

AN ASYMPTOTIC MODEL FOR WATER WAVES

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ABSTRACT. The water wave problem concerns the motion of an incompressible and inviscid fluid with a free surface, and it arises in many areas of fluid dynamics and wave theory. In this project, we derive asymptotic models for water waves starting from the Euler equations under the assumption of irrotational flow. By introducing the velocity potential and using the Hamiltonian framework, the governing equations can be rewritten in terms of surface variables through the Dirichlet-Neumann operator. We also examine asymptotic approximations that lead to reduced nonlinear dispersive equations and include numerical simulations, based on spectral methods, to illustrate the main features of the wave dynamics.

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1. INTRODUCTION

The study of water waves is a classical problem in fluid dynamics with important applications in oceanography, coastal engineering, and nonlinear wave theory. From a mathematical perspective, it provides a rich framework for the study of nonlinear and dispersive partial differential equations.

We consider the motion of an incompressible, inviscid fluid with a free surface. The fluid velocity field $\mathbf{u}(x, y, t)$ satisfies the Euler equations

$$(1.1) \quad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P + \mathbf{g},$$

together with the incompressibility condition

$$(1.2) \quad \nabla \cdot \mathbf{u} = 0.$$

The domain of the fluid is given by

$$(1.3) \quad \Omega(t) = \{(x, y) \in \mathbb{R}^2 : -\infty < y < \eta(x, t)\},$$

where $\eta(x, t)$ denotes the free surface elevation.

Assuming the flow is irrotational, we introduce a velocity potential $\phi(x, y, t)$ such that

$$(1.4) \quad \mathbf{u} = \nabla \phi,$$

$$(1.5) \quad \Delta \phi = 0 \quad \text{in } \Omega(t).$$

The evolution of the free surface is governed by the kinematic and dynamic boundary conditions

$$(1.6) \quad \partial_t \eta = \partial_y \phi - \partial_x \eta \partial_x \phi,$$

$$(1.7) \quad \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g\eta = 0 \quad \text{on } y = \eta(x, t),$$

together with the decay condition

$$(1.8) \quad \nabla \phi \rightarrow 0 \quad \text{as } y \rightarrow -\infty.$$

A major difficulty of this problem lies in the presence of the free boundary. A classical approach is to reformulate the system in terms of surface variables. Defining the surface potential

$$(1.9) \quad \xi(x, t) = \phi(x, \eta(x, t), t),$$

the problem can be expressed using the Dirichlet–Neumann operator $G(\eta)$ as

$$(1.10) \quad \partial_t \eta = G(\eta) \xi.$$

This formulation reveals a Hamiltonian structure, where the evolution is governed by the energy functional

$$(1.11) \quad H(\eta, \xi) = \frac{1}{2} \int (\xi G(\eta) \xi + g\eta^2) dx.$$

In many physical regimes, the full water wave system can be approximated by simpler nonlinear dispersive equations. Under suitable scaling assumptions, one obtains reduced models such as the Korteweg–de Vries (KdV) equation and the nonlinear Schrödinger (NLS) equation, which capture the balance between nonlinearity and dispersion.

The goal of this project is to derive and study asymptotic models for water waves starting from the Euler equations. This involves rewriting the problem within a Hamiltonian framework, applying asymptotic expansions (in particular for the Dirichlet–Neumann operator), and analyzing the resulting reduced nonlinear PDEs. In addition, numerical methods such as Fourier

pseudo-spectral schemes are used to simulate wave propagation and to support the theoretical results.

Overall, the project combines analytical derivations with asymptotic techniques and numerical simulations. It shows how a complicated fluid system can be simplified into more manageable mathematical models, while still retaining the main physical properties of the original problem.

2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

2.1. Mass Conservation. We study the motion of an incompressible fluid with velocity field $\mathbf{u}(x, y, t) = (u(x, y, t), v(x, y, t))$.

Let $V(t) \subset \mathbb{R}^2$ be a control volume that moves together with the fluid. The principle of mass conservation means that the total mass inside $V(t)$ does not change over time, which can be written as

$$(2.1) \quad \frac{d}{dt} \int_{V(t)} \rho \, dV = 0,$$

where ρ denotes the fluid density.

Applying the Reynolds transport theorem, we obtain

$$(2.2) \quad \int_{V(t)} (\partial_t \rho + \nabla \cdot (\rho \mathbf{u})) \, dV = 0.$$

Since this holds for any control volume, it follows that

$$(2.3) \quad \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0.$$

In the case of an incompressible fluid, the density ρ does not vary in time or space. As a result, the continuity equation simplifies to

$$(2.4) \quad \nabla \cdot \mathbf{u} = 0.$$

In two spatial dimensions, this condition takes the form

$$(2.5) \quad \partial_x u + \partial_y v = 0.$$

2.2. Momentum Conservation. We consider the motion of an incompressible, inviscid fluid with velocity field $\mathbf{u}(x, y, t)$ and density ρ .

The conservation of linear momentum is given by Newton's second law applied to a fluid element. In the absence of viscosity, the motion is governed by the Euler equations

$$(2.6) \quad \rho (\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) = -\nabla P + \rho \mathbf{g},$$

where $P(x, y, t)$ denotes the pressure and $\mathbf{g}$ is the gravitational acceleration.

Dividing by ρ , we obtain

$$(2.7) \quad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P + \mathbf{g}.$$

In two spatial dimensions, writing $\mathbf{u} = (u, v)$, the system becomes

$$(2.8) \quad \partial_t u + u \partial_x u + v \partial_y u = -\frac{1}{\rho} \partial_x P,$$

$$(2.9) \quad \partial_t v + u \partial_x v + v \partial_y v = -\frac{1}{\rho} \partial_y P - g.$$

2.3. Irrotational Flow and Boundary Conditions. We consider an incompressible, inviscid fluid with velocity field $\mathbf{u}(x, y, t)$. The flow is assumed to be irrotational, meaning that the vorticity vanishes:

$$(2.10) \quad \nabla \times \mathbf{u} = 0.$$

In two spatial dimensions, this condition becomes

$$(2.11) \quad \partial_y u - \partial_x v = 0.$$

As a consequence, there exists a velocity potential $\phi(x, y, t)$ such that

$$(2.12) \quad \mathbf{u} = \nabla \phi,$$

and therefore

$$(2.13) \quad \Delta \phi = 0 \quad \text{in the fluid domain.}$$

The fluid occupies the domain

$$\Omega(t) = \{(x, y) \in \mathbb{R}^2 : -\infty < y < \eta(x, t)\},$$

where $\eta(x, t)$ denotes the free surface elevation.

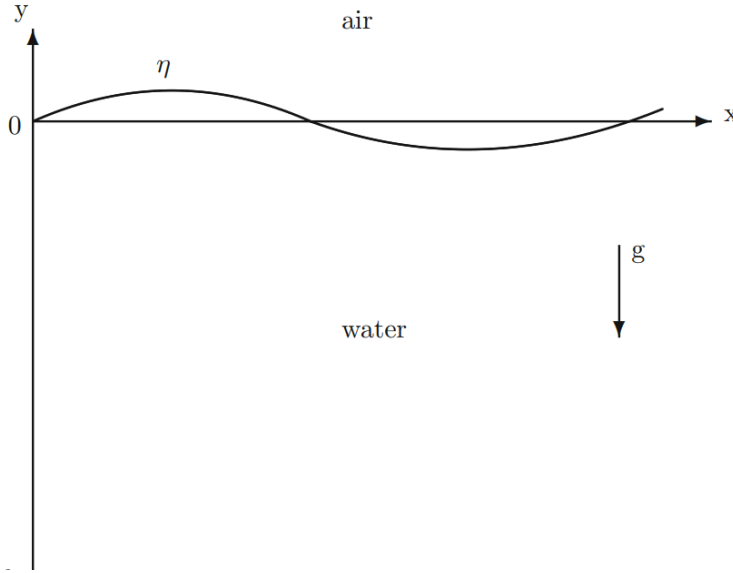


FIGURE 1. Two-dimensional irrotational, incompressible fluid of infinite depth with free surface $y = \eta(x, t)$.

Boundary Conditions. The motion of the free surface is governed by two boundary conditions. Kinematic boundary condition. The free surface moves with the fluid particles, which gives

$$(2.14) \quad \partial_t \eta = \partial_y \phi - \partial_x \eta \partial_x \phi \quad \text{on } y = \eta(x, t).$$

Dynamic boundary condition. At the free surface, the pressure is equal to the atmospheric pressure. Using Bernoulli's equation, this yields

$$(2.15) \quad \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g \eta = 0 \quad \text{on } y = \eta(x, t).$$

Boundary condition at infinite depth. We assume that the fluid has infinite depth, so that the domain extends to $y \rightarrow -\infty$. Physically, this means that the influence of the free surface vanishes far below the fluid, and the motion becomes negligible.

Accordingly, the velocity potential satisfies the decay condition

$$(2.16) \quad \nabla \phi \rightarrow 0 \quad \text{as } y \rightarrow -\infty.$$

Equivalently,

$$(2.17) \quad \partial_x \phi \rightarrow 0, \quad \partial_y \phi \rightarrow 0 \quad \text{as } y \rightarrow -\infty.$$

3. HAMILTONIAN STRUCTURE AND FORMULATION OF WATER WAVES

3.1. Hamiltonian Structure: Finite-Dimensional Systems. To illustrate the Hamiltonian structure, we first consider a simple finite-dimensional system. Let $(x(t), y(t))$ satisfy

$$(3.1) \quad \dot{x} = y, \quad \dot{y} = -x.$$

This system can be written in Hamiltonian form by introducing the energy function

$$(3.2) \quad H(x, y) = \frac{1}{2}(x^2 + y^2).$$

Then the equations of motion take the canonical form

$$(3.3) \quad \dot{x} = \partial_y H, \quad \dot{y} = -\partial_x H,$$

or equivalently,

$$(3.4) \quad \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix}.$$

Moreover, the Hamiltonian is conserved along solutions:

$$(3.5) \quad \frac{d}{dt} H(x(t), y(t)) = 0.$$

3.2. Hamiltonian Structure for Partial Differential Equations. This structure extends naturally to infinite-dimensional systems such as partial differential equations. For a field variable $u(x, t)$, the Hamiltonian becomes a functional

$$(3.6) \quad H[u] = \int \mathcal{H}(u, \partial_x u, \dots) dx,$$

where $\mathcal{H}$ is the energy density. The evolution equation takes the form

$$(3.7) \quad \partial_t u = J \frac{\delta H}{\delta u},$$

where $\delta H / \delta u$ denotes the variational derivative and J is a skew-symmetric operator.

3.3. Example: The Wave Equation. As an important example, we consider the one-dimensional wave equation

$$(3.8) \quad \partial_t^2 p = \partial_x^2 p.$$

Introducing the auxiliary variable

$$(3.9) \quad q = \partial_t p,$$

we rewrite the equation as a first-order system:

$$(3.10) \quad \partial_t p = q, \quad \partial_t q = \partial_x^2 p.$$

This system can be written in Hamiltonian form with the energy functional

$$(3.11) \quad H[p, q] = \int \left(\frac{1}{2} q^2 + \frac{1}{2} (\partial_x p)^2 \right) dx.$$

To obtain the corresponding equations of motion, we compute the variational derivatives of H . Let v be a test function. Considering a perturbation of the form $p \mapsto p + sv$, we get

$$(3.12) \quad \frac{d}{ds} H[p + sv, q] \Big|_{s=0} = \int (\partial_x p) (\partial_x v) dx = - \int (\partial_x^2 p) v dx,$$

which implies

$$(3.13) \quad \frac{\delta H}{\delta p} = -\partial_x^2 p.$$

In the same way, varying $q \mapsto q + sv$ gives

$$(3.14) \quad \frac{\delta H}{\delta q} = q.$$

With these expressions, the system can be written in Hamiltonian form as

$$(3.15) \quad \partial_t \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta H / \delta p \\ \delta H / \delta q \end{pmatrix}.$$

Substituting the variational derivatives, we obtain

$$(3.16) \quad \partial_t p = q, \quad \partial_t q = \partial_x^2 p,$$

which leads back to the classical wave equation.

This example shows how the Hamiltonian framework, first introduced for finite-dimensional systems, naturally extends to partial differential equations and can be used to describe wave dynamics in a unified way.

3.4. Zakharov's Hamiltonian Formulation for Water Waves. We now return to the water wave problem. Zakharov showed that the system can be reformulated as a Hamiltonian system in terms of surface variables.

We introduce the surface potential

$$(3.17) \quad \xi(x, t) = \phi(x, \eta(x, t), t),$$

obtained by restricting the velocity potential to the free surface.

Using the chain rule, we obtain

$$(3.18) \quad \partial_t \xi = \partial_t \phi + \partial_y \phi \partial_t \eta,$$

$$(3.19) \quad \partial_x \xi = \partial_x \phi + \partial_y \phi \partial_x \eta.$$

The Dirichlet-Neumann operator is defined by

$$(3.20) \quad G(\eta) \xi = (\partial_y \phi - \partial_x \eta \partial_x \phi) \Big|_{y=\eta(x, t)}.$$

Using this operator, the kinematic boundary condition becomes

$$(3.21) \quad \partial_t \eta = G(\eta) \xi.$$

The Hamiltonian of the system is given by

$$(3.22) \quad H(\eta, \xi) = \frac{1}{2} \int (\xi G(\eta) \xi + g \eta^2) dx,$$

which represents the total energy of the fluid.

The equations of motion can be written in canonical Hamiltonian form as

$$(3.23) \quad \partial_t \eta = \frac{\delta H}{\delta \xi}, \quad \partial_t \xi = -\frac{\delta H}{\delta \eta}.$$

Computing the variational derivatives yields

$$(3.24) \quad \frac{\delta H}{\delta \xi} = G(\eta)\xi,$$

$$(3.25) \quad \frac{\delta H}{\delta \eta} = g\eta + \frac{1}{2}|\nabla \phi|^2 \Big|_{y=\eta(x,t)}.$$

Thus, the evolution equations become

$$(3.26) \quad \partial_t \eta = G(\eta)\xi,$$

$$(3.27) \quad \partial_t \xi = -g\eta - \frac{1}{2}|\nabla \phi|^2 \Big|_{y=\eta(x,t)},$$

which is Zakharov's Hamiltonian formulation of the water wave problem.

4. DIRICHLET-NEUMANN OPERATOR AND ITS ASYMPTOTIC EXPANSION

The Dirichlet–Neumann operator $G(\eta)$ plays a central role in the Hamiltonian formulation of water waves. It maps the surface potential $\xi(x, t)$ to the normal derivative of the velocity potential at the free surface.

Let $\phi(x, y, t)$ solve Laplace's equation in the fluid domain

$$(4.1) \quad \Delta \phi = 0, \quad -h < y < \eta(x, t),$$

with bottom boundary condition

$$(4.2) \quad \partial_y \phi(x, -h, t) = 0,$$

and Dirichlet condition

$$(4.3) \quad \phi(x, \eta(x, t), t) = \xi(x, t).$$

The Dirichlet–Neumann operator is defined by

$$(4.4) \quad G(\eta)\xi = (\partial_y \phi - \partial_x \eta \partial_x \phi) \Big|_{y=\eta(x,t)}.$$

4.1. Linear Dirichlet–Neumann Operator. To determine the leading-order operator, we consider the flat-surface case $\eta = 0$. For a Fourier mode e^{ikx} , the harmonic extension satisfying the bottom condition is

$$(4.5) \quad \phi_k(x, y) = \frac{\cosh((y+h)|k|)}{\cosh(h|k|)} \widehat{\xi}(k) e^{ikx}.$$

Evaluating at $y = 0$, we obtain

$$(4.6) \quad \partial_y \phi_k(x, 0) = |k| \tanh(h|k|) \widehat{\xi}(k) e^{ikx}.$$

Thus, the linear Dirichlet–Neumann operator is the Fourier multiplier

$$(4.7) \quad G_0 = |D| \tanh(h|D|),$$

where $|D|$ has symbol $|k|$.

In the deep-water limit $h \rightarrow \infty$, this simplifies to

$$(4.8) \quad G_0 = |D|.$$

4.2. Asymptotic Expansion of $G(\eta)$. For small free-surface elevation η , the operator admits the expansion

$$(4.9) \quad G(\eta) = G_0 + G_1(\eta) + G_2(\eta) + \mathcal{O}(\eta^3).$$

The first-order term is

$$(4.10) \quad G_1(\eta)\xi = -G_0(\eta G_0\xi) - \partial_x(\eta \partial_x \xi).$$

The second-order term can be written as

$$(4.11) \quad G_2(\eta)\xi = -\frac{1}{2} (G_0(\eta^2 G_0^2 \xi) + \partial_x^2(\eta^2 G_0 \xi) - 2G_0(\eta G_0(\eta G_0 \xi))).$$

These expressions describe how the nonlocal operator depends on the free surface.

4.3. Expansion of the Hamiltonian. The Hamiltonian of the water wave system is

$$(4.12) \quad H(\eta, \xi) = \frac{1}{2} \int (g\eta^2 + \xi G(\eta)\xi) dx.$$

Substituting the expansion of $G(\eta)$ yields

$$(4.13) \quad H = H^{(2)} + H^{(3)} + H^{(4)} + \dots$$

The quadratic part is

$$(4.14) \quad H^{(2)} = \frac{1}{2} \int (g\eta^2 + \xi G_0 \xi) dx,$$

which corresponds to linear wave energy.

The cubic term is

$$(4.15) \quad H^{(3)} = \frac{1}{2} \int \eta ((\partial_x \xi)^2 - (G_0 \xi)^2) dx,$$

representing nonlinear wave interactions.

Higher-order terms describe more complex nonlinear effects.

4.4. Remark. The expansion of the Dirichlet-Neumann operator allows the free-boundary problem to be reduced to a Hamiltonian system involving only surface variables. Truncating this expansion at different orders leads to asymptotic models such as the Korteweg-de Vries and nonlinear Schrödinger equations.

5. HAMILTONIAN EXPANSION FOR WEAKLY NONLINEAR WATER WAVES

The asymptotic expansion of the Hamiltonian developed in this section is based on the Dirichlet-Neumann operator expansion introduced in Section 4. We consider the Hamiltonian

$$(5.1) \quad H(\eta, \xi) = \frac{1}{2} \int (g\eta^2 + \xi G(\eta)\xi) dx,$$

and expand it for small free-surface elevation η .

Using the expansion

$$(5.2) \quad G(\eta) = G_0 + G_1(\eta) + G_2(\eta) + \dots,$$

we obtain

$$(5.3) \quad H = H^{(2)} + H^{(3)} + H^{(4)} + \dots$$

5.1. Quadratic Hamiltonian. The leading-order term is

$$(5.4) \quad H^{(2)} = \frac{1}{2} \int (g\eta^2 + \xi G_0 \xi) dx,$$

which describes linear wave motion.

5.2. Cubic Hamiltonian. The first nonlinear correction is

$$(5.5) \quad H^{(3)} = \frac{1}{2} \int \xi G_1(\eta) \xi \, dx,$$

where

$$(5.6) \quad G_1(\eta) \xi = -|D|(\eta|D|\xi) - \partial_x(\eta \partial_x \xi).$$

Substituting and integrating by parts, we obtain

$$(5.7) \quad H^{(3)} = \frac{1}{2} \int \eta ((\partial_x \xi)^2 - (|D|\xi)^2) \, dx.$$

This term describes three-wave interactions.

5.3. Quartic Hamiltonian. The next-order contribution is

$$(5.8) \quad H^{(4)} = \frac{1}{2} \int \xi G_2(\eta) \xi \, dx,$$

which accounts for higher-order nonlinear effects. Although its explicit form is more involved, it consists of quadratic combinations of η and derivatives of ξ .

5.4. Remark. This expansion provides the basis for weakly nonlinear wave models. Truncating the Hamiltonian at higher orders leads to more precise asymptotic equations.

6. NUMERICAL SIMULATION AND STRUCTURE-PRESERVING SCHEME

To simulate the water wave system, we use a Fourier pseudo-spectral method. This choice is motivated by the presence of the nonlocal operator $|D|$, which is most naturally handled in Fourier space.

The operator $|D|$ is defined through its Fourier symbol:

$$(6.1) \quad \widehat{|D|f}(k) = |k| \hat{f}(k).$$

which makes it convenient to compute numerically using the Fast Fourier Transform (FFT). In practice, the computation proceeds as follows:

- (1) Compute the Fourier transform $\hat{f}(k)$,
- (2) Multiply by $|k|$,
- (3) Apply the inverse Fourier transform.

We work on a finite interval $x \in [-L, L]$ and impose periodic boundary conditions. The system is then evolved in time in its Hamiltonian form.

6.1. Numerical Evolution of the Surface. We simulate the evolution of an initial Gaussian profile

$$\eta_0(x) = 0.1e^{-x^2}.$$

Figure 2 shows the comparison between the initial profile $\eta_0(x)$ and the numerical solution $\eta(x, T)$ at a later time T .

The solution spreads and decreases in amplitude over time, illustrating the dispersive nature of the model.

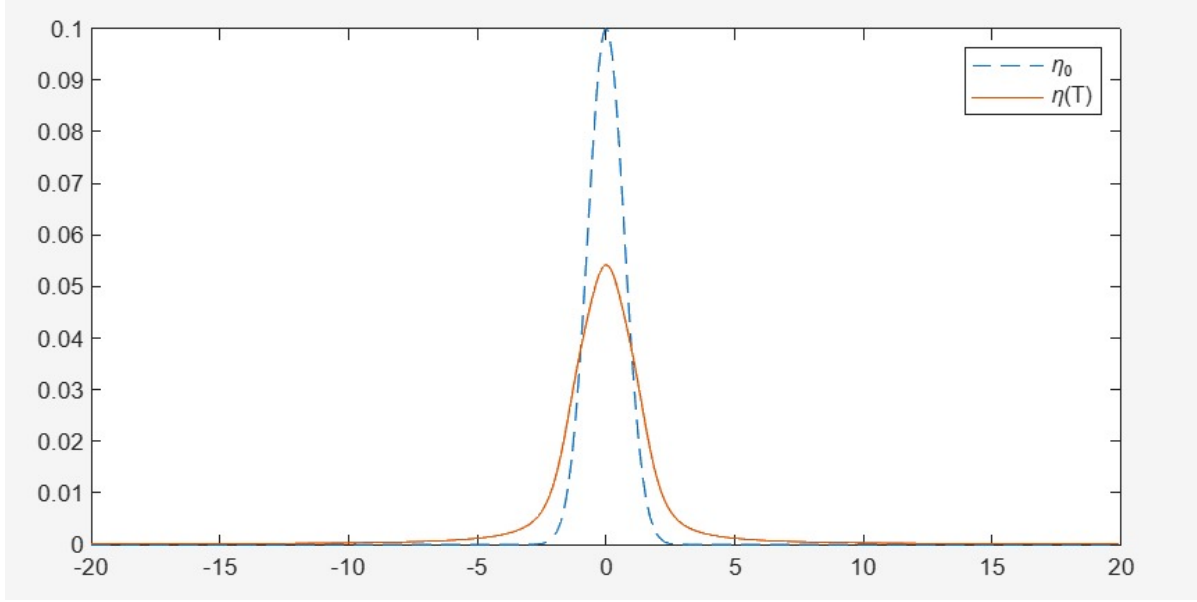


FIGURE 2. Numerical evolution of the surface elevation $\eta(x, t)$ computed using a Fourier pseudo-spectral method. The dashed curve represents the initial condition $\eta_0(x)$, while the solid curve shows the solution $\eta(x, T)$ at time T . The initially localized Gaussian decreases in amplitude and spreads due to dispersive effects generated by the nonlocal operator $|D|$.

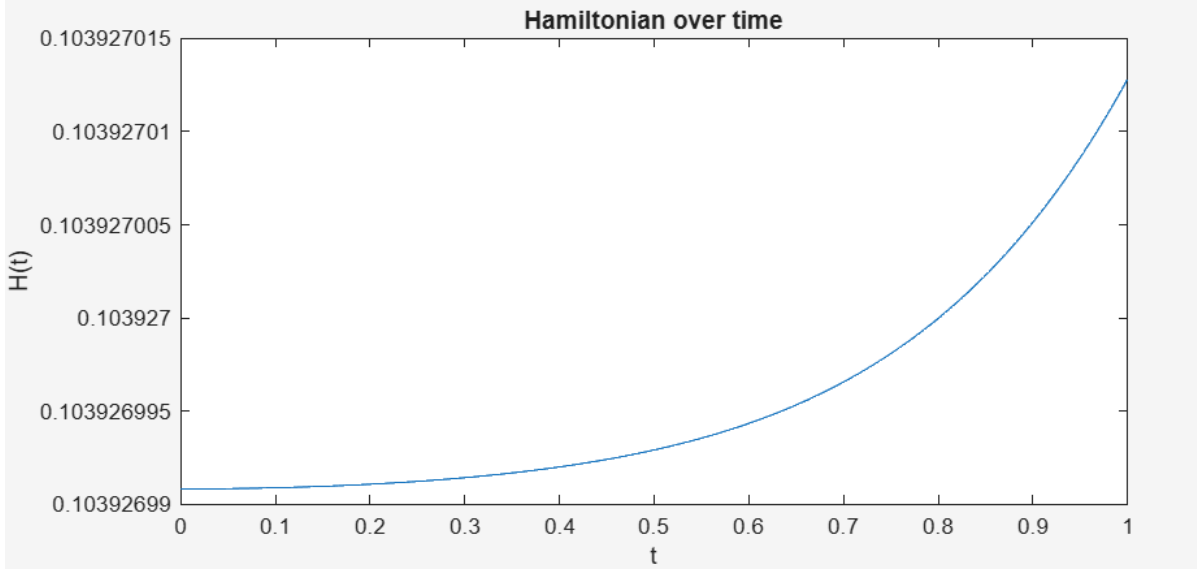


FIGURE 3. Time evolution of the truncated Hamiltonian $H(t)$. The Hamiltonian remains nearly constant over time, with only a small drift caused by time discretization.

6.2. Hamiltonian Conservation. To verify the accuracy of the numerical method, we monitor the truncated Hamiltonian

$$(6.2) \quad \tilde{H}(\eta, \xi) = \frac{1}{2} \int (g\eta^2 + \xi|D|\xi + \eta(\xi_x)^2 - \eta(|D|\xi)^2) dx.$$

Figure 3 shows the time evolution of the Hamiltonian.

The Hamiltonian remains approximately constant, confirming that the numerical scheme preserves the underlying structure of the system.

6.3. Relative Error Analysis. To quantify the conservation error, we compute the relative Hamiltonian error.

Figure 4 shows the evolution of this error over time.

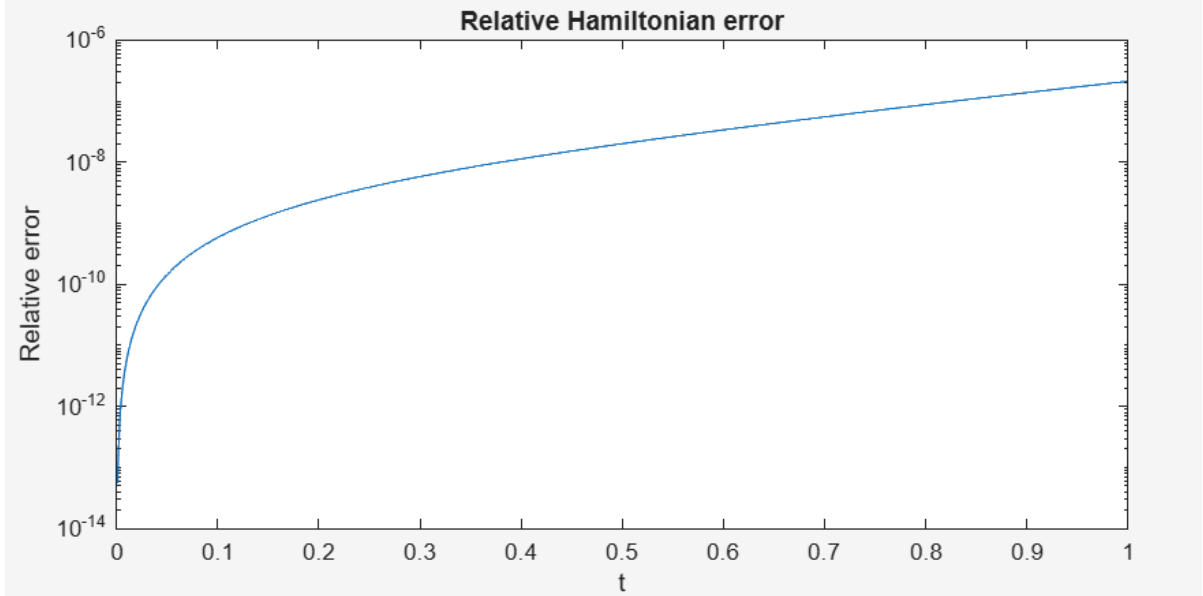


FIGURE 4. Relative error of the Hamiltonian during the numerical simulation. The error remains very small, indicating good preservation of the Hamiltonian structure.

The error stays very small throughout the simulation, which indicates that the Fourier pseudo-spectral method is able to capture the dynamics with good accuracy.

6.4. Comparison of Initial and Final Profiles. Figure 5 shows a comparison between the initial and final surface elevations over the entire domain.

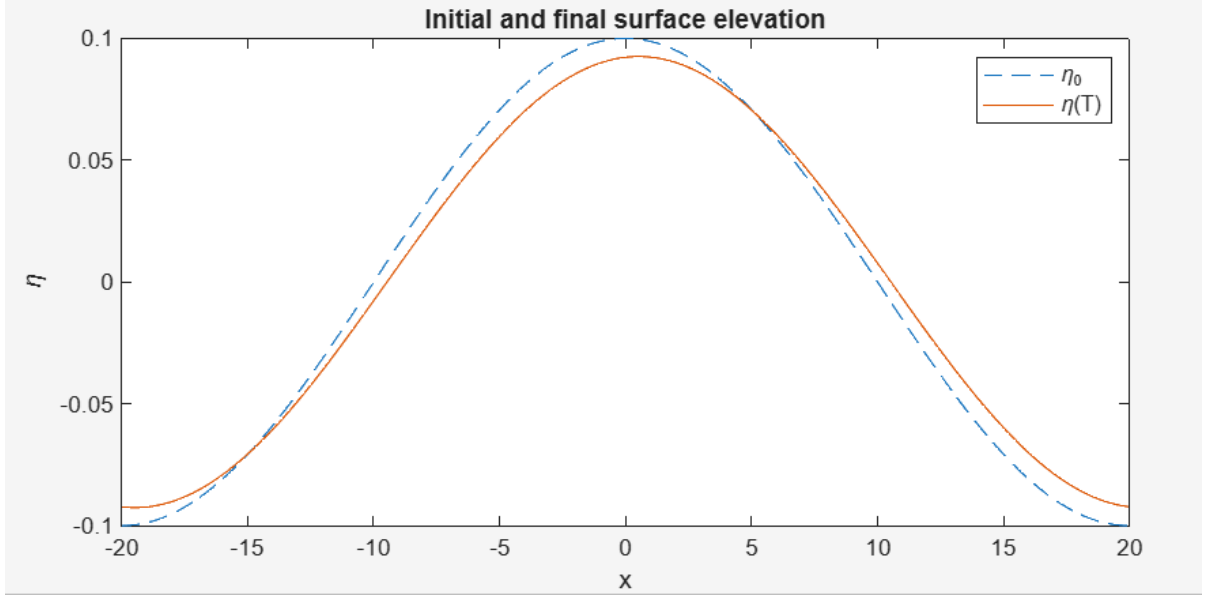


FIGURE 5. Comparison between the initial surface elevation $\eta_0(x)$ and the numerical solution $\eta(x, T)$ at time T . The solution remains smooth and periodic, with a small decrease in amplitude and some spreading, which can be attributed to nonlinear dispersive effects.

The numerical results show that the solution stays smooth and stable over time, while gradually spreading due to dispersion and weak nonlinear effects. Overall, this behavior agrees well with what is expected from the theoretical model.

7. CONCLUSION

In this project, we developed a mathematical framework for studying water waves based on the principles of fluid dynamics and Hamiltonian structure. Starting from the governing equations for incompressible, inviscid, and irrotational flow, the problem was reformulated in terms of surface variables using the Dirichlet–Neumann operator, leading to Zakharov’s Hamiltonian formulation. An asymptotic expansion of the operator allowed the derivation of reduced models describing weakly nonlinear wave dynamics. In addition, a Fourier pseudo-spectral method was implemented to simulate the evolution of the system, and the numerical results demonstrated dispersive wave behavior and approximate conservation of the Hamiltonian. As a direction for further research, one may consider higher-order asymptotic models, more advanced structure-preserving numerical schemes, or extensions to multidimensional wave problems.

APPENDIX A. NUMERICAL CODE

The numerical simulations presented in this project were implemented in MATLAB. For completeness, the main code used to compute the numerical evolution of the truncated water-wave system is included below.

```

1  % Nonlocal Hamiltonian PDE (pseudo-spectral, RK4)
2  N=1024; L=20; g=1; dt=1e-3; T=1;
3  x=linspace(-L,L,N+1)'; x(end)=[];
4  k=(pi/L)*[0:N/2 -N/2+1:-1]'; absD=abs(k); ik=1i*k;
5
6
7  eta = 0.1*exp(-x.^2);
8  xi = zeros(size(x));
9
10 rhs = @(eta,xi) deal( ...
11     real(ifft(absD.*fft(xi))) ...
12     - real(ifft(absD.*fft( eta .* real(ifft(absD.*fft(xi))) ))) ...
13     - real(ifft(ik.*fft( eta .* real(ifft(ik.*fft(xi))) ))), ...
14     -g*eta + (real(ifft(absD.*fft(xi))).^2 - (real(ifft(ik.*fft(xi))))
15         .^2 );
16
17 nsteps=round(T/dt);
18 for n=1:nsteps
19     [k1e,k1x]=rhs(eta,xi);
20     [k2e,k2x]=rhs(eta+0.5*dt*k1e, xi+0.5*dt*k1x);
21     [k3e,k3x]=rhs(eta+0.5*dt*k2e, xi+0.5*dt*k2x);
22     [k4e,k4x]=rhs(eta+dt*k3e, xi+dt*k3x);
23     eta = eta + (dt/6)*(k1e+2*k2e+2*k3e+k4e);
24     xi = xi + (dt/6)*(k1x+2*k2x+2*k3x+k4x);
25 end
26
27 % Plot results
28 eta0 = 0.1*exp(-x.^2);
29 plot(x,eta0,'--',x,eta,'-'), legend('\eta_0','\eta(T)')
30
31 % Hamiltonian computation
32 dx=2*L/N;
33 Ham = @(eta,xi) 0.5*dx*sum( ...
34     g*eta.^2 ...
35     + xi .* real(ifft(absD.*fft(xi))) ...
36     + eta .* (real(ifft(ik.*fft(xi))).^2 ...
37     - eta .* (real(ifft(absD.*fft(xi))).^2 );
38
39 Hhist=zeros(nsteps+1,1);
40 thist=zeros(nsteps+1,1);
41 Hhist(1)=Ham(eta,xi);
42
43 for n=1:nsteps
44     [k1e,k1x]=rhs(eta,xi);
45     [k2e,k2x]=rhs(eta+0.5*dt*k1e, xi+0.5*dt*k1x);
46     [k3e,k3x]=rhs(eta+0.5*dt*k2e, xi+0.5*dt*k2x);
47     [k4e,k4x]=rhs(eta+dt*k3e, xi+dt*k3x);

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```

48     eta = eta + (dt/6)*(k1e+2*k2e+2*k3e+k4e);
49     xi  = xi  + (dt/6)*(k1x+2*k2x+2*k3x+k4x);
50
51     Hhist(n+1)=Ham(eta,xi);
52     thist(n+1)=n*dt;
53 end
54
55 relerr = abs(Hhist - Hhist(1))/abs(Hhist(1));
56
57 % Plots
58 figure; plot(thist,Hhist);
59 xlabel('t'); ylabel('H(t)');
60 title('Hamiltonian over time');
61
62 figure; semilogy(thist,relerr);
63 xlabel('t'); ylabel('Relative error');
64 title('Relative Hamiltonian error');

```

LISTING 1. MATLAB code for Fourier pseudo-spectral simulation of the water-wave system

APPENDIX B. STABILITY OF AN EXPLICIT FINITE DIFFERENCE SCHEME

For comparison, we briefly recall the stability condition for a standard explicit finite difference method applied to a diffusion-type equation. Consider the model problem

$$(B.1) \quad u_t = u_{xx},$$

discretized by the forward Euler method in time and the central difference method in space:

$$(B.2) \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}.$$

This can be rewritten as

$$(B.3) \quad u_j^{n+1} = u_j^n + \lambda (u_{j+1}^n - 2u_j^n + u_{j-1}^n), \quad \lambda = \frac{\Delta t}{(\Delta x)^2}.$$

A standard von Neumann stability analysis shows that the explicit scheme is stable only if

$$(B.4) \quad \lambda \leq \frac{1}{2}.$$

This condition is important because stability guarantees that small numerical errors do not grow uncontrollably during the iteration. If the condition is violated, the computed solution may become oscillatory or blow up, even when the exact solution remains smooth.

In the present project, the main solver is based on a Fourier pseudo-spectral method, which is more suitable for the nonlocal operator $|D|$. Nevertheless, the explicit finite difference stability condition provides a useful benchmark illustrating the role of timestep restrictions in numerical PDE methods.

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