

Equation-Based Modeling of European Options under the Brownian and CGMY Processes by Using the COMSOL Multiphysics® Software

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Abstract

The Brownian process-based Black-Scholes (BS) PDE model and the nonlinear PDE developed by H.E. Leland, which includes transaction costs are first evaluated numerically by the COMSOL Multiphysics® software. Then, the corresponding partial integro-differential equation (PIDE) models based on Levy process are modeled numerically. The numerical solutions of all four models are compared favorably to those in the literature obtained by the finite difference methods. Unlike the Brownian motion, the Lévy motion allows for jumps and discontinuous paths that might better reflect the observed jump behavior for some assets. Some technical modifications and transformations are introduced to the PDEs and the PIDEs so that the resulting systems of equations are suitable for application for the equation-based modeling framework in the COMSOL Multiphysics® software. The method developed in this work allows us to take the advantage of the prowess of the software and it provides a novel and an efficient procedure for evaluation of the option pricing models.

1 Introduction

During the past century, the global financial market experienced a significant shift driven by the pursuit of more accurate options pricing models. One of the financial tools that has been subjected to the scrutiny of many researchers could be options pricing models. An option is a financial security that gives the holder a right, but not the obligation, to buy or sell the underlying asset given specific conditions for expiration time (T) and the agreed price of the asset (strike price K). There are two main option types: “European” option exercised at the time of expiration; whilst “American” option can be exercised by the time of expiration.

Although pricing options has already become a common practice in the current financial market, it was not until the innovative option pricing model developed by Fischer Black and Myron Scholes in 1973 [2], which has become the milestone in derivative pricing and won the Nobel price in 1997 because of its success. However, the model is not without drawbacks. First of all, the Black-Scholes theory assumes Brownian Motion and lognormal distribution of stock asset prices. The first problem is that this assumption may not always hold, since the asset might exhibit sudden jumps in prices and heavy tails in distribution. Second problem is that the model assumes no transaction costs, which does not reflect real-world practice.

In order to solve the first problem, the standard linear Black-Scholes PDE is modified by incorporating the model developed by Carr, Geman, Madan, and Yor (CGMY) [3]. Unlike the standard Geometric Brownian Motion (GBM), CGMY motion allows for infinite amount of jumps, making it a pure jump process. This process not only captures the real-world asset movement behavior, but also allows a better fit to volatility implied by the asset itself. This project specifically investigates the CGMY process described in terms of its partial integro-differential equation (PIDE). In order to solve the second problem, BS equation is altered according to the Leland model (1985) [9], which includes the transaction costs with the condition of discrete rebalancing instead of continuous.

Asset returns modeled by jump processes were the subject of study for many scholars. Madan and Senata [11] and Madan, Carr, and Chang [10] studied variance gamma model with finite-activity pure jump process, while Geman [8], Cont and Voltchkova [6] studied CGMY model with infinite-activity. For our study, specifically, we will work on the infinite-activity case with infinite variation. To address the second drawback, H. E. Leland [9] developed a nonlinear PDE similar to the classical linear BS equation, in which the constant volatility σ is replaced by an adjusted variance $\hat{\sigma}$. The foremost goal of this project is show that it is possible to use the commonly available software COMSOL Multiphysics® software to numerically solve the four option pricing models efficiently. The most challenging part is to deal with the integral term in the PIDE equation. Integral term has an infinite density near the origin, which poses the major problem in its approximation. In the literature, Wang, Wan and Forsyth [12] transformed the integral term by splitting integration region into 3 parts: infinite density domain, singular domain and smooth integrable domain. Another paper by Almendral and Oosterlee [1] handled the discretization of integral by transforming it into singular Volterra equation using integration by parts. Numerical approximations of their models were mostly based on mesh-dependent finite-difference method. Compared to these two papers, our approach is to convert the PIDE into systems of equations by change of variables, and use Taylor expansion on the option price after jump, thus, revealing an interesting and novel approach to handle the integral term, before meshing the domain by FDM or FEM.

It is important to note that COMSOL Multiphysics was not extensively used in finance literature for the purposes of numerical evaluations of option pricing

models. The only available official literature was an article in the COMSOL Manual, where the linear BS equation was solved as an example for possible use of program in finance [5]. It is necessary to change the original PDEs and PIDEs into systems of equations to match the input format in the COMSOL module, since COMSOL Multiphysics cannot solve the original PDE or PIDE equations directly. The modified models allow us to capitalize on the software capabilities.

The paper is structured as follows. In Section 2, we introduce the original linear Black-Scholes PDE, the nonlinear Leland PDE, and then the CGMY PIDE, as well as the CGMY PIDE with transaction costs. After we introduce all the equations that are planned to be studied in this project, we perform transformations on all the models in Section 3. In Section 4, a technique based on Taylor expansion is introduced for the integral term in the PIDE. In the last Section, we present the numerical solutions of the BS equation along with the transformed models with comparisons to the solutions in literature.

2 Option Pricing Models

In this section, we introduce the original equations of linear Black-Scholes and nonlinear Leland in which the underlying assets are governed by the Brownian motion. We also introduce our main equation of interest which is the PIDE Levy model with an underlying asset that assumes the CGMY process.

2.1 Linear Black-Scholes equation

The classical BS equation for finding the fair price of the European call option with strike price K and mature time at T is given as follows [2]:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (1)$$

where S is the price of the underlying asset, t is the time between $[0, T]$, r is the constant risk-free rate, σ is the volatility of the stock, and most importantly V is the price of the option. The solution of this PDE yields the fair price of the option given the terminal time and boundary conditions below:

$$V(S, T) = \max(S - K, 0), \quad S \geq 0$$

$$V(0, t) = 0, \quad t \geq 0$$

$$V(S, t) = S - K, \quad S \rightarrow \infty$$

Assumptions that were held in the derivation of the BS equation in paper by Black and Scholes [2] are “idealistic”, hence do not necessarily capture the real market behavior that is swayed by disturbances ranging from the transaction costs, varying volatility, behavioral factors, and information asymmetry. In pursuit of improving the model to account for as many real-world factors as possible, researchers have tried to modify the classic BS based on more realistic

assumptions. Thus, scholars developed nonlinear models such that of Leland [9], which will be discussed next.

2.2 The Leland model

In his paper, Leland [9] proposed to allow for a discrete time rebalancing instead of continuous one. Using the δt time-lag between transactions for a European call option with strike price K and expiration time T , he derived the following nonlinear PDE model:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\hat{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (2)$$

where the variance is adjusted as follows:

$$\hat{\sigma}^2 = \sigma^2(1 + Le \cdot \text{sign}(V_{SS})) \quad (3)$$

Here, σ^2 is the constant historical volatility, as used in the linear classical PDE model, and Le is the Leland number: $Le = \sqrt{\frac{2}{\pi}} \frac{c}{\sigma\sqrt{\delta t}}$ with c representing the round tip of transaction costs per dollar. The fact that we have to account for $\text{sign}(V_{SS})$ makes the PDE nonlinear, and no analytic solution is available, and hence looking for its numerical solution becomes more practical.

The corresponding time and boundary conditions are:

$$\begin{aligned} V(S, T) &= \max(S - K, 0), \quad S \geq 0 \\ V(0, t) &= 0, \quad t \geq 0 \\ V(S, t) &= S - K, \quad S \rightarrow \infty \end{aligned}$$

What is important to note is the fact that V_{SS} is positive for the original Black-Scholes model [2]. Thus, if this condition holds, we can rewrite the variance formula (3) as the absolute value of V_{SS} .

$$\tilde{\sigma}^2 = \sigma^2(1 + Le|V_{SS}|) \quad (4)$$

2.3 The CGMY model

In the paper by Carr, Geman, Madan, and Yor (CGMY) [3], CGMY process can be described as an infinite activity pure jump process, which can capture even the smallest fluctuations of the asset in the marketplace. It can exhibit heavier tails, skewness, and kurtosis, as opposed to the linear BS PDE which assumes Brownian process with a lognormal distribution for asset prices, leading to thinner tails and a lower peak. It is important to note that the CGMY model does not rely on diffusion component; thus, in fact, the model represents the real-market behavior. As the empirical results indicate the the market procing index dynamics do not have a diffusion component [3].

The CGMY process is determined by the parameters C, G, M, and Y. Parameter C controls the overall intensity of jumps, G and M dictates the decay

rate of the left and right tail respectively, and $Y < 2$ characterizes the behavior of Levy density in the neighborhood of zero. If $Y < 0$, the measure ν is integrable, leading to a process with finite activity. When $Y \in [0, 1]$, the process exhibits infinite activity but retains finite variation since $\int_{|y| < 1} y\nu(y)dy < \infty$. For $Y \in (1, 2)$, the process not only has infinite activity but also infinite variation.

There are several ways to define the CGMY model, including PIDE, Fractional PDE, and SDE. In this study, we will only investigate the PIDE form that is given as:

$$V_\tau = \frac{\sigma^2}{2}S^2V_{SS} + rSV_S - rV + \int_{\mathbb{R}} \nu(y) [V(Se^y, \tau) - V(S, \tau) - S(e^y - 1)V_S] dy \quad (5)$$

where $\tau = T - t$ is a change of variables which makes the numerical solving the problem easier. The Lévy measure of the CGMY process is defined as

$$\nu(y) = \begin{cases} C \frac{e^{-G|y|}}{|y|^{1+Y}} & \text{for } y < 0, \\ C \frac{e^{-M|y|}}{|y|^{1+Y}} & \text{for } y > 0. \end{cases} \quad (6)$$

The kernel function $\nu(y)$ has specific conditions for convergence as follows:

$$\int_{|y| < 1} y^2 \nu(y) dy < \infty, \quad \int_{|y| \geq 1} \nu(y) dy < \infty.$$

The terminal time and boundary conditions for the model with a European call option is

$$\begin{aligned} V(S, T) &= \max(S - K, 0), \quad S \geq 0 \\ V_\tau(0, \tau) &= -rV, \quad t \geq 0, \quad \tau = T - t \\ V_\tau(S, \tau) &= \frac{\sigma^2}{2}S^2V_{SS} + rSV_S - rV, \quad S \rightarrow \infty \end{aligned}$$

2.4 CGMY Model with Transaction Costs

One of this project's novelty is to fuse the Leland model with the CGMY model. In other words, we introduce transaction costs into the PIDE equation by modifying the variance.

$$V_\tau = \frac{\hat{\sigma}^2}{2}S^2V_{SS} + rSV_S - rV + \int_{\mathbb{R}} \nu(y) [V(Se^y, \tau) - V(S, \tau) - S(e^y - 1)V_S] dy \quad (7)$$

Here $\hat{\sigma}$ (4) represents the variance formula which incorporates the transaction cost per dollar c introduced in the Leland Model subsection. In the next section, we will modify all the nonlinear models presented here into coupled equations.

3 The Transformed Models

Although COMSOL's equation-based module can solve the linear BS equation directly, it presents difficulties for nonlinear models. It is difficult to approximate the Leland nonlinear PDE due to the absolute value of the second derivative with respect to S $|V_{SS}|$ in the PDE. As for the module to solve the PIDEs, additional obstacles occur due to singularity of the kernel $\nu(y)$ and also due to the appearance of both Se^y and S as the first variable in V in the integrand.

In this section, we first present the transformed versions of the models introduced in the previous section except for the Linear BS.

3.1 Leland model

The following change of variables is performed in order to transform the equation (2) from a terminal-boundary problem to a initial-boundary value problem as in the paper by Wei, Erlangga and Zhumakhanova[13].

$$\begin{aligned}\tau &= \frac{1}{2}\sigma^2(T-t) \\ x &= \ln(S) + k\tau \\ u(x, \tau) &= e^{k\tau}V(S, t)\end{aligned}$$

Here, $k = \frac{2r}{\sigma^2}$. The corresponding derivatives are as follows:

$$\begin{aligned}V_t &= \frac{\sigma^2}{2}e^{-k\tau}(-ku_x + ku - u_\tau), \\ V_S &= \frac{1}{S}e^{-k\tau}u_x, \\ V_{SS} &= \frac{1}{S^2}e^{-k\tau}(u_{xx} - u_x).\end{aligned}$$

As the result of change of variables we arrive at the modified coupled equation below which has two dependent variables u and v :

$$\begin{aligned}u_\tau &= v + Le|v|, \\ v &= u_{xx} - u_x.\end{aligned}\tag{8}$$

Accordingly, the time and boundary conditions change in order to align with the change of variables.

$$\begin{aligned}u(x, 0) &= \max(e^x - K, 0) \\ u(x, \tau) &= 0 \quad \text{as } x \rightarrow -\infty \\ u(x, \tau) &= e^x - K \quad \text{as } x \rightarrow \infty.\end{aligned}$$

Here, the coupled equation (8) is the final model that is simulated in the COMSOL software.

3.2 The CGMY model

In order to transform model (5), we perform change of variables:

$$\begin{aligned}x &= \log S, \\ \tau &= T - t,\end{aligned}$$

where $S = e^x$ and $t = T - \tau$. The option price value function transformation is given by:

$$V(S, \tau) = u(x, \tau),$$

Taking derivatives with respect to τ and S :

$$\begin{aligned}V_\tau(S, \tau) &= u_\tau(x, \tau), \\ V_S(S, \tau) &= u_x(x, \tau) \cdot \frac{\partial x}{\partial S}, \\ &= [u_x(x, \tau)] \cdot \frac{1}{S}, \\ V_{SS}(S, \tau) &= \frac{1}{S^2} [u_{xx} - u_x].\end{aligned}$$

The PIDE now takes the new form:

$$\begin{aligned}u_\tau &= \frac{\sigma^2}{2} [u_{xx} - u_x] + ru_x - ru \\ &\quad + \int \nu(y) [V(Se^y, \tau) - u - u_x(e^y - 1)] dy\end{aligned}$$

Defining auxiliary functions:

$$\begin{aligned}f &= u_{xx} - u_x, \\ h &= u_x - u.\end{aligned}$$

Rewriting the equation yields:

$$u_\tau = \frac{\sigma^2}{2} f + rh + \int \nu(y) [V(Se^y, \tau) - h - u_x e^y] dy \quad (9)$$

Boundary conditions for the transformed model is given as follows.

$$\begin{aligned}u(x, 0) &= \max(e^x - K, 0), \quad x \geq 0 \\ u_\tau(0, \tau) &= -ru, \quad t \geq 0, \quad \tau = T - t \\ u_\tau(x, \tau) &= \frac{\sigma^2}{2} u_{xx} + rh, \quad x \rightarrow 0\end{aligned}$$

The integral term in (9) still presents difficulty in its evaluation, because the term $V(Se^y, \tau)$ and $V(S, \tau)$ in the integrand. In the next Section, we will present a modification of the integrand and provide a solution to remove this specific obstacle.

4 Modification of the Integral Term in the PIDE

In this section, we deal with the integral term in (9) and specifically we perform a Taylor expansion on the option price after the jump about $y = 0$.

4.1 CGMY model

Whilst the equation (9) developed in the previous section could be used for the numerical approximation, the term $V(Se^y, \tau)$ is complicated to handle since it is different from $V(S, \tau)$. It is, indeed, the option price function after the jump. Thus, we can perform a Taylor expansion around $y = 0$, where kernel function actually tends to infinity. Recall that the integral term is given as

$$\int_{\mathbb{R}} \nu(y) [V(Se^y, \tau) - V(S, \tau) - S(e^y - 1)V_S] dy$$

We will take a finite computational domain instead of \mathbb{R} , thus we have an interval $[y_{min}, y_{max}]$. Then in order to approximate $V(Se^y, \tau)$, we perform a Taylor expansion around $y = 0$, for fixed S and τ . Let $g(y) = V(Se^y, \tau)$, then

$$g(y) = V(S, \tau) + SV_S(S, \tau)y + \frac{1}{2}S^2V_{SS}(S, \tau)y^2 + \mathcal{O}(y^3)$$

It's also important to remark that COMSOL can take up to the second derivative, which is the reason we did not include the third or higher derivatives in the transformed model. Then inserting this Taylor expansion of $g(y)$ and e^y into the integral yields:

$$\int_{y_{min}}^{y_{max}} \nu(y) \left[\frac{1}{2}y^2(S^2V_{SS} - SV_S) + \mathcal{O}(y^3) \right] dy$$

After we account for the the change of variables and the approximation of g , we arrive at the final transformed equation:

$$\begin{aligned} u_\tau &= \frac{\sigma^2}{2}f + rh + \int_{y_{min}}^{y_{max}} \nu(y) \left[\frac{1}{2}y^2f + \mathcal{O}(y^3) \right] dy \\ u_{xx} - u_x &= f \\ u_x - u &= h \end{aligned} \tag{10}$$

The transformed PIDE model (10) is now suitable for application in the COMSOL module for numerical solutions, which will be shown in the final Section of the paper. Next, we will deal with the PIDE with transaction cost.

4.2 CGMY model with transaction costs

In order to introduce transaction costs into the model (10), we use the same adjusted variance given by Leland as in (4). Using the same technics in the

previous Section, we then have the following system of PIDEs as our new model

$$\begin{aligned}
 u_\tau &= \frac{\sigma^2}{2} f + \frac{\sigma^2}{2} Le|f| + rh + \int_{y_{min}}^{y_{max}} \nu(y) \left[\frac{1}{2} y^2 (f) + \mathcal{O}(y^3) \right] dy \\
 u_{xx} - u_x &= f \\
 u_x - u &= h
 \end{aligned}
 \tag{11}$$

Using these transformed systems of PDE and PIDE models developed, we are able now to solve them numerically by using the COMSOL module as described earlier. The solution procedures and numerical solutions of the transformed models will be obtained by using COMSOL and then backward transformations will be used to obtain the numerical solutions of the original PDE and PIDEs as shown in the next Section.

5 Numerical Results Using COMSOL

In this section, we present the numerical results of the equations (1), (8), (10), and (11) which were computed using the software COMSOL Multiphysics 5.4. Although COMSOL has its own advantages, like user-friendly simulation options for different types of PDEs, it can be a little inconvenient because of the fixed input of equation types as of compared to writing codes using MatLab or Python. However, this inconvenience is negligible compared to writing a full FD or FEM code. The novelty of this study, is to modify the original PDE and PIDEs and obtain systems of PDEs and PIDEs to take advantages of the powerful numerical codes inside the COMSOL box.

The first problem faced by the software was to perform the change of variables for S , t , and V respectively. Initial plan of using *Variables* section was unsuccessful, since COMSOL has fixed arrangements for dependent variable (u) and independent variables (x, t). Thus, one way to overcome this issue was to change the *Interval* of the line component from regular price interval into its logarithmic values, thus performing the change of variables for S . Next, the time component t was time-reversed using the Time-Dependent Study section, where expiration time T and δt were defined. For example, for CGMY model with $T=0.5$ and $\Delta t=0.025$, the interval in the software was defined as $(0.5, -0.025, 0)$. Performing the computation gives us the result for the transformed u , thus the values can later be exported as a dataset and graphed in the Python.

The second problem faced by the software was actually the fact that it cannot directly solve the nonlinear PDE in its entirety. In other words, if it was not for the auxiliary function like v in (8), COMSOL would fail to evaluate the function. For that purpose as well, it was deemed appropriate to transform all the models since they are all nonlinear. On the other hand, COMSOL had no problem solving the linear Black-Scholes model without any kind of transformations.

After addressing the above mentioned issues, COMSOL was able to easily perform the computations we need. For all the models in the study, the standard settings were to use Coefficient form PDE in Physics and Time-Dependent

option in the Study. Since we have a coupled equations, values were entered using the matrix multiplication. In other words, standard Coefficient form PDE is given as follows [4]:

$$e_a \frac{\partial^2 u}{\partial t^2} + d_a \frac{\partial u}{\partial t} - \nabla \cdot (c \nabla u + \alpha u - \gamma) + \beta \cdot \nabla u + a u = f^* \quad (12)$$

where $u = (u_1, u_2)$ and $\nabla = \frac{\partial}{\partial x}$. The notation for f in COMSOL is changed to f^* in order to differentiate between the auxiliary dependent variable that was defined in the Section 3. In order to enter the mass term, e_a in the coupled system, we need to fill out the 2×2 matrix, so that after matrix multiplication, we get the terms we need. The mass term is defined as

$$e_a \frac{\partial^2 u}{\partial t^2} = \begin{bmatrix} e_{a11} & e_{a12} \\ e_{a21} & e_{a22} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial^2 u_2}{\partial t^2} \end{bmatrix} = \begin{bmatrix} e_{a11} \frac{\partial^2 u_1}{\partial t^2} + e_{a12} \frac{\partial^2 u_2}{\partial t^2} \\ e_{a21} \frac{\partial^2 u_1}{\partial t^2} + e_{a22} \frac{\partial^2 u_2}{\partial t^2} \end{bmatrix}$$

Hence, we use this matrix manual given at the COMSOL Reference book [4] to define our coupled equation.

5.1 Linear Black-Scholes model

As mentioned earlier, linear BS model can be solved directly using COMSOL. For this model the following parameters were used to evaluate a European call option:

| σ | K | T | r |
|----------|-----|-----|-----|
| 0.2 | 100 | 1 | 0.1 |

Table 1: Linear Black-Scholes input parameters

These values in Table 1 are chosen similar to the parameters in the paper by Wei, Erlangga and Zhumakhanova [13] in order to compare and check the results of the solution.

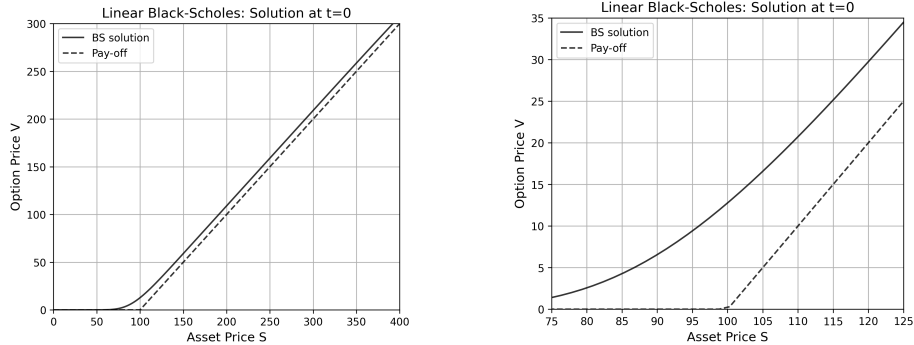


Figure 1: Left: Solid line is the numerical solution obtained by COMSOL. Right: Zoom in at the Strike price of the same figure at left

Results given in Figure 1 closely resemble the solutions of the exact solution as well as the solutions given in paper [13].

Next, we will compare nonlinear Leland model values with that of the linear BS results since they both are used to price the option with the assumption that underlying stock satisfies the lognormal distribution.

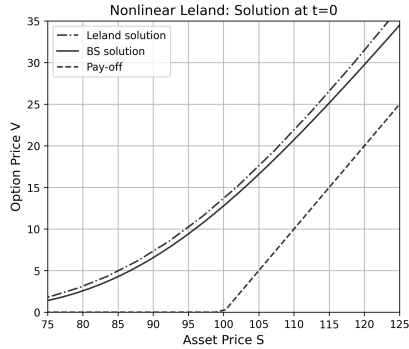
5.2 Leland model

For the numerical solutions of nonlinear Leland model, equation (8) is entered into COMSOL software first. Second step is to export the data and construct the solution using Python code. The details of the procedure are explained in Section 5. Parameters of the European call option is similar to that of the reference paper by Wei, Erlangga, and Zhumakhanova [13].

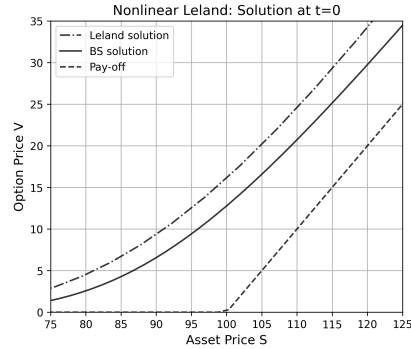
| Cases | σ | K | T | r | δt | c | Le |
|-------|----------|-----|-----|-----|------------|------|--------|
| (1) | 0.2 | 100 | 1 | 0.1 | 0.01 | 0.01 | 0.3989 |
| (2) | 0.2 | 100 | 1 | 0.1 | 0.01 | 0.02 | 0.7978 |
| (3) | 0.2 | 100 | 1 | 0.1 | 0.01 | 0.03 | 1.1967 |

Table 2: Nonlinear Leland input parameters

By choosing the transaction costs for the cases in Table 2 between 0.01-0.03, we can observe the behavior of the model to the introduction of the higher transaction costs.

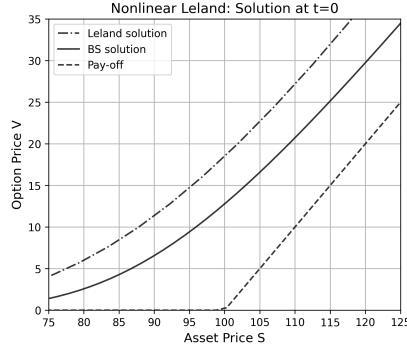


(a) Solution for $c=0.01$ and $Le=0.3989$ for case (1)



(b) Solution for $c=0.02$ and $Le=0.7978$ for case (2)

Figure 2: Solutions to nonlinear Leland coupled equation



(a) Solution for $c=0.03$ and $Le=1.1967$ for case (3)

Figure 3: Solutions to nonlinear Leland coupled equation

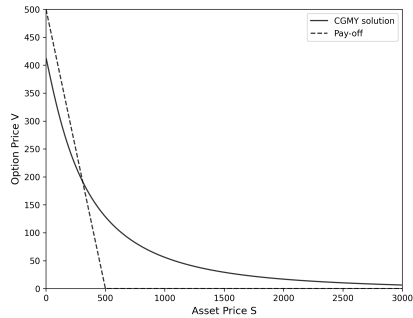
The solutions in the Figure 2 and 3 make logical sense since for the higher transaction cost c , we can observe a higher option price V as of compared to the linear BS case in which $c = 0$.

5.3 CGMY model

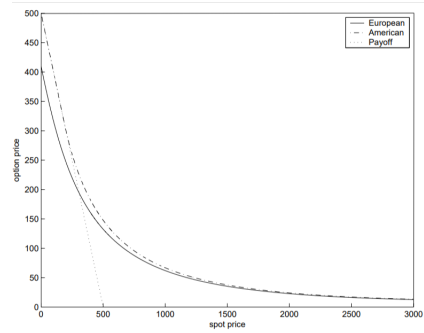
As discussed in the Section 1, CGMY model deals with the option prices of the stock price with a heavy tail distribution while the all the BS models assume lognormal distribution for the asset. Therefore, it is not logical to compare models that assume different distributions. Thus, we will compare solutions of the CGMY to the solutions in the literature. For case 1, we use the same put option parameters as in the paper by Wang, Wan, and Forsyth [12]. For case 2, we use parameters for a call option in the paper by Eberlein and Glau [7]. These parameters can be found below in Table 3.

| Cases | K | T | r | σ | C | G | M | Y |
|-------|-----|-----|------|----------|-----|-----|-----|-----|
| (1) | 500 | 0.5 | 0.4 | 0.2 | 1.0 | 1.4 | 2.5 | 1.4 |
| (2) | 100 | 0.5 | 0.02 | 0.2 | 2.0 | 0.5 | 1.2 | 1.1 |

Table 3: Nonlinear CGMY model Parameters

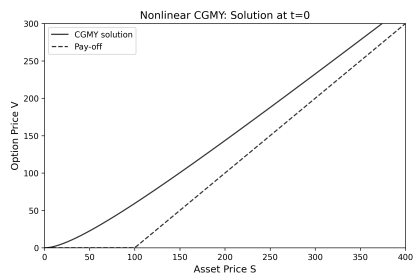


(a) Solution by COMSOL Multiphysics Software

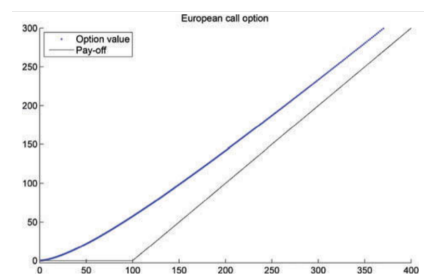


(b) Solution by Wang, Wan and Forsyth [12]

Figure 4: Solution for $C=1.0$, $G=1.4$, $M=2.5$, $Y=1.4$ for case (1)



(a) Solution by COMSOL Multiphysics



(b) Solution by Eberlein and Glau [7]

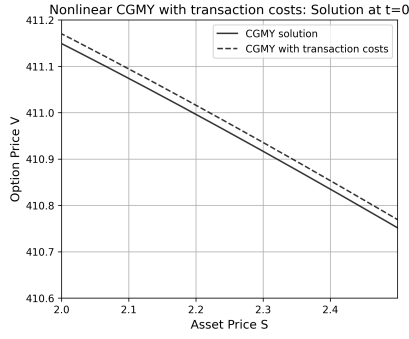
Figure 5: Solution for $C=2.0$, $G=0.5$, $M=1.2$, $Y=1.1$ for case (2)

Solutions found by the equation (10) were similar to that of in the papers [12] and [7]. As shown in Figure 4, results found by COMSOL 4a and results in reference paper 4b for put option in case (1) show close resemblance. The same is true for the Figure 5, where 5a and 5b present similar solution for call option in case (2).

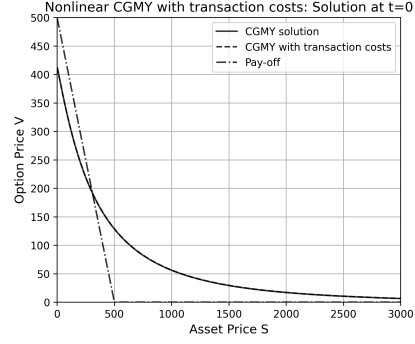
Hence, the next step will be to add transactions costs and evaluate the equation (11). In literature, after the discretization of integral term using the semi-Lagrangian method with Crank-Nicolson timestepping, Wang, Wan, and Forsyth [12] use preconditioned BiCGSTAB, an iterative algorithm. On other hand, Eberlein and Glau [7] employed Wavelet-Galerkin method to find the numerical solutions to the CGMY model. Our study employs the transformed model (10), thus allowing us to use COMSOL's capabilities for discretization and numerical approximation.

5.4 CGMY model with transaction costs

For this model, we solve equation (11) using the same parameters as in Table 3 case 1 but with the $Le = 1.1967$.



(a) Solution at a closer look with $Le=1.1967$



(b) Solution for the CGMY with $Le=1.1967$

Figure 6: Solutions to nonlinear CGMY with transaction costs coupled equation

Although the difference between the regular CGMY model and the model with transaction costs is negligible at first sight, actually there is a subtle difference of approximately 0.3-0.1 which can be seen throughout the interval. A closer look at Figure 6a indeed reveals that there is a difference, although little, in the option price.

Also, for the future reference, the input values for the CGMY model with transaction costs are presented below. The transformed model is typed in the software so that to fit in the standard Coefficient form PDE in the COMSOL, which is the equation (12).

$$d_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad c = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \beta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The values for α , e_a , γ are zero since we do not need those terms according to the transformed model. Lastly, we also have the source term f^* in the (12), which is written as:

$$f^* = \begin{bmatrix} -\left(\frac{\sigma^2}{2}(f + Le|f|) + rh + \frac{1}{2}g \int_{y_{min}}^{y_{max}} \nu(y)y^2(f)dy\right) \\ 0 \\ 0 \end{bmatrix}$$

Inputs for Leland Model as well as the CGMY model without the transaction costs are omitted, since they are trivial and can be done in the same way as the CGMY model with transaction costs.

6 Conclusion

In this project, we investigated the applicability of the COMSOL Multiphysics software for numerical approximation of the nonlinear Leland PDE based on Brownian motion as well as the nonlinear PIDE model based on CGMY process. The main novelty of the paper is in the way integral term and the option price after the jump was handled using Taylor expansion. The next key novelty is the fact that nonlinear models like Leland and CGMY were transformed into novel forms of systems of PDEs and PIDEs so as to use the software capabilities of COMSOL Multiphysics for solutions. Numerical results obtained by COMSOL compare favorably with results in the literature. Thus, this study suggests that COMSOL with its finite element method can be an effective alternative to other numerical methods such as finite difference. This study demonstrates the prospective practical use of COMSOL for financial professionals. The novel forms of the transformed models are also easier to solve by directly applying the FDM and FEM to them compared to the models in their original forms.

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