

EFFECTS OF THE MATTER-GRAVITY COUPLING ON COSMOLOGICAL PERTURBATIONS

by

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ABSTRACT

We investigate the effects of matter–gravity coupling on the evolution of cosmological perturbations within the framework of Markov–Mukhanov modified gravity [1]. First, the continuity, Euler, and Poisson equations are derived within the standard formulation of fluid dynamics, as well as in both Newtonian and relativistic cosmological perturbation theory, with the Einstein equations additionally obtained in the relativistic case. These equations are later reformulated in the context of Markov–Mukhanov modified gravity. Three different models $\chi(\epsilon)$ is chosen for three different models: standard GR, linear and exponential. The latter two models particularly influence the evolution of the gravitational potential at certain eras of the universe.

The results demonstrate that matter–gravity coupling can significantly modify the behavior of cosmological perturbations, particularly at late times, leading to deviations from the standard Λ CDM predictions. These findings suggest that such models may provide an alternative explanation in the realm of modified gravity theories.

CHAPTER 1

INTRODUCTION

The universe today is filled with radiation, baryonic matter, and cold dark matter. It is also expanding at an accelerating rate. Theoretically all the existing matter in the universe should exert enough gravitational pull to slow down the expansion, however, this is not what is observed [2]. Therefore, scientists have suggested an energy density that drives this accelerated expansion. It is called "dark energy" and would need to account for approximately $\approx 70\%$ of the entire energy budget of the universe today to achieve this accelerated expansion [3]. The cosmological constant Λ along with cold dark matter is a reasonable approximation of our current universe, it is called Λ CDM model. In this model, dark energy is a constant energy density filling space homogeneously [4].

Recent observations suggested there is a dynamical nature to dark energy, which challenges the Λ CDM paradigm. They had been successful to varying degree as proven by recent observational data [5], [6].

Modified theories of gravity offer a unique perspective on this issue, very different from other propositions. They suggest that dark energy presents the first evidence that general relativity requires modifications. Any adjustments proposed by such theories must be exceptionally small. However, if these modifications prove accurate, it will eliminate the necessity to postulate any exotic forms of energy density to account for dark energy [7]. One of such modified gravity theories, proposed by Markov and Mukhanov in [1], will be studied in this work.

The theory of cosmological perturbations is used to describe the evolution of large-scale structures such as galaxy clusters from small initial fluctuations [8]. It will be presented, studied and subsequently applied to the Markov-Mukhanov modified gravity theory to see how matter fluctuations evolve at both early and late times.

CHAPTER 2

THEORETICAL BACKGROUND

The Friedmann equations in general relativity are [8]:

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\epsilon - \frac{k}{a^2}, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\epsilon + 3p),\end{aligned}\tag{2.1}$$

where a is the scale factor (distance between two fixed points in the universe, as the universe expands the distance increases), G is the gravitational constant, k is a curvature component, ϵ and p are the energy density and pressure from all contributions in the entire universe. These are the primary equations describing the expansion of the universe. The first equation describes how the expansion rate, denoted by the Hubble parameter $H = \dot{a}/a$, is controlled by the matter component $8\pi G\epsilon/3$ and curvature component $-k/a^2$. The curvature element k slows the expansion when it is positive and speeds it up when it is negative.

The second equation describes the accelerated expansion of the universe. It directly relates the sign of $\ddot{a}$ to the energy density ϵ and pressure p . It is then easy to see that for fluids with positive density $\epsilon > 0$ and positive or vanishing pressure $p \geq 0$, the combination $\epsilon + 3p$ implies that the expansion is always decelerating $\ddot{a} < 0$. In order to have an accelerating universe $\ddot{a} > 0$, requires instead that $\epsilon + 3p < 0$. Introducing the equation-of-state parameter $\omega = p/\epsilon$ [8], the condition can be rewritten as $\omega < -1/3$. As said before, currently the observational data says that the universe is under going accelerated expansion, which can not be explained by any matter field with $\omega > -1/3$. The cosmological constant Λ corresponds to a fluid with $\omega = -1$. More generally, this result implies that the dark energy must exhibit negative pressure. Dynamical dark energy models, such as quintessence, often implement equation of state parameter $\omega > -1$ [9]. While idea of a phantom or ghost field corresponds to $\omega < -1$ [10].

1 Fluid equations

In the late universe, on scales well inside the *Hubble radius*, also called the *Hubble length* cH^{-1} (this about 4420 million parsecs or 14.4 billion light years [11], [12], [13]) and for non-relativistic matter, gravity can be accurately described by the Newtonian limit rather than the full framework of general relativity. In the Newtonian limit, spacetime curvature effects are weak, velocities are much smaller than the speed of light c , and gravitational interactions are governed by the Poisson equation for the potential Φ . This simplification allows one to treat cosmic structure formation using familiar Newtonian fluid equations. The expansion is described by a scale factor $a(t)$, whose dynamics follow from the continuity equation, Euler equation, and Poisson equation applied to a homogeneous and isotropic fluid. Essentially they are the analogs of the Friedmann equations 2.1 presented above. Since the universe is modeled as a fluid, deriving these equations and understanding them in depth is an important part of this work. Below, they are derived from conservation laws and explained.

Consider a moving fluid within some volume of space V . For such a fluid, the definition of mass is given by integral [14]:

$$M = \int_V \epsilon dV, \quad (2.2)$$

where ϵ is the energy density and integrating over the whole volume V gives the total mass M . The rate of change of total mass inside the volume V equals the time derivative of that integral [14]:

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \int_V \epsilon dV. \quad (2.3)$$

The flow of mass is described by a quantity known as the *mass flux density* denoted by $\epsilon \mathbf{u}$, where $\mathbf{u}$ is the velocity of the flow rate. Now using this quantity, we write the expression for the net mass *leaving* the volume per unit time [14]:

$$\oint_S \epsilon \mathbf{u} \cdot d\mathbf{f}, \quad (2.4)$$

where $\oint_S d\mathbf{f}$ denotes a surface integral, in which $d\mathbf{f} = \mathbf{n}dS$ is the vector surface element of the magnitude equal to the area of the surface element dS and its direction is along the normal $\mathbf{n}$. Equating the two expressions 2.3 and 2.4:

$$\frac{\partial}{\partial t} \int_V \epsilon dV = - \oint_S \epsilon \mathbf{u} \cdot d\mathbf{f}. \quad (2.5)$$

This expression states that the rate at which mass changes inside a volume is equal to the rate at which mass leaves the same volume, indicated by the minus sign on the right.

The divergence theorem says that for any vector field $\mathbf{\Upsilon}$ [14]:

$$\oint_S \mathbf{\Upsilon} \cdot d\mathbf{f} = \int_V \nabla \cdot \mathbf{\Upsilon} dV. \quad (2.6)$$

Therefore, we can rewrite the equation 2.4 using the divergence theorem 2.6, where the mass flux density $\epsilon \mathbf{u}$ is treated as the vector field $\mathbf{\Upsilon}$ [14]:

$$\oint_S \epsilon \mathbf{u} \cdot d\mathbf{f} = \int_V \nabla \cdot (\epsilon \mathbf{u}) dV. \quad (2.7)$$

Substitute this back into 2.5:

$$\frac{\partial}{\partial t} \int_V \epsilon dV = - \int_V \nabla \cdot (\epsilon \mathbf{u}) dV. \quad (2.8)$$

Now rewrite both sides under the same volume integral:

$$\int_V \left(\frac{\partial \epsilon}{\partial t} + \nabla \cdot (\epsilon \mathbf{u}) \right) dV = 0. \quad (2.9)$$

The integral vanishes since equation must hold for any volume, rearranging results in the following *continuity equation* [14]:

$$\boxed{\frac{\partial \epsilon}{\partial t} = -\nabla \cdot (\epsilon \mathbf{u})}. \quad (2.10)$$

This is the differential form of the continuity equation for fluid dynamics.

Next, we derive the equation of motion for fluids. The total force $\mathbf{F}$ acting on the volume V is given by [14]:

$$\mathbf{F} = - \oint_S p d\mathbf{f}, \quad (2.11)$$

where p is the pressure. Rewrite the surface integral as a volume integral [14]:

$$- \oint_S p d\mathbf{f} = - \int_V \nabla p dV, \quad (2.12)$$

where $-\nabla p$ is the force acting on the volume element dV of the fluid. This allows us to write the equation of motion for the volume element, using Newton's law [14]:

$$-\nabla p = \epsilon \frac{d\mathbf{u}}{dt}. \quad (2.13)$$

An important subtlety here is that the acceleration $d\mathbf{u}/dt$ is not just $\partial\mathbf{u}/\partial t$, but it is the *material derivative* or *convective time derivative*, meaning it follows the fluid element as

it moves. It can be written in two parts:

$$\frac{d\mathbf{u}}{dt} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}, \quad (2.14)$$

where $\partial\mathbf{u}/\partial t$ is the local change at a point and $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is the change due to moving in space. We plug this back Newton's law and use equation 2.13:

$$-\frac{\nabla p}{\epsilon} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}. \quad (2.15)$$

This is the equation of motion of the fluid. It is called *Euler's equation* and was first obtained by Leonhard Euler in 1755 [14]. Since this work is concerned with the cosmological applications, the *Euler equation for a fluid in a gravitational field* can be obtained by adding an additional force with potential Φ to the left hand side. This leads to the following:

$$\boxed{-\frac{\nabla p}{\epsilon} - \nabla\Phi = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\mathbf{u}}, \quad (2.16)$$

where Φ is the gravitational potential, given by the *Poisson equation* [14]:

$$\boxed{\nabla^2\Phi = 4\pi G\epsilon}. \quad (2.17)$$

2 Newtonian perturbation theory

The primary idea behind perturbation theory is the decomposition of all quantities into background values and perturbations [15]:

$$\epsilon(t, \mathbf{r}) = \bar{\epsilon}(t) + \delta\epsilon(t, \mathbf{r}), \quad p(t, \mathbf{r}) = \bar{p}(t) + \delta p(t, \mathbf{r}), \quad (2.18)$$

$$\mathbf{u}(t, \mathbf{r}) = \bar{\mathbf{u}}(t) + \delta\mathbf{u}(t, \mathbf{r}), \quad \Phi(t, \mathbf{r}) = \bar{\Phi}(t) + \delta\Phi(t, \mathbf{r}). \quad (2.19)$$

where the $\bar{\epsilon}$ denotes the background value of energy density and the $\delta\epsilon$ denotes the perturbation (and similarly for p , $\mathbf{u}$ and Φ). Applying this for the fluid equations derived above, will give us the perturbed versions of them.

2.1 Static fluid without gravity

Let's look at a special case, where the fluid is static and in the absence of gravity. This can give us some insights into the nature of density perturbations. This special case implies that the energy density and pressure are unchanging $\bar{\epsilon} = \text{const.}$, $\bar{p} = 0$, the background velocity is zero $\bar{\mathbf{u}} = 0$ and the gravitational potential is zero $\Phi = 0$. The perturbed continuity equation is:

$$\frac{\partial}{\partial t}(\bar{\epsilon} + \delta\epsilon) + \nabla \cdot [(\bar{\epsilon} + \delta\epsilon)(\bar{\mathbf{u}} + \delta\mathbf{u})] = 0. \quad (2.20)$$

Substituting the values from above:

$$\frac{\partial}{\partial t}\delta\epsilon + \nabla \cdot [\bar{\epsilon}\delta\mathbf{u} + \delta\epsilon\delta\mathbf{u}] = 0, \quad (2.21)$$

the second-order term $\delta\epsilon\delta\mathbf{u}$ can be dropped. After rearranging, the resulting perturbed continuity equation looks like [8]:

$$\boxed{\frac{\partial}{\partial t}\delta\epsilon = -\nabla \cdot (\bar{\epsilon}\delta\mathbf{u}).} \quad (2.22)$$

The perturbed Euler equation is:

$$-\frac{\nabla(\bar{p} + \delta p)}{\bar{\epsilon} + \delta\epsilon} - \nabla(\bar{\Phi} + \delta\Phi) = \left(\frac{\partial}{\partial t} + (\bar{\mathbf{u}} + \delta\mathbf{u}) \cdot \nabla \right) (\bar{\mathbf{u}} + \delta\mathbf{u}). \quad (2.23)$$

Applying the special case properties and dropping the second-order terms to get perturbed Euler equation in this form:

$$-\frac{\nabla(\bar{p} + \delta p)}{\bar{\epsilon} + \delta\epsilon} = \left(\frac{\partial}{\partial t} + \delta\mathbf{u} \cdot \nabla \right) \delta\mathbf{u}. \quad (2.24)$$

At linear order, second and higher-order terms can be dropped. Therefore, we can drop $\nabla\delta p/\delta\epsilon$, $(\delta\mathbf{u} \cdot \nabla)\delta\mathbf{u}$ and $\nabla\bar{p}$ because background values $\bar{p}(t)$ are only functions of time. Rearranging $\bar{\epsilon}$ to the right side gives [8]:

$$\boxed{\frac{\partial}{\partial t}\delta\mathbf{u} = -\frac{\nabla\delta p}{\bar{\epsilon}}.} \quad (2.25)$$

To see how the perturbations behave in this medium, the wave equation needs to be obtained and solved. First step, take the perturbed continuity equation 2.22 and derive with respect to t :

$$\frac{\partial^2}{\partial t^2}\delta\epsilon = -\nabla \cdot \left(\bar{\epsilon} \frac{\partial}{\partial t}\delta\mathbf{u} \right). \quad (2.26)$$

Substitute the perturbed Euler equation 2.25 into the perturbed continuity equation 2.26:

$$\frac{\partial^2}{\partial t^2}\delta\epsilon = -\bar{\epsilon}\nabla \cdot \left(-\frac{\nabla\delta p}{\bar{\epsilon}} \right). \quad (2.27)$$

After simplifying this, the wave equation in the form is obtained:

$$\frac{\partial^2}{\partial t^2}\delta\epsilon = \nabla^2\delta p, \quad (2.28)$$

where the fluctuations are *adiabatic*, meaning the pressure fluctuations are proportional to the density fluctuations $\delta p = c_s^2\delta\epsilon$, where c_s is the *speed of sound*, more on this later.

Considering this, we can rewrite the wave equation as:

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta\epsilon = 0. \quad (2.29)$$

This is a partial differential equation solved by a plane wave:

$$\delta\epsilon = Ae^{i(ft - \mathbf{k} \cdot \mathbf{r})}, \quad (2.30)$$

where $f = c_s |\mathbf{k}|$. Perturbations behave like sound waves. The main take away from this, is that the fluctuations oscillate with constant amplitude in a static fluid with no gravity. This means that hypothetically, in a universe where gravity doesn't exist, no structure would ever form.

2.2 Static fluid with gravity

With gravity turned on, a more interesting picture is obtained. The perturbed continuity equation stays the same. However, the perturbed Euler equation 2.25 now includes a gravitational potential term $\nabla\delta\Phi$:

$$\boxed{\frac{\partial}{\partial t} \delta \mathbf{u} = -\frac{\nabla \delta p}{\bar{\epsilon}} - \nabla \delta \Phi.} \quad (2.31)$$

Repeating the same method to obtain the wave equation, substitute expression above into the equation 2.26:

$$\frac{\partial^2}{\partial t^2} \delta\epsilon = \nabla^2 \delta p + \bar{\epsilon} \nabla^2 \delta\Phi. \quad (2.32)$$

Using the equation of state for adiabatic perturbation $\delta p = c_s^2 \delta\epsilon$ and the perturbed Poisson equation $\nabla\delta\Phi = 4\pi G\delta\epsilon$, we can substitute to get:

$$\frac{\partial^2}{\partial t^2} \delta\epsilon = c_s^2 \nabla^2 \delta\epsilon + \bar{\epsilon} \nabla^2 (4\pi G\delta\epsilon). \quad (2.33)$$

Rearranging some parts to get the wave equation in the form:

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta\epsilon = 4\pi G\bar{\epsilon} \nabla^2 \delta\epsilon. \quad (2.34)$$

This has the same plane wave solution as case without gravity $\delta\epsilon = Ae^{i(ft - \mathbf{k} \cdot \mathbf{r})}$, but with the frequency f having an additional term:

$$f^2 = c_s^2 k^2 - 4\pi G\bar{\epsilon}. \quad (2.35)$$

For sufficiently large wavenumbers k , the first term dominates, so that $f^2 > 0$. This case is identical to the case without gravity, where perturbations behave like sound waves.

For sufficiently small wavenumbers k , the gravitational term dominates, and f^2 becomes

negative. In this case, the frequency f is imaginary, and the solution is no longer oscillatory but exponential:

$$\delta\epsilon \propto e^{|f|t}. \quad (2.36)$$

This signals an instability, meaning that any small overdensity grows rapidly with time. The boundary between these two regimes is defined by the condition $f^2 = 0$, which gives the critical wavenumber:

$$k_J = \sqrt{\frac{4\pi G\bar{\epsilon}}{c_s^2}}. \quad (2.37)$$

This corresponds to a characteristic length scale, the *Jeans length*,

$$\lambda_J = \frac{2\pi}{k_J} = c_s \sqrt{\frac{\pi}{G\bar{\epsilon}}}. \quad (2.38)$$

In this sense, the Jeans length acts like a filter on scales. It determines which fluctuations in the universe can actually collapse to form objects like galaxies, and which ones remain stable and merely oscillate.

2.3 Expanding fluid

For expanding fluid, the distance measured anywhere in space or the physical coordinate $\mathbf{r}$ is given by equation [8]:

$$\mathbf{r}(t) = a(t)\mathbf{x}, \quad (2.39)$$

where $a(t)$ is the scale factor and the $\mathbf{x}$ is the comoving coordinate, which is a coordinate that stays fixed for objects that simply follow the expansion of the universe. The proper velocity changes from $\mathbf{u}(t) = \partial\mathbf{x}/\partial t$ to:

$$\mathbf{u}(t) = \dot{\mathbf{r}} = \dot{a}\mathbf{x} + a\dot{\mathbf{x}}, \quad (2.40)$$

where $\dot{a}\mathbf{x} = \mathbf{r}\dot{a}/a$ is the Hubble flow (where the Hubble parameter $H = \dot{a}/a$) and $a\dot{\mathbf{x}} = \mathbf{v}$ is the proper velocity. This new relationship fundamentally changes how the derivatives are defined because they are no longer independent. The proper spatial derivative $\nabla_{\mathbf{r}}$ with respect to the physical coordinate $\mathbf{r}$ is equal:

$$\nabla_{\mathbf{r}} = \frac{\nabla_{\mathbf{x}}}{a}. \quad (2.41)$$

where a is the scale factor and $\nabla_{\mathbf{x}}$ is the spatial derivative with respect to the comoving coordinate $\mathbf{x}$. Drop the $\mathbf{x}$ index for convenience.

The relationship between time derivatives is also different. The chain rule at fixed $\mathbf{r}$ and at fixed $\mathbf{x}$ states that:

$$\left(\frac{\partial}{\partial t}\right)_{\mathbf{r}} = \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial\mathbf{x}}{\partial t}\right)_{\mathbf{r}} \cdot \nabla_{\mathbf{x}}, \quad (2.42)$$

where the operators are not acting on any functions yet. If $\mathbf{x}$ is expanded, the time

derivative can be rewritten as:

$$\left(\frac{\partial}{\partial t}\right)_r = \left(\frac{\partial}{\partial t}\right)_x + \left(\frac{\partial}{\partial t} \left(\frac{\mathbf{r}}{a(t)}\right)\right) \cdot \nabla_x = \left(\frac{\partial}{\partial t}\right)_x - H\mathbf{x} \cdot \nabla_x. \quad (2.43)$$

Drop the indices $\mathbf{x}$ for both time and spatial derivatives moving on. With everything about the expanding fluid in mind, the fluid equations can be obtained.

The perturbed continuity equation in an expanding fluid shows how the total density $\bar{\epsilon}(1 + \delta)$ (more on this below) changes due to the Hubble flow $H\mathbf{a}\mathbf{x}$ and peculiar velocity v [8]. To obtain it, begin with the perturbed continuity equation in the form:

$$\frac{\partial}{\partial t}[\bar{\epsilon}(1 + \delta)] + \nabla_r \cdot [\bar{\epsilon}(1 + \delta)\mathbf{u}] = 0, \quad (2.44)$$

where $\delta = \delta\epsilon/\bar{\epsilon}$ is the *fractional density perturbation* or just density contrast [8]. Adding substituting the derivatives and the proper velocity into this equation:

$$\left[\frac{\partial}{\partial t} - H\mathbf{x} \cdot \nabla \right] [\bar{\epsilon}(1 + \delta)] + \frac{\nabla}{a} \cdot [\bar{\epsilon}(1 + \delta)(H\mathbf{a}\mathbf{x} + v)] = 0. \quad (2.45)$$

The perturbed Euler equation in an expanding fluid describes the evolution of the velocity v at first-order in perturbations. Begin with the perturbed Euler equation in form:

$$-\frac{\nabla_r \delta p}{\bar{\epsilon}} - \nabla_r \bar{\Phi} - \nabla_r \delta \Phi = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla_r \right) \mathbf{u}. \quad (2.46)$$

The $\nabla_r \bar{p}$ term is dropped due to the background pressure not being a function of r . Expanding the derivatives:

$$-\frac{\nabla \delta p}{a\bar{\epsilon}} - \frac{\nabla \bar{\Phi}}{a} - \frac{\nabla \delta \Phi}{a} = \left[\frac{\partial}{\partial t} - H\mathbf{x} \cdot \nabla + (H\mathbf{a}\mathbf{x} + v) \cdot \frac{\nabla}{a} \right] (H\mathbf{a}\mathbf{x} + v). \quad (2.47)$$

The terms $-H\mathbf{x} \cdot \nabla$ and $H\mathbf{x} \cdot \nabla$ cancel out, resulting in:

$$-\frac{\nabla \delta p}{a\bar{\epsilon}} - \frac{\nabla \bar{\Phi}}{a} - \frac{\nabla \delta \Phi}{a} = \left[\frac{\partial}{\partial t} + v \cdot \frac{\nabla}{a} \right] (H\mathbf{a}\mathbf{x}) + \left[\frac{\partial}{\partial t} + v \cdot \frac{\nabla}{a} \right] v. \quad (2.48)$$

The expanding first bracket on the right-hand side results in component $[\partial/\partial t + v \cdot \nabla/a] H\mathbf{a}\mathbf{x} = (\dot{H} + H^2)\mathbf{a}\mathbf{x} + H\mathbf{v}$ and expanding the second bracket on the right-hand side results in component $[\partial/\partial t + v \cdot \nabla/a] v = \dot{v}$, since $(v \cdot \nabla/a)v$ can be dropped because it is a second-order term.

$$-\frac{\nabla \delta p}{a\bar{\epsilon}} - \frac{\nabla \bar{\Phi}}{a} - \frac{\nabla \delta \Phi}{a} = (\dot{H} + H^2)\mathbf{a}\mathbf{x} + H\mathbf{v} + \dot{v}. \quad (2.49)$$

The expression $(\dot{H} + H^2)\mathbf{a}\mathbf{x}$ is the acceleration of the Hubble flow and cancels out with the background gravity component $\nabla \bar{\Phi}$, resulting in the perturbed Euler equation for an

expanding fluid:

$$\boxed{-\frac{\nabla\delta p}{a\bar{\epsilon}} - \frac{\nabla\delta\Phi}{a} = \dot{v} + Hv,} \quad (2.50)$$

where $\dot{v}$ is the time derivative of the proper velocity v , Hv represents the Hubble drag, which causes v to decay in an expanding space in the absence of pressure and gravitational forces (where $v \propto 1/a$), the terms on the right-hand side are the forces, specifically, a pressure gradient $\nabla\delta p$ and a gravitational gradient ($\nabla\delta\Phi$).

The Poisson equation relates the gravitational potential perturbation $\delta\Phi$ to the density contrast δ :

$$\boxed{\nabla^2\delta\Phi = 4\pi G a^2 \bar{\epsilon} \delta,} \quad (2.51)$$

where $\nabla^2\delta\Phi$ is the Laplacian of the gravitational potential perturbation, the source of the perturbation is proportional to $4\pi G$ (Newton's constant), the average background density $\bar{\epsilon}$, and the fractional density perturbation δ , the factor of a^2 comes from using derivatives with respect to comoving coordinates 2.41.

3 Relativistic perturbation theory

The Newtonian treatment of cosmological perturbations is not good enough for relativistic fluids and for scales beyond the Hubble radius, so the full description requires a general relativistic treatment. This will require a more detailed look at the role of gauge transformations, the construction of gauge-invariant variables, and the physical interpretation of the resulting equations.

3.1 Newtonian gauge

Before explaining the specifics of the relativistic perturbation theory, we need to address the *gauge problem* [16]. In general relativity, coordinates have no physical meaning by themselves. When the metric perturbation is written as:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad (2.52)$$

it introduces redundant degrees of freedom. There are different ways of removing them such as using gauge-invariant transformations, but here we will simply fix the gauge by introducing the *Newtonian gauge*, also called *longitudinal gauge* [8]. The perturbed Friedmann-Lemaître-Robertson-Walker (FLRW) metric is defined as:

$$ds^2 = a^2(\tau)[(1 + 2A)d\tau^2 - 2B_i dx^i d\tau - (\delta_{ij} + h_{ij})dx^i dx^j], \quad (2.53)$$

where A , B_i and h_{ij} are functions of space and time [8]. We will implement scalar-vector-tensor decomposition [17], which splits any 3-vector into the gradient of a scalar and a divergenceless vector. Since the Newtonian gauge is only defined for scalars, decomposi-

tion is carried out for B_i and h_{ij} [16]:

$$B_i = \partial_i B, \quad h_{ij} = 2C\delta_{ij} + 2\partial_{(i}\partial_{j)}E. \quad (2.54)$$

Substituting these into the equation 2.53:

$$ds^2 = a^2(\tau)[(1 + 2A)d\tau^2 - 2\partial_i B dx^i d\tau - ((1 + 2C)\delta_{ij} + 2\partial_{(i}\partial_{j)}E)dx^i dx^j]. \quad (2.55)$$

The Newtonian gauge imposes $B = 0$ and $E = 0$ on equation 2.55 [16]. This removes coordinate freedom and leaves only A and C which are renamed to Φ and Ψ , corresponding to gravitational potentials:

$$ds^2 = a^2(\tau)[(1 + 2\Psi)d\tau^2 - (1 - 2\Phi)\delta_{ij}dx^i dx^j]. \quad (2.56)$$

This is the final form of the perturbed FLRW line element in the Newtonian gauge, and it is completely fixed [16].

3.2 Energy-momentum tensor

With the space component (metric) derived, the matter component is next in line. In general relativity, all information about energy and matter fields is given by the quantity called the *energy-momentum tensor*, or the *stress-energy tensor*, T^μ_ν . Considering the small perturbations of the energy-momentum tensor:

$$T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu, \quad (2.57)$$

where the background component takes the form of the perfect fluid:

$$\bar{T}^\mu_\nu = (\bar{\epsilon} + \bar{p})\bar{u}^\mu\bar{u}_\nu - \bar{p}\delta^\mu_\nu. \quad (2.58)$$

Here the four-velocity components are $\bar{u}_\mu = a\delta_\mu^0$ and $\bar{u}^\mu = \delta_0^\mu/a$ for a comoving observer. Therefore, the background four-velocity components are $\bar{u}^0 = 1/a$, $\bar{u}^i = 0$, $\bar{u}_0 = a$, $\bar{u}_j = 0$. Writing the $\bar{T}^\mu_\nu$ components:

$$\bar{T}_0^0 = (\bar{\epsilon} + \bar{p})\left(\frac{1}{a}\right)(a) - \bar{p}\left(\frac{1}{a}\right)(a) = \bar{\epsilon}, \quad (2.59)$$

$$\bar{T}_i^0 = (\bar{\epsilon} + \bar{p})(0) - \bar{p}(0) = 0, \quad (2.60)$$

$$\bar{T}_0^i = (\bar{\epsilon} + \bar{p})(0) - \bar{p}(0) = 0, \quad (2.61)$$

$$\bar{T}_j^i = (\bar{\epsilon} + \bar{p})(0) - \bar{p}\delta_j^i = -\bar{p}\delta_j^i. \quad (2.62)$$

The perturbation component is a little more complex. The energy density ϵ , the pressure p and the four-velocity u^μ can be functions of position as stated before in the equations 2.18. Moreover, the energy-momentum tensor can now have a contribution from *anisotropic*

stresses, Π_ν^μ . The perturbation component of the energy-momentum tensor therefore is:

$$\delta T_\nu^\mu = (\delta\epsilon + \delta p)\bar{u}^\mu\bar{u}_\nu + (\bar{\epsilon} + \bar{p})(\delta u^\mu\bar{u}_\nu + \bar{u}^\mu\delta u_\nu) - \delta p\delta_\nu^\mu - \Pi_\nu^\mu. \quad (2.63)$$

Anisotropic stress Π_ν^μ describes directional stresses in the energy-momentum tensor beyond simple pressure. It is the trace-free spatial part of δT_ν^μ , capturing shear and viscosity-like effects. In a perfect fluid, pressure is isotropic, so anisotropic stress is zero. Therefore, it can be dropped later on without issue. It does play a key role in Einstein's equations by sourcing the difference between gravitational potentials Φ and Ψ . If anisotropic stress vanishes, one obtains the simplification $\Phi = \Psi$. This will be talked about more later when we derive the perturbed Einstein's equations.

For now, focus on deriving all the components of the perturbation of the energy-momentum tensor δT_ν^μ . To find them, compute the perturbed four-velocity in the perturbed FLRW metric 2.56. In the Newtonian gauge, the metric components are $g_{00} = a^2(1 + 2\Psi)$, $g_{i0} = g_{0j} = 0$ and $g_{ij} = -a^2(1 - 2\Phi)\delta_{ij}$. Since $g_{\mu\nu}u^\mu u^\nu = 1$ and $\bar{g}_{\mu\nu}\bar{u}^\mu\bar{u}^\nu = 1$, at linear order:

$$\delta g_{\mu\nu}\bar{u}^\mu\bar{u}^\nu + 2\bar{u}_\mu\delta u^\mu = 0. \quad (2.64)$$

Now, using $\bar{u}^0 = 1/a$ and $\delta g_{00} = 2a^2\Psi$, find that $\delta u^0 = -\Psi/a$ and $\delta u^i = v^i/a$, where $v^i = dx^i/d\tau$ is the coordinate velocity. Thus, in general:

$$u^\mu = \frac{1}{a}(1 - \Psi, v^i). \quad (2.65)$$

From this, another two components can be derived:

$$u_0 = g_{00}u^0 = a^2(1 + 2\Psi)\frac{1}{a}(1 - \Psi) = a(1 + \Psi), \quad (2.66)$$

$$u_i = g_{ij}u^j = -a^2(1 - 2\Phi)\delta_{ij}\frac{v^j}{a} = -av_i, \quad (2.67)$$

where the higher-order term Φv gets dropped. Summarizing,

$$u_\mu = a(1 + \Psi, -v_i). \quad (2.68)$$

Substituting these components of the four-velocity into the perturbation of the energy-

momentum tensor, gives the components:

$$\delta T_0^0 = (\delta\epsilon + \delta p)(1) + (\bar{\epsilon} + \bar{p}) \left[\left(-\frac{\Psi}{a} \right) (a) + \left(\frac{1}{a} \right) (a\Psi) \right] - \delta p \delta_0^0 = \delta\epsilon, \quad (2.69)$$

$$\delta T_0^i = (\delta\epsilon + \delta p)(0) + (\bar{\epsilon} + \bar{p}) \left[\left(\frac{1}{a} \right) (-av^i) \right] - \delta p \delta_0^i = -(\bar{\epsilon} + \bar{p})v^i, \quad (2.70)$$

$$\delta T_j^0 = (\delta\epsilon + \delta p)(0) + (\bar{\epsilon} + \bar{p}) \left[\left(\frac{v^i}{a} \right) a \right] - \delta p \delta_j^0 = (\bar{\epsilon} + \bar{p})v_j, \quad (2.71)$$

$$\delta T_j^i = (\delta\epsilon + \delta p)(0) + (\bar{\epsilon} + \bar{p})(0) - \delta p \delta_j^i = -\delta p \delta_j^i. \quad (2.72)$$

We will denote the expression $(\bar{\epsilon} + \bar{p})v^i$ as the *momentum density* q^i .

3.3 Adiabatic fluctuations

We mentioned before that adiabatic fluctuations imply relationship $\delta p = c_s^2 \delta\epsilon$, where c_s is the speed of sound. In the perturbed universe, adiabatic perturbations mean that all components (matter, radiation, etc.) are perturbed in a coordinated way, as if the time was locally shifted. Instead of density fluctuating randomly, at position x the universe is just slightly ahead or behind in its evolution $t \rightarrow t + \delta\tau(x)$. So the local state is [8]:

$$\epsilon_I(\tau, x) \equiv \bar{\epsilon}_I(\tau + \delta\tau(x)). \quad (2.73)$$

If the universe is filled with multiple fluids, adiabatic perturbations correspond to perturbations induced by a common, local shift in time of all background quantities, or in other words adiabatic density perturbations are defined as [8]:

$$\delta\epsilon_I(\tau, x) \equiv \bar{\epsilon}_I(\tau + \delta\tau(x)) - \bar{\epsilon}_I(\tau) = \dot{\bar{\epsilon}}_I \delta\tau(x), \quad (2.74)$$

where $\delta\tau$ is the same for all species I . This implies:

$$\delta\tau = \frac{\delta\epsilon_I}{\dot{\bar{\epsilon}}_I} = \frac{\delta\epsilon_J}{\dot{\bar{\epsilon}}_J} \quad \text{for all species } I \text{ and } J. \quad (2.75)$$

Using the Friedmann equation $\dot{\bar{\epsilon}}_I = -3H(1 + \omega_I)\bar{\epsilon}_I$, this can be rewritten as:

$$\frac{\delta_I}{1 + w_I} = \frac{\delta_J}{1 + w_J} \quad \text{for all species } I \text{ and } J, \quad (2.76)$$

where $\delta_I = \delta\epsilon_I/\bar{\epsilon}_I$ is the fractional density contrast. For example, if equation of state for radiation components ($\omega_r = 1/3$) and the equation of state for matter components ($w_m \approx 0$) are substituted into this expression, the following relation is obtained:

$$\frac{\delta_r}{1 + \frac{1}{3}} = \frac{\delta_m}{1 + 0} \quad \rightarrow \quad \delta_r = \frac{4}{3} \delta_m. \quad (2.77)$$

3.4 Continuity equation

The perturbed continuity equation for general relativity is derived from the perturbed energy momentum conservation $\nabla_\mu T^\mu_\nu = 0$:

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\sigma} T^\sigma_\nu - \Gamma^\sigma_{\mu\nu} T^\mu_\sigma = 0, \quad (2.78)$$

where the $\Gamma^\sigma_{\mu\nu}$ are the *connection coefficients*, also called *Christoffel symbols*. They are given by equation:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\mu\lambda} (\partial_\mu g_{\lambda\sigma} + \partial_\sigma g_{\lambda\mu} - \partial_\lambda g_{\mu\sigma}). \quad (2.79)$$

These can be derived using our perturbed FLRW metric 2.56. The inverse metric terms are $g^{00} = 1/a^2(1 - 2\Psi)$, $g^{i0} = g^{0j} = 0$ and $g^{ij} = 1/a^2(1 - 2\Phi)\delta^{ij}$. Step by step breaking down each component of the Christoffel symbols, beginning with Γ^0_{00} :

$$\Gamma^0_{00} = \frac{1}{2} g^{00} (\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}) = \frac{1}{2a^2} (1 - 2\Psi) \partial_0 [a^2(1 + 2\Psi)]. \quad (2.80)$$

Calculating the derivative ∂_0 gives:

$$\Gamma^0_{00} = \frac{1}{2a^2} (1 - 2\Psi) (2a\dot{a} + 2a^2\dot{\Psi}) = H + \Psi, \quad (2.81)$$

at linear order, where $H = \dot{a}/a$ is the Hubble parameter. From now on, we will write H instead of $\dot{a}/a$. The second term Γ^0_{0i} :

$$\Gamma^0_{0i} = \frac{1}{2} g^{00} (\partial_0 g_{0i} + \partial_i g_{00} - \partial_0 g_{0i}) = \frac{1}{2a^2} (1 - 2\Psi) \partial_i [a^2(1 + 2\Psi)]. \quad (2.82)$$

Calculating the derivative ∂_i gives:

$$\Gamma^0_{0i} = \frac{1}{2a^2} (1 - 2\Psi) (2a^2 \partial_i \Psi) = \partial_i \Psi, \quad (2.83)$$

at linear order. The background part here is zero and only the perturbation $\partial_i \Psi$ remains. The third term Γ^i_{00} :

$$\Gamma^i_{00} = \frac{1}{2} g^{ij} (\partial_0 g_{j0} + \partial_0 g_{0j} - \partial_j g_{00}) = -\frac{1}{2a^2} (1 + 2\Phi) \delta^{ij} \partial_j [a^2(1 + 2\Psi)]. \quad (2.84)$$

Calculating the derivative ∂_j gives:

$$\Gamma^i_{00} = \frac{1}{2a^2} (1 + 2\Phi) \delta^{ij} (2a^2 \partial_j \Psi) = \delta^{ij} \partial_j \Psi, \quad (2.85)$$

at linear order. Similarly, the background term here is zero. The fourth term Γ^0_{ij} :

$$\Gamma^0_{ij} = \frac{1}{2} g^{00} (\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij}) = -\frac{1}{2a^2} (1 - 2\Psi) \partial_0 [-a^2(1 - 2\Phi)\delta_{ij}]. \quad (2.86)$$

Calculating the derivative ∂_0 gives:

$$\Gamma_{ij}^0 = \frac{1}{2a^2}(1 - 2\Psi)[2a\dot{a}(1 - 2\Phi) - 2a^2\dot{\Phi}]\delta_{ij} = H\delta_{ij} - [\dot{\Phi} + 2H(\Phi + 2\Psi)]\delta_{ij}, \quad (2.87)$$

at linear order. The fifth term Γ_{j0}^i :

$$\Gamma_{j0}^i = \frac{1}{2}g^{ik}(\partial_j g_{k0} + \partial_0 g_{jk} - \partial_k g_{j0}) = -\frac{1}{2a^2}(1 + 2\Phi)\delta^{ik}\partial_0[-a^2(1 - 2\Phi)\delta_{kj}]. \quad (2.88)$$

Calculating the derivative ∂_0 gives:

$$\Gamma_{j0}^i = \frac{1}{2a^2}(1 + 2\Phi)[2a\dot{a}(1 - 2\Phi) - 2a^2\dot{\Phi}]\delta_j^i = (H - \dot{\Phi})\delta_j^i, \quad (2.89)$$

at linear order. Finally, the last term Γ_{jk}^i :

$$\Gamma_{jk}^i = \frac{1}{2}g^{i\lambda}(\partial_j g_{k\lambda} + \partial_k g_{j\lambda} - \partial_\lambda g_{jk}) = -\frac{1}{2a^2}(1 + 2\Phi)\delta^{i\lambda}[2a^2(\partial_j \Phi \delta_{\lambda k} + \partial_k \Phi \delta_{\lambda j} - \partial_\lambda \Phi \delta_{jk})]. \quad (2.90)$$

At linear order, the term takes form:

$$\Gamma_{jk}^i = -(\delta^{i\lambda})(\partial_j \Phi \delta_{\lambda k} + \partial_k \Phi \delta_{\lambda j} - \partial_\lambda \Phi \delta_{jk}) = -(\partial_j \Phi \delta_k^i + \partial_k \Phi \delta_j^i - \partial^i \Phi \delta_{jk}). \quad (2.91)$$

Introduce symmetrization notation $\delta_{(j}^i \delta_{k)}\Phi = 1/2(\partial_j \Phi \delta_k^i + \partial_k \Phi \delta_j^i)$, this can be rewritten as:

$$\Gamma_{jk}^i = -2\delta_{(j}^i \delta_{k)}\Phi + \partial^i \Phi \delta_{jk} \quad \text{or} \quad \Gamma_{jk}^i = -2\delta_{(j}^i \delta_{k)}\Phi + \delta_{jk}\delta^{i\lambda}\partial_\lambda \Phi. \quad (2.92)$$

Now, return to the continuity equation. Consider first the $\nu = 0$ component [8]:

$$\nabla_\mu T_0^\mu = \partial_\mu T_0^\mu + \Gamma_{\mu\sigma}^\mu T_0^\sigma - \Gamma_{\mu 0}^\sigma T_\sigma^\mu = 0. \quad (2.93)$$

Expanding every term individually, gives:

$$\partial_\mu T_0^\mu = \partial_0 T_0^0 + \partial_i T_0^i, \quad (2.94)$$

$$\Gamma_{\mu\sigma}^\mu T_0^\sigma = \Gamma_{00}^0 T_0^0 + \Gamma_{i0}^i T_0^0 + \Gamma_{0i}^0 T_0^i + \Gamma_{ji}^j T_0^i, \quad (2.95)$$

$$\Gamma_{\mu 0}^\sigma T_\sigma^\mu = -\Gamma_{00}^0 T_0^0 - \Gamma_{i0}^0 T_0^i - \Gamma_{00}^i T_i^0 - \Gamma_{j0}^i T_i^j. \quad (2.96)$$

The terms $\Gamma_{00}^0 T_0^0$ and $-\Gamma_{00}^0 T_0^0$ cancel out, so do $\Gamma_{0i}^0 T_0^i$ and $-\Gamma_{i0}^0 T_i^0$. Higher order terms $\Gamma_{ji}^j T_0^i$ and $\Gamma_{00}^i T_i^0$ can be dropped. The remaining terms are:

$$\nabla_\mu T_0^\mu = \partial_0 T_0^0 + \partial_i T_0^i + \Gamma_{i0}^i T_0^0 - \Gamma_{j0}^i T_i^j = 0. \quad (2.97)$$

The Christoffel symbol $\Gamma_{i0}^i = 3(H - \dot{\Phi})$ because $\delta_i^i = 3$. Substituting the perturbed energy-momentum tensor components and the Christoffel symbols gives:

$$\partial_0(\bar{\epsilon} + \delta\epsilon) + \partial_i q^i + 3(H - \dot{\Phi})(\bar{\epsilon} + \delta\epsilon) - (H - \dot{\Phi})\delta_j^i [-(\bar{p} + \delta p)\delta_i^j] = 0. \quad (2.98)$$

Since $\delta_j^i \delta_i^j = 3$, further simplification:

$$\dot{\bar{\epsilon}} + \delta\dot{\epsilon} + \partial_i q^i + 3H(\bar{\epsilon} + \delta\epsilon) - 3\dot{\Phi}(\bar{\epsilon} + \bar{p}) + 3H(\bar{p} + \delta p) = 0. \quad (2.99)$$

This equation can be separated into two parts, the zeroth-order and first-order parts:

$$\dot{\bar{\epsilon}} = -3H(\bar{\epsilon} + \bar{p}), \quad (2.100)$$

$$\delta\dot{\epsilon} = -3H(\delta\epsilon + \delta p) + 3\dot{\Phi}(\bar{\epsilon} + \bar{p}) - \partial_i q^i, \quad (2.101)$$

The zeroth-order part is the conservation of energy in the homogeneous background. The first-order part describes the evolution of the density perturbation. In the first-order part, the first term on the right-hand side is just the dilution due to the background expansion, $\partial_i q^i$ can be rewritten as $\nabla \cdot q$, which denotes the local fluid flow and the $\dot{\Phi}$ term is a purely relativistic effect corresponding to the density changes caused by perturbations to the local expansion rate. This equation can be rewritten in terms of the fractional overdensity $\delta \equiv \delta\epsilon/\bar{\epsilon}$ and the 3-velocity $v = q/(\bar{\epsilon} + \bar{p})$. The equation above then becomes:

$$\boxed{\delta + \left(1 + \frac{\bar{p}}{\bar{\epsilon}}\right) \left(\nabla \cdot v - 3\dot{\Phi}\right) + 3H \left(\frac{\delta p}{\delta\epsilon} - \frac{\bar{p}}{\bar{\epsilon}}\right) \delta = 0.} \quad (2.102)$$

This is the final relativistic version of the continuity equation.

3.5 Euler equation

Next, consider the $\nu = i$ component of the energy-momentum conservation:

$$\partial_\mu T_i^\mu + \Gamma_{\mu\sigma}^\mu T_i^\sigma - \Gamma_{\mu i}^\sigma T_\sigma^\mu = 0. \quad (2.103)$$

Expanding every term individually, gives:

$$\partial_\mu T_i^\mu = \partial_0 T_i^0 + \partial_j T_i^j, \quad (2.104)$$

$$\Gamma_{\mu\sigma}^\mu T_i^\sigma = \Gamma_{00}^0 T_i^0 + \Gamma_{0j}^0 T_i^j + \Gamma_{j0}^j T_i^0 + \Gamma_{jk}^j T_i^k, \quad (2.105)$$

$$\Gamma_{\mu i}^\sigma T_\sigma^\mu = -\Gamma_{0i}^0 T_0^0 - \Gamma_{ji}^0 T_0^j - \Gamma_{0i}^j T_j^0 - \Gamma_{ki}^j T_j^k. \quad (2.106)$$

Christoffel symbols have different indices here, but they all were computed before and the indices correspond to the three spatial coordinates $i, j, k \approx x, y, z$. Substituting ev-

everything, except the last Christoffel symbol Γ_{ki}^j :

$$\partial_\mu T_i^\mu = -\partial_0 q_i - \partial_j [(\bar{p} + \delta p) \delta_i^j], \quad (2.107)$$

$$\Gamma_{\mu\sigma}^\mu T_i^\sigma = -(H + \dot{\Phi}) q_i - (\partial_j \Psi) [(\bar{p} + \delta p) \delta_i^j] - 3(H + \dot{\Phi}) q_i + (-3\partial_k \Phi) [-(\bar{p} + \delta p) \delta_i^k], \quad (2.108)$$

$$\Gamma_{\mu i}^\sigma T_\sigma^\mu = -(\partial_i \Psi)(\bar{\epsilon} + \delta\epsilon) - (H\delta_{ij} - [\dot{\Phi} + 2H(\Phi + 2\Psi)]\delta_{ji}) q^j + [(H - \dot{\Phi}) \delta_i^j] q_j + \quad (2.109)$$

$$+ \Gamma_{ki}^j (\bar{p} + \delta p) \delta_j^k, \quad (2.110)$$

where the last term can be rewritten as $\Gamma_{ki}^j (\bar{p} + \delta p) \delta_j^k = (\bar{p} + \delta p) \Gamma_{ji}^j = -3\partial_i \Phi (\bar{p} + \delta p)$. Dropping higher-order terms gives:

$$\partial_\mu T_i^\mu = -\partial_0 q_i - \partial_i \delta p, \quad (2.111)$$

$$\Gamma_{\mu\sigma}^\mu T_i^\sigma = -H q_i - \bar{p} \partial_i \Psi - 3H q_i + 3\bar{p} \partial_i \Phi, \quad (2.112)$$

$$\Gamma_{\mu i}^\sigma T_\sigma^\mu = -\bar{\epsilon} \partial_i \Psi - H q_i + H q_i - 3\bar{p} \partial_i \Phi. \quad (2.113)$$

Combine and substitute back into the equation, after canceling some terms:

$$-\dot{q}_i - \partial_i \delta p - 4H q_i - \partial_i \Psi (\bar{\epsilon} + \bar{p}) = 0. \quad (2.114)$$

Using expression $\dot{q}_i = (\dot{\bar{\epsilon}} + \dot{\bar{p}}) v_i + (\bar{\epsilon} + \bar{p}) \dot{v}_i = [-3H(\bar{\epsilon} + \bar{p}) + \dot{\bar{p}}] v_i + (\bar{\epsilon} + \bar{p}) \dot{v}_i$ that utilizes the zero-order part of the continuity equation 2.100, rewrite the equation above:

$$(\bar{\epsilon} + \bar{p}) \dot{v}_i + H(\bar{\epsilon} + \bar{p}) \dot{v}_i + \dot{\bar{p}} v_i + \partial_i \delta p + \partial_i \Psi (\bar{\epsilon} + \bar{p}) = 0. \quad (2.115)$$

Divide everything by $\bar{\epsilon} + \bar{p}$:

$$\dot{v}_i + H \dot{v}_i + \frac{\dot{\bar{p}} v_i}{\bar{\epsilon} + \bar{p}} + \frac{\partial_i \delta p}{\bar{\epsilon} + \bar{p}} + \partial_i \Psi = 0. \quad (2.116)$$

Using the zeroth-order part of the continuity equation 2.100 again, to rewrite the expression $\dot{\bar{p}}/(\bar{\epsilon} + \bar{p}) = -3H(\dot{\bar{p}}/\dot{\bar{\epsilon}})$ and change the notation for the $\partial_i = \nabla$:

$$\boxed{\dot{v}_i + H \dot{v}_i - 3H \frac{\dot{\bar{p}}}{\dot{\bar{\epsilon}}} \dot{v}_i + \frac{\nabla \delta p}{\bar{\epsilon} + \bar{p}} + \nabla \Psi = 0.} \quad (2.117)$$

This is the final relativistic version of the Euler equation for a fluid. Pressure gradients $\nabla \delta p$ and gravitational infall $\nabla \Psi$ drive v_i [8]. The equation captures the redshifting of the velocities $\dot{v}_i$ and includes a small correction for relativistic fluids $\dot{\bar{p}}/\dot{\bar{\epsilon}}$. Adiabatic fluctuations satisfy $\bar{p}/\bar{\epsilon} = c_s^2$. Non-relativistic matter fluctuations have a very small sound speed, so the relativistic correction in the Euler equation is much smaller than the redshifting term [8].

3.6 Einstein equations

The most important equations of general relativity are the Einstein field equations, which relate the spacetime and matter components of the universe. We derive the linearised Einstein equation, which requires the perturbation to the Einstein tensor [8]:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \quad (2.118)$$

The perturbed Ricci tensor $R_{\mu\nu}$ and scalar R need to be derived to obtain the Einstein tensor. Recall that the Ricci tensor can be expressed in terms of the Christoffel symbols [8]:

$$R_{\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda + \Gamma_{\sigma\lambda}^\lambda \Gamma_{\mu\nu}^\sigma - \Gamma_{\sigma\nu}^\lambda \Gamma_{\mu\lambda}^\sigma. \quad (2.119)$$

The R_{00} component of the Ricci tensor is:

$$R_{00} = \partial_\lambda \Gamma_{00}^\lambda - \partial_0 \Gamma_{0\lambda}^\lambda + \Gamma_{\sigma\lambda}^\lambda \Gamma_{00}^\sigma - \Gamma_{\sigma 0}^\lambda \Gamma_{0\lambda}^\sigma. \quad (2.120)$$

Expanding every Christoffel symbol inside R_{00} , individually:

$$\partial_\lambda \Gamma_{00}^\lambda = \partial_0 \Gamma_{00}^0 + \partial_i \Gamma_{00}^i = \partial_0(H + \dot{\Psi}) + \partial_i(\partial^i \Psi) = \dot{H} + \ddot{\Psi} + \nabla^2 \Psi, \quad (2.121)$$

$$\partial_0 \Gamma_{0\lambda}^\lambda = \partial_0 \Gamma_{00}^0 + \partial_0 \Gamma_{0i}^i = \partial_0(H + \dot{\Psi}) + \partial_0(3H - 3\dot{\Phi}), \quad (2.122)$$

$$\Gamma_{\sigma\lambda}^\lambda \Gamma_{00}^\sigma = \Gamma_{00}^0 \Gamma_{00}^0 + \Gamma_{0i}^i \Gamma_{00}^0 + \Gamma_{i0}^0 \Gamma_{00}^i + \Gamma_{ij}^j \Gamma_{00}^j = (H + \dot{\Psi})(H + \dot{\Psi}) + \quad (2.123)$$

$$+ (3H - 3\dot{\Phi})(H + \dot{\Psi}) + (\partial_i \Psi)(\partial^i \Psi) + (-2\delta_{(i}^j \partial_{j)} \Phi + \delta_{ij} \delta^{j\lambda} \partial_\lambda \Phi)(\partial^j \Psi), \quad (2.124)$$

$$\Gamma_{\sigma 0}^\lambda \Gamma_{0\lambda}^\sigma = \Gamma_{00}^0 \Gamma_{00}^0 + \Gamma_{0j}^j \Gamma_{00}^j + \Gamma_{00}^i \Gamma_{0i}^0 + \Gamma_{0j}^i \Gamma_{0i}^j = (H + \dot{\Psi})(H + \dot{\Psi}) + \quad (2.125)$$

$$+ (\partial_j \Psi)(\partial^j \Psi) + (\partial^i \Psi)(\partial_i \Psi) + (H - \dot{\Phi})\delta_j^i (H - \dot{\Phi})\delta_i^j. \quad (2.126)$$

At linear order, the only remaining terms are:

$$\partial_\lambda \Gamma_{00}^\lambda = \dot{H} + \ddot{\Psi} + \nabla^2 \Psi, \quad (2.127)$$

$$\partial_0 \Gamma_{0\lambda}^\lambda = -4\dot{H} - \ddot{\Psi} - 3\ddot{\Phi}, \quad (2.128)$$

$$\Gamma_{\sigma\lambda}^\lambda \Gamma_{00}^\sigma = 4H^2 + 5H\dot{\Psi} - 3H\dot{\Phi}, \quad (2.129)$$

$$\Gamma_{\sigma 0}^\lambda \Gamma_{0\lambda}^\sigma = -4H^2 - 2H\dot{\Psi} + 6H\dot{\Phi}. \quad (2.130)$$

Combining everything, the only remaining terms left, after the cancellation, are:

$$R_{00} = -3\dot{H} + \nabla^2 \Psi + 3H\dot{\Phi} + 3H\dot{\Psi} + 3\ddot{\Phi}. \quad (2.131)$$

The R_{i0} component of is:

$$R_{0i} = \partial_\lambda \Gamma_{0i}^\lambda - \partial_i \Gamma_{0\lambda}^\lambda + \Gamma_{\sigma\lambda}^\lambda \Gamma_{0i}^\sigma - \Gamma_{\sigma i}^\lambda \Gamma_{0\lambda}^\sigma. \quad (2.132)$$

Expanding every Christoffel symbol inside R_{0i} , individually:

$$\partial_\lambda \Gamma_{0i}^\lambda = \partial_0 \Gamma_{0i}^0 + \partial_j \Gamma_{0i}^j = \partial_i \dot{\Psi} + \partial_i (H - \dot{\Phi}), \quad (2.133)$$

$$\partial_i \Gamma_{0\lambda}^\lambda = \partial_i \Gamma_{00}^0 + \partial_i \Gamma_{0j}^j = \partial_i (H + \dot{\Psi}) + \partial_i (3H - 3\dot{\Phi}), \quad (2.134)$$

$$\Gamma_{\sigma\lambda}^\lambda \Gamma_{0i}^\sigma = \Gamma_{00}^0 \Gamma_{0i}^0 + \Gamma_{0k}^0 \Gamma_{0i}^k + \Gamma_{j0}^j \Gamma_{0i}^0 + \Gamma_{jk}^j \Gamma_{0i}^k = (H + \dot{\Psi})(\partial_i \Psi) + (\partial_k \Psi)(H - \dot{\Phi})\delta_i^k + \quad (2.135)$$

$$+ (H - \dot{\Phi})\delta_j^j(\partial_i \Psi) + (-2\delta_{(j}^j \partial_{k)} \Phi + \delta_{jk} \partial^j \Phi)(H - \dot{\Phi})\delta_i^k, \quad (2.136)$$

$$\Gamma_{\sigma i}^\lambda \Gamma_{0\lambda}^\sigma = \Gamma_{00}^0 \Gamma_{i0}^0 + \Gamma_{0j}^0 \Gamma_{i0}^j + \Gamma_{00}^k \Gamma_{ik}^0 + \Gamma_{0j}^k \Gamma_{ik}^j = (H + \dot{\Psi})(\partial_i \Psi) + (\partial_j \Psi)(H - \dot{\Phi})\delta_i^j + \quad (2.137)$$

$$+ (\partial^k \Psi)[H\delta_{ik} - (\dot{\Phi} + 2H(\Phi + \Psi))\delta_{ik}] + (H - \dot{\Psi})\delta_j^k(-2\delta_{(i}^j \partial_{k)} \Phi + \delta_{ik} \partial^j \Phi). \quad (2.138)$$

At linear order, the only remaining terms are:

$$\partial_\lambda \Gamma_{0i}^\lambda = \partial_i \dot{\Psi} - \partial_i \dot{\Phi}, \quad (2.139)$$

$$\partial_i \Gamma_{0\lambda}^\lambda = \partial_i \dot{\Psi} - 3\partial_i \dot{\Phi}, \quad (2.140)$$

$$\Gamma_{\sigma\lambda}^\lambda \Gamma_{0i}^\sigma = 5H\partial_i \Psi - 3H\partial_i \Phi, \quad (2.141)$$

$$\Gamma_{\sigma i}^\lambda \Gamma_{0\lambda}^\sigma = 3H\partial_i \Psi - 3H\partial_i \Phi. \quad (2.142)$$

Combining everything, most terms vanish besides:

$$R_{0i} = 2\partial_i \dot{\Phi} + 2H\partial_i \Psi. \quad (2.143)$$

The R_{ij} component of is:

$$R_{ij} = \partial_\lambda \Gamma_{ij}^\lambda - \partial_j \Gamma_{i\lambda}^\lambda + \Gamma_{\sigma\lambda}^\lambda \Gamma_{ij}^\sigma - \Gamma_{\sigma i}^\lambda \Gamma_{j\lambda}^\sigma. \quad (2.144)$$

Expanding every term inside R_{ij} individually:

$$\partial_\lambda \Gamma_{ij}^\lambda = \partial_0 \Gamma_{ij}^0 + \partial_k \Gamma_{ij}^k = \partial_0 [H \delta_{ij} - (\dot{\Phi} + 2H(\Phi + \Psi)) \delta_{ij}] + \quad (2.145)$$

$$+ \partial_k (-2\delta_{(i}^k \partial_{j)} \Phi + \delta_{ij} \partial^k \Phi), \quad (2.146)$$

$$\partial_j \Gamma_{i\lambda}^\lambda = \partial_j \Gamma_{i0}^0 + \partial_j \Gamma_{ik}^k = \partial_j (\partial_i \Psi) + \partial_j (-2\delta_{(i}^k \partial_{k)} \Phi + \delta_{ik} \partial^k \Phi), \quad (2.147)$$

$$\Gamma_{\sigma\lambda}^\lambda \Gamma_{ij}^\sigma = \Gamma_{00}^0 \Gamma_{ij}^0 + \Gamma_{0l}^0 \Gamma_{ij}^l + \Gamma_{k0}^k \Gamma_{ij}^0 + \Gamma_{kl}^k \Gamma_{ij}^l = (H + \dot{\Psi}) \quad (2.148)$$

$$[H \delta_{ij} - (\dot{\Phi} + 2H(\Phi + \Psi)) \delta_{ij}] + (\partial_l \Psi) (-2\delta_{(i}^l \partial_{j)} \Phi + \delta_{ij} \partial^l \Phi) + \quad (2.149)$$

$$+ (H - \dot{\Phi}) \delta_{ij}^k [H \delta_{ij} - (\dot{\Phi} + 2H(\Phi + \Psi)) \delta_{ij}] + \quad (2.150)$$

$$(-2\delta_{(k}^l \partial_{l)} \Phi + \delta_{kl} \partial^l \Phi) (-2\delta_{(i}^l \partial_{j)} \Phi + \delta_{ij} \partial^l \Phi), \quad (2.151)$$

$$\Gamma_{\sigma i}^\lambda \Gamma_{j\lambda}^\sigma = \Gamma_{j0}^0 \Gamma_{i0}^0 + \Gamma_{jk}^0 \Gamma_{i0}^k + \Gamma_{j0}^l \Gamma_{il}^0 + \Gamma_{jk}^l \Gamma_{il}^k = (\partial_j \Psi) (\partial_i \Psi) + \quad (2.152)$$

$$+ [H \delta_{jk} - (\dot{\Phi} + 2H(\Phi + \Psi)) \delta_{jk}] (H - \dot{\Phi}) \delta_i^k + (H - \dot{\Phi}) \delta_j^l \quad (2.153)$$

$$[H \delta_{il} - (\dot{\Phi} + 2H(\Phi + \Psi)) \delta_{il}] + (-2\delta_{(j}^l \partial_{k)} \Phi + \delta_{jk} \partial^l \Phi) \quad (2.154)$$

$$(-2\delta_{(i}^k \partial_{m)} \Phi + \delta_{im} \partial^k \Phi). \quad (2.155)$$

At linear order, the only remaining terms:

$$\partial_\lambda \Gamma_{ij}^\lambda = [\dot{H} - \ddot{\Phi} - 2\dot{H}(\Phi + \Psi) - 2H(\dot{\Phi} + \dot{\Psi})] \delta_{ij} - 2\delta_i \delta_j \Phi + \delta_{ij} \nabla^2 \Phi, \quad (2.156)$$

$$\partial_j \Gamma_{i\lambda}^\lambda = \partial_j \partial_i \Psi - 3\partial_i \partial_j \Phi, \quad (2.157)$$

$$\Gamma_{\sigma\lambda}^\lambda \Gamma_{ij}^\sigma = [4H^2 - 4H\dot{\Phi} + H\ddot{\Phi} - 8H^2(\Phi + \Psi)] \delta_{ij}, \quad (2.158)$$

$$\Gamma_{\sigma i}^\lambda \Gamma_{j\lambda}^\sigma = [-2H^2 + 4H\dot{\Phi} + 4H^2(\Phi + \Psi)] \delta_{ij}. \quad (2.159)$$

Combining everything, the R_{ij} component is:

$$R_{ij} = \partial_i \partial_j (\Phi - \Psi) + [\dot{H} + 2H^2 - \ddot{\Phi} - 2H\dot{\Phi} - H\dot{\Psi} - 2\dot{H}(\Phi + \Psi) - 4H^2(\Phi + \Psi) + \nabla^2 \Phi] \delta_{ij}. \quad (2.160)$$

The Ricci scalar is just a combination the Ricci tensors multiplied by the metric 2.56:

$$R = g^{00} R_{00} + 2g^{0i} R_{0i} + g^{ij} R_{ij}. \quad (2.161)$$

Since the diagonal terms are zero, substituting the $g^{00} R_{00}$ and $g^{ij} R_{ij}$ terms gives:

$$R = \frac{1}{a^2} (1 - 2\Psi) (-3\dot{H} + \nabla^2 \Psi + 3H\dot{\Phi} + 3H\dot{\Psi} + 3\ddot{\Phi}) - \frac{1}{a^2} (1 + 2\Phi) \delta^{ij} [\partial_i \partial_j (\Phi - \Psi) + \quad (2.162)$$

$$- \Psi) + (\dot{H} + 2H^2 - \ddot{\Phi} - 2H\dot{\Phi} - H\dot{\Psi} - 2\dot{H}(\Phi + \Psi) - 4H^2(\Phi + \Psi) + \nabla^2 \Phi) \delta_{ij}]. \quad (2.163)$$

Multiply the entire equation by a^2 . Now, expanding the brackets $1 - 2\Psi$ and $1 + 2\Phi$ gives a lot of higher-order terms that can be dropped. At linear order, the expression becomes:

$$a^2 R = -3\dot{H} + \nabla^2 \Psi + 3H\dot{\Phi} + 3H\dot{\Psi} + 3\ddot{\Phi} + 6\dot{H}\Psi - 3\dot{H} - 6H^2 + 3\ddot{\Phi} - 3\nabla^2 \Phi + \quad (2.164)$$

$$+ 6(\dot{H} + 2H^2)\Phi + 6(\dot{H} + 2H^2)\Psi + 3H\dot{\Psi} + 15H\dot{\Phi} - 6\dot{H}\Phi - 12H^2\Phi + \quad (2.165)$$

$$+ \nabla^2(\Phi - \Psi). \quad (2.166)$$

Grouping similar terms results in:

$$a^2 R = -6(\dot{H} + H^2) + 2\nabla^2 \Psi - 4\nabla^2 \Phi + 12(\dot{H} + H^2)\Psi + 6\ddot{\Phi} + 6H(\dot{\Psi} + 3\dot{\Phi}). \quad (2.167)$$

The Einstein tensor can now be obtained by collecting the Ricci tensor and the Ricci scalar together. The G_{00} component is:

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R = -3\dot{H} + \nabla^2 \Psi + 3H(\dot{\Phi} + \dot{\Psi}) + 3\ddot{\Phi} + 3(1 + 2\Psi)(\dot{H} + H^2) - \quad (2.168)$$

$$- \frac{1}{2}[2\nabla^2 \Psi - 4\nabla^2 \Phi + 12(\dot{H} + H^2)\Psi + 6\ddot{\Phi} + 6H(\dot{\Psi} + 3\dot{\Phi})]. \quad (2.169)$$

Most of the terms cancel leaving the simple result:

$$G_{00} = 3H^2 + 2\nabla^2 \Phi - 6H\dot{\Phi}. \quad (2.170)$$

The G_{0i} component of the Einstein tensor is simply R_{0i} since $g_{0i} = 0$:

$$G_{0i} = 2\partial_i(\dot{\Phi} + H\Psi). \quad (2.171)$$

The G_{ij} component is:

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R = [\dot{H} + 2H^2 - \ddot{\Phi} + \nabla^2 \Phi - 2(\dot{H} + 2H^2)(\Phi + \Psi) - \quad (2.172)$$

$$- H\dot{\Psi} - 5H\dot{\Phi}]\delta_{ij} + \partial_i\partial_j(\Phi - \Psi) - 3(1 - 2\Phi)(\dot{H} + H^2)\delta_{ij} + \quad (2.173)$$

$$+ \frac{1}{2}[2\nabla^2 \Psi - 4\nabla^2 \Phi + 12(\dot{H} + H^2)\Psi + 6\ddot{\Phi} + 6H(\dot{\Psi} + 3\dot{\Phi})]\delta_{ij}. \quad (2.174)$$

This simplifies to:

$$G_{ij} = [-2\dot{H} - H^2 - 2\ddot{\Phi} + \nabla^2(\Psi - \Phi) + 2(2\dot{H} + H^2)(\Phi + \Psi) + 2H\dot{\Psi} + \quad (2.175)$$

$$+ 4H\dot{\Phi}]\delta_{ij} + \partial_i\partial_j(\Phi - \Psi). \quad (2.176)$$

The Einstein equation is:

$$G_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (2.177)$$

Since we have dropped anisotropic stress there is no source on the right-hand side $\partial_i\partial_j(\Phi - \Psi) = 0$. Had the anisotropic stress been kept, the right-hand side would be $-8\pi Ga^2\Pi_{ij}$

[8]. Substituting the perturbed Einstein tensor, metric and energy-momentum tensor into the Einstein equation above gives the equations of motion for the metric perturbations and the zeroth-order Friedmann equations 2.1. In the absence of anisotropic stress $\Phi = \Psi$. Now, consider the G_{00} equation, $G_{00} = 8\pi GT_{00}$:

$$3H^2 + 2\nabla^2\Phi - 6H\dot{\Phi} = 8\pi Gg_{0\mu}T_0^\mu = 8\pi G(g_{00}T_0^0 + g_{0i}T_0^i) = \quad (2.178)$$

$$= 8\pi Ga^2(1 + 2\Phi)(\bar{\epsilon} + \delta\epsilon) = 8\pi Ga^2\bar{\epsilon}(1 + 2\Phi + \delta). \quad (2.179)$$

This can be separated into two parts, the zeroth-order part and first-order parts:

$$H^2 = \frac{8\pi G}{3}a^2\bar{\epsilon}, \quad (2.180)$$

$$\nabla^2\Phi - 3H(\dot{\Phi} + H\Phi) = 4\pi Ga^2\bar{\epsilon}\delta.$$

The zeroth-order is the first Friedmann equation 2.1. The first-order part is this Poisson-like equation. Moving on to G_{0i} equation, $G_{0i} = 8\pi GT_{0i}$, with:

$$2\partial_i(\dot{\Phi} + H\Phi) = 8\pi G(-a^2q_i), \quad (2.181)$$

where $T_{0i} = g_{0\mu}T_i^\mu = g_{00}T_i^0 = -a^2q_i$. Using $q_i = (\bar{\epsilon} + \bar{p})\partial_iv$ and then integrating to get:

$$\dot{\Phi} + H\Phi = -4\pi Ga^2(\bar{\epsilon} + \bar{p})v. \quad (2.182)$$

This is the momentum equation of the perturbed Einstein equations. Finally, consider the G_{ij} component. Instead of taking $G_{ij} = 8\pi GT_{ij}$ directly, we will take $G_i^i = 8\pi GT_i^i$. This will kill a traceless part, significantly making the resulting equation easier. Consider the G_i^i component as:

$$G_i^i = g^{ij}G_{ij} = g^{ik}G_{ki} = -a^{-2}(1 + 2\Phi)\delta^{ik}[-(2\dot{H} + H^2)\delta_{ki} + (2\ddot{\Phi} + 6H\dot{\Phi} + \quad (2.183)$$

$$4(2\dot{H} + H^2)\Phi)\delta_{ki}] = -3a^{-2}[-(2\dot{H} + H^2) + 2(\ddot{\Phi} + 3H\dot{\Phi} + (2\dot{H} + H^2)\Phi)]. \quad (2.184)$$

Substitute this into $G_i^i = 8\pi GT_i^i$:

$$-3a^{-2}[-(2\dot{H} + H^2) + 2(\ddot{\Phi} + 3H\dot{\Phi} + (2\dot{H} + H^2)\Phi)] = 8\pi G[-3(\bar{p} + \delta p)], \quad (2.185)$$

where $-3(\bar{p} + \delta p) = T_i^i$. This equation can be split into two parts:

$$\begin{aligned} 2\dot{H} + H^2 &= -8\pi Ga^2\bar{p}, \\ \ddot{\Phi} + 3H\dot{\Phi} + (2\dot{H} + H^2)\Phi &= 4\pi Ga^2\delta p. \end{aligned} \quad (2.186)$$

The zeroth-order part is just the second Friedmann equation 2.1 and the first-order part is a pressure equation. To summarize our Einstein equations are:

$$\nabla^2\Phi - 3H(\dot{\Phi} + H\Phi) = 4\pi Ga^2\delta\epsilon, \quad (2.187)$$

$$\dot{\Phi} + H\Phi = -4\pi Ga^2(\bar{\epsilon} + \bar{p})v, \quad (2.188)$$

$$\ddot{\Phi} + 3H\dot{\Phi} + (2\dot{H} + H^2)\Phi = 4\pi Ga^2\delta p. \quad (2.189)$$

where just like in the Friedmann equations 2.1, the energy density ϵ and pressure p are sums from all contributions in the entire universe. For adiabatic perturbations, a closed evolution equation for the gravitational potential can be obtained from these. Rewrite one of the Einstein equations as 2.187:

$$\delta\epsilon = \frac{\nabla^2\Phi - 3H(\dot{\Phi} + H\Phi)}{4\pi Ga^2}. \quad (2.190)$$

Using adiabaticity $\delta p = \omega\delta\epsilon$, substitute this into the third equation 2.187:

$$\ddot{\Phi} + 3H\dot{\Phi} + (2\dot{H} + H^2)\Phi = \omega[\nabla^2\Phi - 3H(\dot{\Phi} + H\Phi)]. \quad (2.191)$$

For a single fluid with $p = \omega\epsilon$, the Hubble time derivative is $\dot{H} = (-1 + 3\omega)H^2/2$ [8]. In Fourier space, $\nabla^2 = k^2$ therefore the expression simplifies into:

$$\ddot{\Phi} + 3(1 + \omega)H\dot{\Phi} + \omega k^2\Phi = 0, \quad (2.192)$$

where $\ddot{\Phi}$ is the time evolution of the gravitational potential itself, $3(1 + \omega)H\dot{\Phi}$ is the Hubble friction term that is responsible for damping, $\omega k^2\Phi$ is the pressure term that makes Φ oscillate like a wave.

CHAPTER 3

MARKOV-MUKHANOV MODEL

As stated before, Markov-Mukhanov model is a modified gravity theory. The defining feature of the Markov-Mukhanov gravity is a multiplicative coupling between matter and gravity. Markov-Mukhanov action is [18]:

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{8\pi G} + 2\chi(\epsilon)\mathcal{L} \right), \quad (3.1)$$

where g is the determinant of the metric, $\mathcal{L}$ is the matter Lagrangian, $\chi(\epsilon)$ is a function of energy density ϵ . The effective energy-momentum tensor for this model is [18]:

$$\tilde{T}_{\mu\nu} = (\chi\epsilon)_{,\epsilon} T_{\mu\nu} + (\epsilon^2 \chi_{,\epsilon}) g_{\mu\nu}, \quad (3.2)$$

where $T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + pg_{\mu\nu}$ is energy-momentum tensor for the perfect fluid. This model also possess two running constants:

$$G(\epsilon) = G(\chi\epsilon)_{,\epsilon}, \quad (3.3)$$

$$\Lambda(\epsilon) = -8\pi G\epsilon^2 \chi_{,\epsilon}, \quad (3.4)$$

where $G(\epsilon)$ is a running gravitational constant and $\Lambda(\epsilon)$ is a running cosmological constant [1].

The effective energy density $\tilde{\epsilon}$ and pressure $\tilde{p}$ are denoted using the *tilde* above them:

$$\begin{cases} \tilde{\epsilon} = (\chi\epsilon)_{,\epsilon} \epsilon - \epsilon^2 \chi_{,\epsilon}, \\ \tilde{p} = (\chi\epsilon)_{,\epsilon} p + \epsilon^2 \chi_{,\epsilon}. \end{cases} \quad (3.5)$$

where $\chi_{,\epsilon}$ denotes the first-order derivative with respect to the energy density ϵ . The background of the energy density:

$$\tilde{\bar{\epsilon}} = \chi \bar{\epsilon}, \quad (3.6)$$

$$\tilde{\bar{p}} = (\chi + \bar{\epsilon} \chi_{,\epsilon}) \bar{p} + \bar{\epsilon}^2 \chi_{,\epsilon}. \quad (3.7)$$

The perturbation of the energy density is:

$$\delta\tilde{\epsilon} = (\chi + \bar{\epsilon}\chi_{,\epsilon})\delta\epsilon. \quad (3.8)$$

The perturbation of the pressure is:

$$\delta\tilde{p} = (\chi + \bar{\epsilon}\chi_{,\epsilon})\delta p + (\bar{p} + \bar{\epsilon})(2\chi_{,\epsilon} + \bar{\epsilon}\chi_{,\epsilon\epsilon})\delta\epsilon. \quad (3.9)$$

Using these corrections, the effective energy-momentum tensor can be rewritten in terms:

$$\tilde{T}_{\mu\nu} = (\epsilon\chi)_{,\epsilon}T_{\mu\nu} + (\epsilon^2\chi_{,\epsilon})g_{\mu\nu} = (\tilde{\epsilon} + \tilde{p})u_\mu u_\nu + \tilde{p}g_{\mu\nu}. \quad (3.10)$$

1 Continuity equation

To derive the continuity equation simply substitute in the effective pressure and energy density into the relativistic continuity equations 2.102:

$$\dot{\tilde{\delta}} + \left(1 + \frac{\tilde{p}}{\tilde{\epsilon}}\right) (\nabla \cdot v - 3\dot{\Phi}) + 3H \left(\frac{\delta\tilde{p}}{\delta\tilde{\epsilon}} - \frac{\tilde{p}}{\tilde{\epsilon}}\right) \tilde{\delta} = 0. \quad (3.11)$$

Or if taking the zeroth-order and first-order versions 2.100:

$$\begin{aligned} \dot{\tilde{\epsilon}} &= -3H(\tilde{\epsilon} + \tilde{p}), \\ \delta\dot{\tilde{\epsilon}} &= -3H(\delta\tilde{\epsilon} + \delta\tilde{p}) + 3\dot{\Phi}(\bar{\epsilon} + \bar{p}) - \partial_i q^i. \end{aligned} \quad (3.12)$$

2 Euler equation

Repeat the similar method for the relativistic Euler equation 2.117:

$$\dot{v}_i + H\dot{v}_i - 3H\frac{\dot{\tilde{p}}}{\tilde{\epsilon}}v_i + \frac{\nabla\delta\tilde{p}}{\tilde{\epsilon} + \tilde{p}} + \nabla\Psi = 0. \quad (3.13)$$

3 Einstein equations

Substituting the effective energy density and pressure into the Einstein equations 2.187:

$$\nabla^2\Phi - 3H(\dot{\Phi} + H\Phi) = 4\pi Ga^2\delta\tilde{\epsilon}, \quad (3.14)$$

$$\dot{\Phi} + H\Phi = -4\pi Ga^2(\tilde{\epsilon} + \tilde{p})v, \quad (3.15)$$

$$\ddot{\Phi} + 3H\dot{\Phi} + (2\dot{H} + H^2)\Phi = 4\pi Ga^2\delta\tilde{p}. \quad (3.16)$$

This gives the Einstein equations of the form:

$$\nabla^2\Phi - 3H(\dot{\Phi} + H\Phi) = 4\pi Ga^2(\chi + \bar{\epsilon}\chi_{,\epsilon})\delta\epsilon, \quad (3.17)$$

$$\dot{\Phi} + H\Phi = -4\pi Ga^2(\chi\bar{\epsilon} + \chi\bar{p} + \bar{\epsilon}\bar{p}\chi_{,\epsilon} + \bar{\epsilon}^2\chi_{,\epsilon})v, \quad (3.18)$$

$$\ddot{\Phi} + 3H\dot{\Phi} + (2\dot{H} + H^2)\Phi = 4\pi Ga^2[(\chi + \bar{\epsilon}\chi_{,\epsilon})\delta p + (\bar{p} + \bar{\epsilon})(2\chi_{,\epsilon} + \bar{\epsilon}\chi_{,\epsilon\epsilon})\delta\epsilon]. \quad (3.19)$$

4 Evolution of fluctuations

Coming back to the closed evolution equation for the gravitational potential 2.192, the Markov-Mukhanov version can be derived from the Einstein equation above 3.17. Start with the Einstein equation of the form:

$$\delta\epsilon = \frac{k^2\Phi - 3H(\dot{\Phi} + H\Phi)}{4\pi Ga^2(\chi + \bar{\epsilon}\chi_{,\epsilon})}. \quad (3.20)$$

For the adiabatic perturbations $\delta p = \omega\delta\epsilon$, substituting the energy density perturbation $\delta\epsilon$ into the Einstein equation 3.17 gives:

$$\ddot{\Phi} + 3H\dot{\Phi} + (2\dot{H} + H^2)\Phi = \left[\frac{(2\chi_{,\epsilon} + \bar{\epsilon}\chi_{,\epsilon\epsilon})\bar{p} + 2\bar{\epsilon}\chi_{,\epsilon} + \bar{\epsilon}^2\chi_{,\epsilon\epsilon}}{\chi + \bar{\epsilon}\chi_{,\epsilon}} + \omega \right] \quad (3.21)$$

$$(-k^2\Phi - 3H(\dot{\Phi} + H\Phi)). \quad (3.22)$$

Expanding and grouping the terms from the right-hand side to the left-hand side transforms the evolution equation as:

$$\ddot{\Phi} + 3(1 + \omega)H\dot{\Phi} + \omega k^2\Phi = \zeta(\chi)[-k^2\Phi - 3H(\dot{\Phi} + H\Phi)], \quad (3.23)$$

where $\zeta(\chi)$ is the notation for the terms with χ as:

$$\zeta(\chi) = \frac{(2\chi_{,\epsilon} + \bar{\epsilon}\chi_{,\epsilon\epsilon})\bar{p} + 2\bar{\epsilon}\chi_{,\epsilon} + \bar{\epsilon}^2\chi_{,\epsilon\epsilon}}{\chi + \bar{\epsilon}\chi_{,\epsilon}}. \quad (3.24)$$

4.1 Standard model

Clearly, when the parameter $\chi = 1$ the right-hand side of the equation 3.23 is zero. This simplifies the equation back to its general relativistic form, the same as 2.192. Plotting this equation with respect to the scale factor a , gives the figure depicted on 3.1.

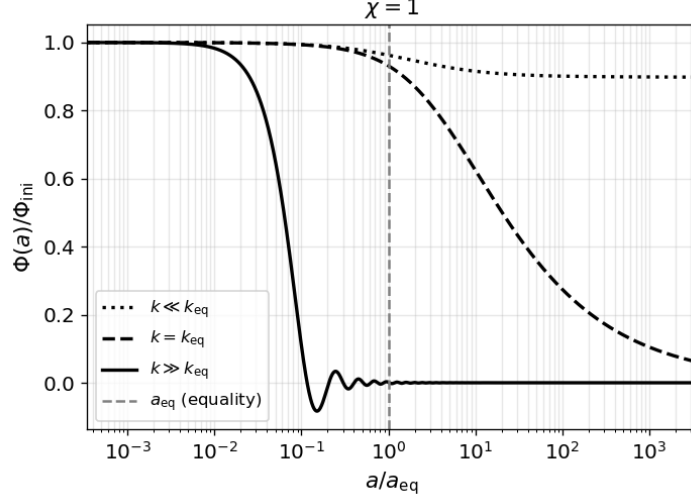


Figure 3.1: Linear evolution of gravitational potential relative to its initial value $\Phi(a)/\Phi_{\text{ini}}$ with $\chi = 1$.

Specifically, it depicts how the gravitational potential evolves relative to its primordial value $\Phi(a)/\Phi_{\text{ini}}$. For $a/a_{\text{eq}} \ll 1$, the universe is radiation dominated, while for $a/a_{\text{eq}} \gg 1$ it is matter dominated. The vertical dashed line at $a/a_{\text{eq}} = 1$ marks the moment of equality, when the energy densities of radiation and matter are equal. The vertical axis measures the ratio $\Phi(a)/\Phi_{\text{ini}}$, so that all modes begin with the same normalized amplitude, reflecting the primordial initial conditions set by inflation [19], [20]. A feature of the plot is that all curves coincide at early times, forming a horizontal plateau at $\Phi(a)/\Phi_{\text{ini}} \approx 1$. This behavior corresponds to the superhorizon regime, where the physical wavelength of a perturbation is larger than the Hubble radius $k \ll H$ [8]. In this limit, the evolution equation for the gravitational potential 2.192 simplifies because the gradient term is negligible. The growing-mode solution is then simply $\dot{\Phi} \approx 0$ and $\Phi = \text{const.}$ Physically, this reflects the absence of causal microphysics on superhorizon scales, gravity cannot reorganize matter, and perturbations remain frozen. The subsequent evolution depends crucially on when each mode enters the horizon, defined by the condition $k = H$. After this, the three curves in the plot begin to diverge, each representing a different ratio of wavenumber to the equality scale. The solid curve correspond to small scales $k \gg k_{\text{eq}}$, the dashed curve corresponds to intermediate scales $k = k_{\text{eq}}$ and the dotted curve corresponds to large scales $k \ll k_{\text{eq}}$.

For the smallest scales $k \gg k_{\text{eq}}$, horizon entry occurs deep in the radiation-dominated era. Once inside the horizon, the evolution is governed by the radiation-era equation:

$$\ddot{\Phi} + \frac{4}{\tau}\dot{\Phi} + \frac{k^2}{3}\Phi = 0, \quad (3.25)$$

where the solution behaves, in the subhorizon limit, as $\Phi \approx \cos[(k\tau)/\sqrt{3}]/(k\tau)^2$. This expression reveals that $\Phi \propto a^{-2}$. In the plot, this manifests as a rapid drop in the

amplitude shortly after horizon entry, accompanied by small oscillations.

The intermediate mode $k = k_{\text{eq}}$, enters the horizon near the time of matter–radiation equality. Initially, it behaves like a superhorizon mode and remains constant. As it approaches equality, it begins to decay, but more gradually and without strong oscillations. This smoother behavior reflects the diminishing role of radiation pressure at this era.

For the largest-scale mode $k \ll k_{\text{eq}}$, the plot remains outside the horizon until after equality. It never experiences the radiation-driven decay. Its only evolution comes from the change in the background equation of state ω during the transition from radiation to matter domination. This effect leads to a modest reduction in amplitude, given by:

$$\Phi_{\text{MD}} = \frac{9}{10} \Phi_{\text{RD}} \quad (3.26)$$

In the plot, this appears as a small drop from unity to approximately 0.9, after which the curve becomes flat again. Once the universe is matter dominated, the evolution equation simplifies to:

$$\ddot{\Phi} + \frac{6}{\tau} \dot{\Phi} = 0. \quad (3.27)$$

4.2 Linear model

When the parameter $\chi(\epsilon)$ is chosen to be $\chi(\epsilon) = 1 + \alpha\epsilon + \beta\epsilon^2$ the universe behaves differently than the standard model at the early times. This model introduces a correction that scales linearly with energy density ϵ . This can be seen on the figure 3.2.

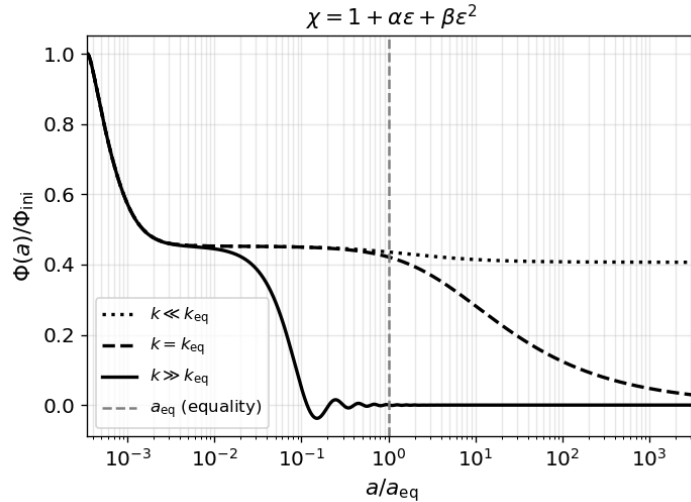


Figure 3.2: Linear evolution of gravitational potential relative to its initial value $\Phi(a)/\Phi_{\text{ini}}$ with $\chi(\epsilon) = 1 + \alpha\epsilon + \beta\epsilon^2$.

The immediate and significant initial drop in the potential occurs at roughly $a/a_{\text{eq}} < 10^{-2}$. Because ϵ is extremely high in the early universe, the modification term $\alpha\epsilon$ dominates, effectively suppressing the initial gravitational potential to a lower equilibrium value around 0.45 before the standard evolution resumes.

4.3 Exponential model

When the parameter $\chi(\epsilon)$ is set to take an exponential form $\chi(\epsilon) = 1 + Ae^{-\gamma/\epsilon}$, the universe behaves differently at the late times as depicted in the figure 3.3.

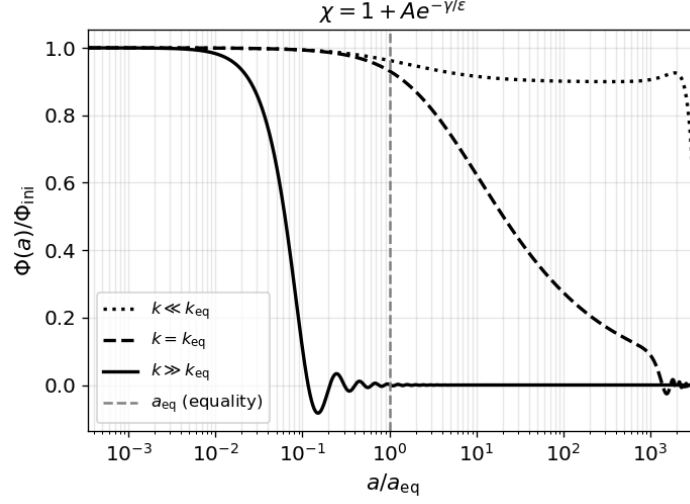


Figure 3.3: Linear evolution of gravitational potential relative to its initial value $\Phi(a)/\Phi_{\text{ini}}$ with $\chi(\epsilon) = 1 + Ae^{-\gamma/\epsilon}$.

At high densities/early times, the $e^{-\gamma/\epsilon}$ term is active, but as the universe expands and ϵ drops, this term vanishes rapidly. This creates a late-time instability at roughly $a/a_{\text{eq}} > 10^3$. The potential Φ , which seemed stable, suddenly collapses. This suggests that as the density hits a certain threshold, the modified gravity effects "turn off" or transition sharply, leading to a breakdown of the potential's stability [18].

CHAPTER 4

CONCLUSION

The evolution of cosmological perturbations has been studied in both general relativity and within the framework of the Markov–Mukhanov modified gravity. The plots obtained for the gravitational potential evolution, show that matter–gravity coupling leads to clear deviations from standard cosmological evolution. While the linear model $\chi(\epsilon) = 1 + \alpha\epsilon + \beta\epsilon^2$ produces moderate changes, the exponential model $\chi(\epsilon) = 1 + Ae^{-\gamma/\epsilon}$ results in chaotic changes to the evolution of the gravitational potential. The analysis does lack in certain aspects. For example, we didn’t do any comparison with observational data. Further investigation is required to determine the physical viability of these results.

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