \to P_{2e-1,2e-1}, & P_{12}^{(e)} \to P_{2e-1,2e}, & P_{13}^{(e)} \to P_{2e-1,2e+1}, & P_{14}^{(e)} \to P_{2e-1,2e+2} \\
P_{21}^{(e)} \to P_{2e,2e-1}, & P_{22}^{(e)} \to P_{2e,2e}, & P_{23}^{(e)} \to P_{2e,2e+1}, & P_{24}^{(e)} \to P_{2e,2e+2} \\
P_{31}^{(e)} \to P_{2e+1,2e-1}, & P_{32}^{(e)} \to P_{2e+1,2e}, & P_{33}^{(e)} \to P_{2e+1,2e+1}, & P_{34}^{(e)} \to P_{2e+1,2e+2} \\
P_{41}^{(e)} \to P_{2e+2,2e-1}, & P_{42}^{(e)} \to P_{2e+2,2e}, & P_{43}^{(e)} \to P_{2e+2,2e+1}, & P_{44}^{(e)} \to P_{2e+2,2e+2}
\end{array}
$$

The global finite element matrix system of equations for the Leland model has the following matrix form which was derived in the previous section

$$
\begin{cases}
\mathbf{K}U + \mathbf{P}U + \mathbf{N}V = \mathbf{Q} \\
\mathbf{M}\dot{U} = k\mathbf{P}U + \mathbf{N}V + \tilde{k}\mathbf{N}|V|
\end{cases}
\tag{3.20}
$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{M}$ is the global mass matrix.

$$\begin{aligned}
K_{ij} &= \int_{-L}^{L} N_i' N_j' \, dx \\
P_{ij} &= \int_{-L}^{L} N_i' N_j \, dx \\
M_{ij} = N_{ij} &= \int_{-L}^{L} N_i N_j \, dx \\
Q_j &= u_x N_j \Big|_{-L}^{L}
\end{aligned} \tag{3.21}$$

In the following section we will apply adapted Crank-Nicolson to solve differential algebraic system (DAE) (3.20) with initial and boundary conditions. By standard calculation, we have local linear finite element matrix equation for the system (2.23):

$$\begin{aligned}
\frac{l^{(e)}}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}\begin{Bmatrix} \dot{u}_1^{(e)} \\ \dot{u}_2^{(e)} \\ \dot{u}_3^{(e)} \\ \dot{u}_4^{(e)} \end{Bmatrix} &= \begin{bmatrix} \frac{1}{l^{(e)}} - \frac{1}{2} & \frac{l^{(e)}}{3} & -\frac{1}{l^{(e)}} + \frac{1}{2} & \frac{l^{(e)}}{6} \\ -\frac{k}{2} & \frac{l^{(e)}}{3} & \frac{k}{2} & \frac{l^{(e)}}{6} \\ -\frac{1}{l^{(e)}} - \frac{1}{2} & \frac{l^{(e)}}{6} & \frac{1}{l^{(e)}} + \frac{1}{2} & \frac{l^{(e)}}{3} \\ -\frac{k}{2} & \frac{l^{(e)}}{6} & \frac{k}{2} & \frac{l^{(e)}}{3} \end{bmatrix}\begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \\ u_4^{(e)} \end{Bmatrix} + \\
&+ \frac{\tilde{k} l^{(e)}}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} |u_1^{(e)}| \\ |u_2^{(e)}| \\ |u_3^{(e)}| \\ |u_4^{(e)}| \end{Bmatrix} - \begin{Bmatrix} -u_x(-L,\tau) \\ 0 \\ u_x(L,\tau) \\ 0 \end{Bmatrix}
\end{aligned} \tag{3.22}$$

and local quadratic finite element matrix equation for the system (2.33)

$$\frac{l^{(e)}}{30}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 16 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 4 & 0 \end{bmatrix}\begin{Bmatrix} \dot{u_1}^{(e)} \\ \dot{u_2}^{(e)} \\ \dot{u_3}^{(e)} \\ \dot{u_4}^{(e)} \\ \dot{u_5}^{(e)} \\ \dot{u_6}^{(e)} \end{Bmatrix} = \frac{\tilde{k}l^{(e)}}{30}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 16 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & 4 \end{bmatrix}\begin{Bmatrix} |u_1^{(e)}| \\ |u_2^{(e)}| \\ |u_3^{(e)}| \\ |u_4^{(e)}| \\ |u_5^{(e)}| \\ |u_6^{(e)}| \end{Bmatrix} +$$

$$+\begin{bmatrix} \frac{7}{3l^{(e)}}-\frac{1}{2} & \frac{2l^{(e)}}{15} & -\frac{8}{3l^{(e)}}-\frac{2}{3} & \frac{l^{(e)}}{15} & \frac{1}{3l^{(e)}}-\frac{1}{6} & -\frac{l^{(e)}}{30} \\ -\frac{k}{2} & \frac{2l^{(e)}}{15} & -\frac{2k}{3} & \frac{l^{(e)}}{15} & -\frac{k}{6} & -\frac{l^{(e)}}{30} \\ -\frac{8}{3l^{(e)}}-\frac{2}{3} & \frac{l^{(e)}}{15} & \frac{16}{3l^{(e)}} & \frac{8l^{(e)}}{15} & -\frac{8}{3l^{(e)}}+\frac{2}{3} & \frac{l^{(e)}}{15} \\ -\frac{2k}{3} & \frac{l^{(e)}}{15} & 0 & \frac{8l^{(e)}}{15} & -\frac{2k}{3} & \frac{l^{(e)}}{15} \\ \frac{1}{3l^{(e)}}+\frac{1}{6} & -\frac{l^{(e)}}{30} & -\frac{8}{3l^{(e)}}-\frac{2}{3} & \frac{l^{(e)}}{15} & \frac{7}{3l^{(e)}}+\frac{1}{2} & \frac{2l^{(e)}}{15} \\ \frac{k}{6} & -\frac{l^{(e)}}{30} & -\frac{2k}{3} & \frac{l^{(e)}}{15} & \frac{k}{2} & \frac{2l^{(e)}}{15} \end{bmatrix}\begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \\ u_4^{(e)} \\ u_5^{(e)} \\ u_6^{(e)} \end{Bmatrix} - \begin{Bmatrix} -u_x(-L,\tau) \\ 0 \\ 0 \\ 0 \\ u_x(L,\tau) \\ 0 \end{Bmatrix}$$

For demonstration purpose, we illustrate an assembling of 1 linear and 1 quadratic element of equal length $l^{(1)} = l^{(2)} = l$ for the system (2.23). The connectivity matrix is used to assemble local and quadratic elements into the global finite element matrix of mixed elements:

$$
\frac{l}{30}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 14 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 16 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & 4 & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \\ \dot{u}_7 \\ \dot{u}_8 \end{Bmatrix} = \frac{\tilde{k}l}{30}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 14 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 16 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & 4 \end{bmatrix} \begin{Bmatrix} |u_1| \\ |u_2| \\ |u_3| \\ |u_4| \\ |u_5| \\ |u_6| \\ |u_7| \\ |u_8| \end{Bmatrix}
$$

$$
+\begin{bmatrix} \frac{1}{l}-\frac{1}{2} & \frac{l}{3} & -\frac{1}{l}+\frac{1}{2} & \frac{l}{6} & 0 & 0 & 0 & 0 \\ -\frac{k}{2} & \frac{l}{3} & \frac{k}{2} & \frac{l}{6} & 0 & 0 & 0 & 0 \\ -\frac{1}{l}-\frac{1}{2} & \frac{l}{6} & \frac{l}{6}+\frac{7}{3l} & \frac{7l}{15} & -\frac{8}{3l}-\frac{2}{3} & \frac{l}{15} & \frac{1}{3l}-\frac{1}{6} & -\frac{l}{30} \\ -\frac{k}{2} & \frac{l}{6} & 0 & \frac{7l}{15} & -\frac{2k}{3} & \frac{l}{15} & -\frac{k}{6} & -\frac{l}{30} \\ 0 & 0 & -\frac{8}{3l}-\frac{2}{3} & \frac{l}{15} & \frac{16}{3l} & \frac{8l}{15} & -\frac{8}{3l}+\frac{2}{3} & \frac{l}{15} \\ 0 & 0 & -\frac{2k}{3} & \frac{l}{15} & 0 & \frac{8l}{15} & -\frac{2k}{3} & \frac{l}{15} \\ 0 & 0 & \frac{1}{3l}+\frac{1}{6} & -\frac{l}{30} & -\frac{8}{3l}-\frac{2}{3} & \frac{l}{15} & \frac{7}{3l}+\frac{1}{2} & \frac{2l}{15} \\ 0 & 0 & \frac{k}{6} & -\frac{l}{30} & -\frac{2k}{3} & \frac{l}{15} & \frac{k}{2} & \frac{2l}{15} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} - \begin{Bmatrix} -u_x(-L,\tau) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u_x(L,\tau) \\ 0 \end{Bmatrix}
$$

The initial and boundary conditions (2.25) are then applied to the above global finite element system, then we obtain

$$\frac{30}{l}\begin{bmatrix} \frac{l}{3} & -\frac{1}{l}+\frac{1}{2} & \frac{l}{6} & 0 & 0 & 0 \\ \frac{l}{6} & \frac{l}{6}+\frac{7}{3l} & \frac{7l}{15} & -\frac{8}{3l}-\frac{2}{3} & \frac{l}{15} & \frac{1}{3l}-\frac{1}{6} \\ 0 & -\frac{8}{3l}-\frac{2}{3} & \frac{l}{15} & \frac{16}{3l} & \frac{8l}{15} & -\frac{8}{3l}+\frac{2}{3} \\ 0 & \frac{1}{3l}+\frac{1}{6} & -\frac{l}{30} & -\frac{8}{3l}-\frac{2}{3} & \frac{l}{15} & \frac{7}{3l}+\frac{1}{2} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 2 \\ 4 \end{bmatrix} e^L + \begin{Bmatrix} u_9 \\ 0 \\ 0 \\ u_{10} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 14 & 0 & 2 & 0 & -1 \\ 0 & 2 & 0 & 16 & 0 & 2 \\ 0 & -1 & 0 & 2 & 0 & 4 \end{bmatrix}\begin{Bmatrix} \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \\ \dot{u}_7 \end{Bmatrix} = \tilde{k}\begin{bmatrix} 10 & 0 & 5 & 0 & 0 & 0 \\ 5 & 0 & 14 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 16 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 \end{bmatrix}\begin{Bmatrix} |u_2| \\ |u_3| \\ |u_4| \\ |u_5| \\ |u_6| \\ |u_7| \end{Bmatrix} + \begin{bmatrix} 0 \\ -\tilde{k}-\frac{l}{30} \\ 2\tilde{k}+\frac{l}{15} \\ 4\tilde{k}+\frac{2l}{15} \end{bmatrix} e^L +$$

$$+\frac{30}{l}\begin{bmatrix} \frac{l}{3} & \frac{k}{2} & \frac{l}{6} & 0 & 0 & 0 \\ \frac{l}{6} & 0 & \frac{7l}{15} & -\frac{2k}{3} & \frac{l}{15} & -\frac{k}{6} \\ 0 & -\frac{2k}{3} & \frac{l}{15} & 0 & \frac{8l}{15} & -\frac{2k}{3} \\ 0 & \frac{k}{6} & -\frac{l}{30} & -\frac{2k}{3} & \frac{l}{15} & \frac{k}{2} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix} \tag{3.23}$$

where $u_9 = -u_x(-L,\tau)$ and $u_{10} = u_x(L,\tau)$. The last equation (3.23) is a DAE with semi-explicit or Hessenberg Index-1 [1], which is a special form of the following

$$U' = f(\tau, U, V) \text{ (differential equation)}$$

$$0 = g(\tau, U, V) \text{ (algebraic equation)}$$

U is the differential variable, V is the index-1 algebraic variable.

In the next chapter we will present a modified Crank-Nicolson numerical scheme for the finite element DAE system.

Chapter 4

Numerical solutions of the Leland model

In this Chapter Crank-Nicolson numerical scheme is adopted for solving the DAE system for option pricing from the Leland model. A Matlab code is developed to implement a numerical example and numerical results are presented.

4.1 Crack-Nicolson scheme for solving DAE

Recall the global DAE finite element matrix has the following matrix form:

$$\begin{cases} \mathbf{K}U + \mathbf{P}U + \mathbf{N}V = \mathbf{Q} \\ \mathbf{M}\dot{U} = k\mathbf{P}U + \mathbf{N}V + \tilde{k}\mathbf{N}|V| \end{cases} \tag{4.1}$$

We use the θ-scheme at the two consecutive time steps τ_n and τ_{n+1}. It is defined by

$$\frac{U_{n+1} - U_n}{\triangle\tau} = \theta\dot{U}_{n+1} + (1-\theta)\dot{U}_n, \quad 0 \le \theta \le 1 \tag{4.2}$$

where $\triangle\tau_{n+1} = \tau_{n+1} - \tau_n$. Substituting (4.2) into the system of equations (4.1), we obtain

$$\begin{cases} \frac{1}{2}\mathbf{K}(U_{n+1} + U_n) + \frac{1}{2}\mathbf{P}(U_{n+1} + U_n) + \frac{1}{2}\mathbf{N}(V_{n+1} + V_n) = \mathbf{Q} \\ \mathbf{M}\ \frac{U_{n+1} - U_n}{\triangle\tau} = \frac{k}{2}\ \mathbf{P}(U_{n+1} + U_n) + \frac{1}{2}\mathbf{N}(V_{n+1} + V_n) + \frac{\tilde{k}}{2}\ \mathbf{N}\Big|(V_{n+1} + V_n)\Big| \end{cases} \tag{4.3}$$

In the system (4.3), the second equation can be expressed through the first equation, then we have

$$\begin{aligned}\mathbf{M}\frac{U_{n+1}-U_n}{\triangle\tau} =& \frac{k}{2}\ \mathbf{P}(U_{n+1}+U_n)+\mathbf{Q}-\frac{1}{2}\mathbf{K}(U_{n+1}+U_n)-\frac{1}{2}\mathbf{P}(U_{n+1}+U_n)+\\ &+\tilde{k}\Big|\mathbf{Q}-\frac{1}{2}(\mathbf{K}+\mathbf{P})(U_{n+1}+U_n)\Big|\end{aligned} \tag{4.4}$$

It can be seen that the expression $(\mathbf{K}+\mathbf{P})\big|(U_{n+1}+U_n)\big|$ cannot be separated in terms of U_{n+1} and U_n. Now we suppose that $(\mathbf{K}+\mathbf{P})\big|(U_{n+1}+U_n)\big| = (\mathbf{K}+\mathbf{P})|U_n|$ and Q is positively defined, then the equation (4.4) has the following form

$$\begin{aligned}\mathbf{M}\frac{U_{n+1}-U_n}{\triangle\tau} = \frac{k}{2}\ \mathbf{P}(U_{n+1}+U_n)+\mathbf{Q}-\frac{1}{2}\mathbf{K}(U_{n+1}+U_n)-\frac{1}{2}\mathbf{P}(U_{n+1}+U_n)+\\ +\tilde{k}\mathbf{Q}+\frac{\tilde{k}}{2}(\mathbf{K}+\mathbf{P})\big|U_n\big|\end{aligned} \tag{4.5}$$

The equation (4.5) can be simplified to

$$\hat{\mathbf{M}}U_{n+1} = \hat{\mathbf{K}}U_n + \hat{\mathbf{N}}|U_n| + \hat{\mathbf{Q}} \tag{4.6}$$

where

$$\begin{aligned}\hat{\mathbf{M}} &= \mathbf{M}+\frac{\triangle\tau}{2}\mathbf{K}-(k-1)\frac{\triangle\tau}{2}\mathbf{P}\\ \hat{\mathbf{K}} &= \mathbf{M}-\frac{\triangle\tau}{2}\mathbf{K}+(k-1)\frac{\triangle\tau}{2}\mathbf{P}\\ \hat{\mathbf{N}} &= \frac{\tilde{k}}{4}(\mathbf{K}+\mathbf{P})\\ \hat{\mathbf{Q}} &= \triangle\tau(\tilde{k}+1)\mathbf{Q}\end{aligned} \tag{4.7}$$

(4.6) and (4.7) is our modified Crank-Nicolson method. The equation (4.6) determines the solution of (4.1) at time $\tau = \tau_{n+1}$ with the known solution at time $\tau = \tau_n$. Then, we use the initial condition at $\tau_0 = 0$ and boundary conditions for (4.6). Now, once the algebraic system (4.6) is solved for each time τ_i, the approximate solution $V(S,t)$ in terms of the original variables can be obtained using inverse transformation of variable via (2.20).

4.2 Comparison study

In this Section, we use the following parameters from paper [10] as an example of our comparison study

Interest rate	r	0.1
Historical volatility	σ	0.2
Strike price	K	100
Maturity date, years	T	1
Time-lag between portfolio adjustments	δt	0.01
Transaction cost per unit dollar	c	0.05

Table 4.1: Parameters

The nonlinear Leland model with transaction costs is compared with the standard Black-Scholes model without transaction costs. The plot of the difference $V_{Leland}(S,t)-V_{linear}(S,t)$ can be shown in the following Figure 4-1.

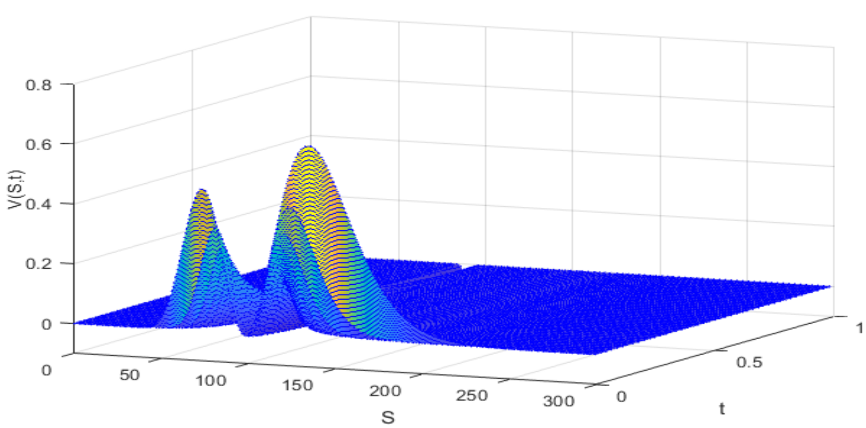

Figure 4-1: The impact of transactions costs

The numerical results show an economically considerable price deviation between the linear Black-Scholes model and Leland model. As shown in the Figure 4-1, the difference is not symmetric, but decreases closer to the maturity date. This is a result of the reducing continuous portfolio adjustments and hence lower transaction costs closer to the expiration date.

Figure 4-2 indicates the price of the European call option for the Leland model obtained by the finite element method and the price of the linear Black-Scholes model. The difference is maximum at $S \approx 90$, where the price of the Leland model computed by the FEM (≈ 15) is remarkably higher than the price of the Black-Scholes model (≈ 7). However, the difference becomes smaller and both graphs show similar trend after the strike price ($K = 100$). The

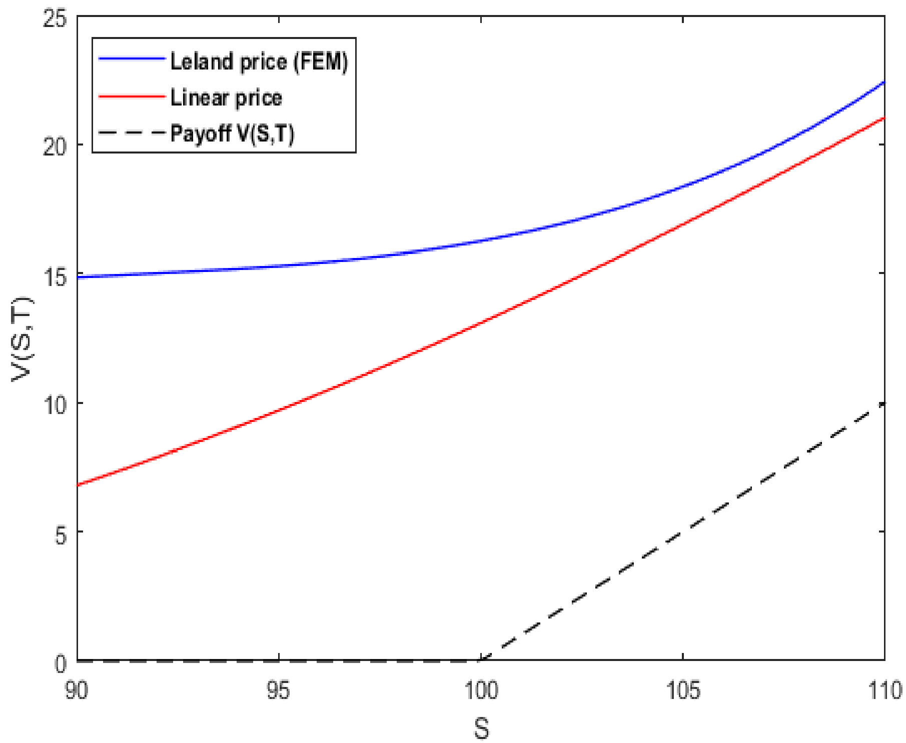

Figure 4-2: Finite element results of the European call option for the Leland model vs. the linear Black-Scholes model

nonlinear and linear prices are compared applying a formula of the Black-Scholes option price based on the modified variance for the Leland model in the original paper of Leland [9] (see Figure 4-3). From the above graph, we see the price of the European call option for the nonlinear Leland model is approximately equal 13, when the strike price $K = 100$. The Table 4.2 taken from [9] shows similar prices of the European call option and total transactions costs (TC). Based on the Matlab code, numerical solutions of 1 quadratic and 1 linear elements were found for the Leland model using the finite element method, see Figures 4-4.

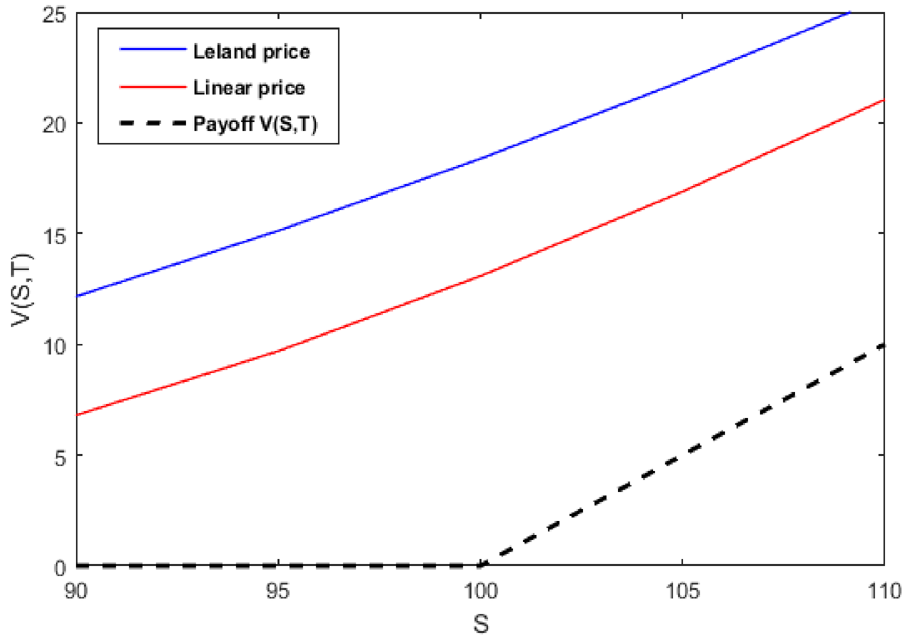

Figure 4-3: Price of the European call option for the Leland model vs. the price of the linear Black-Scholes model

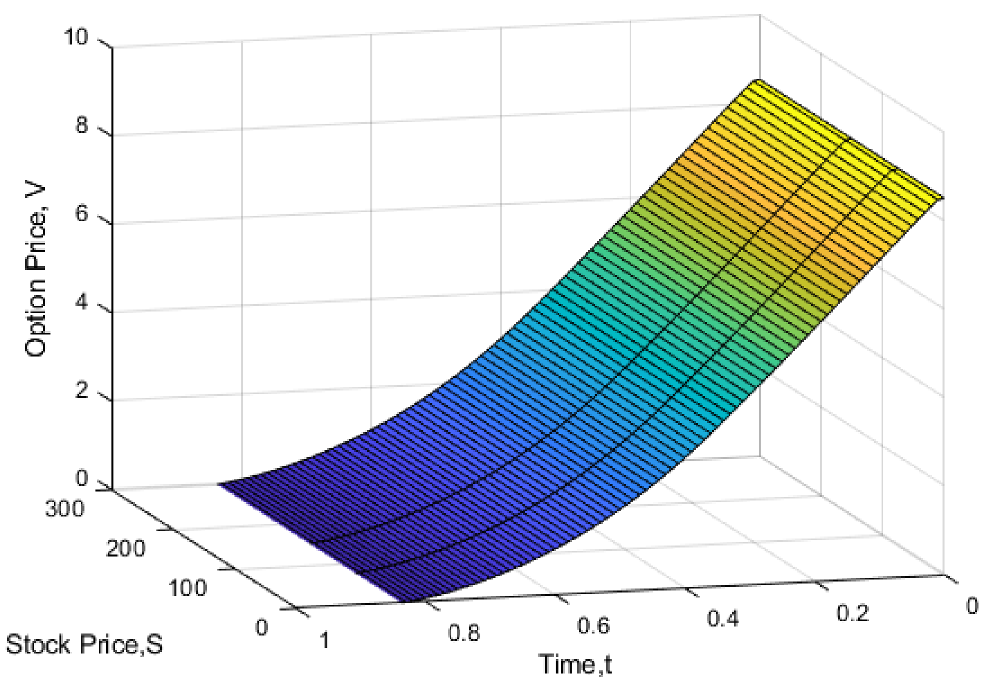

Figure 4-4: Price of the call option based on the Leland model

Striking Price(K)	Call Price	Total TC	Turnover
	Transactions Costs= 0.00%		
80	27.67	0.000	28.01
90	19.68	0.000	63.18
100	12.99	0.000	97.16
110	7.97	0.000	113.25
120	4.55	0.000	107.59
	Transactions Costs= 0.25%		
80	27.67	0.070	28.15
90	19.68	0.156	62.45
100	12.99	0.240	95.81
110	7.97	0.280	112.20
120	4.55	0.267	106.89
	Transactions Costs= 1.00%		
80	27.67	0.300	29.96
90	19.68	0.621	62.12
100	12.99	0.922	92.18
110	7.97	1.069	106.91
120	4.55	1.027	102.68
	Transactions Costs= 4.00%		
80	27.67	1.352	33.80
90	19.68	2.377	59.43
100	12.99	3.259	81.47
110	7.97	3.694	92.34
120	4.55	3.616	90.40

Table 4.2: Transactions Costs and Implied Turnover of Option Replicating Strategies: 1 Year Options, Weekly Revision

Chapter 5

Conclusion

This thesis provides an overview over the nonlinear Leland model for European call options. The partial differential equation of the Leland model was transformed into a system of equations to which Galerkin finite element method can be applied. The global finite element system is a differential algebraic system which was solved by modifying Crank-Nicolson scheme with data from literature. A code of the numerical example was generated on Matlab programming language. For the future work, a general finite element code of the nonlinear Black-Scholes models for other types of options will be generated.

Appendix A

Matlab code

```
function T = myfunc(QNE, LNE, L, u0,  u_BC, det_t, theta, res_t)
% QNE: number of quadratic element;
% LNE: number of linear element;
% L: total length;
% u0: Initial condition;
% u_BC: boundary conditions
% det_t: is the time for one step
% theta: can be equal to 0, 1, 1/2, 2/3;
% res_t: Time to maturity from dataset;
T = 1;
k = 0.5;
k1 = 5.6;
L=2*QNE+LNE;

%creat the element number matrix
if (QNE>0)
    nn=2*QNE+1+LNE;
    Node=1:1:nn;
elseif (QNE≤0)
    nn=LNE+1;
    Node=1:1:nn;
end
```

```
%creat the global node number matrix, nn represent number of nodes
LM =(l/6)*[2 1; 1 2];
LP = -(1/2)*[1 -1;1 -1];
LK = (1/l)*[1 -1;-1 1];
QM = (l/30)*[4 2 -1; 2 16 2;  -1 2 4];
QP = -(1/6)*[3 4 1; 4 0 -4; -1 4 -3];
QK = -(1/3*l)*[7 -8 1;  -8 16 -8; 1 -8 7];
%the program below is for assemble the final M matrix:
M=zeros(nn,nn);
%creat the n by n zero matrix
i=1;
s=1;
n=1;
if (QNE>=1)
    QC=zeros(QNE,3);
    while (i<=QNE)
        while (s<=3)
            QC(i,s)=n;
            n=n+1;
            s=s+1;
        end
        n=n-1;
        i=i+1;
        s=1;
    end
end
%creat connectivity matrix (C) for quadratic elements
if (LNE>=1)
LC=zeros(LNE,2);
i=1;
s=1;
while (i<=LNE)
    while (s<=2)
        LC(i,s)=n;
        n=n+1;
        s=s+1;
```

```
    end
    n=n-1;
    i=i+1;
    s=1;
end
end
%creat C matrix for linear elements

if (QNE≥1)
i=1;
s=1;
n=1;
while (i≤QNE)
    while (s≤3)
        while (n≤3)
            M(QC(i,s),QC(i,n))=M(QC(i,s),QC(i,n))+QM(s,n);
            n=n+1;
        end
        s=s+1;
        n=1;
    end
    i=i+1;
    s=1;
    n=1;
end
end
%assemble quadratic local matrix to global matrix
if (LNE≥1)
i=1;
s=1;
n=1;
while (i≤LNE)
    while (s≤2)
        while (n≤2)
            M(LC(i,s),LC(i,n))=M(LC(i,s),LC(i,n))+LM(s,n);
            n=n+1;
```

```
        end
        s=s+1;
        n=1;
    end
    i=i+1;
    s=1;
    n=1;
end
end
%assemble linear local matrix to global matrix
%The following program is to assemble global K matrix
K=zeros(nn,nn); & %creat the n by n zero matrix
if (QNE≥1)
i=1;
s=1;
n=1;
while (i≤QNE)
    while (s≤3)
        while (n≤3)
            K(QC(i,s),QC(i,n))=K(QC(i,s),QC(i,n))+QK(s,n);
            n=n+1;
        end
        s=s+1;
        n=1;
    end
    i=i+1;
    s=1;
    n=1;
end
end
%assemble quadratic local matrix to global matrix
if (LNE≥1)
i=1;
s=1;
n=1;
while (i≤LNE)
```

```
    while (s≤2)
        while (n≤2)
            K(LC(i,s),LC(i,n))=K(LC(i,s),LC(i,n))+LK(s,n);
            n=n+1;
        end
        s=s+1;
        n=1;
    end
    i=i+1;
    s=1;
    n=1;
end
end
% Assemble matrix P
P=zeros(nn,nn); &  %create the zero P matrix
if (QNE≥1)
i=1;
s=1;
while(i≤QNE)
    while(s≤3)
        P(QC(i,s),1)=P(QC(i,s),1)+QP(s,1);
        s=s+1;
    end
    s=1;
    i=i+1;
end
end
%assemble the quadratic P matrix to the global P matrix
if (LNE≥1)
i=1;
s=1;
while(i≤LNE)
    while(s≤2)
        P(LC(i,s),1)=P(LC(i,s),1)+LP(s,1);
        s=s+1;
    end
```

```
    s=1;
    i=i+1;
end
end
clear u
n=u_BC(1,1)/det_t;
m=res_t/det_t;
u=zeros(nn,n+m);
u(:,1)=ones(nn,1)*u0;
u(1,1:n)=ones(1,n)*u_BC(1,2);
%input the initial condition and boundary condition
i=1;
if(res_t>0)
i=1;
while(i<=n)
    v(:,i)= inv(M)*(-K*u(:,i)-P*u(:,i));
    u(:,i+1) = inv(M-det_t*theta*k*P)*((M+theta*det_t*k*P)*u(:,i))+
    +(theta*M*det_t)*v(:,i)+k1*theta*det_t*M*abs(v(:,i));
    i=i+1;
    u(1,i)=u_BC(1,2);
end
end
while(i<n+m)
    v(:,i)= inv(M)*(-K*u(:,i)-P*u(:,i));
    u(:,i+1) = inv(M-det_t*theta*k*P)*((M+theta*det_t*k*P)*u(:,i))+
    +(theta*M*det_t)*v(:,i)+k1*theta*det_t*M*abs(v(:,i));
    i=i+1;
end
Q=zeros(1,n);
du=zeros(nn,n);
i=1;
while(i<=n)
    du(2:nn,i)=inv(M(2:end,2:end))*(K(2:end,2:end)*u(2:end,i)+
  +M(2:end,2:end)*abs(u(2:end,i)));
    du(1,i)=0;
    i=i+1;
```

```
end
i=1;
while(i≤n)
    Q(1,i)=M(1,:)*du(:,i)+K(1,:)*u(:,i)+M(1,:)*abs(u(:,i));
    i=i+1;
end
%The construction of linear and quadratic shape functions.
x_1l = log(50);
x_2l = log(100);
x_1q = log(100);
x_2q = log(150);
x_3q = log(200);
xr = log(100);
ll = x_2l-x_1l;
lq = x_3q-x_1q;
N1l = (x_2l-xr)/ll;
N2l = (xr-x_1l)/ll;
psi = (2*xr-(x_1q+x_3q))/lq;
N1q = 1/2*(psi^2-psi);
N2q = 1 - psi^2;
N3q = 1/2*(psi^2+psi);
%%%%%%%
H = u(1:end,:);
INH = size(H);
inh = INH(1,1);
% Linear shape functions
NLH = ones(nn,inh);
NLHO = size(NLH(:,1:end));
NLHE = size(NLH(:,2:end));
NLH(:,1:end) = ones(nn,NLHO(1,2))*N1l;
NLH(:,2:end) = ones(nn,NLHE(1,2))*N2l;
NQH = ones(n,inh);
 for o = 1:inh
 if mod(o,3) == 1
    NQH(1,o) = N1q;
  elseif mod(o,3) == 2
```

```
    NQH(1,o) = N2q;
  elseif mod(o,3) == 0
    NQH(1,o) = N3q;
  end
 end
ufem = NLH*H+NQH*H; %Finite element solution of the Leland model
U = exp(0.5*(-0.02))*ufem; %Reversely transformed Leland model
A=0:l/2:l*QNE;
B=l*QNE+l:l:L;
C=[A,B];
[x,y]=meshgrid(C,1:1:m+n);
surf(x,y,U')
ylabel('timesteps');
xlabel('Stock Price');
zlabel('Option Price');
title('Leland Model');
end
```

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