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EXISTENCE, UNIQUENESS, AND NUMERICAL ANALYSIS OF SOLUTIONS OF A QUASILINEAR PARABOLIC PROBLEM*

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Abstract. A quasilinear parabolic problem is studied. By using the method of lines, the existence and uniqueness of a solution to the initial boundary value problem with sufficiently smooth initial conditions are shown. Also given are L^2 error estimates for the error between the extended fully discrete finite element solutions and the exact solution.

Key words. method of lines, finite element method, L^2 estimates, quasilinear parabolic problem

AMS(MOS) subject classifications. 65N30, 35J65

1. Introduction. In this work, we show that, by using the method of lines, the quasilinear parabolic problem governed by the p-harmonic operator has a unique weak solution which is more "classical" than the weak solution obtained by applying the theory of Kačur [4], in the sense that it satisfies the equation pointwise with respect to time. Therefore, in finding numerical solutions to this problem, integration can be carried out only on the spatial domain. In the formulation of this problem integration over the time interval is not needed while it was needed in the formulation used in [4]. With this formulation, L^2 error estimates for the error between the true solution and its fully discrete approximations are obtained. In [7] and [10]–[12], the method of lines is extensively used.

2. An existence and uniqueness result. Throughout this paper, we shall assume that Ω is a bounded convex domain in R^n with smooth boundary $\partial\Omega$, and $p \geqq 2$. We also use $u(t)$ or simply u to denote function $u(x, t)$ which is defined on $\Omega \times [0, \mathrm{T}]$, $T > 0$. We use the following notation

$$\|u\| = \left[\int_\Omega |\nabla u|^p \, dx\right]^{1/p}, \qquad \|u\|_2 = \left[\int_\Omega |u|^2 \, dx\right]^{1/2}.$$

$\|\cdot\|_2$ is the usual $L^2(\Omega)$ norm and $\|\cdot\|$ the seminorm for $W^{1,p}(\Omega)$ which is a norm for $W_0^{1,p}(\Omega)$.

Let $A: W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ be the operator defined by

$$(Au, v) = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla v) \, dx \quad \text{for } v \in W^{1,p}(\Omega).$$

For definitions of Sobolev spaces $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$, and $(W_0^{1,p}(\Omega))^*$, see [2], [5].

We quote the following lemma from [3].

LEMMA 1. *There exist constants* $\alpha > 0$ *and* $\beta > 0$, *such that, for* $p \geqq 2$,

$$\alpha \|u - v|^p \leqq (Au - Av, u - v)$$

and

$$\|Au - Av\|^* \leqq \beta(\|u\| + \|v\|)^{p-2} \cdot \|u - v\| \quad \textit{for any } u, v \in W^{1,p}(\Omega).$$

Note. For $p \geqq 2$, $L^2(\Omega) \supset W^{1,p}(\Omega)$. In following $\langle \cdot, \cdot \rangle$ is understood as the usual inner product in $L^2(\Omega)$ and $(\cdot, \cdot)$ as the duality for a pair in $W_0^{1,p}(\Omega) \times (W_0^{1,p}(\Omega))^*$.

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LEMMA 2. *For any* $g \in W^{1,p}(\Omega)$, *the problem*

$$(Au, v) = \langle g, v \rangle \quad \textit{for any } v \in W_0^{1,p}(\Omega), \qquad u|_{\partial\Omega} = u_0|_{\partial\Omega}$$

has a unique solution $u \in W^{1,p}(\Omega)$, *where* $u_0 \in W^{1,p}(\Omega)$.

Proof. Since Ω is a bounded set, we have

$$(W^{1,p}(\Omega))^* \supset W^{1,q}(\Omega) \supset W^{1,p}(\Omega),$$

where $q = p/(p-1)$, and thus $g \in (W^{1,p}(\Omega))^*$. And by Lemma 1, A is a strictly monotone operator. Therefore A satisfies all the conditions in Theorem 29.5 [2, pp. 242–243]. By the conclusion of this theorem, the problem has a unique solution.

Consider the following nonlinear evolution problem

$$\text{(1)} \qquad \frac{du}{dt} + Au = f, \qquad x \in \Omega, \quad t \in (0, T],$$

$$\text{(2)} \qquad u(x, t) = \phi(x), \qquad x \in \partial\Omega, \quad t \in (0, T],$$

$$\text{(3)} \qquad u(x, 0) = u_0(x), \qquad x \in \Omega,$$

where $u_0 \in W^{1,p}(\Omega)$, $u_0|_{\partial\Omega} = \phi$ and $f: [0, T] \to L^2(\Omega)$ is Lipschitz continuous, i.e., there exists a positive constant L such that $\|f(t) - f(t')\|_2 \leqq L|t - t'|$ for any $t, t' \in [0, T]$.

Note. Here we only consider fixed boundary conditions since the method of lines does not apply to this problem with time-dependent boundary conditions. This is clear since (8) requires $u(t_i) - u(t_{i-1}) \in W_0^{1,p}(\Omega)$.

DEFINITION 1. Let $u(x, t)$: $[0, T] \to L^2(\Omega)$. If there exists a function $g(x, t)$ such that

$$\lim_{\Delta t \to 0} \left\| \frac{u(t + \Delta t) - u(t)}{\Delta t} - g(t) \right\|_2 = 0,$$

we then say that u is differentiable at t, and $g(x, t)$ is called the derivative of $u(x, t)$ at t, which is denoted by $du(x, t)/dt$.

DEFINITION 2. We say that u is a solution of (1)–(3) if $u(x, t) \in W^{1,p}(\Omega)$ for all $t \in (0, T]$,

$$\text{(4)} \qquad \left\langle \frac{du}{dt}, v \right\rangle + (Au, v) = \langle f, v \rangle,$$

$$\text{(5)} \qquad \langle u(0), v \rangle = \langle u_0, v \rangle \quad \text{for any } v \in W_0^{1,p}(\Omega),$$

and

$$\text{(6)} \qquad u(x, t) = \phi(x), \qquad x \in \partial\Omega, \quad t \in (0, T],$$

where $du(x, t)/dt$ is the derivative in the sense of Definition 1, $u_0 \in W^{1,p}(\Omega)$, $u_0|_{\partial\Omega} = \phi(x)$.

THEOREM 1. *Suppose that* $u_0 \in W^{1,p}(\Omega)$ *and* $\nabla \cdot (|\nabla u_0|^{p-2} \nabla u_0) \in L^2(\Omega)$, *then problem* (1)–(3) *has a unique solution* u *in the sense of Definition* 2. *Furthermore,* $u \in C[0, T; W^{1,p}(\Omega)]$ *and* $du/dt \in C[0, T; W_0^{1,p}(\Omega)]$.

Let $\{t_i\}_{i=0,n}$ *be uniform partition of* $[0, T]$, $\Delta t = T/n$, *and* $t_i = i\Delta t$. *Consider the following recursive nonlinear elliptic problems.*

Given u_{i-1}, *find* u_i *such that*

$$\text{(7)} \qquad \left\langle \frac{u_i - u_{i-1}}{\Delta t}, v \right\rangle + (Au_i, v) = \langle f_i, v \rangle,$$

$$\text{(8)} \qquad u_i = u_{i-1}, \qquad \text{on } \partial\Omega \quad \textit{for any } v \in W_0^{1,p}(\Omega),$$

where $u_i = u(x, t_i)$, $f_i = f(x, t_i)$, $i = 1, n$.

Lemma 2 above assures that for each such partition $\{t_i\}_{i=0,n}$, (7), (8) can generate a unique sequence $\{u_i\}_{i=0,n}$ in $w^{1,p}(\Omega)$.

To prove Theorem 1, we first establish several lemmas, namely Lemmas 3–7, under the hypothesis of the theorem, i.e., $\nabla\cdot(|\nabla u_0|^{p-2}\nabla u_0)\in L^2(\Omega)$. In the following $C(u_0,f)$ denotes a generic constant depending only on u_0 and f.

LEMMA 3. *For the above sequence $\{u_i\}_{i=0,n}$, there exists a constant $C(u_0,f)$ such that*

$$\left\|\frac{u_i-u_{i-1}}{\Delta t}\right\|_2 \leqq C(u_0,f)). \tag{9}$$

Proof. In (7), let $i=1$, $v=(u_1-u_0)/\Delta t$. We have

$$\left\|\frac{u_1-u_0}{\Delta t}\right\|_2^2+\frac{1}{\Delta t}(Au_1-Au_0,\,u_1-u_0)=\left\langle f_1,\frac{u_1-u_0}{\Delta t}\right\rangle-\left(Au_0,\frac{u_1-u_0}{\Delta t}\right),$$

which implies that

$$\left\|\frac{u_1-u_0}{\Delta t}\right\|_2^2\leqq\|f_1\|_2\|\left\|\frac{u_1-u_0}{\Delta t}\right\|_2+\left|\left(Au_0,\frac{u_1-u_0}{\Delta t}\right)\right|, \tag{10}$$

since, by Lemma 1, $(1/\Delta t)(Au_1-Au_0,\,u_1-u_0)\geqq 0$.

Applying the divergence theorem to the second term in the right-hand side of (10) and using the fact that $u_1-u_0\in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}\left\|\frac{u_1-u_0}{\Delta t}\right\|_2^2&\leqq\|f_1\|_2\|\left\|\frac{u_1-u_0}{\Delta t}\right\|_2+\left|\int_\Omega\nabla\cdot(|\nabla u_0|^{p-2}\nabla u_0)\left(\frac{u_1-u_0}{\Delta t}\right)dx\right|\\&\leqq(\|f_1\|_2+\|\nabla\cdot(|\nabla u_0|^{p-2}\nabla u_0)\|_2)\left\|\frac{u_1-u_0}{\Delta t}\right\|_2,\end{aligned}$$

and hence obtain

$$\left\|\frac{u_1-u_0}{\Delta t}\right\|_2\leqq(\|f_1\|_2+\|\nabla\cdot(|\nabla u_0|^{p-2}\nabla u_0)\|_2). \tag{11}$$

Since, by letting $v=u_i-u_{i-1}$ for $i\geqslant 2$ in (7),

$$\left\langle\frac{u_i-u_{i-1}}{\Delta t},\,u_i-u_{i-1}\right\rangle+(Au_i,\,u_i-u_{i-1})=\langle f_i,\,u_i-u_{i-1}\rangle,$$

$$\left\langle\frac{u_{i-1}-u_{i-2}}{\Delta t},\,u_i-u_{i-1}\right\rangle+(Au_{i-1},\,u_i-u_{i-1})=\langle f_{i-1},\,u_{i-}u_{i-1}\rangle,$$

we have

$$\begin{aligned}&\left\langle\frac{u_i-u_{i-1}}{\Delta t},\,u_i-u_{i-1}\right\rangle+(Au_i-Au_{i-1},\,u_i-u_{i-1})\\&\qquad=\left\langle\frac{u_{i-1}-u_{i-2}}{\Delta t},\,u_i-u_{i-1}\right\rangle+\langle f_i-f_{i-1},\,u_i-u_{i-1}\rangle,\end{aligned}$$

which implies, by Lemma 1 again,

$$\begin{aligned}&\left\|\frac{u_i-u_{i-1}}{\Delta t}\right\|_2\|u_i-u_{i-1}\|_2+\alpha\|u_i-u_{i-1}\|^p\\&\qquad\leqq\left(\left\|\frac{u_{i-1}-u_{i-2}}{\Delta t}\right\|_2+\|f_i-f_{i-1}\|_2\right)\|u_i-u_{i-1}\|_2.\end{aligned} \tag{12}$$

And hence, by (12) and the Lipschitz continuity of f, we have

$$\left\|\frac{u_i-u_{i-1}}{\Delta t}\right\|_2 \leqq \left\|\frac{u_{i-1}-u_{i-2}}{\Delta t}\right\|_2 + \|f_i - f_{i-1}\|_2 \tag{13}$$
$$\leqq \left\|\frac{u_{i-1}-u_{i-2}}{\Delta t}\right\|_2 + \Delta t L.$$

By (11) and (13), we finally have

$$\left\|\frac{u_i-u_{i-1}}{\Delta t}\right\|_2 \leqq \left\|\frac{u_1-u_0}{\Delta t}\right\|_2 + TL \tag{14}$$
$$\leqq (\|f_1\|_2 + \|\nabla\cdot(|\nabla u_0|^{p-2}\nabla u_0)\|_2) + TL.$$

By (14) and the regularity hypothesis on u_0, i.e., $\nabla\cdot(|\nabla u_0|^{p-2}\nabla u_0)\in L^2(\Omega)$, we then have (9), with $C(u_0,f)=\mathrm{Max}_{0\leqq t\leqq T}\|f(t)\|_2+\|\nabla\cdot(|\nabla u_0|^{p-2}\nabla u_0)\|_2+TL$. The proof is completed.

As a consequence of Lemma 3, we have the following.

COROLLARY 1. *For the sequence* $\{u_i\}_{i=0,n}$ *in Lemma* 3, *there exists a constant* $C(u_0,f)$ *such that* $\|u_i\|_2\leqq C(u_0,f)$, $i=1, n$.

LEMMA 4. *There exists a* $u_i^*\in L^2(\Omega)$ *for each i, such that*

$$(Au_i, v)=\langle u_i^*, v\rangle \quad \textit{for any } v\in W_0^{1,p}(\Omega)$$

and $\|Au_i\|_2^*=\|u_i^*\|_2$, *where* $i=0, n$. *Also* $\|u_i^*\|_2\leqq C(u_0,f)$ *for some constant* $C(u_0,f)$.

Proof. By (7), we have, for $i=1, n$,

$$(Au_i, v)=\left\langle\frac{u_{i-1}-u_i}{\Delta t}, v\right\rangle+\langle f_i, v\rangle \quad \text{for any } v\in W_0^{1,p}(\Omega).$$

By (9) we know that Au_i is a bounded linear operator on $W_0^{1,p}(\Omega)$ with respect to the $L^2(\Omega)$ norm; in fact, $\|Au_i\|_2^*=\|((u_i-u_{i-1})/\Delta t)+f_i\|_2\leqq C(u_0,f)$. Also $W_0^{1,p}(\Omega)$ is a subspace of $L^2(\Omega)$; in fact, this is a compact imbedding. Therefore by the Hahn–Banach theorem [8, p. 111], Au_i can be extended to a bounded linear operator F_i on $L^2(\Omega)$ so that $\|F_i\|_2^*=\|Au_i\|_2^*$. Hence, there exists a $u_i^*\in L^2(\Omega)$ with $\|F_i\|_2^*=\|u_i^*\|_2$, and $F_i(v)=\langle u_i^*, v\rangle$ for any $v\in L^2(\Omega)$. In particular, $(Au_i, v)=F_i(v)=\langle u_i^*, v\rangle$ for any $v\in W_0^{1,p}(\Omega)$, and $\|u_i^*\|_2=\|Au_i\|_2^*\leqq C(u_0,f)$.

COROLLARY 2. *For the sequence* $\{u_i\}_{i=0,n}$ *in Lemma* 3, *there exists a constant* $C(u_0,f)$ *such that*

$$\|u_i\|\leqq C(u_0,f), \qquad i=1, n.$$

Proof. By Lemma 1, Corollary 1, and Lemma 4, we have

$$\alpha\|u_i-u_0\|^p\leqq(Au_i-Au_0, u_i-u_0)=\langle u_i^*-u_0^*, u_i-u_0\rangle \tag{15}$$
$$\leqq(\|u_i^*\|_2+\|u_0^*\|)\|u_i-u_0\|_2\leqq C(u_0,f), \qquad i=1, n.$$

By the convexity of $\|\cdot\|^p$, we have

$$\|u_i\|^p\leqq 2^{p-1}(\|u_i-u_0\|^p+\|u_0\|^p),$$

which, together with (15), gives the result of this lemma.

Now, let $\{t_i\}_{i=0,n}$ and $\{t_k\}_{k=0,m}$ be two uniform partitions of $[0, T]$,

$$u_n(t)=\frac{t-t_i}{\Delta t_i}u_{i+1}+\frac{t_{i+1}-t}{\Delta t_i}u_i, \qquad t_i<t\leqq t_{i+1}, \quad i=0,1,\cdots,n-1,$$

$$u_m(t)=\frac{t-t_k}{\Delta t_k}u_{k+1}+\frac{t_{k+1}-t}{\Delta t_k}u_k, \qquad t_k<t\leqq t_{k+1}, \quad k=0,1,\cdots,m-1,$$

$$u_n(0)=u_m(0)=u(0), \qquad \Delta t_i=i\frac{T}{n}, \quad \Delta t_k=k\frac{T}{m}.$$

Let

$$\underline{u}_n(t)=u_{i+1} \quad \text{for } t_i<t\leqq t_{i+1}, \quad i=0,1,2,\cdots,n-1, \quad \underline{u}_n(0)=u_0,$$

$$\underline{u}_m(t)=u_{k+1} \quad \text{for } t_k<t\leqq t_{k+1}, \quad k=0,1,2,\cdots,m-1, \quad \underline{u}_m(0)=u_0.$$

Obviously,

$$\frac{du_n(t)}{dt}=\frac{u_{i+1}-u_i}{\Delta t_i}, \qquad t_i<t\leqq t_{i+1},$$

$$\frac{du_m(t)}{dt}=\frac{u_{k+1}-u_k}{\Delta t_k}, \qquad t_k<t\leqq t_{k+1}.$$

Remark 1. By Lemma 3, $du_n(t)/dt$, $u_n(t)$, and $\underline{u}_n(t)$ are uniformly bounded with respect to n and t, in $L^2(\Omega)$ norm. In fact, they are all less than or equal to some constant $C(u_0,f)$.

LEMMA 5. *For $u_n(t)$ and $\underline{u}_n(t)$ defined above, we have*

$$\|u_n(t)-\underline{u}_n(t)\|_2\leqq\frac{TC(u_0,f)}{n}.$$

Proof. By Lemma 3, we have

$$\begin{aligned}\|u_n(t)-\underline{u}_n(t)\|_2&=\left\|\frac{(t-t_i)u_{i+1}+(t_{i+}t_1-t)u_i-(t_{i+1}-t_i)u_{i+1}}{\Delta t_i}\right\|_2\\&=\left\|\frac{(t_{i+1}-t)(u_{i+1}-u_i)}{\Delta t_i}\right\|_2\\&\leqq(t_{i+1}-t)\left\|\frac{(u_{i+1}-u_i)}{\Delta t_i}\right\|_2\\&\leqq\frac{TC(u_0,f)}{n}.\end{aligned}$$

This proves Lemma 5.

By the definition of u_n and (7), we have, for $0\leqq i\leqq n$, $0\leqq k\leqq m$,

$$\left\langle\frac{du_n}{dt},v\right\rangle+(Au_{i+1},v)=\langle f_i,v\rangle, \qquad t_i<t\leqq t_{i+1}, \tag{16}$$

and

$$\left\langle\frac{du_m}{dt},v\right\rangle+(Au_{k+1},v)=\langle f_k,v\rangle, \qquad t_k<t\leqq t_{k+1}. \tag{17}$$

Let $v=u_n-u_m$, and subtract (16) from (17). We have, for $t\in(t_i,t_{i+1}]\cap(t_k,t_{k+1}]$,

$$\left\langle\frac{d(u_n-u_m)}{dt},u_n(t)-u_m(t)\right\rangle+(Au_{i+1}-Au_{k+1},u_n(t)-u_m(t))=\langle f_i-f_k,u_n(t)-u_m(t)\rangle,$$

which gives

$$\frac{1}{2}\frac{d}{dt}(\|u_n(t)-u_m(t)\|_2^2)+(Au_{i+1}-Au_{k+1}, u_n(t)-u_m(t))=\langle f_i-f_k, u_n(t)-u_m(t)\rangle.$$

Hence we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u_n(t)-u_m(t)\|_2^2)+(Au_{i+1}-Au_{k+1}, u_{i+1}-u_{k+1})\\ &\quad+(Au_{i+1}-Au_{k+1}, u_n(t)-\underline{u}_n(t))-(Au_{i+1}-Au_{k+1}, u_m(t)-\underline{u}_m(t))\\ &=\langle f_i-f_k, u_n(t)-u_m(t)\rangle\end{aligned}\tag{18}$$

for $t\in(t_{i+1}]\cap(t_k, t_{k+1}]$, since $\underline{u}_n(t)=u_{i+1}$ and $\underline{u}_m(t)=u_{k+1}$.

Using Lemma 1 and (18), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u_n(t)-u_m(t)\|_2^2)+\alpha\|u_{i+1}-u_{k+1}\|^p\\ &\leqq|(Au_{i+1}-Au_{k+1}, u_n(t)-\underline{u}_n(t))|\\ &\quad+|(Au_{i+1}-Au_{k+1}, u_m(t)-\underline{u}_m(t))|+|\langle f_i-f_k, u_n(t)-u_m(t)\rangle|.\end{aligned}\tag{19}$$

By Lemmas 4–5, we have

$$\begin{aligned}|(Au_{i+1}-Au_{k+1}, u_n(t)-\underline{u}_n(t))|&\leqq(\|Au_{i+1}\|_2^*+\|Au_{k+1}\|_2^*)\|u_n(t)-\underline{u}_n(t)\|_2\\ &=(\|u_{i+1}^*\|_2+\|u_{k+1}^*\|_2)\|u_n(t)-\underline{u}_n(t)\|_2\\ &\leqq\frac{TC(u_0,f)}{n}.\end{aligned}\tag{20}$$

Similarly,

$$|(Au_{i+1}-Au_{k+1}, u_m(t)-\underline{u}_m(t))|\leqq\frac{TC(u_0,f)}{m}.\tag{21}$$

By Lipschitz continuity of f, the definition of $u_n(t)$, and Remark 1, we have

$$\begin{aligned}|\langle f_i-f_k, u_n(t)-u_m(t)\rangle|&\leqq L|t_i-t_k|\,\|u_n(t)-u_m(t)\|_2\\ &\leqq 2LTC(u_0,f)\left(\frac{1}{n}+\frac{1}{m}\right).\end{aligned}\tag{22}$$

Using (19)–(22) we get

$$\frac{1}{2}\frac{d}{dt}(\|u_n(t)-u_m(t)\|_2^2)\leqq[2T(1+L)C(u_0,f)]\left(\frac{1}{n}+\frac{1}{m}\right).\tag{23}$$

Integrating (23) over $[0, T]$, and noting that $u_n(0)=u_m(0)$, we obtain

$$\|u_n(t)-u_m(t)\|_2^2\leqq 4T^2(1+L)C(u_0,f)]\left(\frac{1}{n}+\frac{1}{m}\right).\tag{24}$$

Hence by (24) we have proved the following.

LEMMA 6. *$\{u_n\}$ is a Cauchy sequence in $C(0, T; L^2(\Omega))$, and it converges to an element $u\in C(0, T; L^2(\Omega))$.*

The following lemma is a direct result of Lemmas 5 and 6.

LEMMA 7. *$\{\underline{u}_n\}$ converges to u in $L^\infty(0, T; W_0^{1,p}(\Omega))$, and $\lim_{n\to\infty}(A(\underline{u}_n(t)), v)=(A(u(t)), v)$ for any $v\in W_0^{1,p}(\Omega)$, uniformly over $[0, T]$.*

Proof. By using Lemma 4 and the definition of $\underline{u}_n(t)$, we have

$$\begin{aligned}\alpha\|\underline{u}_n(t)-\underline{u}_m(t)\|^p &\leqq (A\underline{u}_n(t)-A\underline{u}_m(t),\, \underline{u}_n(t)-\underline{u}_m(t))\\ &\leqq (\|\underline{u}_n^*(t)\|_2+\|\underline{u}_m^*(t)\|_2)\|\underline{u}_n(t)-\underline{u}_m(t)\|_2\\ &\leqq 2C(u_0,f)\|\underline{u}_n(t)-\underline{u}_m(t)\|_2\\ &\leqq 2C(u_0,f)(\|\underline{u}_n(t)-u_n(t)\|_2+\|u_n(t)-u_m(t)\|_2\\ &\quad +\|u_m(t)-\underline{u}_m(t)\|_2).\end{aligned}\tag{25}$$

Applying Lemmas 5 and 6 to (25), we see that $\{\underline{u}_n(t)\}$ is a Cauchy sequence in $W_0^{1,p}(\Omega)$, and hence it converges to some limit in $W_0^{1,p}(\Omega)$. But this limit must be the same as the limit u of $\{u_n(t)\}$, since by Lemma 5 both $\{\underline{u}_n(t)\}$ and $\{u_n(t)\}$ converge to the same limit in $L^2(\Omega)$. By Corollary 2 and the definition of $\underline{u}_n(t)$ we know that $\|\underline{u}_n(t)\|\leqq C(u_0,f)$ and hence $\|u(t)\|\leqq C(u_0,f)$, i.e., $u\in L^\infty(0,\, T;\, W_0^{1,p}(\Omega))$.

Furthermore, by Lemma 1, we have, for each $v\in W_0^{1,p}(\Omega)$

$$\begin{aligned}|(A(\underline{u}_n(t)),v)-(A(u(t)),v)| &\leqq \beta(\|\underline{u}_n(t)\|+\|u(t)\|)^{p-2}\|\underline{u}_n(t)-u(t)\|\,\|v\|\\ &\leqq C(u_0,f)\|v\|\,\|\underline{u}_n(t)-u(t)\|.\end{aligned}\tag{26}$$

Therefore, the second assertion of Lemma 7 follows from (26) since $\|\underline{u}_n(t)-u(t)\|$ converges to zero uniformly over $[0,\, T]$. Lemma 7 is proved.

Now, let us prove our main result, Theorem 1. Recall that by (7) and the definition of u_n, $\underline{u}_n$,

$$\left\langle\frac{du_n}{dt},\, v\right\rangle+(A\underline{u}_n,\, v)=\langle f,\, v\rangle \quad \text{for any } v\in W_0^{1,p}(\Omega).$$

Taking limits, and applying Lemma 7, we have, for any $v\in W_0^{1,p}(\Omega)$,

$$\lim_{n\to\infty}\left\langle\frac{du_n}{dt},\, v\right\rangle+(Au,\, v)=\langle f,\, v\rangle,\tag{27}$$

uniformly in $[0,\, T]$.

For each $t\in[0,\, T]$, by Remark 1, $\{du_n(t)/dt\}$ is a uniformly bounded sequence, with respect to t, in the reflexive Banach space $L^2(\Omega)$ and hence has a subsequence which converges weakly to an element $w(t)\in L^2(\Omega)$. Thus, we have, by (27), that

$$\langle w(t),\, v\rangle+(Au(t),\, v)=\langle f,\, v\rangle \quad \text{for any } v\in W_0^{1,p}(\Omega).\tag{28}$$

This $w(t)$ is independent of the subsequence, since for fixed u and f, (28) has only one solution. Since the weak limit of a uniformly bounded sequence is also uniformly bounded [2, p. 193], $w\in L^\infty(0,\, T;\, L^2(\Omega))$. Therefore, again by the Hahn–Banach theorem, (28) can be extended to hold for any $v\in L^2(\Omega)$.

Let $t,\, t'\in[0,\, T]$. Using (28), we have

$$\langle w(t)-w(t'),\, v\rangle=(Au(t)-Au(t'),\, v)+\langle f(t)-f(t'),\, v\rangle,\tag{29}$$

which also holds for any $v\in L^2(\Omega)$.

Let $v=u(t)-u(t')$. By (29), Lemma 1, and the boundedness of w in $L^2(\Omega)$ norm, we get

$$\begin{aligned}\alpha\|u(t)-u(t')\|^p &\leqq (Au(t)-Au(t'),\, u(t)-u(t'))\\ &=\langle w(t)-w(t')-f(t)+f(t'),\, u(t)-u(t')\rangle\\ &\leqq c(u_0,f)\|u(t)-u(t')\|_2.\end{aligned}\tag{30}$$

By Lemma 6 and (30), we get

$$\lim_{t\to t'} \|u(t)-u(t')\|=0. \tag{31}$$

Thus, $u\in C[0, T; W^{1,p}(\Omega)]$.

We next show that $w\in C(0, T; L^2(\Omega))$. By Lemma 1, we have

$$\|Au(t)-Au(t')\|^* \leqq \beta(\|u(t)\|+\|u(t')\|)^{p-2}\|u(t)-u(t')\|.$$

Therefore, $\lim_{t\to t'} \|Au(t)-Au(t')\|^*=0$, since $u\in C[0, T; W^{1,p}(\Omega)]$.

Since $w, f\in L^\infty(0, T; L^2(\Omega))$, by the Hahn–Banach theorem and (29), there exist $u^*(t)$, $u^*(t')\in L^2(\Omega)$ so that $\langle w(t)-w(t'), v\rangle = \langle u^*(t)-u^*(t'), v\rangle + \langle f(t)-f(t'), v\rangle$, for any $v\in L^2(\Omega)$. And

$$\lim_{t\to t'} \|u^*(t)-u^*(t')\|_2 = \|Au(t)-Au(t')\|^*=0. \tag{32}$$

Let $v=w(t)-w(t')$ in (29). We get

$$\begin{aligned}\|w(t)-w(t')\|_2^2 &\leqq |(Au(t)-Au(t'), w(t)-w(t'))|+|\langle f(t)-f(t'), w(t)-w(t')\rangle| \\ &\leqq (\|u^*(t)-u^*(t')\|_2+\|f(t)-f(t')\|_2)\|w(t)-w(t')\|_2,\end{aligned}$$

which gives

$$\|w(t)-w(t')\|_2 \leqq (\|u^*(t)-u^*(t')\|_2+\|f(t)-f(t')\|_2). \tag{33}$$

Therefore, (33) and the continuity of f imply that $w\in C(0, T; L^2(\Omega))$.

Let $u^*(t)=\int_0^t w(s)\,ds+u_0$. Using Fubini's theorem, we have

$$\begin{aligned}\langle u_n(t)-u^*(t), v\rangle &= \int_\Omega \int_0^t \left(\frac{du_n}{dt}-w\right) v\,ds\,dx = \int_0^t \int_\Omega \left(\frac{du_n}{dt}-w\right) v\,dx\,ds \\ &= \int_0^t \left\langle \frac{du_n}{dt}-w, v\right\rangle ds \\ &= \int_0^t \left[\left\langle \frac{du_n}{dt}, v\right\rangle -(Au, v)+\langle f, v\rangle\right] ds.\end{aligned} \tag{34}$$

Thus, by (27), $\lim_{n\to 0}(u_n(t)-u^*(t), v)=0$ for any $v\in W_0^{1,p}(\Omega)$, uniformly over $[0, T]$. We have $u(t)=u^*(t)=\int_0^t w(s)\,ds+u_0$, since the weak limit is unique.

We now show that u is differentiable in the sense of Definition 1. In fact, without loss of generality, let $\Delta t>0$. Then, we have

$$\begin{aligned}\left\|\frac{u(t+\Delta t)-u(t)}{\Delta t}-w(t)\right\|_2^2 &= \left\|\frac{1}{\Delta t}\int_t^{t+\Delta t} w(s)\,ds-w(t)\right\|_2^2 \\ &= \int_\Omega \left[\frac{1}{\Delta t}\int_t^{t+\Delta t} (w(x, s)-w(x, t))\,ds\right]^2 dx \\ &\leqq \int_\Omega \left[\frac{1}{\Delta t}\int_t^{t+\Delta t} |w(x, s)-w(x, t)|\,ds\right]^2 dx.\end{aligned}$$

By Jonsen's inequality [8, p. 63], we get

$$\begin{aligned}\left\|\frac{u(t+\Delta t)-u(t)}{\Delta t}-w(t)\right\|_2^2 &\leqq \int_\Omega \frac{1}{\Delta t}\left(\int_t^{t+\Delta t} (w(x, s)-w(x, t))^2\,ds\right) dx \\ &= \frac{1}{\Delta t}\int_t^{t+\Delta t} \left(\int_\Omega (w(x, s)-w(x, t))^2\,dx\right) ds \\ &= \frac{1}{\Delta t}\int_t^{t+\Delta t} \|w(s)-w(t)\|_2^2\,ds \\ &= \|w(\xi)-w(t)\|_2, \quad \text{where } t\leqq \xi \leqq t+\Delta t.\end{aligned} \tag{35}$$

Hence by (35), $\lim_{\Delta t\to 0} \|(u(t+\Delta t)-u(t)/\Delta t)-w(t)\|_2^2=0$, since $w\in C(0, T; L^2(\Omega))$. We get $du/dt=w$. Finally, by (28) and Definition 1, we get

$$\left\langle \frac{du}{dt}, v\right\rangle+(Au, v)=\langle f, v\rangle \quad \text{for any } v\in W_0^{1,p}(\Omega), \quad \text{in } [0, T].$$

This completes the proof of the existence of a solution.

For uniqueness, let us assume that u and $\hat{u}$ are two solutions to the problem. Then,

$$\left\langle \frac{du}{dt}, v\right\rangle+(Au, v)=\langle f, v\rangle \quad \text{for any } v\in W_0^{1,p}(\Omega), \quad \text{in } [0, T], \tag{36}$$

and

$$\left\langle \frac{d\hat{u}}{dt}, v\right\rangle+(A\hat{u}, v)=\langle f, v\rangle \quad \text{for any } v\in W_0^{1,p}(\Omega), \quad \text{in } [0, T]. \tag{37}$$

Subtracting (37) from (36), we get

$$\left\langle \frac{du}{dt}-\frac{d\hat{u}}{dt}, v\right\rangle+(Au-A\hat{u}, v)=0 \quad \text{for any } v\in W_0^{1,p}(\Omega).$$

Let $v=u-\hat{u}$. Then, we have

$$\left\langle \frac{d(u-\hat{u})}{dt}, u-\hat{u}\right\rangle+(Au-A\hat{u}, u-\hat{u})=0,$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}(\|u-\hat{u}\|_2^2)+(Au-A\hat{u}, u-\hat{u})=0.$$

Since, by Lemma 1, $(Au-A\hat{u}, u-\hat{u})\geqq 0$, we have

$$\frac{d}{dt}(\|u-\hat{u}\|_2^2)\leqq 0.$$

$\|u(t)-\hat{u}(t)\|_2^2$ is therefore a decreasing function in $[0, T]$, and therefore

$$\|u(t)-\hat{u}(t)\|_2^2\leqq \|u(0)-\hat{u}(0)\|_2^2=\|u_0-u_0\|_2^2=0,$$

for all t in $[0, T]$. This completes the proof of Theorem 1.

3. L^2 error estimates for the fully discrete scheme. Let $S_h(\Omega)$ be a conformal finite element space of $W^{1,p}(\Omega)$ as constructed in [1, (5.3.5), p. 313], and let $\Pi_h: W^{1,p}(\Omega)\to S_h(\Omega)$ be defined by $\Pi_h u=\sum_{i=1}^m l_i(u)N_i$ [6, Vol. iv, pp. 63–64]. Π_h is known as the finite element interpolation operator; $\{N_i\}_{i=1,m}$ are the global basis functions for $S_h(\Omega)$ and $\{l_i(u)\}_{i=1,m}$ correspond to the global degrees of freedom.

A classical theorem on global interpolation error estimates in the finite element theory [1] leads immediately to the following.

LEMMA 8. *Suppose that $\{T_h\}_h$ is a regular family of triangulation of Ω. We then have, for $p\geqq 2$, the following interpolation error estimate*:

$$\|u-\Pi_h u\|\leqq Ch|u|_2 \quad \textit{for } u\in W^{2,p}(\Omega),$$

where $|u|_2$ is the L^2 norm of the second derivatives of u, C is a constant independent of u, h is the maximum of the diameters of all the elements in $\{T_h\}_h$, and $\Pi_h u$ is the finite element interpolation operator.

Remark 2. If Π_h is the interpolation operator defined in [9, (2.12)], then we have

$$\|u-\Pi_h u\|_p \leqq Ch\|u\| \quad \text{for } u \in W^{1,p}(\Omega).$$

Again for simplicity, let $\{t_i\}_{i=0,n}$ be a uniform partition of $[0, T]$ and $\Delta t = T/n$. Let $\{u_i\}_{i=0,n}$ be the sequence generated by (7), (8). For each i consider the following problem.

Find $W_i \in S_h(\Omega)$, such that

$$\text{(38)} \quad \begin{aligned} &(AW_i, V) = (Au_i, V) \quad \text{for any } V \in S_h(\Omega) \cap W_0^{1,p}(\Omega), \quad i = 0, n, \\ &W_i|_{\partial\Omega} = \Pi_h u_i|_{\partial\Omega}. \end{aligned}$$

By Theorem 29.5 of [2], for each i, problem (38) has a unique solution.

LEMMA 9. $\|W_i\| \leqq C(u_0, f)$, $i = 0, n$.

Proof. In (38), let $V = W_i - \Pi_h u_0$. Then

$$(AW_i, W_i - \Pi_h u_0) = (Au_i, W_i - \Pi_h u_0),$$

i.e.,

$$\begin{aligned} \int_\Omega |\nabla W_i|^p\, dx &- \int_\Omega |\nabla W_i|^{p-2}(\nabla W_i, \nabla\Pi_h u_0)\, dx \\ &= \int_\Omega |\nabla u_i|^{p-2}(\nabla u_i, \nabla W_i)\, dx - \int_\Omega |\nabla u_i|^{p-2}(\nabla u_i, \nabla\Pi_h u_0)\, dx. \end{aligned}$$

We hence get

$$\begin{aligned} \int_\Omega |\nabla W_i|^p\, dx &\leqq \int_\Omega |\nabla u_i|^{p-1}|\nabla W_i|\, dx + \int_\Omega |\nabla w_i|^{p-1}|\nabla\Pi_h u_0|\, dx \\ &\quad + \int_\Omega |\nabla u_i|^{p-1}|\nabla\Pi_h u_0|\, dx \\ &\leqq \left[\int_\Omega |\nabla u_i|^p\, dx\right]^{(p-1)/p}\left[\int_\Omega |\nabla W_i|^p\, dx\right]^{1/p} \\ &\quad + \left\{\left[\int_\Omega |\nabla W_i|^p\, dx\right]^{(p-1)/p} + \left[\int_\Omega |\nabla u_i|^p\, dx\right]^{(p-1)/p}\right\}\left[\int_\Omega |\nabla\Pi_h u_0|^p\, dx\right]^{1/p}, \end{aligned}$$

i.e.,

$$\text{(39)} \quad \|W_i\|^p \leqq \|\Pi_h u_0\|(\|W_i\|^{p-1} + \|u_i\|) + \|u_i\|^{p-1}\|W_i\|.$$

From (39) and Lemma 3, the conclusion of this lemma can be obtained.

LEMMA 10. $\|u_i - W_i\| \leqq C(u_0, f)(\|u_i - \Pi_h u_i\|)^{1/(p-1)}$, $i = 0, n$.

Proof. By (38), we have

$$(AW_i - Au_i, V) = 0 \quad \text{for any } V \in S_h(\Omega) \cap W_0^{1,p}(\Omega), \quad i = 0, n.$$

In particular,

$$\text{(40)} \quad (AW_i - Au_i, \Pi_h u_i - W_i) = 0, \qquad i = 0, n.$$

By (40), we get

$$\text{(41)} \quad \begin{aligned} (Au_i - AW_i, u_i - W_i) &= (Au_i - AW_i, u_i - \Pi_h u_i + \Pi_h u_i - W_i) \\ &= (Au_i - AW_i, u_i - \Pi_h u_i). \end{aligned}$$

By Lemma 1 and (41), we get

$$\alpha\|u_i - W_i\|^p \leqq (Au_i - AW_i, u_i - \Pi_h u_i)$$
$$\leqq \|Au_i - AW_i\|^* \|u_i - \Pi_h u_i\| \leqq \beta(\|u_i\| + \|W_i\|)^{p-2}\|u_i - W_i\| \|u_i - \Pi_h u_i\|,$$

which gives

$$\alpha\|u_i - W_i\|^{p-1} \leqq \beta(\|u_i\| + \|W_i\|)^{p-2}\|u_i - \Pi_h u_i\|. \tag{42}$$

By Lemma 9 and (42) we get the result.

Now, we consider the fully discrete scheme: Let $U_0 = W_0$, where W_0 is defined by (38). Find $U_i \in S_h(\Omega)$, such that

$$\left\langle \frac{U_i - U_{i-1}}{\Delta t}, V \right\rangle + (AU_i, V) = \langle f_i, V \rangle \quad \text{for any } V \in S_h(\Omega) \cap W_0^{1,p}(\Omega), \tag{43}$$
$$U_i|_{\partial\Omega} = W_i|_{\partial\Omega}, \ i = 1, n.$$

LEMMA 11. $\|(U_i - U_{i-1}/\Delta t)\|_2 \leqq C(u_0, f)$.

Proof. In (43), let $i = 1$ and $V = (U_1 - U_0/\Delta t)$. We then get

$$\left\|\frac{U_1 - U_0}{\Delta t}\right\|_2^2 + \left(AU_1, \frac{U_1 - U_0}{\Delta t}\right) = \left\langle f_1, \frac{U_1 - U_0}{\Delta t}\right\rangle. \tag{44}$$

By Lemma 1 and (44), we have

$$\left\|\frac{U_1 - U_0}{\Delta t}\right\|_2^2 \leqq \left\langle f_1, \frac{U_1 - U_0}{\Delta t}\right\rangle - \left(AU_0, \frac{U_1 - U_0}{\Delta t}\right).$$

Thus

$$\left\|\frac{U_1 - U_0}{\Delta t}\right\|_2^2 \leqq \|f_1\|_2 \left\|\frac{U_1 - U_0}{\Delta t}\right\|_2 + \left|\left(Au_0, \frac{U_1 - U_0}{\Delta t}\right)\right|, \tag{45}$$

since

$$\left(AU_0, \frac{U_1 - U_0}{\Delta t}\right) = \left(Au_0, \frac{U_1 - U_0}{\Delta t}\right).$$

Equation (45) is identical to (11) if we replace U_1 and U_0 by u_1 and u_0, respectively. Hence the rest of the proof of this Lemma can be obtained along the lines of the proof of Lemma 3. By (7) and (38), we have

$$\left\langle \frac{u_i - u_{i-1}}{\Delta t}, V \right\rangle + (AW_i, V) = \langle f_i, V \rangle \quad \text{for any } V \in S_h(\Omega) \cap W_0^{1,p}(\Omega), \quad i = 1, n. \tag{46}$$

Subtract (43) from (46), and let $V = W_i - U_i$. We get

$$\left\langle \frac{u_i - u_{i-1}}{\Delta t} - \frac{U_i - U_{i-1}}{\Delta t}, W_i - U_i \right\rangle + (AW_i - AU_i, W_i - U_i) = 0. \tag{47}$$

We extend the fully discrete solution $\{U_i\}_{i=0,n}$ to $[0, T]$ by

$$U_n(t) = \frac{t - t_i}{\Delta t_i} U_{i+1} + \frac{t_{i+1} - t}{\Delta t_i} U_i, \quad U_n(0) = U_0, \quad t_i < t \leqq t_{i+1}, \quad i = 0, 1, \cdots, n-1, \tag{48}$$

similar to the definition of $u_n(t)$. Then

$$\begin{aligned}
&\left\langle \frac{u_i-u_{i-1}}{\Delta t}-\frac{U_i-U_{i-1}}{\Delta t},\, u_n(t)-U_n(t)\right\rangle \\
&\quad=\left\langle \frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt},\, u_n(t)-U_n(t)\right\rangle \\
&\quad=\left\langle \frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt},\, u_n(t)-W_i\right\rangle+\left\langle \frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt},\, W_i-U_i\right\rangle \\
&\qquad+\left\langle \frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt},\, U_i-U_n(t)\right\rangle, \qquad t_i<t\leqq t_{i+1}, \quad i=0,1,\cdots,n-1.
\end{aligned}\tag{49}$$

By (47) and (49), we get, for $t_i<t\leqq t_{i+1}$, $i=0,1,\cdots,n-1$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|u_n(t)-U_n(t)\|_2^2)&=\left\langle \frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt},\, u_n(t)-U_n(t)\right\rangle \\
&=\left\langle \frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt},\, u_n(t)-W_i\right\rangle-(AW_i-AU_i,\, W_i-U_i) \\
&\quad+\left\langle \frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt},\, U_i-U_n(t)\right\rangle \\
&\leqq\left\langle \frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt},\, u_n(t)-W_i\right\rangle \\
&\quad+\left\langle \frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt},\, U_i-U_n(t)\right\rangle,
\end{aligned}\tag{50}$$

since $(AW_i-AU_i,\, W_i-U_i)\geqq 0$.

We now estimate the right-hand side of (50): By Lemmas 3 and 11, we have

$$\left\|\frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt}\right\|_2\leqq C(u_0,f).\tag{51}$$

Thus, by (51) and Lemma 10

$$\begin{aligned}
\left|\left\langle \frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt},\, u_n(t)-W_i\right\rangle\right|&\leqq C(u_0,f)\|u_n(t)-W_i\|_2 \\
&=C(u_0,f)\left\|\frac{t-t_i}{\Delta t}(u_{i+1}-u_i)+u_i-W_i\right\|_2 \\
&\leqq C(u_0,f)\left[|(t-t_i)|\left\|\frac{u_{i+1}-u_i}{\Delta t}\right\|_2+\|u_i-W_i\|_2\right] \\
&\leqq C(u_0,f)[\Delta t+\|u_i-W_i\|_2].
\end{aligned}\tag{52}$$

Similarly, we have

$$\begin{aligned}
\left|\left\langle \frac{du_n(t)}{dt}-\frac{dU_n(t)}{dt},\, U_i-U_n(t)\right\rangle\right|&\leqq C(u_0,f)\|U_i-U_n(t)\|_2 \\
&=C(u_0,f)\left\|\frac{t-t_i}{\Delta t}(U_{i+1}-U_i)\right\|_2 \\
&\leqq C(u_0,f)(t-t_i)\left\|\frac{U_{i+1}-U_i}{\Delta t}\right\|_2 \\
&\leqq C(u_0,f)\Delta t.
\end{aligned}\tag{53}$$

By (50), (52), and (53), we have, for $t_i < t \leqq t_{i+1}$, $i = 0, 1, \cdots, n-1$

$$(54)\quad \begin{aligned} \frac{1}{2}\frac{d}{dt}(\|u_n(t) - U_n(t)\|_2^2) &\leqq C(u_0, f)[\Delta t + \|u_i - W_i\|_2] \\ &\leqq C(u_0, f)\left[\Delta t + \underset{0\leqq i\leqq n}{\operatorname{Max}} \|u_i - W_i\|_2\right]. \end{aligned}$$

Integrating (54) over $[0, t]$, we get

$$(55)\quad \|u_n(t) - U_n(t)\|_2^2 \leqq C_1\Delta t + C_2 \underset{0\leqq i\leqq n}{\operatorname{Max}} \|u_i - W_i\|_2 + \|u_0 - W_0\|_2^2,$$

where C_1 and C_2 depend only on $C(u_0, f)$. Therefore, we have the following.

THEOREM 2. *Let $u(t)$ be the true solution of problem* (1)–(3) *obtained in Theorem* 1, *and let $U_n(t)$ be the extended fully discrete solution defined by* (48). *We then have L^2 error estimates*

$$\|u(t) - U_n(t)\|_2^2 \leqq C_1\Delta t + C_2 \underset{0\leqq i\leqq n}{\operatorname{Max}} \|u_i - W_i\|_2 + \|u_0 - W_0\|_2^2.$$

Proof. By (24) and (55), we have

$$\begin{aligned} \|u(t) - U_n(t)\|_2^2 &\leqq 2(\|u(t) - u_n(t)\|_2^2 + \|u_n(t) - U_n(t)\|_2^2) \\ &\leqq C_1\Delta t + C_2 \underset{0\leqq i\leqq n}{\operatorname{Max}} \|u_i - W_i\|_2 + 2\|u_0 - W_0\|_2^2. \end{aligned}$$

Remark 3. By [1, Thm. 5.3.2, p. 317], without assuming "higher regularity" on u, we have

$$\lim_{h\to 0} \|u_i - W_i\|_2 = 0 \quad \text{for each } i, 0 \leqq i \leqq n.$$

Therefore, this and Theorem 2 imply convergence:

$$\lim_{\Delta t\to 0}\left(\lim_{h\to 0} \|u(t) - U_n(t)\|_2\right) = 0.$$

Remark 4. If we assume that for each t, $u(t) \in |W^{2,p}(\Omega)$. Then by Lemma 8, Remark 2, Lemma 10, and Theorem 2, we have L^2 error estimates

$$\begin{aligned} \|u(t) - U_n(t)\|_2^2 &\leqq C_1\Delta t + C_2 \underset{0\leqq i\leqq n}{\operatorname{Max}} \|u_i - W_i\|_2 + \|u_0 - W_0\|_2^2 \\ &\leqq C_1\Delta t + C_2 \underset{0\leqq i\leqq n}{\operatorname{Max}} (\|u_i - \Pi_h u_i\|)^{1/(p-1)} + (\|u_0 - \Pi_h u_0\|)^{2/(p-1)} \\ &\leqq C_1\Delta t + C_2\left(\underset{0\leqq i\leqq n}{\operatorname{Max}} |u_i|_2\right) h^{1/(p-1)} + C_3|u_0|_2 h^{2/(p-1)}. \end{aligned}$$

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